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Research article

Periodic stationarity conditions for mixture periodic INGARCH models

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Abstract: This paper proposes strict periodic stationarity and periodic ergodicity conditions for a finite mixture integer-valued GARCH model with S-periodic time-varying parameters that depend on the state of an independent and periodically distributed regime sequence. In this model, the past conditional mean values depend on the past of the regime variable in the same order, so the model is characterized by path-regime dependence. We also propose sufficient conditions for periodic stationarity when the conditional means are nonlinear of past observations. The results are applied to various discrete conditional distributions.

Keywords: integer-valued time series models; finite mixture models; periodic time-varying models; ergodic properties; path dependence
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1. Introduction

Integer-valued GARCH (INGARCH) models have proved to be useful in modeling count time series which are characterized by specific patterns that cannot be accounted for by standard ARMA models ([1, 5, 18–20, 23, 25, 30, 32, 37–39]). Among the most apparent features of count time series are low values, overdispersion (the variance is larger than the mean), persistence, asymmetry, and a positive autocorrelation structure (cf. [19] and [27]). In its general form, the INGARCH model is defined through a discrete conditional distribution (e.g., Poisson, negative binomial, etc.) with a stochastic time-varying conditional mean, which depends on past observations through time-invariant parameters.

For count data that also exhibit a seasonal behavior, the periodic INGARCH model – in which the parameters are taken to be periodic over time – has been found to be attractive ([7, 14, 16, 29]). Aknouche et al. [7] proposed periodic stationarity and periodic ergodicity conditions for the first-order periodic INGARCH(1, 1) model with conditional distributions belonging to the class of Poisson mixtures. Almohaimeed in [11] extended the result to a higher-order periodic INGARCH(p, q) in a larger family of distributions whose stochastic order is identical to the mean order ([3]). These general results can be used to support asymptotic inference for periodic INGARCH models used in studying the asymptotic properties of the Poisson quasi-maximum likelihood estimate (QMLE) and the negative binomial QMLE ([1] and [8]).

In spite of the high-level of generality of the periodic INGARCH model in modeling a wide range of count data, it seems that it is unable to model some other pathological features such as multimodality and heavy tailedness of the conditional distributions ([21, 22, 40]). A well-known approach to model these features is to utilize a finite mixture of distributions (e.g., [4, 10]) leading to a mixture periodic INGARCH representation. Ouzzani and Bentarzi [35] proposed a Poisson mixture periodic INARCH model and studied its periodic stationarity in mean properties. However, strict periodic stationarity and periodic ergodicity have not been considered. Moreover, the model only deals with the Poisson mixture and the ARCH forms, which can be restrictive.

This paper provides strict periodic stationarity and periodic ergodicity conditions for a general mixture periodic INGARCH(p, q) model. This model is defined via a finite mixture of discrete distributions with conditional means depending on past observations through periodic time-varying parameters. The mixture feature allows for finitely many component specifications corresponding to the conditional mean and is specified via a finite independent and periodically distributed chain called the switching (or regime) process. Moreover, the lagged values of the conditional mean in each regime are governed by the past values of the regime sequence (cf. [4, 9, 26, 34, 40]). Finally, the conditional mean in each regime is a general (linear or nonlinear) function of past observations. The model thus allows for a large spectrum of distributions and various conditional mean shapes.

The rest of this paper has the following structure. Section 2 defines the model and the tools needed for the subsequent Sections. In Section 3, we propose periodic ergodicity conditions for the mixture periodic INGARCH (henceforth MP-INGARCH). Section 4 extends the results to the case of nonlinear Lipschitz conditional mean forms. Section 5 illustrates the general results on specific mixture periodic distributions such as the periodic Poisson mixture INGARCH, and the periodic negative binomial mixture INGARCH. In addition, a simulation study is carried out to compare the theoretical seasonal expectations generated by the model and their sample counterparts computed using simulated series. Main conclusions are indicated in Section 6.

2. Mixture periodic INGARCH model

In the sequel, it is assumed that all random variables and sequences are defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in subsets of the integer set $\mathbb{N} = \{0, 1, ...\}$.

Let us recall some probability properties for periodic processes such as periodic stationarity and periodic ergodicity. A sequence of random variables $\{X_t,\ t \in \mathbb{Z}\}$ is said to be independent and S-periodically distributed (ipd_S) if: i) $\{X_t,\ t \in \mathbb{Z}\}$ is independent, and ii) $X_t \overset{d}{=} X_{t+kS}$ for all $k, t \in \mathbb{Z}$, where $S \geq 1$ is a positive integer called the period, and the symbol "$\overset{d}{=}$" stands for equality in distribution. When $S = 1$, an ipd_1 sequence is just an independent and identically distributed (*iid*) process.

A stochastic process $\{Y_t,\ t \in \mathbb{Z}\}$ is said to be *strictly periodically stationary* with period $S \geq 1$ if all processes $\{Y_{nS+s},\ n \in \mathbb{Z}\}$ $(1 \leq s \leq S)$ are strictly stationary in the ordinary sense. In addition, $\{Y_t,\ t \in \mathbb{Z}\}$ is called S*-periodically stationary* in mean if $E(Y_t)$ is finite (for all $t \in \mathbb{Z}$) and is S-periodic over t. Naturally, strict periodic stationarity implies periodic stationarity in the mean whenever the seasonal means of the process are finite. The process $\{Y_t,\ t \in \mathbb{Z}\}$ is said to be *periodically ergodic* with period S if all sub-processes $\{Y_{nS+s},\ n \in \mathbb{Z}\}$ ($s \in \{1, ..., S\}$) are ergodic in the usual sense (cf. [6, 17]). The simplest strictly periodically stationary and periodically ergodic process (with period S) is an ipd_S sequence.

