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Research article

A novel high accurate numerical approach for the time-delay optimal control problems with delay on both state and control variables

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Abstract: In this study, we intend to present a numerical method with highly accurate to solve the time-delay optimal control problems with delay on both the state and control variables. These problems can be seen in many sciences such as medicine, biology, chemistry, engineering, etc. Most of the methods used to work out time delay optimal control problems have high complexity and cost of computing. We extend a direct Legendre-Gauss-Lobatto spectral collocation method for numerically solving the issues mentioned above, which have some difficulties with other methods. The simple structure, convergence, and high accuracy of our approach are the advantages that distinguish it from different processes. At first, by replacing the delay functions of state and control variables in the dynamical method, we propose an equivalent system. Then discretizing the problem at the collocation points, we achieve a nonlinear programming problem. We can solve this discrete problem to obtain the approximate solutions for the main problem. Moreover, we prove the gained approximate solutions convergent to the exact optimal solutions when the number of collocation points increases. Finally, we show the capability and the superiority of the presented method by solving some numeral examples and comparing the results with those of others.

Keywords: time delay optimal control problem; Legendre spectral method; nonlinear programming; Legendre-Gauss-Lobatto points; convergence analysis
Mathematics Subject Classification: 65M70, 49J15, 90C30

1. Introduction

Time delay systems are one of the essential systems which occur in applied models such as biological, chemical, transportation models [1–3]. In recent years by many researchers, these systems have been studied (see [4–7] and references therein).

Optimal control (OC) of time-delay systems [8–11] is a class of control problems that can not usually solve exactly. Hence, some usually direct and indirect method has been suggested to solve these problems numerically. Inoue et al. [12] have proposed a sensitivity approach to obtain the suboptimal control for linear systems with slight delays in the state. Guinn [13] has sketched a simple method for obtaining necessary conditions for control problems with a constant delay in the state variable. In [14], an optimal linear-quadratic regulator for a linear system with multiple time delays in the control input is propounded. Wu et al. in [15] have extended a computational method for solving an OC problem governed by a switched dynamical system with time delay. Jamshidi et al. in [16], the interaction prediction method of large-scale systems is developed for nonlinear systems with multi-delays in both control and state variables. In [17], an efficient numerical way for finding the solution of piecewise constant delay systems based on hybrid block-pulse functions and Chebyshev polynomials is suggested. Hashemi and Asadi [18] have introduced a wavelet collocation approach for finding the approximate OC of the nonlinear time-delay systems. A suboptimal control was designed using functional analysis and linear programming theories and optimizing an appropriate cost function by Kushkouei et al. [19]. In [20], an approximate scheme using Haar wavelets for solving time delayed OC (TDOC) problems with terminal inequality constraints is presented. Sun and Huang [21] applied a brand-new nonlinear programming method and line-up competition algorithm based on the principle of evolution to solve TDOC problems. Kushner and Barnea in [22] proposed a state feedback control for TDOC problems, including quadratic objective functional and linear delay integro-differential constraint.

Spectral and pseudospectral methods have been used extensively for smooth and nonsmooth non-delayed OC problems, and convergence analysis has been proposed by many researchers for these problems. These methods were developed mainly in the 1970s for solving partial differential equations arising in fluid dynamics and meteorology [23]. Over the past decade, pseudospectral methods have been used as one of the most efficient numerical methods for solving constrained nonlinear optimal control problems [24–26]. The main point in pseudospectral methods is that they avoid the poor behavior of the classical polynomial interpolation method by removing the restriction to equally spaced interpolation points. In the pseudospectral method, orthogonal polynomials or interpolating polynomials are utilized to approximate the control and state variables. The collocation points are considered the roots of some orthogonal polynomials or their derivatives. In these approximations, Lagrange polynomials are the trial functions, and the control and state variables at the collocation points are the unknown coefficients. These methods have some crucial advantages over other methods. One noticeable advantage of using pseudospectral methods is the high accuracy of pseudospectral approximations [27, 28]. In [23], it has been demonstrated the pseudospectral methods presented a convergence rate that is faster than any convergence rate for approximate analytic functions.

Indirect spectral methods for solving OC problems depend on the initial guesses for unknown variables, and hence a relatively accurate initial guess is needed to converge the technique. Also, if the OC problem has non-equal constraints, we need to know about the intervals where the optimal solution lies on the boundary of the limitations. So, in this paper, we propose a direct Legendre-Gauss-Lobatto spectral collocation (LGLSC) method for the numerical solution of the TDOC problems and prove the convergence of the present method by providing some theorems. The considered TDOC problem includes a dynamical system with delay on both the state and control variables.

This article contains the following sections: In Section 2, the TDOC problem with delay on both the state and control variables is given. A technique is suggested to convert the TDOC problem into an equivalent continuous-time problem. We present a direct LGLSC method for numerically solving the obtained problem in Section 3. Nonlinear programming is also suggested. Section 4 deals with the existence and convergence of gained approximate solutions to the exact optimal solutions. In Section 5, four examples are presented to evaluate the method's performance. The Conclusions are explained in Section 6.

