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Research article

The image of polynomials in one variable on 2×2 upper triangular matrix algebras

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Abstract: In the present paper, we give a description of the image of polynomials in one variable on 2×2 upper triangular matrix algebras over an algebraically closed field. As consequences, we give concrete descriptions of the images of polynomials of degrees up to 4 in one variable on 2×2 upper triangular matrix algebras over an algebraically closed field.

Keywords: polynomial; upper triangular matrix algebra; algebraically closed field
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1. Introduction

Let K be a field. By $K\langle x_1, \ldots, x_n\rangle$ we denote the free K-algebra generated by non-commuting indeterminate $x_1, \ldots, x_n$ and refer to the elements of $K\langle x_1, \ldots, x_n\rangle$ as polynomials. Without special explanation we always assume that every polynomial over K is a polynomial with zero constant term.

Images of polynomials evaluated on algebras play an important role in non-commutative algebras. The old and famous Lvov-Kaplansky conjecture asserts: The image of a multilinear polynomial in non-commutative variables over a field K on the matrix algebra $M_n(K)$ is a vector space (see [6] for details). The parallel topic in group theory (the images of words in groups) has been studied extensively (see [2, 17]).

In 2012, Kanel-Belov, Malev and Rowen [6] made a major breakthrough and solved the Lvov-Kaplansky conjecture for $n = 2$. In 2016, Kanel-Belov, Malev and Rowen [7] solved the Lvov-Kaplansky conjecture for $n = 3$. We remark that in [6, 7] the authors considered multilinear elements only. Some results on the Lvov-Kaplansky conjecture have been obtained in [8, 9, 12].

We remark that the images of multilinear polynomials of small degree on Lie algebras [1, 13] and Jordan algebras [11] have been discussed.

In 2019, Fagundes [3] gave a complete description of the image of multilinear polynomials on strictly upper triangular matrix algebras. In 2019, Fagundes and Mello [4] discussed the image of multilinear polynomials of degree up to four on upper triangular matrix algebras. They proposed the following variation of the Lvov-Kaplansky conjecture:

Conjecture 1.1. *The image of a multilinear polynomial over a field K on the upper trinagular matrix algebra $T_n(K)$ is always a vector space.*

In 2019, Wang [14] gave a positive answer of Conjecture 1.1 for $n = 2$. We remark that there exists a gap in the proof of [14, Theorem 1]. In 2019, Wang, Liu and Bai [15] gave a correct proof of [14, Theorem 1]. It should be mentioned that Conjecture 1.1 has been answered (see [5, 10]).

In 2021, Zhou and Wang [18] gave a complete description of the image of completely homogeneous polynomials on 2×2 upper triangular matrix algebras over an algebraically closed field. In the same year, Wang, Zhou and Luo [16] gave the Zariski topology structure of the image of polynomials on 2×2 upper triangular matrix algebras over an algebraically closed field.

In the present paper, we give a description of the image of polynomials in one variable on 2×2 upper triangular matrix algebras over an algebraically closed field. As consequences, we give concrete descriptions of the images of polynomials of degrees up to 4 in one variable on 2×2 upper triangular matrix algebras over an algebraically closed field.

2. Single elements of polynomials in one variable

Let K be a field. Let $p(x)$ be a polynomial in one variable over K. We now give the following definition, which is crucial for the proof of our main result.

Definition 2.1. *Let K be a field. Let $p(x)$ be a polynomial in one variable over K. An element $c \in K$ is said to be a single element of p if $p(x) - c$ has a simple root in K.*

By $S(p)$ we denote the set of all simple elements of p.

The following examples give complete descriptions of the set of all simple elements of polynomials of degree up to 4 in one variable. We omit the proofs of both Examples 2.1 and 2.2.

Example 2.1. *Let K be an algebraically closed field. Let*

$$p(x) = x^2 + \beta x,$$

where $\beta \in K$. Then one of the following statements holds:

(i) Suppose $char(K) \neq 2$. Then $S(p) = K \setminus \{-\frac{1}{4}\beta^2\}$;
(ii) Suppose $char(K) = 2$ and $\beta = 0$. Then $S(p) = \emptyset$;
(iii) Suppose $char(K) = 2$ and $\beta \neq 0$. Then $S(p) = K$.