Let F_λ be a cumulative distribution function (cdf) with discrete support and mean $\lambda = \int_0^{+\infty} x dF_\lambda(x) > 0$. Suppose that F_λ satisfies the following "equal stochastic and mean orders" property introduced by [3]

$$\lambda \leq \lambda^* \quad \Rightarrow \quad F_\lambda^-(u) \leq F_{\lambda^*}^-(u),\ \forall u \in (0,1), \tag{2.1}$$

where F_λ^- is the generalized inverse of F_λ. The family of distributions satisfying (2.1) is quite large and contains the one-parameter exponential family such as the Poisson and negative binomial distributions, and also other interesting distributions (cf. [3]). In [11] proposed a large class of periodic INGARCH(p, q) models described as follows. An integer-valued process $\{Y_t, t \in \mathbb{Z}\}$ is said to be a periodic INGARCH(p, q) of orders p and q ($p, q \in \mathbb{N}$), and positive integer period $S \geq 1$, if its conditional distributions are given by

$$Y_t|\mathcal{F}_{t-1} \sim F_{t,\lambda_t},\ t \in \mathbb{Z}, \tag{2.2}$$

where $\mathcal{F}_{t-1}$ is the σ-algebra generated by $\{Y_{t-1}, Y_{t-2}, ...\}$, the cdf $F_{t,\lambda_t} := F_{\lambda_t}$ satisfies (2.1) for all $t \in \mathbb{Z}$, and the conditional mean λ_t is given by

$$\lambda_t = \omega_t + \sum_{i=1}^{q} \alpha_{ti} Y_{t-i} + \sum_{j=1}^{p} \beta_{tj} \lambda_{t-j},\ t \in \mathbb{Z}. \tag{2.3}$$

The parameters $\omega_t > 0$, $\alpha_{ti} \geq 0$ ($i = 1, ..., q$) and $\beta_{tj} \geq 0$ ($j = 1, ..., p$) are periodic in t with period S in the sense $\omega_t = \omega_{t+kS}$, $\alpha_{ti} = \alpha_{t+kS,i}$ and $\beta_{tj} = \beta_{t+kS,j}$ for all $k, t \in \mathbb{Z}$. In a more compact form, Eq (2.3) can be written as follows

$$\lambda_{nS+s} = \omega_s + \sum_{i=1}^{q} \alpha_{si} Y_{nS+s-i} + \sum_{j=1}^{p} \beta_{sj} \lambda_{nS+s-j},\ n \in \mathbb{Z},\ 1 \leq s \leq S.$$

In this paper we consider a finite mixture generalization of the periodic INGARCH(p, q) model (2.2)–(2.3). Let a positive integer L, which refers to the number of regimes (or components). Let also $\{\Delta_t, t \in \mathbb{Z}\}$ be an ipd_S sequence of random variables, valued in the finite set $\{1, ..., L\}$ with distribution $P(\Delta_t = l) = \pi_t(l)$, where $\pi_t(l) \geq 0$, $\sum_{l=1}^{L} \pi_t(l) = 1$, and $\pi_t(l)$ is S-periodic in t in the sense $\pi_{t+hS}(l) = \pi_t(l)$ for all t, h and l. The values assumed by Δ_t are called components or regimes and the probability $\pi_t(l)$ is called the mixing proportion along the season or channel t.

A stochastic process $\{Y_t, t \in \mathbb{Z}\}$ is said to be a mixture periodic INGARCH(p, q) (henceforth MP-INGARCH(p, q)) model if its conditional distribution is a finite mixture of L mixing distributions, that is

$$Y_t|\mathcal{F}_{t-1} \sim \pi_t(1) F_{t,1,\lambda_{1,t}} + \cdots + \pi_t(L) F_{t,L,\lambda_{L,t}}, \tag{2.4}$$

where $F_\lambda = F_{t,l,\lambda}$ satisfies (2.1) and $\lambda_t := \lambda_{\Delta_t,t}$ is given by

$$\lambda_t = \omega_t(\Delta_t) + \sum_{i=1}^{q} \alpha_{ti}(\Delta_t) Y_{t-i} + \sum_{j=1}^{p} \beta_{tj}(\Delta_t) \lambda_{t-j}. \tag{2.5}$$

For all l, the parameters $\omega_t(l) > 0$, $\alpha_{ti}(l) \geq 0$ and $\beta_{tj}(l) \geq 0$ are S-periodic over t in the above sense. Equation (2.5) can be written in the periodic form

$$\lambda_{nS+s} = \omega_s(\Delta_{nS+s}) + \sum_{i=1}^{q} \alpha_{si}(\Delta_{nS+s}) Y_{nS+s-i} + \sum_{j=1}^{p} \beta_{sj}(\Delta_{nS+s}) \lambda_{nS+s-j},\ n \in \mathbb{Z},\ 1 \leq s \leq S,$$

where, for instance, $\omega_s(l)$ $(1 \leq s \leq S,\ 1 \leq l \leq L)$ denotes the intercept at season s and regime l. Naturally, when $L = 1$, the MP-INGARCH model given by (2.4)–(2.5) reduces to the periodic INGARCH model defined by (2.2)–(2.3) (cf. [11]).

For identifiability purposes, one can assume that for all $1 \leq v \leq S$, $\pi_v(1) \geq \pi_v(2) \geq \cdots \geq \pi_v(L)$. See also [2, 34, 40] for time-invariant mixtures corresponding to $S = 1$. Note, however, that the identifiability of the mixture INGARCH model is rather important for asymptotic estimation theory but is not required from a probabilistic point of view.

