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Research article

The structure of minimally 2-subconnected graphs

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Abstract: A graph G with at least $2k$ vertices is called k-subconnected if, for any $2k$ vertices in G, there are k independent paths $P_1, P_2, \cdots, P_k$ joining the $2k$ vertices in pairs. A graph G is minimally 2-subconnected if G is 2-subconnected and $G - e$ is not 2-subconnected for any edge e in G. The concept of k-subconnected graphs is introduced in the research of matching theory, and this concept has been found to be related with connectivity of graphs. It is of theorectical interests to characterize the structure of minimally k-subconnected graphs. In this paper, we characterize the structure of minimally 2-subconnected graphs.

Keywords: k-subconnected graph; structure of minimally 2-subconnected graphs; independent paths; number of deleted edges
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1. Introduction and terminology

In this paper, we only study undirected, connected, finite and simple graphs.

Let $G = (V, E)$ be a graph with at least $2k$ vertices. Then G is called *k-subconnected* if, for any $2k$ vertices $u_1, u_2, \ldots, u_{2k}$ in G, there are k independent paths $P_1, P_2, \ldots, P_k$ in G joining the $2k$ vertices in pairs. If G is k-subconnected and G has at least $3k-1$ vertices, then G is called *properly k-subconnected*.

Let G be a connected graph with at least one cut vertex. Let $B_1, B_2, \ldots, B_m$ be all blocks of G, and $c_1, c_2, \ldots, c_n$ be all cut vertices in G. Then the *block graph* $B(G)$ of G is such a graph $B(G) = (V, E)$ that $V = \{b_i, c_j | b_i$ corresponds to $B_i, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$ and $E = \{b_i c_j | B_i$ contains c_j, $1 \le i \le m, 1 \le j \le n\}$. By Lemma 1 in the following, $B(G)$ is a tree. If a block B of G corresponds to a leaf vertex of $B(G)$, then B is called a *leaf block* of G. The concept of the block graph of G is from [1] (see [1], p. 44). In [2], Hung studies optimal vertex ranking of block graphs, but in his paper, a block graph is a graph of which each block is a clique (complete graph). Our concept is different from his concept.

Let $G = (V, E)$ be a graph. We denote $|V|$ by ν and $|E|$ by ε. We also denote the number of the components of G by $\omega(G)$. Let $C = u_0u_1u_2 \dots u_nu_0$ be a cycle, suppose $u_0, u_1, u_2, \dots, u_n$ appear on C in turn in clockwise orientation. If $u_i, u_j \in V(C)$, we denote by $C[u_i, u_j]$ the path from u_i to u_j on C in clockwise orientation. For any vertex $u \in V(G)$, $d_G(u)$ is the degree of u in G. If H is a subgraph of G, and $u \in V(H)$, then $d_H(u)$ is the degree of u in H.

Let H be a graph and P be a (u, v) path such that $V(H) \cap V(P) = \{u, v\}$ and $E(H) \cap E(P) = \emptyset$, then $H' = H + P$ is such a graph that $V(H') = V(H) \cup V(P)$ and $E(H') = E(H) \cup E(P)$.

Let G be a connected graph, an *H-path* P of G is a path $P = u_0, u_1, \dots, u_k$ in G of length $k \geq 1$ such that $d_G(u_i) = 2$ $(i = 1, 2, \dots, k-1)$, $d_G(u_0) \geq 3$ and $d_G(u_k) \geq 3$.

A connected graph G is k-connected, if deleting any r $(0 \leq r < k)$ vertices from G, G is still connected. A graph G is called *minimally k-connected* if G is k-connected but, for any $e \in E(G)$, $G - e$ is no longer k-connected. A graph G is called *minimally k-subconnected* if G is k-subconnected but, for any $e \in E(G)$, $G - e$ is not k-subconnected.

The concept of k-subconnected graphs is introduced in the research of matching theory. In 1980, Plummer [3] introduced the concept of n-extendable graphs. A graph G with $\nu(G) \geq 2n + 2$ is called n-extendable if G has a matching of n edges, and any matching M of n edges in G is contained in a perfect matching of G. Since this concept is proposed, an extensive research has been done. Yu [4] and Faveron [5] propose a related concept k-critical (or k-factor-critical) graphs, extending the concepts of factor critical graphs and bicritical graphs in matching theory. A graph G with $\nu(G) \geq k$ and $\nu(G) \equiv k \pmod 2$ is called k-critical if, for any subset S of k vertices of $V(G)$, $G - S$ has a perfect matching. Obviously, a $2k$-critical graph is also k-extendable. Aldred, Holton, Lou and Zhong [6] characterize $2k$-critical garaphs as following: A graph G with a perfect matching M is $2k$-critical if and only if, for any $2k$ vertices $u_1, u_2, \cdots, u_{2k}$ in G, there are k independent M-alternating paths $P_1, P_2, \cdots, P_k$ starting and ending with edges in M, joining the $2k$ vertices in pairs. To design an efficient algorithm to determine $2k$-criticality of G, we shall find the largest number k of the M-alternating paths in G. This problem is still unsolved. As a model to study this problem, Qin, Lou, Zhu and Liang [7] introduce the concept of k-subconnected graphs to study k normal paths connecting any given $2k$ vertices in G. To obtain an efficient algorithm to determine k-subconnectivity of a graph may help to design an efficient algorithm to determine $2k$-criticality of graphs.

Since the concept of k-subconnected graphs is proposed, we find its strong relation with connectivity of graphs.

Connectivity is an important property of graphs, it has been extensively studied (see [8]).

In recent years, conditioned connectivities attract researchers' attention. For example, Peroche [9] studied several sorts of connectivities, including cyclic edge (vertex) connectivity, and their relation. A *cyclic edge (vertex) cutset* S of G is an edge (vertex) cutset whose deletion disconnects G such that at least two of the components of $G - S$ contain a cycle respectively. The *cyclic edge (vertex) connectivity*, denoted by $c\lambda(G)$ $(c\kappa(G))$, is the cardinality of a minimum cyclic edge (vertex) cutset of G. Dvořák, Kára, Král and Pangrác [10] obtained the first efficient algorithm to determine the cyclic edge connectivity of cubic graphs. Lou and Wang [11] obtained the first efficient algorithm to determine the cyclic edge connectivity for k-regular graphs. Lou [12] also obtained a square time algorithm to determine the cyclic edge connectivity of planar graphs.

Another related concept is linkage. Let G be a graph with at least $2k$ vertices. If, for any $2k$ vertices $u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_k$, there are k disjoint paths P_i from u_i to v_i $(i = 1, 2, \dots, k)$ in G, then G

is called *k-linked*. Thomassen [13] mentioned that a necessary condition for G to be k-linked is that G is $(2k-1)$-connected. But this condition is not sufficient unless $k = 1$. He also gave a complete characterization of 2-linked graphs. Bollobás and Thomason [14] proved that if $\kappa(G) \geq 22k$, then G is k-linked. Kawarabayashi, Kostochka and Yu [15] proved that every $2k$-connected graph with average degree at least $12k$ is k-linked.