Example 2.2. *Let K be an algebraically closed field. Let*

$$p(x) = x^3 + \beta_1 x^2 + \beta_2 x,$$

where $\beta_1, \beta_2 \in K$. Then one of the following statements holds:

(i) Suppose $char(K) \neq 3$ and $\beta_1^2 = 3\beta_2$. Then $S(p) = K \setminus \{-\frac{1}{27}\beta_1^3\}$;

(ii) Suppose char(K) ≠ 3 and $\beta_1^2 \neq 3\beta_2$. *Then* $S(p) = K$*;*
(iii) Suppose char(K) = 3 and $\beta_1 = \beta_2 = 0$. *Then* $S(p) = \emptyset$*;*
(iv) Suppose char(K) = 3 and either $\beta_1 \neq 0$ *or* $\beta_2 \neq 0$. *Then* $S(p) = K$.

Example 2.3. *Let K be an algebraically closed field. Let*

$$p(x) = x^4 + \beta_1 x^3 + \beta_2 x^2 + \beta_3 x,$$

where $\beta_1, \beta_2, \beta_3 \in K$. *Then one of the following statements holds:*

(i) Suppose char(K) = 2 and $\beta_1 = \beta_3 = 0$. *Then* $S(p) = \emptyset$*;*
(ii) Suppose char(K) = 2 and either $\beta_1 \neq 0$ *or* $\beta_3 \neq 0$. *Then* $S(p) = K$*;*
(iii) Suppose char(K) ≠ 2 and $\beta_1 = \beta_3 = 0$. *Then* $S(p) = \emptyset$*;*
(iv) Suppose char(K) ≠ 2, $\beta_1 = 0$ *and* $\beta_3 \neq 0$. *Then* $S(p) = K$*;*
(v) Suppose char(K) ≠ 2, $\beta_1 \neq 0$, $\beta_2 = \frac{1}{4}\beta_1^2 + 2\beta_3\beta_1^{-1}$. *Then* $S(p) = K \setminus \{-(\beta_1^{-1}\beta_3)^2\}$*;*
(vi) Suppose char(K) ≠ 2, $\beta_1 \neq 0$, $\beta_2 \neq \frac{1}{4}\beta_1^2 + 2\beta_3\beta_1^{-1}$. *Then* $S(p) = K$.

Proof. We just give the proof of (i). The other statements can be proved analogously. For $c \in K$, we set

$$f(x) = p(x) - c.$$

It is easy to check that $f(x)$ has no simple roots in K if and only if

$$f(x) = (x-\alpha)^2(x-\beta)^2, \tag{2.1}$$

where $\alpha, \beta \in K$. Expanding (2.1) and comparing the coefficients of (2.1) we obtain

$$\begin{aligned} \beta_1 &= -2(\alpha+\beta), \\ \beta_2 &= \alpha^2 + 4\alpha\beta + \beta^2, \\ \beta_3 &= -2\alpha\beta(\alpha+\beta), \\ c &= -\alpha^2\beta^2. \end{aligned} \tag{2.2}$$

Suppose first that char(K) = 2 and $\beta_1 = \beta_3 = 0$. Let $\omega_1, \omega_2 \in K$ be a solution of the following equation:

$$x^2 + \beta_2 x - c = 0.$$

It follows that

$$\begin{aligned} \omega_1 + \omega_2 &= -\beta_2, \\ \omega_1\omega_2 &= -c. \end{aligned}$$

Let $\gamma_1 \in K$ be a solution of $x^2 = \omega_1$. Let $\gamma_2 \in K$ be a solution of $x^2 = \omega_2$. We have

$$\begin{aligned} \gamma_1^2 + \gamma_2^2 &= -\beta_2, \\ \gamma_1^2\gamma_2^2 &= -c. \end{aligned}$$

It follows that

$$f(x) - c = x^4 + \beta_2 x^2 - c = (x-\gamma_1)^2(x-\gamma_2)^2.$$

In view of Definition 2.1, we get that $c \notin S(p)$. Hence $S(p) = \emptyset$. This proves (i). □

3. The main result

Set $K^* = K \setminus \{0\}$. Let $T_2(K)$ be the set of all 2×2 upper triangular matrices over K.

We give a description of the image of polynomials in one variable on 2×2 upper triangular matrix algebras over an algebraically closed field.

Theorem 3.1. *Let $d \geq 1$ be integer. Let K be an algebraically closed field. Let*

$$p(x) = \beta_d x^d + \beta_{d-1}x^{d-1} + \cdots + \beta_1 x,$$

where $\beta_i \in K$ for $i = 1, \ldots, d$ with $\beta_d \neq 0$. We have

$$p(T_2(K)) = T_2(K) \setminus \left\{ \begin{pmatrix} c & K^* \\ 0 & c \end{pmatrix} \mid c \notin S(p) \right\}.$$

In particular, $p(T_2(K))$ is not a vector space if $S(p) \neq K$.