It is worth noting that the past recent values of $\lambda_{\Delta_t,t}$ in (2.5) depend on the past values of the regime variable Δ_t (see also [4, 9, 24, 34]). The likelihood of (2.4)–(2.5) is therefore not simple to obtain as it depends on the whole path information concerning Δ_t. A more general specification is

$$\lambda_t = g_{t,\Delta_t}(Y_{t-1}, \ldots, Y_{t-q}, \lambda_{t-1}, \ldots, \lambda_{t-p}), \tag{2.6}$$

where $g_{t,l}$ $(1 \leq l \leq L)$ is $[0, \infty)$-valued, and is S-periodic over t.

Denote by $\mathcal{F}_t^{\Delta}$ the sigma-field generated by $\{Y_i, \Delta_{i+1}, i \leq t\}$. The distribution given by (2.4) can be rewritten in function of Δ_t as follows

$$Y_t \mid \mathcal{F}_{t-1}^{\Delta} \sim F_{t,\Delta_t,\lambda_t}. \tag{2.7}$$

Model (2.4)–(2.5) encompasses several important mixtures of distributions such as:

i) The Poisson mixture

$$\pi_t(1)\mathcal{P}(\lambda_{1,t}) + \cdots + \pi_t(L)\mathcal{P}(\lambda_{L,t}). \tag{2.8}$$

ii) The quadratic negative binomial mixture (Aknouche and Francq, 2022)

$$\pi_t(1)\mathcal{NB}\left(r_t(1), \tfrac{r_t(1)}{r_t(1)+\lambda_{1,t}}\right) + \cdots + \pi_t(L)\mathcal{NB}\left(r_t(L), \tfrac{r_t(L)}{r_t(L)+\lambda_{L,t}}\right), \tag{2.9}$$

where $r_t(L) > 0$ $(l = 1, ..., L)$ is S-periodic over t.

iii) The linear negative binomial mixture

$$\pi_t(1)\mathcal{NB}\left(r_t(1)\lambda_{1,t}, \tfrac{r_t(1)}{r_t(1)+1}\right) + \cdots + \pi_t(L)\mathcal{NB}\left(r_t(L)\lambda_{L,t}, \tfrac{r_t(L)}{r_t(L)+1}\right). \tag{2.10}$$

iv) The mixture of different distributions such as, for example with $L = 3$,

$$\pi_t(1)\mathcal{P}(\lambda_{1t}) + \pi_t(2)\mathcal{NB}\left(r_t(2), \tfrac{r_t(2)}{r_t(2)+\lambda_{2,t}}\right) + \pi_t(3)\mathcal{NB}\left(r_t(3)\lambda_{3,t}, \tfrac{r_t(3)}{r_t(3)+1}\right).$$

3. Periodic ergodicity conditions: the linear conditional mean case

Conditions under which the MP-INGARCH process defined by (2.4) and (2.5) is strictly periodically stationary and periodically ergodic are now given. Let $m = \max(p, q)$ and set

$$c_{ti} = \sum_{l=1}^{L} \pi_t(l)\left(\alpha_{ti}(l) + \beta_{ti}(l)\right),\ i = 1, ..., m.$$

Consider the $m \times m$ companion matrix A_t given by

$$A_t = \begin{pmatrix} c_{t1} & c_{t2} & \cdots & c_{t,m-1} & c_{tm} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{3.1}$$

Let $\rho(B)$ denote the spectral radius of the matrix B, i.e., the maximum absolute eigenvalues of B.

Theorem 3.1. *Under*

$$\rho\left(\prod_{s=1}^{S} A_{S-s+1}\right) < 1, \tag{3.2}$$

there exists a strictly periodically stationary, and periodically ergodic process $\{Y_t,\ t \in \mathbb{Z}\}$ *satisfying*

$$P\left(Y_t \leq y \mid \mathcal{F}_{t-1}^{\Delta}\right) = F_{t,\Delta_t,\lambda_t}(y), \quad y \in \mathbb{R}, \tag{3.3}$$

where F_{t,Δ_t,λ_t} *fulfills* (2.1) *for all* $t \in \mathbb{Z}$, *and* λ_t *satisfies* (2.5).

Conversely, if there is a mean periodically stationary process $\{Y_t, t \in \mathbb{Z}\}$ *satisfying* (3.3) *with* $E(Y_t) < \infty$ *for all* $t \in \mathbb{Z}$, *then* (3.2) *holds.*

Proof. Let us prove the necessary part of the Theorem. If there exists a mean periodically stationary process $\{Y_t, t \in \mathbb{Z}\}$ satisfying (3.2) with $E(Y_t) = E(\lambda_t)$ for all t, then we have

$$E(Y_t) = \sum_{l=1}^{L} \pi_t(l)\,\omega_t(l) + \sum_{i=1}^{m}\sum_{l=1}^{L} \pi_t(l)\left(\alpha_{ti}(l) + \beta_{tj}(l)\right) E(Y_{t-i}). \tag{3.4}$$

Setting $\underline{Y}_t = (E(Y_t), ..., E(Y_{t-m+1}))'$ and $B_t = \left(\sum_{l=1}^{L} \pi_t(l)\,\omega_t(l), 0, ..., 0\right)'$, equality (3.4) can be rewritten in the following matrix equation

$$\underline{Y}_t = A_t \underline{Y}_{t-1} + B_t.$$

By iterating the latter equation S times while using the S-periodicity in-mean of the process, which implies that $E\left(\underline{Y}_{t-S}\right) = E\left(\underline{Y}_t\right)$, we get the following equation

$$\underline{Y}_t = (A_t \cdots A_{t-S+1})\,\underline{Y}_t + C_t, \tag{3.5}$$

where

$$C_t = \sum_{j=0}^{S-1}\prod_{i=0}^{j-1} A_{t-i} B_{t-j}.$$

Using Lemma A.1 in [4] and Corollary 8.1.29 in [31], equality (3.5) entails

$$\rho(A_t \cdots A_{t-S+1}) < 1,$$

which in turns is equivalent to (3.2).