In [7], Qin, Lou, Zhu and Liang introduced the new concept of k-subconnected graphs as defined at the beginning of this paper, and in [16], Qin and Lou defined a properly k-subconnected graph to be a k-subconnected graph G with $\nu(G) \geq 3k-1$. In [7], Qin et. al. showed that a properly k-subconnected graph is also a properly $(k-1)$-subconnected graph. But only when $\nu(G) \geq 3k-1$, that G is k-subconnected implies that G is $(k-1)$-subconnected. Qin et al. [7] also gave a sufficient condition for a graph to be k-subconnected and a necessary and sufficient condition for a graph to be a properly k-subconnected graph (see Lemma 3 of this paper).

If G has at least $2k$ vertices, that G is k-linked implies that G is k-connected, while that G is k-connected implies that G is k-subconnected (see Lemma 4 in this paper). Also by [17], in a k-connected graph G, deleting arbitrarily $k-1$ edges from G, the resulting graph H is still k-subconnected. So a graph H to be k-subconnected is a spanning substructure of a k-connected graph G. To study k-subconnected graphs may help to know more properties in the structure of k-connected graphs. Notice that a k-connected graph may have a spanning substructure to be m-subconnected for $m > k$.

For other terminology and notation not defined in this paper, the reader is referred to [18].

2. Preliminary results

In this section, we show some known results or some immediate results on k-connected graphs or k-subconnected graphs, which will be used in the proof of our main results on the structure of minimally 2-subconnected graphs.

Lemma 1. (Proposition 3.1.1 of [1]) The block graph of a connected graph is a tree.

Lemma 2. (Proposition 3.1.2 of [1]) A graph G is 2-connected if and only if $G = C_0 + P_1 + P_2 + \ldots + P_n$ $(n \geq 0)$, where C_0 is a cycle and P_i is a path of length at least 1 connecting two different vertices of $G_{i-1} = C_0 + P_1 + P_2 + \ldots + P_{i-1}$ $(i = 1, 2, \ldots, n)$ and P_i is internally disjoint with G_{i-1}.

Lemma 3. (Theorem 3 of [7]) A graph G with $\nu(G) \geq 3k-1$ is k-subconnected if and only if, for any cutset $S \subseteq V(G)$ with $|S| \leq k-1$, $\omega(G-S) \leq |S|+1$.

Let us give examples of graphs satisfying Lemmas 1–3 respectively. Let $B_0 = a_0a_1\cdots a_5a_0$, $B_1 = a_5b_1$, $B_2 = b_0b_1b_2b_3b_0$, $B_3 = c_0c_1(= b_2)c_2c_3c_0$, $B_4 = d_0d_1(= b_3)d_2d_3d_4d_0$, then B_i is a block $(i = 0, 1, 2, 3, 4)$ of graph G and $x_1 = a_5, x_2 = b_1, x_3 = b_2$ and $x_4 = b_3$ are cut vertices of G. Notice that a_i, b_j, c_k, d_l are different vertices except the cases that we specify that they are the same vertices as above. Then $V(B(G)) = \{b_i^{'}, x_j | b_i^{'}$ corresponds to B_i, $i = 0, 1, 2, 3, 4;\ j = 1, 2, 3, 4\ \}$, $E(B(G)) = \{b_0^{'}x_1, b_1^{'}x_1, b_1^{'}x_2, b_2^{'}x_2, b_2^{'}x_3, b_2^{'}x_4, b_3^{'}x_3, b_4^{'}x_4\}$. Then $B(G)$ is a tree, satisfying Lemma 1.

Let $C_0 = a_0a_1a_2\cdots a_7a_0, P_1 = b_0b_1\cdots b_6, P_2 = c_0c_1c_2c_3, P_3 = d_0d_1d_2$, where $a_1 = b_0, a_3 = b_6, c_0 = b_1, c_3 = b_5, d_0 = b_2$ and $d_2 = b_4$, and a_i, b_j, c_k, d_l are different vertices except the cases that we specify that they are the same vertices as above. Then $G_1 = C_0 + P_1 + P_2 + P_3$ is a 2-connected graph satisfying Lemma 2. In fact, G_1 is also a minimally 2-connected graph.

Let $G_2 = v_1v_2\cdots v_n$ $(n \geq 3k-1$ and $k \geq 1)$, then G_2 is a Hamilton path and for any cutset $S \subseteq V(G_2)$ with $|S| \leq k-1, \omega(G_2 - S) \leq |S|+1$, so G_2 satisfies Lemma 3. Let $G_2^{'} = K_{n,n+1}$ with $2n+1 \geq 3k-1$

for a $k \geq 1$ be a complete bipartite graph, then $G_2^{'}$ also satisfies Lemma 3.
Lemma 4. (Lemma 6 of [16]) A k-connected graph with at least $2k$ vertices is k-subconnected.

In the following, Lemma 5 will give a necessary and sufficient condition of 1-subconnected graphs, and Lemma 6 will describe the structure of minimally 1-subconnected graphs.

Lemma 5. A graph G with $\nu(G) \geq 2$ is 1-subconnected if and only if G is connected.

Proof. By the definition of 1-subconnected graph, the result follows. ■

Lemma 6. A graph G with $\nu(G) \geq 2$ is a minimally 1-subconnected graph if and only if G is a tree.

Proof. Since G is 1-subconnected if and only if G is connected with $\nu(G) \geq 2$ by Lemma 5, G is minimally 1-subconnected if and only if G is minimally connected. But a minimally connected graph is a tree. Lemma 6 follows. ■

Theorem 7. If G is a minimally 2-connected graph, then $G = C_0 + P_1 + P_2 + \ldots + P_n$, where P_i is a path of length at least 2, connecting two different vertices of $G_{i-1} = C_0 + P_1 + P_2 + \ldots + P_{i-1}$, $1 \leq i \leq n$ and $G_0 = C_0$ is a cycle, and, for any edge $uv \in E(G)$, if $d_G(u) \geq 3$ and $d_G(v) \geq 3$, then $G - uv$ has a cut vertex.

Proof. By Lemma 2, if G is 2-connected, then $G = C_0 + P_1 + P_2 + \ldots + P_n$. If P_i is a path of length 1, i.e., P_i is an edge, then $G - P_i = C_0 + P_1 + \ldots + P_{i-1} + P_{i+1} + \ldots + P_n$, by Lemma 2, $G - P_i$ is still a 2-connected graph. So G is not a minimally 2-connected graph, a contradiction. So every P_i has length at least 2 ($1 \leq i \leq n$). Now we only need to prove that if G is a minimally 2-connected graph, then, for any edge $uv \in E(G)$, $G - uv$ has a cut vertex. For any $uv \in E(G)$, then $uv \in E(C_0)$ or $uv \in E(P_i)$ ($1 \leq i \leq n$). If $d_G(u) = 2$ or $d_G(v) = 2$, without loss of generality, assume $d_G(u) = 2$, then in $G - uv$, u is connected by a path P to a vertex x of degree at least 3; or $G - uv$ is a path P (now $G = C_0$), and we assume w is the vertex on P adjacent to u (now $d_{G-uv}(w) \geq 2$). So w is a cut vertex of $G - uv$. If $d_G(u) \geq 3$ and $d_G(v) \geq 3$, since G is minimally 2-connected, $G - uv$ is not 2-connected, so $G - uv$ has a cut vertex. Hence the structure of minimally 2-connected graph G is as described in this theorem. ■

Lemma 8. Let G be a minimally 2-connected graph. Then, for any edge $e = uv \in E(G)$, $G' = G - e$ has a cut vertex, and the block graph of G' is a path P, and u and v are contained respectively in the two blocks corresponding to the two end vertices of P.