2. Problem statement

We consider the smooth TDOC problem

$$\text{Minimize} \quad J(\zeta(\cdot), \vartheta(\cdot)) = \Lambda(\zeta(T), T) + \int_0^T h(\zeta(t), \vartheta(t), t)\mathrm{d}t, \tag{2.1}$$

subject to

$$\dot{\zeta}(t) = g(\zeta(t), \zeta(t-\mu_1), \vartheta(t), \vartheta(t-\mu_2), t), \quad 0 \le t \le T, \tag{2.2}$$
$$\zeta(t) = \Phi(t), \quad -\mu_1 \le t \le 0, \tag{2.3}$$
$$\vartheta(t) = \Psi(t), \quad -\mu_2 \le t \le 0, \tag{2.4}$$
$$e(\zeta(t), \vartheta(t)) \le 0, \quad 0 \le t \le T, \tag{2.5}$$

where $\zeta : [0, T] \to \Omega_1 \subseteq \mathbb{R}^m$, $\vartheta : [0, T] \to \Omega_2 \subseteq \mathbb{R}^p$ are the state and control variables, respectively, $\Lambda : \mathbb{R}^m \times [0, T] \to \mathbb{R}$, $h : \mathbb{R}^m \times \mathbb{R}^p \times [0, T] \to \mathbb{R}$, $e : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^d$, $g : \mathbb{R}^{2m} \times \mathbb{R}^{2p} \times [0, T] \to \mathbb{R}^m$ are given continuously differentiable functions, μ_1 and μ_2 are two constant positive scalars such that $\mu_1 < \mu_2$ and $\Phi : [0, T] \to \mathbb{R}^m$, $\Psi : [0, T] \to \mathbb{R}^p$ are two arbitrary continuous functions. Every piecewise continuous function $\vartheta(\cdot)$ on $[0, T]$ is said an admissible control for TDOC problems (2.1)–(2.5). Also, related to the existence of optimal solution for OC problems we refer the reader to [29–31].

The aim is to find state $\zeta(t) = (\zeta_1(t), \zeta_2(t), \ldots, \zeta_m(t))$ and admissible control $\vartheta(t) = (\vartheta_1(t), \vartheta_2(t), \ldots, \vartheta_p(t))$ on interval $[0, T]$ so that objective functional (2.1) be minimized. We assume that sets Ω_1 and Ω_2 are compact. Here, we propose a LGLSC method to solve TDOC problems (2.1)–(2.5). At first, by replacing the delay functions $\Phi(t)$ for $t \in [0, \mu_1]$, and $\Psi(\cdot)$ for $t \in [0, \mu_2]$ in dynamical systems, we receive the following equivalent system.

$$\text{Minimize} \quad J(\zeta(\cdot), \vartheta(\cdot)) = \Lambda(\zeta(T), T) + \int_0^T h(\zeta(t), \vartheta(t), t)\mathrm{d}t, \tag{2.6}$$

subject to

$$\dot{\zeta}(t)=\begin{cases} g(\zeta(t),\Phi(t-\mu_1),\vartheta(t),\Psi(t-\mu_2),t), & 0\le t\le \mu_1,\\ g(\zeta(t),\zeta(t-\mu_1),\vartheta(t),\Psi(t-\mu_2),t), & \mu_1< t\le \mu_2,\\ g(\zeta(t),\zeta(t-\mu_1),\vartheta(t),\vartheta(t-\mu_2),t), & \mu_2< t\le T,\end{cases} \tag{2.7}$$

$$e(\zeta(t),\vartheta(t))\le 0,\quad 0\le t\le T, \tag{2.8}$$

$$\zeta(0)=\Phi(0),\quad \vartheta(0)=\Psi(0). \tag{2.9}$$

It is easy to show that TDOC problems (2.6)–(2.9) is equivalent to problems (2.1)–(2.5).

3. Implementation of the LGLSC method

Now, suppose $\tau_0<\tau_1<\cdots<\tau_M$ are the Legendre-Gauss-Lobatto (LGL) points on the interval $[-1,1]$, which they are roots of $(\tau^2-1)\dot{P}_M(\tau)$, wherein $P_M(\cdot)$ denotes the Legendre polynomial of degree M which defined by the following recursive relation

$$(i+1)P_{i+1}(\tau)=(2i+1)\tau P_i(\tau)-iP_{i-1}(\tau),\quad i\ge 1,$$

where $P_0(\tau)=1$ and $P_1(\tau)=\tau$ for $\tau\in[-1,1]$.

To discretize the problems (2.6)–(2.9), we transform the LGL points to the interval $[0,T]$ by transformation

$$t_j=\frac{T(\tau_j+1)}{2},\quad j=0,1,\dots,M. \tag{3.1}$$

Also, we approximate the state and control variables on the interval $[0,T]$, as follows

$$\zeta(t)\simeq\zeta_M(t)=\sum_{i=0}^{M}\bar{\zeta}_i L_i(t),\ \ \vartheta(t)\simeq\vartheta_M(t)=\sum_{i=0}^{M}\bar{\vartheta}_i L_i(t),\quad 0\le t\le T, \tag{3.2}$$

where

$$\zeta(t_i)\simeq\zeta_M(t_i)=\bar{\zeta}_i,\quad \vartheta(t_i)\simeq\vartheta_M(t_i)=\bar{\vartheta}_i, \tag{3.3}$$

are known coefficients and $L_i(\cdot)$ is the Lagrange polynomial of degree M, defined by

$$L_i(t)=\prod_{j=0,j\ne i}^{M}\frac{t-t_j}{t_i-t_j},\quad i=0,1,\dots M. \tag{3.4}$$