We now prove the sufficiency part of the theorem. Let $\{U_t, t \in \mathbb{Z}\}$ be an iid uniformly distributed sequence in $[0,1]$, independent of $\{\Delta_t, t \in \mathbb{Z}\}$. For all $t \in \mathbb{Z}$ put

$$\lambda_t^{(k)} = \begin{cases} \omega_t(\Delta_t) + \sum\limits_{i=1}^{q} \alpha_{ti}(\Delta_t)\, Y_{t-i}^{(k-i)} + \sum\limits_{j=1}^{p} \beta_{tj}(\Delta_t)\,\lambda_{t-j}^{(k-j)} & \text{if } k \geq 1 \\ 0 & \text{if } k \leq 0, \end{cases} \tag{3.6}$$

and

$$Y_t^{(k)} = \begin{cases} F^{-}_{t,\Delta_t,\lambda_t^{(k)}}(U_t) & \text{if } k \geq 1 \\ 0 & \text{if } k \leq 0. \end{cases} \tag{3.7}$$

If $k \geq 2$, combining (3.6) and (3.7) we have

$$\lambda_t^{(k)} = \varphi_{t,k}(U_{t-1}, \ldots, U_{t-k+1}, \Delta_t, ..., \Delta_{t-k+1}),$$

where in view of the S-periodicity of the model parameters, the measurable function $\varphi_{t,k} : [0,1]^k \times \{1, ..., L\}^k \to [0, \infty)$ is S-periodic in t in the sense $\varphi_{t,k} = \varphi_{hS+t,k}$ for all $t, h \in \mathbb{Z}$. Therefore, the processes $\left\{\lambda_t^{(k)}, t \in \mathbb{Z}\right\}$ and $\left\{Y_t^{(k)}, t \in \mathbb{Z}\right\}$ are strictly periodically stationary and periodically ergodic (e.g., [6]) for all k. Now, let $\mathcal{F}_{t-1}^{(k)}$ and $\mathcal{F}_{t-1}^{*}$ be the σ-algebras generated by $\left\{Y_{t-i}^{(k-i)}, \Delta_{t-i+1}, i > 0\right\}$ and $\{U_i, \Delta_{i+1}, i < t\}$, respectively. Since the variable $F_\lambda^{-}(U)$ has the cdf F_λ when U is uniformly distributed in $[0, 1]$, it follows that

$$E\left(Y_t^{(k)} \mid \mathcal{F}_{t-1}^{(k)}\right) = E\left(Y_t^{(k)} \mid \mathcal{F}_{t-1}^{*}\right) = \lambda_t^{(k)}$$
$$P\left(Y_t^{(k)} \leq y \mid \mathcal{F}_{t-1}^{(k)}\right) = P\left(F^{-}_{t,\Delta_t,\lambda_t^{(k)}}(U_t) \leq y \mid \mathcal{F}_{t-1}^{*}\right) = F_{t,\Delta_t,\lambda_t^{(k)}}(y).$$

If we establish the following limiting result

$$\lambda_t = \lim_{k\to\infty} \lambda_t^{(k)}, \text{ a.s.,} \tag{3.8}$$

then the existence of a process satisfying (3.3), with $\mathcal{F}_{t-1}^{*}$ in place of $\mathcal{F}_{t-1}^{\Delta}$, is proved. Taking the limit as $k \to \infty$ on the two sides of the equalities (3.6) and (3.7), we obtain a.s.

$$Y_t = \lim_{k\to\infty} Y_t^{(k)} = F^{-}_{t,\Delta_t,\lambda_t}(U_t).$$

As λ_t is $\mathcal{F}_{t-1}^{\Delta}$-measurable, the distribution of Y_t given $\mathcal{F}_{t-1}^{*}$ is the same as that of Y_t given $\mathcal{F}_{t-1}^{\Delta}$. To show that (3.2) implies (3.8), let us first prove that the sequence $\left(\lambda_t^{(k)}\right)_k$ is increasing, i.e.,

$$0 \leq \lambda_t^{(k-1)} \leq \lambda_t^{(k)}, \text{ for all } k \tag{3.9}$$

and that

$$E\left(Y_t^{(k)}\right) \geq E\left(Y_t^{(k-1)}\right), \text{ for all } k. \tag{3.10}$$

When $k \leq 0$, the inequalities (3.9) and (3.10) are obviously satisfied. For $k \geq 1$, (3.9) is shown by induction. If

$$... \leq \lambda_t^{(k_0-1)} \leq \lambda_t^{(k_0)}$$

is satisfied then from (2.1) we obtain

$$\lambda_t^{(k_0+1)} = \omega_t(\Delta_t) + \sum_{i=1}^{q} \alpha_{ti}(\Delta_t) F^{-}_{\lambda_{t-i}^{(k_0+1-i)}}(U_{t-i}) + \sum_{j=1}^{p} \beta_{tj}(\Delta_t) \lambda_{t-j}^{(k_0+1-j)}$$
$$\geq \omega_t(\Delta_t) + \sum_{i=1}^{q} \alpha_{ti}(\Delta_t) F^{-}_{\lambda_{t-i}^{(k_0-i)}}(U_{t-i}) + \sum_{j=1}^{p} \beta_{tj}(\Delta_t) \lambda_{t-j}^{(k_0-j)}$$