Proof. For $\begin{pmatrix} a & m \\ & a \end{pmatrix} \in T_2(K)$, we claim that

$$p\begin{pmatrix} a & m \\ & a \end{pmatrix} = \begin{pmatrix} p(a) & p'(a)m \\ & p(a) \end{pmatrix}, \tag{3.1}$$

where $p'(x)$ is the derivation of $p(x)$. Indeed, we have

$$\begin{aligned} p\begin{pmatrix} a & m \\ & a \end{pmatrix} &= \sum_{i=1}^{d} \beta_i \begin{pmatrix} a & m \\ & a \end{pmatrix}^i \\ &= \sum_{i=1}^{d} \beta_i \begin{pmatrix} a^i & ia^{i-1}m \\ & a^i \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^{d} \beta_i a^i & \sum_{i=1}^{d} \beta_i i a^{i-1} m \\ & \sum_{i=1}^{d} \beta_i a^i \end{pmatrix} \\ &= \begin{pmatrix} p(a) & p'(a)m \\ & p(a) \end{pmatrix}. \end{aligned}$$

For $\begin{pmatrix} a & m \\ & b \end{pmatrix} \in T_2(K)$, where $a \neq b$, we claim that

$$p\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p(a) & \frac{p(a)-p(b)}{a-b} m \\ & p(b) \end{pmatrix}. \tag{3.2}$$

Indeed, we have

$$\begin{aligned}
p\begin{pmatrix} a & m \\ & b \end{pmatrix} &= \sum_{i=1}^{d}\beta_i\begin{pmatrix} a & m \\ & b \end{pmatrix}^i \\
&= \sum_{i=1}^{d}\beta_i\begin{pmatrix} a^i & \sum\limits_{s=1}^{i} a^{i-s}b^{s-1}m \\ & b^i \end{pmatrix} \\
&= \sum_{i=1}^{d}\beta_i\begin{pmatrix} a^i & \frac{a^i-b^i}{a-b}m \\ & b^i \end{pmatrix} \\
&= \begin{pmatrix} p(a) & \sum\limits_{i=1}^{d}\beta_i\frac{a^i-b^i}{a-b}m \\ & p(b) \end{pmatrix} \\
&= \begin{pmatrix} p(a) & \frac{\sum_{i=1}^{d}\beta_i a^i-\sum_{i=1}^{d}\beta_i b^i}{a-b}m \\ & p(b) \end{pmatrix} \\
&= \begin{pmatrix} p(a) & \frac{p(a)-p(b)}{a-b}m \\ & p(b) \end{pmatrix}.
\end{aligned}$$

For any $\begin{pmatrix} a' & m' \\ & b' \end{pmatrix} \in T_2(K)$, where $a' \neq b'$, we have that there exist $a, b \in K$ such that

$$p(a) = a' \quad \text{and} \quad p(b) = b'.$$

Note that $a \neq b$. Set

$$\lambda = \left(\frac{a'-b'}{a-b}\right)^{-1}.$$

Take $u = \begin{pmatrix} a & \lambda m' \\ & b \end{pmatrix}$. It follows from (3.2) that

$$\begin{aligned}
p(u) &= \begin{pmatrix} p(a) & \frac{p(a)-p(b)}{a-b}\lambda m' \\ & p(b) \end{pmatrix} \\
&= \begin{pmatrix} a' & \frac{a'-b'}{a-b}\lambda m' \\ & b' \end{pmatrix} \qquad (3.3)\\
&= \begin{pmatrix} a' & m' \\ & b' \end{pmatrix}.
\end{aligned}$$

This implies that

$$\left\{\begin{pmatrix} a' & K \\ & b' \end{pmatrix} \mid a \neq b\right\} \subseteq p(T_2(K)). \qquad (3.4)$$