Moreover, the derivative of the state variable is approximated at the LGL points as

$$\dot{\zeta}(t_i)\simeq\sum_{j=0}^{M}\zeta(t_j)\dot{L}_j(t_i)=\sum_{j=0}^{M}\bar{\zeta}_j D_{ij},\quad i=0,1,\dots M, \tag{3.5}$$

where D is the derivative matrix, defined as follows (see [32] for details)

$$
D_{ij} = \begin{cases} \dfrac{2}{T}.\dfrac{-M(M+1)}{4}, & i = j = 0, \\ \dfrac{2}{T}.\dfrac{M(M+1)}{4}, & i = j = M, \\ \dfrac{\hat{P}_M(t_i)}{\hat{P}_M(t_j)}.\dfrac{1}{t_i - t_j}, & i \neq j, \\ 0, & \text{otherwise.} \end{cases}
$$

where $\hat{P}_M(t_p) = P_M(\frac{T}{2}\tau_p + \frac{T}{2})$ for $p = 0, 1, \ldots, M$.

Now, the integral term in the cost functional (2.1) can be approximated with the following lemma.

Lemma 3.1. *(see Theorem 3.29 in [33] and Lemma 2 in [35]) Assume that* $\{t_i\}_{i=0}^M$ *are the shifted LGL points on interval* $[0, T]$. *Then for any continuous function* $\eta(\cdot)$ *on* $[0, T]$ *we have*

$$
\int_0^T \eta(t)\mathrm{d}t = \lim_{M\to\infty} \frac{T}{2} \sum_{i=0}^{M} w_i \eta(t_i),
$$

where $\{w_i\}_{i=0}^M$ *are the LGL weights as*

$$
w_i = \frac{2}{M(M+1)[P_M(\frac{2t_i}{T} - 1)]^2}, \quad i = 0, 1, \ldots, M. \tag{3.6}
$$

By applying relations (3.2), (3.3), (3.5), and Lemma (3.1), the TDOC problems (2.1)–(2.5), are converted into the following nonlinear programming (NLP) problem:

$$
\text{Minimize} \quad \bar{J}_M(\bar{\zeta}, \bar{\vartheta}) = \Lambda(\bar{\zeta}_M, t_M) + \frac{T}{2} \sum_{l=0}^{M} w_l h(\bar{\zeta}_l, \bar{\vartheta}_l, t_l), \tag{3.7}
$$

subject to

$$
\sum_{j=0}^{M} \bar{\zeta}_j D_{ij} = g(\bar{\zeta}_i, \Phi(t_i - \mu_1), \bar{\vartheta}_i, \Psi(t_i - \mu_2), t_i), \quad i = 0, 1, \ldots, l_{\mu_1}, \tag{3.8}
$$

$$
\sum_{j=0}^{M} \bar{\zeta}_j D_{ij} = g(\bar{\zeta}_i, \sum_{j=0}^{M} \bar{\zeta}_j L_j(t_i - \mu_1), \bar{\vartheta}_i, \Psi(t_i - \mu_2), t_i), \quad i = l_{\mu_1} + 1, \ldots, l_{\mu_2}, \tag{3.9}
$$

$$
\sum_{j=0}^{M} \bar{\zeta}_j D_{ij} = g(\bar{\zeta}_i, \sum_{j=0}^{M} \bar{\zeta}_j L_j(t_i - \mu_1), \bar{\vartheta}_i, \sum_{j=0}^{M} \bar{\vartheta}_j L_j(t_i - \mu_2), t_i), \quad i = l_{\mu_2} + 1, \ldots, M, \tag{3.10}
$$

$$
e(\bar{\zeta}_i, \bar{\vartheta}_i) \leq 0, \quad i = 0, 1, \ldots, M, \tag{3.11}
$$

$$
\zeta(0) = \Phi(0), \quad \vartheta(0) = \Psi(0). \tag{3.12}
$$

Here, we choose the indexes l_{μ_1} and l_{μ_2} such that satisfies in $t_{l_{\mu_1}} \leq \mu_1 < t_{l_{\mu_1}+1} < t_{l_{\mu_2}} \leq \mu_2 < t_{l_{\mu_2}+1} < t_M = T$.

We solve the NLP problems (3.7)–(3.12) and find an approximate optimal solution to the TDOC problems (2.1)–(2.5).

4. Convergence analysis

In the following, we intend to prove the convergence of the proposed method. We can show that the sequence of obtained interpolating polynomials is converged to an optimal solution.