$$= \lambda_t^{(k_0)}.$$

Since $E\left(Y_t^{(k)}\right) = E\left(\lambda_t^{(k)}\right)$ exists for any fixed k, we have

$$E\left(Y_t^{(k+1)}\right) = E\left(\lambda_t^{(k+1)}\right) \geq E\left(\lambda_t^{(k)}\right) = E\left(Y_t^{(k)}\right),$$

which establishes (3.10). Using (3.9) and (3.10), we have for all $k \geq 1$

$$E\left(\left|\lambda_t^{(k)} - \lambda_t^{(k-1)}\right|\right) \leq \sum_{i=1}^{m} \sum_{l=1}^{L} \pi_t(l)(\alpha_{ti}(l) + \beta_{ti}(l))E\left(\left|\lambda_{t-i}^{(k-i)} - \lambda_{t-i}^{(k-i-1)}\right|\right). \tag{3.11}$$

Letting $v_t^{(k)} = E\left(\left|\lambda_t^{(k)} - \lambda_t^{(k-1)}\right|\right)$ and $V_t^{(k)} = \left(v_t^{(k)}, ..., v_{t-m+1}^{(k-m+1)}\right)'$, inequality (3.11) can be written in the following matrix form

$$\begin{aligned} V_t^{(k)} &\leq A_t V_{t-1}^{(k-1)} \\ &\leq (A_t \cdots A_{t-S+1}) V_{t-S}^{(k-S)} \\ &= (A_t \cdots A_{t-S+1}) V_t^{(k-S)}, \end{aligned} \tag{3.12}$$

where A_t is given by (3.1), and the equality $V_{t-S}^{(k-S)} = V_t^{(k-S)}$, which follows from the S-periodicity of the coefficients, is used. Under (3.2) and (3.12), we get for all t

$$V_t^{(k)} \to 0 \text{ as } k \to \infty.$$

Thus the sequence $\left\{\lambda_t^{(k)}\right\}_k$ converges in L^1 and a.s. In addition, since

$$\begin{aligned} \lambda_t &= \lim_{k\to\infty} \varphi_{t,k}(U_{t-1}, \ldots, U_{t-k+1}, \Delta_t, , ...\Delta_{t-k+1}) \\ &: = \varphi_t(U_{t-1}, U_{t-2}, \ldots; \Delta_t, \Delta_{t-1}, ...), \end{aligned}$$

where $\varphi_t : [0, 1]^\infty \times \{1, ..., L\}^\infty \to [0, \infty)$ is measurable and S-periodic in t, the sequence $\{\lambda_t, t \in \mathbb{Z}\}$ is therefore strictly stationary and ergodic in the periodic meaning. □

In view of (3.5) and the S-periodicity of the model parameters, the S unconditional (seasonal) means of the MP-INGARCH$_S$ (p, q) model are given, under the periodic ergodicity condition (3.2), by

$$\begin{cases} \underline{\mu}_v = \left(I_m - \prod_{s=1}^{S} A_{S-s+1}\right)^{-1} \sum_{j=0}^{S-1} \prod_{i=0}^{j-1} A_{v-i} B_{v-j} \\ E(Y_v) = (1, 0, ..., 0)'_{1\times m}\, \underline{\mu}_v \end{cases}, \quad v \in \{1, ..., S\}, \tag{3.13}$$

where I_m stands for the identity matrix of order $m = \max(p, q)$.

4. MP-INGARCH(p, q) model with non-linear conditional means

This Section extends the periodic ergodicity conditions for the MP-INGARCH(p, q) model when the conditional mean λ_t has the more general nonlinear form (2.6). Suppose that for all $t \in \mathbb{Z}$, the

S-periodic function $g_{t,l}(y_1,\ldots,y_q,\lambda_1,\ldots,\lambda_p)$ is Lipschitz, that is for all $\left(y_i,y_i'\right)$, $i=1,\ldots,q$ and all $\left(\lambda_j,\lambda_j'\right)$, $j=1,\ldots,p$,

$$\left|g_{t,l}(y_1,\ldots,y_q,\lambda_1,\ldots,\lambda_p)-g_{t,l}(y_1',\ldots,y_q',\lambda_1',\ldots,\lambda_p')\right|\leq\sum_{i=1}^{q}\alpha_{ti}\,(l)\,|y_i-y_i'|+\sum_{j=1}^{p}\beta_{tj}\,(l)\,|\lambda_j-\lambda_j'|,\quad(4.1)$$

where $\alpha_{ti}\,(l)\geq 0$ and $\beta_{tj}\,(l)\geq 0$ are S-periodic over t. See also [34] and [4] for the non-periodic INGARCH setting. Let A_t be defined as in (3.1). The following result shows that condition (3.3) in which the coefficients of the matrix A_t are replaced by those of the Lipschitz inequality (4.1), is still sufficient for the existence of a periodically ergodic process verifying (2.7) and (2.6) (or equivalently (2.4) and (2.6))