Proof. Since G is a minimally 2-connected graph, G is 2-connected and has no cut vertex, but $G - e$ is 1-connected and has a cut vertex for any edge $e = uv \in E(G)$. So $G' = G - e$ has a block graph $B(G')$. If $B(G')$ has a vertex of degree at least 3, by Lemma 1, $B(G')$ is a tree, and $B(G')$ has at least 3 leaves corresponding 3 leaf blocks of G' of which one contains neither u nor v. So, in $G = G' + e$, that block still contains a cut vertex which contradicts the hypothesis that G is 2-connected. So every vertex in $B(G')$ has degree at most 2, and $B(G')$ is a path P. Then we prove that u and v are contained respectively in the two blocks corresponding to the two end vertices of P. Suppose P has one end vertex b_1 corresponding to a leaf block B_1 in G' containing neither u nor v. Then the vertex x' adjacent to b_1 in $B(G')$ is a cut vertex in G', and x' also corresponds to a cut vertex in $G = G' + e$, which contradicts the hypothesis that G is 2-connected. So u is contained in the block B_1 in G' corresponding to the end vertex b_1 of P and u is not the cut vertex x' contained by B_1. By the same reason, v is contained in the block B_n in G' corresponding to the other end vertex b_n of P, and v is not the cut vertex x'' contained by B_n in G'. Hence Lemma 8 is proved. ■

Lemma 9. A connected graph G with $\nu(G) \geq 5$ is 2-subconnected if and only if, for any subset $S \subseteq V(G)$ with $|S| = 1$, $\omega(G - S) \leq 2$.

Proof. Let $k = 2$, by Lemma 3, we have the conclusion of this lemma. ■

3. The structure of minimally 2-subconnected graphs

In this section, we prove the structure of minimally 2-subconnected graphs to be as described in Theorem 10.

Theorem 10. A connected graph G with $\nu(G) \geq 5$ is minimally 2-subconnected if and only if G has at least one cut vertex and every cut vertex is contained in exactly two blocks in G, and
(1) Each leaf block of G is a K_2; and
(2) Each block of G which is not K_2 is a minimally 2-connected graph $B = C_0 + P_1 + P_2 + \ldots + P_m$.

If $B = C_0 = u_0u_1 \ldots u_nu_0$, then C_0 has two vertices u_i and u_j such that each of u_i and u_j is a cut vertex of G, and is contained in a block different from B and $i < i+2 \leq j < j+2 \leq i$, where the subscripts are reduced modulo $n+1$; or $C_0 = u_0u_1u_2u_0$, where each of u_0, u_1, u_2 is a cut vertex of G and is contained in a block different from B.

If $B = C_0 + P_1 + P_2 + \ldots + P_m (m \geq 1)$, for any H-path $P = (u =)u_ru_{r+1} \ldots u_s(= v)$ of length at least 1 in B connecting two vertices u and v of degree at least 3 in B, P is contained in a segment $u_1u_2 \ldots u_ru_{r+1} \ldots u_s \ldots u_n$ of C_0 connecting two end vertices of P_1 (here the subscripts of u_i are different from those of C_0 in the above), or P is contained in a $P_i = u_1u_2 \ldots u_ru_{r+1} \ldots u_s \ldots u_n$ in B $(1 \leq i \leq m)$ such that (i) there is a cut vertex x of $B - u_ru_{r+1}$ on $u_1u_2 \ldots u_{r-1}$ and x is also a cut vertex of G; or (ii) there is a cut vertex y of $B - u_{s-1}u_s$ on $u_{s+1}u_{s+2} \ldots u_n$ and y is also a cut vertex of G; or when (i) and (ii) do not hold, we have (iii) P is an H-path with $s - r \geq 2$ and P has two vertices u_i and u_j such that each of u_i and u_j is a cut vertex of G and is contained in a block different from B, and $r \leq i < i+2 \leq j \leq s$; and
(3) Besides the cut vertices described in (2), any other vertex of B may or may not be a cut vertex of G.

Proof. We prove sufficiency first. Since each cut vertex x of G is contained in exactly 2 blocks, $\omega(G - x) = 2$. By Lemma 9, G is 2-subconnected. In the following, we prove that G is a minimally 2-subconnected graph, that is , for any edge $e \in E(G)$, $G - e$ is no longer 2-subconnected.

For any edge $e = uv \in E(G)$, $e \in E(B)$ for some block B in G.

Case 1. B is a K_2.

Then e is the only edge in the K_2. Then $G-e$ is not connected, and there is a cutset $S = \varnothing \subseteq V(G-e)$ with $|S| \leq 2 - 1 = 1$ such that $\omega((G - e) - S) \geq |S| + 2 = 2$. By Lemma 3, $G - e$ is not 2-subconnected.

Case 2. B is a minimally 2-connected graph.

Case 2.1. B is a cycle $C_0 = u_0u_1u_2 \cdots u_nu_0$.

By condition (2) of this theorem, there exist cut vertices u_i and u_j in G such that u_i and u_j are contained respectively in blocks B_i and B_j besides B and $i < i+2 \leq j < j+2 \leq i$ where subscripts i and j are reduced modulo $n + 1$; or $C_0 = u_0u_1u_2u_0$ such that each of u_0, u_1, u_2 is a cut vertex of G and they are contained respectively in blocks B_0, B_1, B_2 besides B.

In the first case, if $e = u_iu_{i+1}$, then in $G - e$, the segment $u_{i+1}u_{i+2} \cdots u_{j-1}$ of $C_0 - e$ is not empty, and $(G - e) - u_j$ contains 3 components containing $P = u_{i+1}u_{i+2} \cdots u_{j-1}$, $B_j - u_j$ and $Q = u_{j+1}u_{j+2} \cdots u_i$ on C_0 respectively. So $\omega((G - e) - u_j) \geq 3$, by Lemma 9, $G - e$ is not 2-subconnected.

If $e = u_{j-1}u_j$, then segment $u_{i+1}u_{i+2} \cdots u_{j-1}$ of $C_0 - e$ is not empty, and $(G - e) - u_i$ contains 3 components containing $P = u_{i+1}u_{i+2} \cdots u_{j-1}$, $B_i - u_i$ and $Q = u_ju_{j+1} \cdots u_{i-1}$ on C_0 respectively. So $\omega((G - e) - u_i) \geq 3$, by Lemma 9, $G - e$ is not 2-subconnected.

If $e = u_tu_{t+1}$ and $i + 1 \leq t \leq t + 1 \leq j - 1$, then on $C_0 - e$, $P = u_{t+1}u_{t+2} \cdots u_{j-1}$ is not empty, and $(G - e) - u_j$ contains 3 components containing $P = u_{t+1}u_{t+2} \cdots u_{j-1}$, $B_j - u_j$ and $Q = u_{j+1}u_{j+2} \cdots u_iu_{i+1} \cdots u_t$

on C_0 respectively. So $\omega((G-e)-u_j)\geq 3$, by Lemma 9, $G-e$ is not 2-subconnected.

For e on segment $u_ju_{j+1}\cdots u_i$ on C_0, the discussion is similar.