The Sobolev space $W_m^{n,\infty}$ for $n \geq 2$ and $m \geq 1$ with its norm that defined below, contains all functions $\xi : [0, T] \to \mathbb{R}^m$ whose the derivative $\xi^{(j)}(\cdot)$ for every $0 \leq j \leq n$ lie in L^∞ space. The norm of space $W_m^{n,\infty}$ is as

$$\left\|\xi(\cdot)\right\|_{n,\infty} = \sup_{j=0,1,\ldots,n} \left\|\xi^{(j)}(\cdot)\right\|_{L^\infty} = \sup_{j=0,1,\ldots,n} \left(\sup_{0\leq t\leq T} \left\|\xi^{(j)}(t)\right\|_\infty \right) = \sup_{j=0,1,\ldots,n} \left(\sup_{0\leq t\leq T} \left(\sup_{i=1,2,\ldots,m} \left|\xi_i^{(j)}(t)\right| \right) \right),$$

where $\xi_i^{(j)}(\cdot)$ is the $i-th$ component of vector function $\xi^{(j)}(\cdot)$. In fact, the Sobolev space is defined as

$$W_m^{n,\infty} = \{\xi : [0, T] \to \mathbb{R}^m, \quad \left\|\xi(\cdot)\right\|_{n,\infty} < \infty\}.$$

Lemma 4.1. *([34]) For any given function $\xi(\cdot) \in W_m^{n,\infty}$, there is a polynomial $s_M(\cdot)$ of degree M or less, so that*

$$\left\|\xi(\cdot) - s_M(\cdot)\right\|_{n,\infty} \leq BB_0 M^{-n},$$

where B is a constant independent of M and $B_0 = \left\|\xi(\cdot)\right\|_{n,\infty}$.

The NLP problems (3.7)–(3.12) can be easily to guarantee the feasibility as follows

$$\text{Minimize} \quad \bar{J}_M(\bar{\zeta}, \bar{\vartheta}) = \Lambda(\bar{\zeta}_M, t_M) + \frac{T}{2} \sum_{l=0}^{M} w_l h(\bar{\zeta}_l, \bar{\vartheta}_l, t_l), \tag{4.1}$$

subject to

$$\left\| \sum_{j=0}^{M} \bar{\zeta}_j D_{ij} - g(\bar{\zeta}_i, \Phi(t_i - \mu_1), \bar{\vartheta}_i, \Psi(t_i - \mu_2), t_i) \right\|_\infty \leq (M-1)^{\frac{3}{2}-n}, \quad i = 0, 1, \ldots, l_{\mu_1}, \tag{4.2}$$

$$\left\| \sum_{j=0}^{M} \bar{\zeta}_j D_{ij} - g(\bar{\zeta}_i, \sum_{j=0}^{M} \bar{\zeta}_j L_j(t_i - \mu_1), \bar{\vartheta}_i, \Psi(t_i - \mu_2), t_i) \right\|_\infty \leq (M-1)^{\frac{3}{2}-n}, \quad i = l_{\mu_1} + 1, \ldots, l_{\mu_2}, \tag{4.3}$$

$$\left\| \sum_{j=0}^{M} \bar{\zeta}_j D_{ij} - g(\bar{\zeta}_i, \sum_{j=0}^{M} \bar{\zeta}_j L_j(t_i - \mu_1), \bar{\vartheta}_i, \sum_{j=0}^{M} \bar{\vartheta}_j L_j(t_i - \mu_2), t_i) \right\|_\infty \leq (M-1)^{\frac{3}{2}-n}, \quad i = l_{\mu_2} + 1, \ldots, M, \tag{4.4}$$

$$e(\bar{\zeta}_i, \bar{\vartheta}_i) \leq (M-1)^{\frac{3}{2}-n}.\mathbf{1}, \quad i = 0, 1, \ldots, M, \tag{4.5}$$

$$\left\|\bar{\zeta}_0 - \Phi(0)\right\|_\infty \leq (M-1)^{\frac{3}{2}-n}, \ \left\|\bar{\vartheta}_0 - \Psi(0)\right\|_\infty \leq (M-1)^{\frac{3}{2}-n}, \tag{4.6}$$

where $\mathbf{1}$ denotes $[1, 1, \ldots, 1]^T$.

Theorem 4.1. *([35]) Suppose that $(\zeta(\cdot), \vartheta(\cdot))$ for $n \geq 2$ is a feasible solution for the problems* (3.7)–(3.12) *where $\zeta(\cdot) \in W_m^{n,\infty}$. Then there exists $M_1 \in \mathbb{N}$ so that for any $M > M_1$, problems* (4.1)–(4.6) *has* *a feasible solution* $\bar{\zeta} = (\bar{\zeta}_0, \bar{\zeta}_1, \ldots, \bar{\zeta}_M)$ *and* $\bar{\vartheta} = (\bar{\vartheta}_0, \bar{\vartheta}_1, \ldots, \bar{\vartheta}_M)$*. Furthermore, the feasible solution satisfies*

$$\bar{\vartheta}_i = \vartheta(t_i), \quad \left\|\zeta(t_i) - \bar{\zeta}_i\right\|_\infty \leq L(M-1)^{1-n}, \quad i = 0, 1, \ldots M, \tag{4.7}$$

where $\{t_i\}_{i=0}^M$ *are the LGL points and* L *is a positive constant independent of* M*.*

Proof. According to Lemma (4.1), there exists a polynomial $s(\cdot)$ of degree $(M-1)$ and a constant B_1 independent of M, so that

$$\left\|\dot{\zeta}(\cdot) - s(\cdot)\right\|_{n,\infty} \leq B_1(M-1)^{1-n}.$$

We defined

$$\zeta_M(t) = \int_0^t s(\eta)\mathrm{d}\eta + \phi(0), \quad \vartheta_M(t) = \vartheta(t), \quad 0 \leq t \leq T.$$