Theorem 4.1. *There exists a strictly periodically stationary and periodically ergodic process* $\{Y_t,t\in\mathbb{Z}\}$ *with a conditional distribution given by* (2.1) *and* (2.7)*, where* λ_t *satisfies* (2.6) *and* (4.1)*, if*

$$\rho\left(A_tA_{t-1}\cdots A_{t-S+1}\right)<1.\quad(4.2)$$

Proof. Using similar arguments in the proof of Theorem 3.1, let $\{U_t,t\in\mathbb{Z}\}$ be defined as above. Define for all $t\in\mathbb{Z}$

$$\lambda_t^{(k)}=\begin{cases}g_{t,\Delta_t}(Y_{t-1}^{(k-1)},\ldots,Y_{t-q}^{(k-q)},\lambda_{t-1}^{(k-1)},\ldots,\lambda_{t-p}^{(k-p)}) & \text{if } k\geq 1\\ 0 & \text{if } k\leq 0\end{cases}\quad(4.3)$$

and let $Y_t^{(k)}$ be given as in (3.7). As for the proof of Theorem 3.1, under (4.2) we show the existence of a periodically ergodic process satisfying (2.1), (2.7), (2.6), and (4.1). This amounts to show the a.s. convergence of $\lambda_t^{(k)}$ given by (4.3) to the limit λ_t, which is given by (2.6). From (2.7) we have

$$\begin{aligned}E\left(|Y_t^{(k)}-Y_t^{(k-1)}|\right)&=EE\left(|Y_t^{(k)}-Y_t^{(k-1)}||\lambda_t^{(k)},\lambda_t^{(k-1)}\right)\\&=E\left(|\lambda_t^{(k)}-\lambda_t^{(k-1)}|\right).\end{aligned}$$

From (4.1) it follows that

$$E\left|\lambda_t^{(k)}-\lambda_t^{(k-1)}\right|\leq\sum_{i=1}^{m}\sum_{l=1}^{L}\pi_t\,(l)\,(\alpha_{ti}\,(l)+\beta_{ti}\,(l))E\left|\lambda_{t-i}^{(k-i)}-\lambda_{t-i}^{(k-i-1)}\right|\quad\forall k\geq 1.$$

The latter can be rewritten in the following inequality

$$V_t^{(k)}\leq A_t\cdots A_{t-S+1}V_t^{(k-S)},$$

where $V_t^{(k)}$ is defined as in (3.12). Under (4.2), $V_t^{(k)}\to 0$ from which the Theorem is established. □

5. Illustrations

5.1. On particular mixture periodic distributions

In this subsection, we apply the very general results given by Theorem 3.1 to various special cases.

Example 5.1. (First-order Poisson MP-INGARCH$(1,1)$ model)

For model (2.8) and (2.5), taking $p = q = 1$, condition (3.2) becomes

$$\prod_{s=1}^{S}\sum_{k=1}^{L}\pi_t(k)\left(\alpha_{t-s,1}(k)+\beta_{t-s,1}(k)\right)<1.$$

If in addition $L = 1$, the latter reduces to the periodic ergodicity condition given by [7] for the first-order periodic INGARCH(1, 1) model with a Poisson mixture conditional distribution. See also [13, 33, 35].

Example 5.2. (First-order Negative binomial MP-INGARCH(1, 1) model)

The condition (3.2) also applies to a vaster class of distributions given by (2.1) such as the negative binomial mixture. We consider here two instances:

- The quadratic negative binomial mixture ([5]) given by (2.9).
- The linear negative binomial mixture given by (2.10).

Example 5.3. (Non-mixed periodic INGARCH(p, q) model)

When $L = 1$, the matrix A_t in condition (3.2) reduces to

$$A_t=\begin{pmatrix}\alpha_{t1}+\beta_{t1} & \alpha_{t2}+\beta_{t2} & \cdots & \alpha_{t,m-1}+\beta_{t,m-1} & \alpha_{tm}+\beta_{tm}\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix},$$

which corresponds to the same periodic ergodicity condition given by [11].

Example 5.4. (The iid mixture INGARCH(p, q) model)

When $S = 1$ the coefficient c_{ti} in the matrix given by (3.1) becomes time-invariant, i.e.,

$$c_i=\sum_{l=1}^{L}\pi(l)\left(\alpha_i(l)+\beta_i(l)\right),\ i=1,...,m,$$

and therefore condition (3.2) reduces to

$$\sum_{l=1}^{L}\pi(l)\sum_{i=1}^{m}\left(\alpha_i(l)+\beta_i(l)\right)<1,$$

which is the condition (3.6) given by [4].

Example 5.5. (The standard INGARCH(p, q) model)

When $S = L = 1$, condition (3.2) reduces to

$$\rho(A)<1, \tag{5.1}$$

where

$$A=\begin{pmatrix}\alpha_1+\beta_1 & \alpha_2+\beta_2 & \cdots & \alpha_{m-1}+\beta_{m-1} & \alpha_m+\beta_m\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix},$$

and α_i ($i = 1, ..., q$) and β_j ($j = 1, ..., p$) are the conditional mean parameters of the time-invariant INGARCH(p, q) model corresponding to $S = 1$. According to Corollary 2.2 in [27], the condition (5.1) is equivalent to

$$\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1,$$

which is the standard ergodicity condition given by [4].

Example 5.6. (Mixture periodic INARCH(q) models)

When $p = 0$ in the general model (2.4)–(2.5), we obtain the particular mixture periodic INARCH model considered by [35]. So our condition (3.2) reduces to their periodic stationarity-in-mean condition. Note, however, that condition (3.2) also ensures the periodic ergodicity of the model, which has not been studied by [35]. Moreover, our condition also applies to larger classes of distributions, to general *GARCH* lags, and to non-linear conditional mean forms.