For any $\begin{pmatrix} a' & 0 \\ & a' \end{pmatrix} \in T_2(K)$, we have that there exists $a \in K$ such that $p(a) = a'$. Take $u = \begin{pmatrix} a & 0 \\ & a \end{pmatrix} \in T_2(K)$. It follows from (3.1) that

$$p(u) = \begin{pmatrix} a' & 0 \\ & a' \end{pmatrix}.$$

This implies that
$$K \cdot I_2 \subseteq p(T_2(K)), \tag{3.5}$$
where I_2 is the identity matrix of $T_2(K)$. For $c \in S(p)$, we set
$$f(x) = p(x) - c.$$
In view of Definition 2.1, we have that $f(x)$ has a simple root $\omega \in K$. We have
$$f(\omega) = p(\omega) - c = 0$$
and
$$f'(\omega) = p'(\omega) \neq 0,$$
where f' and p' are the derivations of f and p, respectively. Set
$$u = \begin{pmatrix} \omega & p'(\omega)^{-1}m \\ & \omega \end{pmatrix}$$
for $m \in K$. It follows from (3.1) that
$$\begin{aligned} p(u) &= p\begin{pmatrix} \omega & p'(\omega)^{-1}m \\ & \omega \end{pmatrix} \\ &= \begin{pmatrix} p(\omega) & p'(\omega)p'(\omega)^{-1}m \\ & p(\omega) \end{pmatrix} \\ &= \begin{pmatrix} c & m \\ & c \end{pmatrix}. \end{aligned}$$
This implies that
$$\left\{\begin{pmatrix} c & K \\ & c \end{pmatrix} \mid c \in S(p)\right\} \subseteq p(T_2(K)). \tag{3.6}$$
We get from (3.4)–(3.6) that
$$T_2(K) \setminus \left\{\begin{pmatrix} c & K^* \\ 0 & c \end{pmatrix} \mid c \notin S(p)\right\} \subseteq p(T_2(K)).$$

For any $u = \begin{pmatrix} a & m \\ & a \end{pmatrix} \in T_2(K)$, we get from (3.1) that
$$p(u) = \begin{pmatrix} p(a) & p'(a)m \\ & p(a) \end{pmatrix}.$$
Suppose first $p'(a) = 0$. We have
$$p(u) \in K \cdot I_2. \tag{3.7}$$
Suppose next $p'(a) \neq 0$. Set
$$f(x) = p(x) - p(a).$$

It is clear that

$$f(a) = 0 \quad \text{and} \quad f'(a) = p'(a) \neq 0.$$

This implies that $a \in K$ is a simple root of $f(x)$.

In view of Definition 2.1, we get $p(a) \in S(p)$. We have

$$p(u) \in \left\{ \begin{pmatrix} c & K \\ & c \end{pmatrix} \mid c \in S(p) \right\}. \tag{3.8}$$

For any $u = \begin{pmatrix} a & m \\ & b \end{pmatrix} \in T_2(K)$, where $a \neq b$, we get from (3.2) that

$$p(u) = \begin{pmatrix} p(a) & \frac{p(a)-p(b)}{a-b} m \\ & p(b) \end{pmatrix}.$$

Suppose first $p(a) \neq p(b)$. We have

$$p(u) \in \left\{ \begin{pmatrix} a' & K \\ & b' \end{pmatrix} \mid a' \neq b' \in K \right\}. \tag{3.9}$$

Suppose next $p(a) = p(b)$. We have that

$$p(u) \in K \cdot I_2. \tag{3.10}$$

We get from (3.7)–(3.10) that

$$p(T_2(K)) \subseteq T_2(K) \setminus \left\{ \begin{pmatrix} c & K^* \\ 0 & c \end{pmatrix} \mid c \notin S(p) \right\}.$$

We obtain

$$p(T_2(K)) = T_2(K) \setminus \left\{ \begin{pmatrix} c & K^* \\ 0 & c \end{pmatrix} \mid c \notin S(p) \right\}.$$

Suppose $S(p) \neq K$. We claim that $p(T_2(K))$ is not a vector space.

Take $a \in K \setminus S(p)$. Suppose first $0 \in S(p)$. Take

$$\begin{pmatrix} a & 0 \\ & a \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \in p(T_2(K)).$$

We have

$$\begin{pmatrix} a & 0 \\ & a \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} = \begin{pmatrix} a & 1 \\ & a \end{pmatrix} \notin p(T_2(K)).$$

This implies that $p(T_2(K))$ is not a vector space. Suppose next $0 \notin S(p)$. Take $\begin{pmatrix} 1 & 1 \\ & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ & 0 \end{pmatrix} \in p(T_2(K))$. We have

$$\begin{pmatrix} 1 & 1 \\ & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \notin p(T_2(K)).$$

This implies that $p(T_2(K))$ is not a vector space. The proof of the result is complete. □

As a consequence, we give a concrete description of the image of polynomials of degree up to 4 in one variable on 2×2 upper triangular matrix algebras over an algebraically closed field.