In the second case, $C_0 = u_0u_1u_2u_0$ and each u_i is contained in a block B_i besides $B(i = 0,1,2)$. Assume $e = u_iu_{i+1}(i = 0,1,2$ and the subscripts are reduced modulo 3). Since u_{i+2} is a cut vertex of G, contained in a block B_{i+2} besides B, $(G-e)-u_{i+2}$ has 3 components containing $P = u_{i+1}, B_{i+2}-u_{i+2}$ and $Q = u_i$ respectively. So $\omega((G-e)-u_{i+2})\geq 3$, by Lemma 9, $G-e$ is not 2-subconnected.

Case 2.2. $B = C_0 + P_1 + P_2 + \cdots + P_m(m \geq 1)$.

Case 2.2.1. $e = xy \in E(B)$ and, $d_B(x) = 2$ or $d_B(y) = 2$.

Now e is on an H-path $P = (u =)u_ru_{r+1}\cdots u_s(= v)(s-r\geq 2)$ connecting two vertices u and v of degree at least 3 in B. But, in B, the two ends of each P_i has degree at least 3 and the degree of each internal vertex of P is 2, so P is contained in a $P_i = u_1u_2\cdots u_ru_{r+1}\cdots u_s\cdots u_n(1\leq i\leq m)$; or P is contained in a segment $C_0[u_1,u_n] = u_1u_2\cdots u_r, u_{r+1}\cdots u_s\cdots u_n$ connecting the two ends of P_1 on C_0 (Notice that here the subscripts of $u_i(i = 1,2,\cdots,n)$ are different from those of u_i on $C_0 = u_0u_1\cdots u_nu_0$ before).

Assume $e = u_tu_{t+1}(r\leq t\leq t+1\leq s)$. Now, the cut vertices on $u_1u_2\cdots u_{r-1}$ of $B-e$ are the same cut vertices on $u_1u_2\cdots u_{r-1}$ of $B-u_ru_{r+1}$; and the cut vertices on $u_{s+1}u_{s+2}\cdots u_n$ of $B-e$ are the same cut vertices on $u_{s+1}u_{s+2}\cdots u_n$ of $B-u_{s-1}u_s$. Also the cut vertices of $B-e$ can appear only on P_i-e or $C_0[u_1,u_n]-e$ (We shall prove it later).

If $B-e$ has a cut vertex x on $u_1u_2\cdots u_{r-1}$ or $u_{s+1}u_{s+2}\cdots u_n$ and x is also a cut vertex of G, then x is contained in a block B_x besides B and $(B-e)-x$ has two components C_1 and C_2, so $(G-e)-x$ has 3 components containing C_1, C_2 and B_x-x. Hence $\omega((G-e)-x)\geq 3$, by Lemma 9, $G-e$ is not 2-subconnected. In cases (i) and (ii) of condition (2) of this theorem, the conclusion holds. Suppose cases (i) and (ii) do not hold and case (iii) holds. Then P is an H-path, and $s-r\geq 2$, and then P has at least two cut vertices u_i and u_j of G, contained respectively in blocks B_i and B_j besides B, where $r\leq i < i+2\leq j\leq s$.

Assume $e = u_tu_{t+1}$. If $j\leq t < s$, then $(G-e)-u_i$ has 3 components containing $R = u_{i+1}u_{i+2}\cdots u_t, B_i-u_i$, and $T = u_ru_{r+1}\cdots u_{i-1}$ respectively. (If T is empty, then the third component is the one not containing R and B_i-u_i in $(G-e)-u_i$. So $\omega((G-e)-u_i)\geq 3$, by Lemma 9, $G-e$ is not 2-subconnected.

If $t = j-1$, the proof is the same as last case.

If $t < j-1$, then $(G-e)-u_j$ has 3 components containing $R = u_{t+1}u_{t+2}\cdots u_{j-1}, B_j-u_j$ and $T = u_{j+1}u_{j+2}\cdots u_s$ respectively. If T is empty, then the third component is the one not containing R and B_j-u_j in $(G-e)-u_j$. So $\omega((G-e)-u_j)\geq 3$, by Lemma 9, $G-e$ is not 2-subconnected.

Case 2.2.2. $e = uv \in E(B)$ and $d_B(u)\geq 3$ and $d_B(v)\geq 3$.

Now $P = uv$. P is on a seqment $C_0[u_1,u_n] = u_1u_2\cdots uv\cdots u_n$ on C_0 connecting the two ends of P_1; or P is on a path $P_i = u_1u_2\cdots uv\cdots u_n(1\leq i\leq m)$. Then we have $u = u_t = u_r$ and $v = u_{t+1} = u_s$. By condition (2) of this theorem, only case (i) or case (ii) can happen. By the condition, $B-e$ has a cut vertex x on $u_1u_2\cdots u_{r-1}$ or $u_{s+1}u_{s+2}\cdots u_n$ such that x is also a cut vertex of G and x is contained in a block B_x besides B. Then $(B-e)-x$ has exactly two components C_1 and C_2 and then $(G-e)-x$ has 3 components containing C_1, C_2 and B_x-x respectively. So $\omega((G-e)-x)\geq 3$, by Lemma 9, $G-e$ is not 2-subconnected.

If, according to condition (3), besides the cut vertices in condition (2), B has another cut vertex x of G, since x is contained in exactly two blocks, $\omega(G-x) = 2$, by Lemma 9, G is 2-subconnected. Since the cut vertices required by condition (2) exist, for each block B in G, $G-e$ is not 2-subconnected for every edge $e \in E(B)$. Hence the sufficiency of this theorem is proved.

Now we prove the necessity. Suppose G is a minimally 2-subconnected graph. Now G has two possible cases: (1) G does not contain any cut vertex; (2) G has a cut vertex.

Case 1. G does not contain any cut vertex.

Then G is a minimally 2-connected graph and $G = C_0 + P_1 + P_2 + \cdots + P_m$. Suppose not ,$G$ has an edge e such that $G - e$ is still 2-connected, by Lemma 4, G is 2-subconnected and $G - e$ is still 2-subconnected, contradicting the assumption that G is a minimally 2-subconnected graph.

By Theorem 7, $G = C_0 + P_1 + P_2 + \cdots + P_m$, where P_i is an H-path in G_i connecting two vertices in $G_{i-1} = C_0 + P_1 + P_2 + \cdots + P_{i-1}, 1 \le i \le m$, and $G_0 = C_0$ is a cycle, and for each $uv \in E(G)$, if $d_G(u) \ge 3$ and $d_G(v) \ge 3$, then $G - uv$ has a cut vertex.

If $G = C_0$, then, for any edge e on $C_0, C_0 - e$ is a Hamilton path, hence is 2-subconnected. So G is not a minimally 2-subconnected graph, a contradition to the assumption.

If $G = C_0 + P_1 + P_2 + \cdots + P_m (m \ge 1)$, by Theorem 7, $P_m = u_1u_2 \cdots u_n$ and $n \ge 3$, as $G' = C_0 + P_1 + P_2 + \cdots + P_{m-1}$ is also 2-connected, then $G - u_{n-1}u_n$ contains cut vertices $u_i (i = 1, 2, \cdots, n-2)$, and $(G - u_{n-1}u_n) - u_i$ has exactly two components $P = u_{i+1}u_{i+2} \cdots u_{n-1}$ and the rest part of $(G - u_{n-1}u_n) - u_i$. So, for any cut set $S \subseteq V(G)$ with $|S| = 1$, we have $\omega((G - u_{n-1}u_n) - S) \le 2$. By Lemma 9, $G - u_{n-1}u_n$ is 2-subconnected. Hence G is not minimally 2-subconnected, contradicting the assumption of G. So Case 1 does not hold, and G must have a cut vertex.