5.2. On simulated data: Monte Carlo estimations of the theoretical means

In this Subsection, the veracity of the unconditional mean formula (3.13) (which is based on condition (3.2) of Theorem 3.1) is assessed through a simulation study. Three mixture periodic distributions are considered.

i) The first one is the two-component Poisson MP-INGARCH$_4$(1, 1) model with period $S = 4$,

$$Y_t|\mathcal{F}_{t-1} \sim \pi_t(1)\mathcal{P}(\lambda_{1,t}) + \pi_t(2)\mathcal{P}(\lambda_{2,t}),$$

the parameters of which are reported in Table 1.

ii) The second one is the two-component (quadratic) negative binomial MP-INGARCH$_4$(1, 1) model with period $S = 4$:

$$Y_t|\mathcal{F}_{t-1} \sim \pi_t(1)\mathcal{NB}\left(r_t(1), \tfrac{r_t(1)}{r_t(1)+\lambda_{1,t}}\right) + \pi_t(2)\mathcal{NB}\left(r_t(2), \tfrac{r_t(2)}{r_t(2)+\lambda_{2,t}}\right),$$

where the corresponding parameters are shown in Table 2.

iii) The third case is the three-component Poisson MP-INARCH$_4$(1) model ($p = 0$) with period $S = 4$ (cf. Table 3), given by

$$Y_t|\mathcal{F}_{t-1} \sim \pi_t(1)\mathcal{P}(\lambda_{1,t}) + \pi_t(2)\mathcal{P}(\lambda_{2,t}) + \pi_t(3)\mathcal{P}(\lambda_{3,t}).$$

For each case, 1000 Monte Carlo replications with sample-size $T = 800$ (hence $N = 200$) are simulated, from which the S seasonal sample means $\overline{Y}_v = \frac{1}{N}\sum_{n=0}^{N-1} Y_{nS+v}$, for all $1 \leq v \leq S$, are obtained. Then, these seasonal sample means are compared with their seasonal theoretical counterparts obtained from (3.13) under the periodic stationarity condition (3.2). For the first case, Figure 1 shows the Boxplots of the 4 seasonal sample means and the corresponding theoretical means in solid line. Figures 2 and 3 show the same for cases 2 and 3, respectively.

Table 1. Parameters of the Poisson 2-mixture 4-Periodic INGARCH_4(1,1) model.

l	1				2				$E(Y_v)$
v	$\omega_v(1)$	$\alpha_v(1)$	$\beta_v(1)$	$\pi_v(1)$	$\omega_v(2)$	$\alpha_v(2)$	$\beta_v(2)$	$\pi_v(2)$	
1	$\omega_1(1)$	$\alpha_1(1)$	$\beta_1(1)$	$\pi_1(1)$	$\omega_1(2)$	$\alpha_1(2)$	$\beta_1(2)$	$\pi_1(2)$	
	1	0.1	0.5	0.3	1.5	0.2	0.3	0.7	7.200
2	$\omega_2(1)$	$\alpha_2(1)$	$\beta_2(1)$	$\pi_2(1)$	$\omega_2(2)$	$\alpha_2(2)$	$\beta_2(2)$	$\pi_2(2)$	
	0.5	0.2	0.3	0.7	0.7	0.1	0.5	0.3	4.499
3	$\omega_3(1)$	$\alpha_3(1)$	$\beta_3(1)$	$\pi_3(1)$	$\omega_3(2)$	$\alpha_3(2)$	$\beta_3(2)$	$\pi_3(2)$	
	3	0.3	0.1	0.4	4	0.4	0.4	0.6	6.497
4	$\omega_4(1)$	$\alpha_4(1)$	$\beta_4(1)$	$\pi_4(1)$	$\omega_4(2)$	$\alpha_4(2)$	$\beta_4(2)$	$\pi_4(2)$	
	5	0.4	0.2	0.2	6	0.3	0.6	0.8	11.208

Table 2. Parameters of the 2-component negative binomial MP-INGARCH_4(1,1) model.

l	1				2				$E(Y_v)$
v	$\omega_v(1)$	$\alpha_v(1)$	$\beta_v(1)$	$\pi_v(1)$	$\omega_v(2)$	$\alpha_v(2)$	$\beta_v(2)$	$\pi_v(2)$	
	$r_v(1)$				$r_v(2)$				
1	$\omega_1(1)$	$\alpha_1(1)$	$\beta_1(1)$	$\pi_1(1)$	$\omega_1(2)$	$\alpha_1(2)$	$\beta_1(2)$	$\pi_1(2)$	
	0.50	0.20	0.30	0.40	0.80	0.10	0.50	0.60	8.909
	$r_1(1)$	2.0			$r_1(2)$	1.0			
2	$\omega_2(1)$	$\alpha_2(1)$	$\beta_2(1)$	$\pi_2(1)$	$\omega_2(2)$	$\alpha_2(2)$	$\beta_2(2)$	$\pi_2(2)$	
	7.0	0.10	0.40	0.60	8.0	0.35	0.20	0.40	12.353
	$r_2(1)$	4.0			$r_2(2)$	2.0			
3	$\omega_3(1)$	$\alpha_3(1)$	$\beta_3(1)$	$\pi_3(1)$	$\omega_3(2)$	$\alpha_3(2)$	$\beta_3(2)$	$\pi_3(2)$	
	12.0	0.30	0.20	0.60	10.0	0.45	0.30	0.40	18.512
	$r_3(1)$	0.5			$r_3(2)$	1.0			
4	$\omega_4(1)$	$\alpha_4(1)$	$\beta_4(1)$	$\pi_4(1)$	$\omega_4(2)$	$\alpha_4(2)$	$\beta_4(2)$	$\pi_4(2)$	
	2.0	0.10	0.60	0.80	3.0	0.40	0.20	0.20	14.794
	$r_4(1)$	15.0			$r_4(2)$	10.0			

Table 3. Parameters of the Poisson 3-mixture 4-Periodic INARCH_4(1,1) model.