Corollary 3.1. *Let K be an algebraically closed field. We have*

(1) Let $p(x) = x^2 + \beta x$, where $\beta \in K$. Then one of the following statements holds:

(i) Suppose $char(K) \neq 2$. Then

$$p(T_2(K)) = T_2(K) \setminus \begin{pmatrix} -\frac{1}{4}\beta^2 & K^* \\ & -\frac{1}{4}\beta^2 \end{pmatrix}$$

is not a vector space;

(ii) Suppose $char(K) = 2$ and $\beta = 0$. Then

$$p(T_2(K)) = T_2(K) \setminus \left\{ \begin{pmatrix} c & K^* \\ & c \end{pmatrix} \mid c \in K \right\}$$

is not a vector space;

(iii) Suppose $char(K) = 2$ and $\beta \neq 0$. Then $p(T_2(K)) = T_2(K)$.

(2) Let $p(x) = x^3 + \beta_1 x^2 + \beta_2 x$, where $\beta_1, \beta_2 \in K$. Then one of the following statements holds:

(i) Suppose $char(K) \neq 3$ and $\beta_1^2 = 3\beta_2$. Then

$$p(T_2(K)) = T_2(K) \setminus \begin{pmatrix} -\frac{1}{27}\beta_1^3 & K^* \\ & -\frac{1}{27}\beta_1^3 \end{pmatrix}$$

is not a vector space;

(ii) Suppose $char(K) \neq 3$ and $\beta_1^2 \neq 3\beta_2$. Then $p(T_2(K)) = T_2(K)$;

(iii) Suppose $char(K) = 3$ and $\beta_1 = \beta_2 = 0$. Then

$$p(T_2(K)) = T_2(K) \setminus \left\{ \begin{pmatrix} c & K^* \\ & c \end{pmatrix} \mid c \in K \right\}$$

is not a vector space;

(iv) Suppose $char(K) = 3$ and either $\beta_1 \neq 0$ or $\beta_2 \neq 0$. Then $p(T_2(K)) = T_2(K)$.

(3) Let $p(x) = x^4 + \beta_1 x^3 + \beta_2 x^2 + \beta_3 x$, where $\beta_1, \beta_2, \beta_3 \in K$. Then one of the following statements holds:

(i) Suppose $char(K) = 2$ and $\beta_1 = \beta_3 = 0$. Then

$$p(T_2(K)) = T_2(K) \setminus \left\{ \begin{pmatrix} c & K^* \\ & c \end{pmatrix} \mid c \in K \right\}$$

is not a vector space;

(ii) Suppose $char(K) = 2$ and either $\beta_1 \neq 0$ or $\beta_3 \neq 0$. Then $p(T_2(K)) = T_2(K)$;

(iii) Suppose $char(K) \neq 2$ and $\beta_1 = \beta_3 = 0$. Then

$$p(T_2(K)) = T_2(K) \setminus \left\{ \begin{pmatrix} c & K^* \\ & c \end{pmatrix} \mid c \in K \right\}$$

is not a vector space;

(iv) Suppose $char(K) \neq 2$, $\beta_1 = 0$ *and* $\beta_3 \neq 0$. *Then* $p(T_2(K)) = T_2(K)$*;*

(v) Suppose $char(K) \neq 2$, $\beta_1 \neq 0$, $\beta_2 = \frac{1}{4}\beta_1^2 + 2\beta_3\beta_1^{-1}$. *Then*

$$p(T_2(K)) = T_2(K) \setminus \begin{pmatrix} -(\beta_1^{-1}\beta_3)^2 & K^* \\ & -(\beta_1^{-1}\beta_3)^2 \end{pmatrix}$$

is not a vector space;

(vi) Suppose $char(K) \neq 2$, $\beta_1 \neq 0$, $\beta_2 \neq \frac{1}{4}\beta_1^2 + 2\beta_3\beta_1^{-1}$. *Then* $p(T_2(K)) = T_2(K)$.

Proof. The statement (1) follows from both Example 2.1 and Theorem 3.1. The statement (2) follows from both Example 2.2 and Theorem 3.1. The statement (3) follows from both Example 2.3 and Theorem 3.1. □

4. Conclusions

In this paper, we first defined the set of all single elements of polynomials in one variable. We next gave a description of the image of polynomials in one variable on 2×2 upper triangular matrix algebras over an algebraically closed field. As an application of our main result, we gave concrete descriptions of the images of polynomials of degrees up to 4 in one variable on 2×2 upper triangular matrix algebras over an algebraically closed field.

Conflict of interest

The authors declare no conflicts of interest in this paper.

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