Case 2. G has a cut vertex.

First, every cut vertex of G is contained in exactly two blocks of G. Otherwise, suppose a cut vertex x is contained in at least 3 blocks B_1, B_2, B_3 in G. Then $G - x$ has at least 3 components containing $B_1 - x, B_2 - x$ and $B_3 - x$ respectively. So $\omega(G - x) \ge 3$, by Lemma 9, G is not 2-subconnected, contradicting the assumption of G.

Second, each block not to be K_2 in G must be a minimally 2-connected graph. Suppose B is a block not to be K_2, then B is a 2-connected graph as B is a block and $\nu(B) \ge 3$. Suppose B is not a minimally 2-connected graph, then B has an edge e such that $B - e$ is still 2-connected. Then each block of G is still a block in $G - e$, and each cut vertex of G is still a cut vertex of $G - e$ and $B - e$ does not have any new cut vertex different from those cut vertices in G. Then each cut vertex x in $G - e$ is still contained in exactly two blocks of $G - e$. So $\omega((G - e) - x) \le 2$ for each vertex x in $G - e$. By Lemma 9, $G - e$ is still 2-subconnected. Hence G is not minimally 2-subconnected, a contradiction to the assumption of G.

Now we prove conclusion (1): Every leaf block B of G is K_2. Suppose not. Then B is a minimally 2-connected graph and $B = C_0 + P_1 + P_2 + \cdots + P_m$ by Theorem 7.

In the first case, $B = C_0 = u_0u_1 \cdots u_nu_0$. Since B is a leaf block, B contains exactly one cut vertex x of G. Without loss of generality, assume that $x = u_0$. Let $e = u_nu_0$. In $G - e$, for any cut vertex x, x is originally a cut vertex in G or $x = u_i (i = 0, 1, \cdots, n-1), (G - e) - x$ has exactly two components, i.e., $\omega((G - e) - x) = 2$. By Lemma 9, $G - e$ is still 2-subconnected, and hence G is not minimally 2-subconnected, contradictiong the assumption of G.

In the second case, $B = C_0 + P_1 + P_2 + \cdots + P_m$ $(m \ge 1)$. Let $P_m = u_1u_2 \cdots u_n$ $(n \ge 3)$. If the only cut vertex x of G in B is not on P_m or $x = u_n$, then let $e = u_{n-1}u_n$; if the only cut vertex x of G in B is $x = u_j$ $(1 \le j < n)$, then let $e = u_ju_{j+1}$. Now, $\omega((G - e) - y) = 2$ for each cut vertex y in $G - e$ (y is original cut vertex in G or $y = u_i$ $(i = 1, 2, \cdots, j$; or $i = j + 2, j + 3, \cdots, n)$. Hence $G - e$ is still 2-subconnected, and then G is not minimally 2-subconnected, contradicting the assumption of G. So conclusion (1) of

Theorem 10 is proved.

Now we prove conclusion (2). We have proved that each block B not to be K_2 is a minimally 2-connected graph. If B contains only one cut vertex of G, then B is a leaf block, by the proof before, B must be a K_2, contradicting the above assumption of B. So B contians at least two cut vertices of G.

If $B = C_0 = u_0u_1 \cdots u_nu_0$, then B contains two cut vertices u_i and u_j such that they are separated by at least one vertex on C_0, i.e., i $< i+2 \leq j < j+2 \leq i$ (where the subscripts i and j are reduced modulo n+1); or $C_0 = u_0u_1u_2u_0$, and u_0, u_1 and u_2 are all cut vertices of G each of which is contained in one block of G besides B. Suppose not. Then $B = C_0$ contains only two cut vertices u_i and u_j of G and $j = i+1$, and then let $e = u_iu_{i+1}$. Now every vertex of $C_0 - e$ is a cut vertex of $G - e$, and every cut vertex of G is still a cut vertex of $G-e$. Also $\omega((G-e)-x) = 2$ holds for each cut vertex x of $G-e$. So $G - e$ is still 2-subconnected, which contradicts the assumption that G is a minimally 2-subconnected graph.

Now suppose $B = C_0 + P_1 + P_2 + \cdots + P_m$ $(m \geq 1)$. Let $P = (u =)u_ru_{r+1} \cdots u_s(= v)$ be a path in B connecting two vertices of degree at least 3 in B. If P is not an edge uv, then P is an H-path. Since the degree of each inner vertex of P is 2, but the degree of each end vertex of P_i $(1 \leq i \leq m)$ is at least 3, so P is on the segment $C_0[u_1, u_n] = u_1u_2 \cdots u_ru_{r+1} \cdots u_s \cdots u_n$ of C_0 connecting two end vertices of P_1. (Notice that the subscript of u_i of $C_0[u_1, u_n]$ is different from that of u_i of $C_0 = u_0u_1 \cdots u_nu_0$ before); or P is on a $P_i = u_1u_2 \cdots u_ru_{r+1} \cdots u_s \cdots u_n$ $(1 \leq i \leq m)$.

For each edge $e = u_tu_{t+1}$ of B, if $d_B(u_t) = 2$ or $d_B(u_{t+1}) = 2$, then e is on an H-path P as above, and P is on a segment $C_0[u_1, u_n]$ or a P_i $(1 \leq i \leq m)$.

Since G is a minimally 2-subconnected graph, $G - e$ is not a 2-subconnected graph, by Lemma 9, in the following, we only need to prove that $B - e$ has a cut vertex x, and x is also a cut vertex of G contained in a block B_x besides B. Then $(B-e)-x$ has two components C_1 and C_2, and then $(G-e)-x$ has 3 components containing C_1, C_2 and $B_x - x$, hence we can prove that x satisfies conclusion (2) in this theorem.

Now we firstly prove that the cut vertices of $B - e$ are on $C_0[u_1, u_n] - e$ or on $P_i - e$ $(1 \leq i \leq m)$. If $e = u_tu_{t+1}$ is on $C_0[u_1, u_n]$, then $C_0[u_n, u_1] + P_1$ is a cycle, i.e., a 2-connected graph. The cut vertices of $(C_0 + P_1) - e$ are on $C_0[u_1, u_n] - e$. Since $(C_0 + P_1) - e$ is a connected graph, adding P_j to it, which connects two different vertices of $(C_0 + P_1) - e + P_2, \cdots + P_{j-1} (j = 2, 3, \cdots, n)$, every P_j is contained in a cycle. So each vertex of P_j, except the end vertex (or two end vertices) of P_j on $C_0[u_1, u_n] - e$, is not a cut vertex of $B - e$. Hence the cut vertices of $B - e = (C_0 + P_1) - e + P_2 + P_3 + \cdots + P_m$ are all on $C_0[u_1, u_n] - e$. If $e = u_tu_{t+1}$ is on a P_i, then $C_0 + P_1 + P_2 + \cdots + P_{i-1}$ is 2-connected, then the cut vertices of $((C_0 + P_1 + \cdots + P_{i-1}) + P_i) - e$ are on $P_i - e$. Since $(C_0 + P_1 + \cdots + P_{i-1} + P_i) - e$ is a connected graph, adding P_j to it, which connects two vertices of $(C_0 + P_1 + \cdots + P_i) - e + P_{i+1} + \cdots + P_{j-1}$, every P_j is on a cycle of $((C_0 + P_1 + \cdots + P_{i-1} + P_i) - e + P_{i+1} + \cdots + P_{j-1}) + P_j$ $(j = i+1, i+2, \cdots, m)$. Then each vertex of P_j, except the end vertex (or two end vertices) of P_j on $P_i - e$, is not a cut vertex of $B - e$. Hence the cut vertices of $B - e = (C_0 + P_1 + \cdots + P_i) - e + P_{i+1} + \cdots + P_m$ are all on $P_i - e$.