So we get

$$\dot{\zeta}_M(t) = s(t), \quad \zeta_M(0) = \phi(0).$$

Therefore

$$\begin{aligned} \left\|\zeta(\cdot) - \zeta_M(\cdot)\right\|_{L^\infty} &= \left\|\int_0^t (\dot{\zeta}(v) - s(v))\mathrm{d}v\right\|_{L^\infty} \leq \int_0^t \left\|\dot{\zeta}(\cdot) - s(\cdot)\right\|_{L^\infty}\mathrm{d}v \leq \int_0^t \left\|\dot{\zeta}(\cdot) - s(\cdot)\right\|_{n,\infty}\mathrm{d}v \\ &\leq B_1(M-1)^{1-n}\int_0^t \mathrm{d}v \leq B_1 T(M-1)^{1-n}. \end{aligned} \tag{4.8}$$

Now, we show that $(\bar{\zeta}_i, \bar{\vartheta}_i) = (\bar{\zeta_M}(t_i), \vartheta(t_i))$ for $i = 0, 1, \ldots, M$ satisfy the constraints of problems (4.1)–(4.6). Since functions g and h are continuously differentiable on compact sets $\Omega_1^2 \times \Omega_2^2 \times [0, T]$ and $\Omega_1 \times \Omega_2 \times [0, T]$, respectively, there exist constants N_1, N_2 and N_3 such that

$$\left|g(\xi_1, \xi_2, \xi_3, \xi_4, t) - g(\eta_1, \eta_2, \eta_3, \eta_4, t)\right| \leq N_1\Big(\left\|\xi_1 - \eta_1\right\|_\infty + \left\|\xi_2 - \eta_2\right|_\infty + \left\|\xi_3 - \eta_3\right\|_\infty + \left\|\xi_4 - \eta_4\right\|_\infty\Big), \tag{4.9}$$

$$\left|h(\xi_1, \xi_3, t) - h(\eta_1, \eta_3, t)\right| \leq N_2\Big(\left\|\xi_1 - \eta_1\right\|_\infty + \left\|\xi_3 - \eta_3\right\|_\infty\Big), \tag{4.10}$$

$$\left\|e(\xi_1, \xi_3) - e(\eta_1, \eta_3)\right\|_\infty \leq N_3\Big(\left\|\xi_1 - \eta_1\right\|_\infty + \left\|\xi_3 - \eta_3\right\|_\infty\Big), \tag{4.11}$$

for all $(\xi_1, \xi_2, \xi_3, \xi_4, t)$ and $(\eta_1, \eta_2, \eta_3, \eta_4, t)$ in $\Omega_1^2 \times \Omega_2^2 \times [0, T]$. Further, since $\zeta_M(\cdot)$ is a polynomial of degree less than or equal to M, it's derivative at the LGL points can be written as

$$\dot{\zeta}_M(t_i) = \sum_{j=0}^M \bar{\zeta}_j D_{ij}, \quad i = 0, 1, \ldots, M. \tag{4.12}$$

Hence, by (4.8)–(4.12) for $i = 0, 1, \ldots, l_{\mu_1}$ we have

$$\begin{aligned} &\left\|\sum_{j=0}^M \bar{\zeta}_j D_{ij} - g(\bar{\zeta}_i, \Phi(t_i - \mu_1), \bar{\vartheta}_i, \Psi(t_i - \mu_2), t_i)\right\|_\infty \\ &\quad \leq \left\|\dot{\zeta}_M(t_i) - \dot{\zeta}(t_i)\right\|_\infty + \left\|\dot{\zeta}(t_i) - g(\bar{\zeta}_i, \Phi(t_i - \mu_1), \bar{\vartheta}_i, \Psi(t_i - \mu_2), t_i)\right\|_\infty \end{aligned}$$

$$
\begin{aligned}
&= \left\|s(t_i) - \dot{\zeta}(t_i)\right\|_\infty + \left\|g(\zeta(t_i), \Phi(t_i - \mu_1), \vartheta(t_i), \Psi(t_i - \mu_2), t_i) - g(\bar{\zeta}_i, \Phi(t_i - \mu_1), \bar{\vartheta}_i, \Psi(t_i - \mu_2), t_i)\right\|_\infty \\
&\leq B_1(M-1)^{1-n} + N_1(\left\|\zeta(t_i) - \bar{\zeta}_i\right\|_\infty + \left\|\vartheta(t_i) - \bar{\vartheta}_i\right\|_\infty) \\
&\leq B_1(M-1)^{1-n} + N_1 B_1 T (M-1)^{1-n} \\
&= B_1(M-1)^{1-n}(1 + N_1 T),
\end{aligned}
$$