l	1			2			3			$E(Y_v)$
v	$\omega_v(1)$	$\alpha_v(1)$	$\pi_v(1)$	$\omega_v(2)$	$\alpha_v(2)$	$\pi_v(2)$	$\omega_v(3)$	$\alpha_v(3)$	$\pi_v(3)$	
1	$\omega_1(1)$	$\alpha_1(1)$	$\pi_1(1)$	$\omega_1(2)$	$\alpha_1(2)$	$\pi_1(2)$	$\omega_1(3)$	$\alpha_1(3)$	$\pi_1(3)$	
	0.5	0.2	0.4	0.8	0.1	0.3	0.7	0.1	0.3	9.113
2	$\omega_2(1)$	$\alpha_2(1)$	$\pi_2(1)$	$\omega_2(2)$	$\alpha_2(2)$	$\pi_2(2)$	$\omega_2(3)$	$\alpha_2(3)$	$\pi_2(3)$	
	7.0	0.1	0.2	8.0	0.35	0.2	9.0	0.1	0.6	12.439
3	$\omega_3(1)$	$\alpha_3(1)$	$\pi_3(1)$	$\omega_3(2)$	$\alpha_3(2)$	$\pi_3(2)$	$\omega_3(3)$	$\alpha_3(3)$	$\pi_3(3)$	
	12.0	0.3	0.3	10.0	0.45	0.4	11.0	0.2	0.3	18.775
4	$\omega_4(1)$	$\alpha_4(1)$	$\pi_4(1)$	$\omega_4(2)$	$\alpha_4(2)$	$\pi_4(2)$	$\omega_4(3)$	$\alpha_4(3)$	$\pi_4(3)$	
	2.0	0.1	0.2	3.0	0.4	0.5	4.0	0.1	0.3	14.848

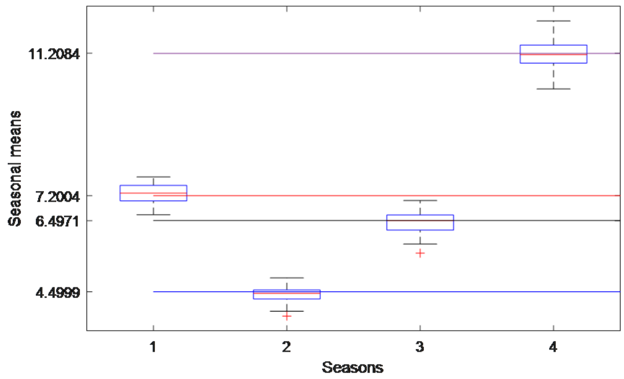

Figure 1. Theoretical and sample seasonal means for the 2-component, 4-periodic Poisson MP-INGARCH$_4(1, 1)$ model.

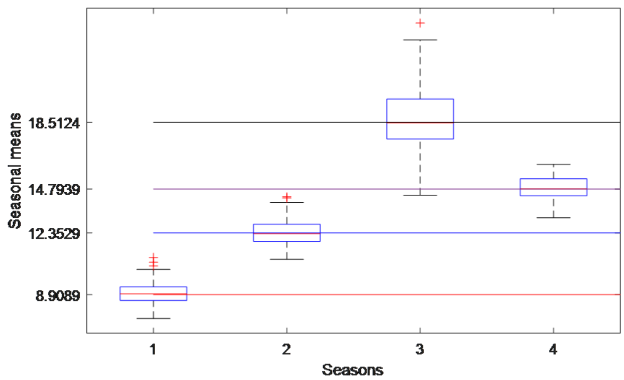

Figure 2. Theoretical and sample seasonal means for the 2-component, 4-periodic negative binomial MP-INGARCH$_4(1, 1)$ model.

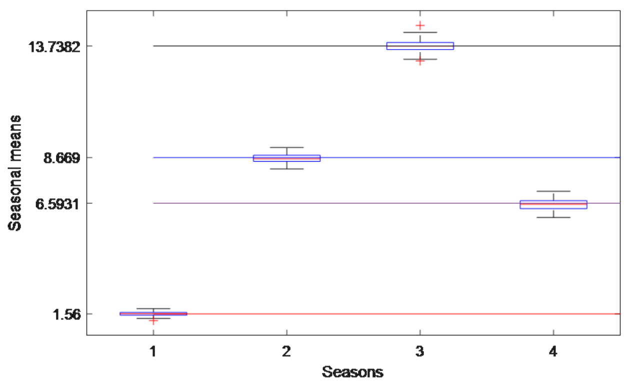

Figure 3. Theoretical and sample seasonal means for the 3-component, 4-periodic Poisson MP-INGARCH$_4(1, 1)$ model.

From Figures 1–3, it can be seen that the theoretical seasonal means are at the center of their corresponding Boxplots, which supports the above theoretical calculations. Moreover, the Boxplots are, in most cases, symmetric and are therefore consistent with the normality assumption, which follows from the central limit theorem for periodically ergodic processes.

6. Conclusions

In this paper, we examined some probability properties of a general mixture periodic integer-valued GARCH model, which can be used to model seasonally-varying integer-valued time series data. More precisely, we proposed strict periodic stationarity (and periodic ergodicity) conditions for a wide family of mixture distributions whose stochastic order is the same as the mean order (cf. [4]). The regime sequence driving the mixture feature is assumed to be independent and periodically distributed. Moreover, the lagged conditional means are governed by the lagged values of the mixture variable, which makes the model depends on the past of the regime variable. Therefore, the likelihood of the model is not easy to calculate, but the estimation of the latter can be done using the generalized method of moments ([28]), Bayesian MCMC methods ([12]), or particle filtering ([36]). On the other hand, the model we studied is also general from the form of the conditional means which can be linear or Lipschitz nonlinear.

Various extensions of the proposed model can be proposed. We mention in particular multivariate mixture periodic models and non-Lipschitz conditional functions such as threshold models.

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Conflict of interest

The author declares no conflicts of interest regarding this article.

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