Now assume that $e = u_tu_{t+1}$ is on $P = u_ru_{r+1} \cdots u_s$, and P is contained in $C_0[u_1, u_n] = u_1u_2 \cdots u_ru_{r+1} \cdots u_s \cdots u_n$ or is contained in $P_i = u_1u_2 \cdots u_ru_{r+1} \cdots u_s \cdots u_n$. By the proof before, $B - e$ has a cut vertex x on $C_0[u_1, u_n] - e$ or on $P_i - e$. As $(B-e)-x$ has only two components, suppose every cut vertex x of $B - e$ is not a cut vertex of G, then $\omega((G-e)-x) = \omega((B-e)-x) = 2$. By Lemma 9, $G-e$ is still 2-subconnected, contradictiong the assumption that G is minimally 2-subconnected. Hence $B - e$ has a cut vertex x and x is also a cut vertex of G contained in a block B_x besides B in G. Notice that the degree of each inner vertex of P is 2, so the cut vertices of $B - e$ on $u_1u_2\cdots u_{r-1}$ are the same as the cut vertices of $B - u_ru_{r+1}$ on $u_1u_2\cdots u_{r-1}$. If $B - e$ has a cut vertex x on $u_1u_2\cdots u_{r-1}$ such that x is also a cut vertex of G, then $B - u_ru_{r+1}$ also has a cut vertex x on $u_1u_2\cdots u_{r-1}$ such that x is also a cut vertex of G. Hence conclusion (2) (i) of this theorem is proved.

By the same reason, since the degree of each inner vertex of P in B is 2, the cut vertices of $B - e$ on $u_{s+1}u_{s+2}\cdots u_n$ are the same as the cut vertices of $B - u_{s-1}u_s$ on $u_{s+1}u_{s+2}\cdots u_n$. If $B - e$ has a cut vertex y on $u_{s+1}u_{s+2}\cdots u_n$ and y is also a cut vertex of G, then $B - u_{s-1}u_s$ also has the cut vertex y on $u_{s+1}u_{s+2}\cdots u_n$ such that y is also a cut vertex of G. Hence conclusion (2) (ii) of the theorem is proved.

Now suppose conclusions (i) and (ii) of (2) do not hold. As $d_B(u_t) = 2$ or $d_B(u_{t+1}) = 2$, and $e = u_tu_{t+1}$ is on $P = u_ru_{r+1}\cdots u_s$, so $s - r \geq 2$. Suppose that P does not satisfy that P has at least two vertices u_i and u_j such that u_i and u_j are cut vertices of G and $r \leq i < i + 2 \leq j \leq s$, then P does not have any cut vertex of G; or P has only one cut vertex u_i of G; or P has exactly two cut vertices u_i and u_j of G with $j = i + 1$.

In the first case, P does not have any cut vertex of G. By the proof as before, $B-e$ has a cut vertex x on $C_0[u_1, u_2]-e$; or on P_i-e such that x is also a cut vertex of G. By the assumption in last paragraph, conclusions (i) and (ii) do not hold, then $u_1u_2\cdots u_{r-1}$ and $u_{s+1}u_{s+2}\cdots u_n$ do not have any cut vertex x of G, so cut vertex x of G can only lie on $P = u_ru_{r+1}\cdots u_s$, it contradicts the assumption of this case.

In the second case, P has exactly one cut vertex u_i of G. If $i = s$, let $e = u_{s-1}u_s$; if $r \leq i \leq s - 1$, let $e = u_iu_{i+1}$. Then, in $G - e$, each cut vertex is x on $u_1u_2\cdots u_{r-1}$; or y on $u_{s+1}u_{s+2}\cdots u_n$; or u_j($j = i = s$ and $j = r, r + 1, \cdots i - 2$; or $j = r, r + 1, \cdots, i$ and $j = i + 2, i + 3, \cdots s$). But since x and y are not cut vertices of G and only u_i is a cut vertex of G on P, each cut vertex z of $G - e$ is contained only in two blocks, then $\omega((G - e) - z) \leq 2$, by Lemma 9, $G - e$ is still 2-subconnected, contradicting the assumption that G is minimally 2-suconnected.

In the third case, P has exactly two cut vertices u_i and u_j respectively contained in two blocks B_i and B_j besides B in G with $j = i + 1$. Let $e = u_iu_{i+1}$. Then in $G - e$, each cut vertex x on $u_1u_2\cdots u_{r-1}$ is not a cut vertex of G; each cut vertex y on $u_{s+1}u_{s+2}\cdots u_n$ is not a cut vertex of G; and u_p is a cut vertex of $G - e$ for $p = r, r + 1, \cdots, i$; and $p = i + 1, i + 2, \cdots, s$. But each cut vertex of $G - e$ is contained in exactly two blocks. So $\omega((G - e) - z) \leq 2$ for each cut vertex z of $G - e$. By Lemma 9, $G-e$ is still 2-subconnected, contradicting the assumption that G is minimally 2-subconnected. Hence conclusion(iii) of (2) holds.

Suppose $e = u_tu_{t+1}$ satisfy that $d_B(u_t) \geq 3$ and $d_B(u_{t+1}) \geq 3$.Then $P = uv = u_ru_{r+1}\cdots u_s$ satisfies that $r = t$ and $s = t + 1$. Since $B = C_0 + P_1 + P_2 + \cdots + P_m$, P is on $C_0[u_1, u_n] = u_1u_2\cdots u_tu_{t+1}\cdots u_n$; or on $P_i = u_1u_2\cdots u_tu_{t+1}\cdots u_n$. As $G - e$ has a cut vertex z such that $\omega((G - e) - z) \geq 3$, by the same proof as before, the cut vertex of $B-e$ is on $u_1u_2\cdots u_{r-1}(= u_{t-1})$; or on $(u_{t+2}=)\ u_{s+1}u_{s+2}\cdots u_n$. Each of u_t and u_{t+1} is possibly a cut vertex of $G - e$, but each of these two possible cut vertices can be contained only in two blocks of $G - e$. To satisfy that $\omega((G - e) - z) \geq 3$, z can be only cut vertex x on $u_1u_2\cdots u_{r-1}$ in $B - e$ such that x is also a cut vertex of G contained in a block B_x besides B; or can be a cut vertex y on $u_{s+1}u_{s+2}\cdots u_n$ in $B - e$ such that y is also a cut vertex of G contained in a block B_y besides B. Then conclusions (i) and (ii) of (2) of this theorem hold.