for $i = l_{\mu_1} + 1, \ldots, l_{\mu_2}$ we get

$$
\begin{aligned}
&\Big\| \sum_{j=0}^{M} \bar{\zeta}_j D_{ij} - g(\bar{\zeta}_i, \sum_{j=0}^{M} \bar{\zeta}_j L_j(t_i - \mu_1), \bar{\vartheta}_i, \Psi(t_i - \mu_2), t_i) \Big\|_\infty \\
&\quad \leq \left\|\dot{\zeta}_M(t_i) - \dot{\zeta}(t_i)\right\|_\infty + \left\|\dot{\zeta}(t_i) - g(\bar{\zeta}_i, \zeta_M(t_i - \mu_1), \bar{\vartheta}_i, \Psi(t_i - \mu_2), t_i)\right\|_\infty \\
&\quad \leq \left\|\dot{\zeta}_M(t_i) - \dot{\zeta}(t_i)\right\|_\infty + \left\|g(\zeta(t_i), \zeta(t_i - \mu_1), \vartheta(t_i), \Psi(t_i - \mu_2), t_i) - g(\bar{\zeta}_i, \zeta_M(t_i - \mu_1), \bar{\vartheta}_i, \Psi(t_i - \mu_2), t_i)\right\|_\infty \\
&\quad \leq B_1(M-1)^{1-n} + N_1\Big(\left\|\zeta(t_i) - \bar{\zeta}_i\right\|_\infty + \left\|\zeta(t_i - \mu_1) - \zeta_M(t_i - \mu_1)\right\|_\infty + \left\|\vartheta(t_i) - \bar{\vartheta}_i\right\|_\infty\Big) \\
&\quad \leq B_1(M-1)^{1-n} + N_1(B_1 T (M-1)^{1-n} + B_1 T (M-1)^{1-n}) \\
&\quad = B_1(M-1)^{1-n}(1 + 2N_1 T),
\end{aligned}
$$

and for $i = l_{\mu_2} + 1, \ldots, M$ we obtain

$$
\begin{aligned}
&\Big\| \sum_{j=0}^{M} \bar{\zeta}_j D_{ij} - g(\bar{\zeta}_i, \sum_{j=0}^{M} \bar{\zeta}_j L_j(t_i - \mu_1), \bar{\vartheta}_i, \sum_{j=0}^{M} \bar{\vartheta}_j L_j(t_i - \mu_2), t_i) \Big\|_\infty \\
&\quad \leq \left\|\dot{\zeta}_M(t_i) - \dot{\zeta}(t_i)\right\|_\infty + \left\|\dot{\zeta}(t_i) - g(\bar{\zeta}_i, \zeta_M(t_i - \mu_1), \bar{\vartheta}_i, \vartheta_M(t_i - \mu_2), t_i)\right\|_\infty \\
&\quad \leq \left\|\dot{\zeta}_M(t_i) - \dot{\zeta}(t_i)\right\|_\infty + \left\|g(\zeta(t_i), \zeta(t_i - \mu_1), \vartheta(t_i), \vartheta(t_i - \mu_2), t_i) - g(\bar{\zeta}_i, \zeta_M(t_i - \mu_1), \bar{\vartheta}_i, \vartheta_N(t_i - \mu_2), t_i)\right\|_\infty \\
&\quad \leq B_1(M-1)^{1-n} + N_1\Big(\left\|\zeta(t_i) - \bar{\zeta}_i\right\|_\infty + \left\|\zeta(t_i - \mu_1) - \zeta_M(t_i - \mu_1)\right\|_\infty + \left\|\vartheta(t_i) - \bar{\vartheta}_i\right\|_\infty + \left\|\vartheta(t_i - \mu_2) - \vartheta_M(t_i - \mu_2)\right\|_\infty \\
&\quad \leq B_1(M-1)^{1-n} + N_1(B_1 T (M-1)^{1-n} + B_1 T (M-1)^{1-n}) \\
&\quad = B_1(M-1)^{1-n}(1 + 2N_1 T).
\end{aligned}
$$

About (4.5), by (2.9) and (4.11) we gain (for $i = 0, 1, \ldots, N$)

$$
e(\bar{\zeta}_i, \bar{\vartheta}_i) = e(\bar{\zeta}_i, \vartheta(t_i)) \leq e(\zeta(t_i), \vartheta(t_i)) + N_3\|\bar{\zeta}_i - \zeta(t_i)\|_\infty.\mathbf{1} \leq N_3\|\bar{\zeta}_i - \zeta(t_i)\|_\infty.\mathbf{1} \leq N_3 B_1 T (M-1)^{1-n}.\mathbf{1}.
$$

Therefore with choosing $M_1 \in \mathbb{N}$, so that $\max\{B_1(1 + 2N_1 T), N_3 B_1 T\} \leq (M-1)^{\frac{1}{2}}$, we get

$$
\begin{aligned}
&\Big\| \sum_{j=0}^{M} \bar{\zeta}_j D_{ij} - g(\bar{\zeta}_i, \Phi(t_i - \mu_1), \bar{\vartheta}_i, \Psi(t_i - \mu_2), t_i) \Big\|_\infty \leq (M-1)^{\frac{3}{2}-n}, \quad i = 0, 1, \ldots, l_{\mu_1}, \\
&\Big\| \sum_{j=0}^{M} \bar{\zeta}_j D_{ij} - g(\bar{\zeta}_i, \sum_{j=0}^{M} \bar{\zeta}_j L_j(t_i - \mu_1), \bar{\vartheta}_i, \Psi(t_i - \mu_2), t_i) \Big\|_\infty \leq (M-1)^{\frac{3}{2}-n}, \quad i = l_{\mu_1} + 1, \ldots, l_{\mu_2}, \\
&\Big\| \sum_{j=0}^{M} \bar{\zeta}_j D_{ij} - g(\bar{\zeta}_i, \sum_{j=0}^{M} \bar{\zeta}_j L_j(t_i - \mu_1), \bar{\vartheta}_i, \sum_{j=0}^{M} \bar{\vartheta}_j L_j(t_i - \mu_2), t_i) \Big\|_\infty \leq (M-1)^{\frac{3}{2}-n}, \quad i = l_{\mu_2} + 1, \ldots, M, \\
&e(\bar{\zeta}_i, \bar{\vartheta}_i) \leq (M-1)^{\frac{3}{2}-n}.\mathbf{1}, \quad i = 0, 1, \ldots, M,
\end{aligned}
$$

for all $M \geq M_1$. Moreover $\bar{\zeta}_0 = \Phi(0)$ and $\bar{\vartheta}_0 = \Psi(0)$. Thus $(\bar{\zeta}, \bar{\vartheta})$ satisfies (4.2)–(4.6). □

Now, assume $\{(\bar{\zeta}_i^*, \bar{\vartheta}_i^*)\}_{i=0}^M$ is an optimal solution to the problems (4.1)–(4.6). Define

$$\zeta_M^*(t) = \sum_{i=0}^{M} \bar{\zeta}_i^* L_i(t), \quad t \in [0, T], \tag{4.13}$$

$$\vartheta_M^*(t) = \sum_{i=0}^{M} \bar{\vartheta}_i^* L_i(t), \quad t \in [0, T], \tag{4.14}$$

where $L_i(t)$ is the Lagrange interpolating polynomial. We have a sequence of direct responses $\{(\bar{\zeta}_i^*, \bar{\vartheta}_i^*), i = 0, 1, \ldots, M\}_{M=M_1}^{\infty}$ and their sequence of interpolating polynomials $\{(\zeta_M^*(\cdot), \vartheta_M^*(\cdot))\}_{M=M_1}^{\infty}$.

Theorem 4.2. *Let* $\{(\bar{\zeta}_i^*, \bar{\vartheta}_i^*), i = 0, 1, \ldots, M\}_{M=M_1}^{\infty}$ *be a sequence of optimal solution of problems* (4.1)–(4.6) *and* $\{(\zeta_M^*(\cdot), \vartheta_M^*(\cdot))\}_{M=M_1}^{\infty}$ *be their interpolating sequence. It is supposed that the sequence* $\{(\bar{\zeta}_0^*, \dot{\zeta}_M^*(\cdot), \vartheta_M(\cdot))\}_{M=M_1}^{\infty}$ *has a subsequence that uniformly converges to* $\{\zeta_0^{\infty}, q(\cdot), \vartheta^*(\cdot)\}$ *where* $\vartheta^*(\cdot)$ *and* $q(\cdot)$ *are continuous functions and* $\zeta_0^{\infty} \in \mathbb{R}^m$*. Then* $(\zeta^*(\cdot), \vartheta^*(\cdot))$ *is an optimal solution to the problems* (2.6)–(2.9)*, where*

$$\zeta^*(t) = \int_0^t q(\tau)\mathrm{d}\tau + \zeta_0^{\infty}, \quad 0 \le t \le T. \tag{4.15}$$

Proof. By assumptions of Theorem (4.2), there is a subsequence $\{(\dot{\zeta}_{M_j}^*(\cdot), \vartheta_{M_j}^*(\cdot))\}_{j=1}^{\infty}$ of sequence $\{(\dot{\zeta}_M^*(\cdot), \vartheta_M^*(\cdot))\}_{M=M_1}^{\infty}$ such that $\lim_{j\to\infty} M_j = \infty$ and

$$\lim_{j\to\infty} (\dot{\zeta}_{M_j}^*(\cdot), \vartheta_{M_j}(\cdot)) = (q(\cdot), \vartheta^*(\cdot)). \tag{4.16}$$

By (4.15) and (4.16), we can get

$$\lim_{j\to\infty} \dot{\zeta}_{M_j}^*(\cdot) = \dot{\zeta}^*(\cdot).$$

The proof has including three steps. At first, we illustrate that $(\zeta^*(\cdot), \vartheta^*(\cdot))$ is a feasible solution to the problems (2.6)–(2.9). Afterward, we prove the convergence of the objective functional $\bar{J}_{M_j}(\bar{\zeta}^*, \bar{\vartheta}^*)$ to the continuous objective functional $J(\zeta^*(\cdot), \vartheta^*(\cdot))$, and eventually illustrate that $(\zeta^*(\cdot), \vartheta^*(\cdot))$ is an optimal solution of problems (2.6)–(2.9).

Step 1. *Suppose that* $\zeta^*(\cdot)$ *and* $\vartheta^*(\cdot)$ *don't satisfy the constraint of the problems* (2.6)–(2.9)*. Hence, there is a time* $\bar{t} \in [0, T]$ *so that*

$$\dot{\zeta}^*(\bar{t}) - \eta\Big(\zeta^*(\bar{t}), \zeta^*(\bar{t} - \mu_1), \Phi(\bar{t} - \mu_1), \vartheta^*(\bar{t}), \vartheta^*(\bar{t} - \mu_2), \Psi(\bar{t} - \mu_2), \bar{t}\Big) \neq 0, \tag{4.17}$$

or

$$\zeta^*(0) - \Phi(0) \neq 0, \tag{4.18}$$

where

$$\eta(\zeta^*(t), \zeta^*(t-\mu_1), \Phi(t-\mu_1), \vartheta^*(t), \vartheta^*(t-\mu_2), \Psi(t-\mu_2), t) = \begin{cases} g(\zeta^*(t), \Phi(t-\mu_1), \vartheta^*(t), \Psi(t-\mu_2), t), \ 0 \le t \le \mu_1, \\ g(\zeta^*(t), \zeta^*(t-\mu_1), \vartheta^*(t), \Psi(t-\mu_2), t), \ \mu_1 < t \le \mu_2, \\ g(\zeta^*(t), \zeta^*(t-\mu_1), \vartheta^*(t), \vartheta^*(t-\mu_2), t), \mu_2 < t \le T. \end{cases}$$