In above discussion, if G is a minimally 2-subconnected graph, we always have conclusions (1) and (2) of this theorem. Since, in logic, if A implies B, then A implies B or B and C, so we also have conclusion (3). Now we prove conclusions (1)–(3) imply that G is a minimally 2-subconnected graphs. Besides the cut vertices required by (2), each other vertex x of a block B not to be K_2 may be a cut vertex of G. Since by conditions (1) and (2), each cut vertex x of G is contained in exactly two blocks, so $\omega(G - x) \leq 2$ for each cut vertex x in G, by Lemma 9, G is 2-subconnected. By the proof of sufficiency, for each edge $e \in E(B)$ and each block B of G, if G satisfies conditions (1)–(3) (and hence (1) and (2)), then G is a minimally 2-subconnected graph. So conclusion (3) holds.

Hence the theorem is proved. ■

Now we give examples of graphs satisfying Theorem 10. Let $C_0 = a_0a_1 \cdots a_8a_0$, $P_1 = b_0b_1 \cdots b_8$, $P_2 = c_0c_1c_2c_3c_4$, $P_3 = d_0d_1d_2d_3$. Let $B_0 = C_0 + P_1 + P_2 + P_3$, where $a_1 = b_0$, $a_4 = b_8$, $b_2 = c_4$, $b_6 = c_0$, $b_1 = d_3$ and $b_7 = d_0$. Notice that a_i, b_j, c_k, d_l are different vertices except the cases that we specify that they are the same vertices as above. Then B_0 is a minimally 2-connected graph and it is a block discussed in Theorem 10. Let $D_0 = x_0x_1x_2x_3x_4x_5x_0$ be a cycle, $D_1 = x_3y_1$, $H = D_0 \cup D_1$. Let $H_1, H_2, \cdots, H_9$ be nine copies of H. Let G_3 be the graph by identifying a_1 in B_0 and x_1 in H_1, a_3 in B_0 and x_1 in H_2, a_5 in B_0 and x_1 in H_3, b_1 in B_0 and x_1 in H_4, b_2 in B_0 and x_1 in H_5, b_5 in B_0 and x_1 in H_6, b_7 in B_0 and x_1 in H_7, c_1 in B_0 and x_1 in H_8, and y_1 in H_8 and x_1 in H_9 respectively. Then G_3 has the structure in Theorem 10. Let G_4 be the graph by identifying a_1 in C_0 and x_1 in H_1, a_4 in C_0 and x_1 in H_2 respectively. Then G_4 consists of C_0, H_1 and H_2, and it also has the structure in Theorem 10.

Theorem 11. A 2-connected graph G has a spanning 2-subconnected subgraph H such that (1) if G has a Hamilton path, then $|E(G)|-|E(H)| \geq 1$; (2) if G does not have a Hamilton path, then $|E(G)|-|E(H)| \geq 2$.

Proof. First, assume that $\nu(G) \geq 5$, and assume that G is a minimally 2-connected graph. Since, by Lemma 4, a 2-connected graph G must be 2-subconnected, so deleting some edges from G,we can obtain a minimally 2-subconnected spanning subgraph H. As the minimum degree of a vertex in 2-connected graph G is at least 2, by Theorem 10, the minimum degree of a vertex in H is 1. To obtain H, we must delete at least 1 edge from G. So $|E(G)| - |E(H)| \geq 1$, by Remark 1 in the following, if G is a cycle (a Hamilton cycle), then H is a Hamilton path and $|E(G)| - |E(H)| = 1$.

Now assume that G does not contain a Hamilton path. By the argument in last paragraph, G has a minimally 2-subconnected spanning subgraph H. By Theorem 10, H has a cut vertex and each leaf block of H is a K_2. Since, by Lemma 1, the block graph $B(H)$ of H is a tree, and H has a cut vertex and at least 2 leaf blocks, so H has at least 2 vertices of degree 1. If $B(H)$ has at least 3 leaves, i.e., H has at least 3 leaf blocks K_2, then H has at least 3 vertices of degree 1. Since the degree of each vertex of G is at least 2, to obtain H, we have to delete at least two edges from G. Then $|E(G)| - |E(H)| \geq 2$.

Now assume that the block graph $B(H)$ of H has exactly two leaves. Then $B(H)$ is a path, and H has exactly two leaf blocks K_2, each of which has a vertex u(or v) of degree 1. Since $\nu(H) = \nu(G) \geq 5$, by the assumption at the beginning of this proof, besides the two blocks K_2, H has another block B which contains exactly two cut vertices of H. If H is not obtained by deleting edge uv from G, that is, G does not contain edge uv, since $\delta(G) \geq 2$, to obtain the two vertices u and v of degree 1, we have to delete at least two edges from G. Then $|E(G)| - |E(H)| \geq 2$. Now assume that $E(G) = E(H) \cup \{uv\}$ and $uv \notin E(H)$.

If we go from u to v in H and every blocks gone through is K_2, then H is a Hamiltian path, contradicting the former assumption that G has no Hamiltonian path. So at least one block B gone through from u to v in H is not K_2.

If $B = C_0 = u_0u_1 \cdots u_nu_0$ and C_0 has exactly two cut vertices u_i and u_j of $H(i < i+2 \leq j < j+2 \leq i$, where the subscripts i and j are reduced modulo $n + 1$) by Theorem 10. Then we delete $u_{i-1}u_i$ and $u_{j-1}u_j$ from $G = H + uv$ to obtain H', and H' is still minimally 2-subconnected, so $|E(G)| - |E(H')| \geq 2$.

This is because, in $H' = G - \{u_{i-1}u_i, u_{j-1}u_j\}$, except blocks B and uv, every block is the same as it is in H and has the same cut vertices as in H, the two leaf blocks of H and edge uv form a path in H', and each of the blocks in this path is a K_2, and now the two leaf blocks $u_{i-2}u_{i-1}$ and $u_{j-2}u_{j-1}$ are K_2, and $u_iu_{i+1}\cdots u_{j-1}$ and $u_ju_{j+1}\cdots u_{i-1}$ are two paths containing cut vertices u_i and u_j respectively on which each edge is a block K_2. So H' satisfies the hypotheses of minimally 2-subconnected graphs in Theorem 10, and hence H' is a minimally 2-subconnected graph.

Suppose $B = C_0 + P_1 + P_2 + \cdots + P_m$ $(m \geq 1)$. Let $C_0 = w_0w_1\cdots w_nw_0$, and two end vertices of P_1 on C_0 be $x = w_i$ and $y = w_j$ and $i < i+2 \leq j < j+2 \leq i$, where the subscripts i and j are reduced modulo $n+1$. Now $P_m = u_1u_2\cdots u_k$ be an H-path in B connecting two vertices of degree at least 3. Now cases (i) and (ii) of conclusion (2) in Theorem 10 do not hold since $r = 1$ and $s = k$. So only case (iii) happens, that is, P_m has two vertices u_i and u_j to be cut vertices of H contained in two blocks respectively besides B and $1 \leq i < i+2 \leq j \leq k$. Since B contains exactly 2 cut vertices of H, u_i and u_j are the only two cut vertices of H in B.

If at least one of u_i and u_j is not an end vertex of P_m, without loss of generality, assume $u_i \neq u_1$, and then u_i is a vertex of degree 2 in B. Now not both x and y are cut vertices of H since both x and y are of degree at least 3 in B. Without loss of generality, assume that x is not a cut vertex of H. Assume that $Q_1 = a_1a_2\cdots a_{k_1}$ and $Q_2 = b_1b_2\cdots b_{k_2}$ be the two segments of C_0 from y to x, and $Q_3 = P_1 = c_1c_2\cdots c_{k_3}$, where $a_1 = b_1 = c_1 = y$ and $a_{k_1} = b_{k_2} = c_{k_3} = x$. Since u_i is a vertex of degree 2 in B, it can be contained in only one of Q_1, Q_2 and Q_3. So one of Q_1, Q_2 and Q_3 (without loss of generality, assume Q_1) does not contain any cut vertex of H except y. As B is a minimally 2-connected graph, the block graph of $B - a_1a_2$ is a path. Let $B' = C_0 + P_1$. Then all cut vertices of $B' - a_1a_2$ are on $Q_1 - a_1$. Also since $B - a_1a_2 = (B' - a_1a_2) + P_2 + P_3 + \cdots + P_m$, each P_i connects two different vertices of connected graph $(B' - a_1a_2) + P_2 + \cdots + P_{i-1}$, and P_i is contained in a cycle of $B - a_1a_2$, so except the cut vertices on $Q_1 - a_1$ in $B' - a_1a_2$, $B - a_1a_2$ can not have other cut vertices, hence all cut vertices of $B - a_1a_2$ are on $Q_1 - a_1$. As $Q_1 - a_1$ does not contain any cut vertex of H and $Q_1 - a_1$ is a path, so $\omega(H - a_1a_2 - w) \leq 2$ for each cut vertex w of $B - a_1a_2$ on $Q_1 - a_1$. For any other cut vertex w of $H - a_1a_2$, w is also a cut vertex of H, and w satisfies that $\omega(H - a_1a_2 - w) \leq 2$. By Lemma 9, $G - uv - a_1a_2 = H - a_1a_2$ is still a 2-subconnected graph which contains a minimally 2-subconnected spanning subgraph H', hence $|E(G)| - |E(H')| \geq 2$.

Suppose both u_i and u_j are end vertices of P_m, i.e., $u_i = u_1$ and $u_j = u_k$. Suppose $u_1, u_k \neq x, y$, without loss of generality, assume $x \notin u_1, u_k$, i.e., x is not a cut vertex of H. By the same argument as above, we can get a minimally 2-subconnected spanning subgraph H' of G such that $|E(G)| - |E(H')| \geq 2$.

Now suppose $u_1 = x$ and $u_k = y$. Then B does not contain any other cut vertex besides x and y, and the block graph $B(H)$ of H is a path containing the two vertices corresponding to x and y in H. Assume that $Q_1 = a_1a_2\cdots a_{k_1}$ and $Q_2 = b_1b_2\cdots b_{k_2}$ are the two segments of C_0 from y to x, and $Q_3 = P_1 = c_1c_2\cdots c_{k_3}$, where $a_1 = b_1 = c_1 = y$, $a_{k_1} = b_{k_2} = c_{k_3} = x$. Let $B' = C_0 + P_1$. Let G' be the graph from G by replacing B by B'. Then G' is also a minimally 2-connected graph. Let $H' = G' - a_1a_2 - b_{k_2}b_{k_2-1}$. Then all cut vertices of H' are on $Q_1 - a_1$ and $Q_2 - b_{k_2}$. Since B does contain any other cut vertex of H besides x and y, and $B - a_1a_2 - b_{k_2}b_{k_2-1} = (B' - a_1a_2 - b_{k_2}b_{k_2-1}) + P_2 + P_3 + \cdots + P_m$, each P_i connects two different vertices in connected graph $B' - a_1a_2 - b_{k_2}b_{k_2-1} + P_2 + \cdots + P_{i-1} (2 \leq i \leq m)$, then each P_i is contained in a cycle, so besides the vertices on $Q_1 - a_1$ and $Q_2 - b_{k_2}$, $B - a_1a_2 - b_{k_2}b_{k_2-1}$ does not contain any other cut vertex of $G - a_1a_2 - b_{k_2}b_{k_2-1}$. Hence every cut vertex of $G - a_1a_2 - b_{k_2}b_{k_2-1}$

is on $Q_1 - a_1$ and $Q_2 - b_{k_2}$. But each of $Q_1 - a_1$ and $Q_2 - b_{k_2}$ is a path, so $\omega(G - a_1a_2 - b_{k_2}b_{k_2-1} - w) \leq 2$ for each cut vertex w of $G - a_1a_2 - b_{k_2}b_{k_2-1}$ on $Q_1 - a_1$ or $Q_2 - b_{k_2}$. Besides the above w, $G - a_1a_2 - b_{k_2}b_{k_2-1}$ does not have any other cut vertex. By Lemma 9, $G - a_1a_2 - b_{k_2}b_{k_2-1}$ is a 2-subconnected graph containing a minimally 2-subconnected spanning subgraph H'' such that $|E(G)| - |E(H'')| \geq |\{a_1a_2, b_{k_2}b_{k_2-1}\}| = 2$.

Then the conclusion of this theorem is proved.

If $v(G) \leq 4$, it is easy to verify that the conclusion of this theorem also holds. ■

The examples of graphs satisfying Theorem 11 are the same as those in Theorem 10, and some examples are illustrated in Remark 1.

Remark 1. In [17], we prove that, in a k-connected graph G, by deleting arbitrarily $k - 1$ edges, the resulting graph is still k-subconnected. If we choose edges to be deleted properly, the number of deleted edges to keep k-subconnectedness would be much more. For example, if G has a Hamiltonian path P, then we can delete all edges except those on P, the resulting graph is still k-subconnected. But for $k = 2$, the number of edges to be deleted will not increase. For example, if G is a cycle C, then G is a 2-connected graph. Deleting arbitrarily $k - 1 = 1$ edge, G is still 2-subconnected. But deleting any 2 edges, no matter how to choose the 2 edges, the resulting graph is not connected, by Lemma 5, it is not 1-subconnected and hence not 2-subconnected.

If G is the union of four internally disjoint paths $P = a_1a_2 \cdots a_n$, $Q = b_1b_2 \cdots b_n$, $R = c_1c_2 \cdots c_n$ and $S = d_1d_2 \cdots d_n$ with $x = a_1 = b_1 = c_1 = d_1$ and $y = a_n = b_n = c_n = d_n$ $(n \geq 3)$, then G is a minimally 2-connected graph without Hamiltonian path. Then $G - a_1a_2 - b_{n-1}b_n$ is a minimally 2-subconnected graph, and deleting any three edges from G, the resulting graph is not 2-subconnected. So the lower bounds of edges to be deleted in Theorem 11 are sharp.

4. Conclusions

As we mentioned in the introduction of this paper, to study k-subconnected graphs may help to solve the computation problem of $2k$-critical graphs. Also every k-connected graph has a minimally k-subconnected spanning subgraph. To characterize the structure of minimally k-subconnected graphs may help us to know more about the structure of k-connected graphs.

To start, in this paper, we characterize the structure of minimally 2-subconnected graphs.

But for $k > 2$, the structure of minimally k-subconnected graphs is still difficult to characterize. We also do not know how many edges can be deleted in a k-connected graph to keep k-subconnectedness if we choose the edges to be deleted properly.

Conflict of interest

The authors declare no conflict of interest.

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