Since collocation points $\{t_i\}_{i=0}^{\infty}$ *are dense in* $[0,T]$*, there is a subsequence* i_{M_j} *such that* $0 < i_{M_j} < M_j$ *and* $\lim_{j\to\infty} t_{i_{M_j}} = \bar{t}$*. Thus*

$$\dot{\zeta}^*(\bar{t}) - \eta\Big(\zeta^*(\bar{t}), \zeta^*(\bar{t}-\mu_1), \Phi(\bar{t}-\mu_1), \vartheta^*(\bar{t}), \vartheta^*(\bar{t}-\mu_2), \Psi(\bar{t}-\mu_2), \bar{t}\Big)$$
$$= \lim_{j\to\infty}\Big(\dot{\zeta}_{M_j}(t_{i_{M_j}}) - \eta\Big(\zeta_{M_j}(t_{i_{M_j}}), \zeta_{M_j}(t_{i_{M_j}}-\mu_1), \Phi(t_{i_{M_j}}-\mu_1), \vartheta_{M_j}(t_{i_{M_j}}), \vartheta_{M_j}(t_{i_{M_j}}-\mu_2), \Psi(t_{i_{M_j}}-\mu_2), t_{i_{M_j}}\Big)\Big) \neq 0. \tag{4.19}$$

On the other hand, $\lim_{j\to\infty}(M_j-1)^{\frac{3}{2}-n} = 0$*. Therefore with constraints of the problems* (4.2)–(4.6)*, we have*

$$\lim_{j\to\infty}\Big(\dot{\zeta}_{M_j}(t_{i_{M_j}}) - \eta\Big(\zeta_{M_j}(t_{i_{M_j}}), \zeta_{M_j}(t_{i_{M_j}}-\mu_1), \Phi(t_{i_{M_j}}-\mu_1), \vartheta_{M_j}(t_{i_{M_j}}), \vartheta_{M_j}(t_{i_{M_j}}-\mu_2), \Psi(t_{i_{M_j}}-\mu_2), t_{i_{M_j}}\Big)\Big) = 0,$$

which is a contradiction to (4.19)*. By a similar process, we can show that* $\zeta^*(0) = \Phi(0)$*. We have*

$$0 = \lim_{M\to\infty}(M-1)^{\frac{3}{2}-n} \geq \lim_{M\to\infty}\left\|\bar{\zeta}_0^* - \Phi(0)\right\| = \left\|\lim_{M\to\infty}(\bar{\zeta}_0^* - \Phi(0))\right\| = \left\|\zeta_0^\infty - \Phi(0)\right\| \geq 0. \tag{4.20}$$

Hence, $\zeta^*(0) = \Phi(0)$*. Thus* $(\zeta^*(\cdot), \vartheta^*(\cdot))$ *is a feasible solution for the problems* (2.6)–(2.9)*. The path constraint can be proved by the same contradiction argument, so*

$$e(\zeta^*(\cdot), \vartheta^*(\cdot)) \leq 0. \tag{4.21}$$

Step 2. *In Step* 2*, we attend to demonstrate that*

$$\lim_{j\to\infty}\bar{J}_{M_j}(\bar{\zeta}^{**}, \bar{\vartheta}^{**}) = J(\zeta^*(\cdot), \vartheta^*(\cdot)), \tag{4.22}$$

wherein

$$\bar{J}_{M_j}(\bar{\zeta}^{**}, \bar{\vartheta}^{**}) = \Lambda(\bar{\zeta}_{M_j}^{**}, t_{M_j}) + (\frac{T}{2})\sum_{l=0}^{M_j} w_l h(\bar{\zeta}_l^{**}, \bar{\vartheta}_l^{**}, t_l),$$
$$J(\zeta^*(\cdot), \vartheta^*(\cdot)) = \Lambda(\zeta^*(T), T) + \int_0^T h(\zeta^*(t), \vartheta^*(t), t)\mathrm{d}t.$$

Since $(\zeta_{M_j}(\cdot), \vartheta_{M_j}(\cdot))$ *converges to* $(\zeta^*(\cdot), \vartheta^*(\cdot))$ *uniformly, we have*

$$\lim_{j\to\infty}\left\|\zeta_{M_j}(t_i) - \zeta^*(t_i)\right\|_\infty = \lim_{j\to\infty}\left\|\bar{\zeta}_i^{**} - \zeta^*(t_i)\right\|_\infty = 0, \tag{4.23}$$
$$\lim_{j\to\infty}\left\|\vartheta_{M_j}(t_i) - \vartheta^*(t_i)\right\|_\infty = \lim_{j\to\infty}\left\|\bar{\vartheta}_i^{**} - \vartheta^*(t_i)\right\|_\infty = 0, \tag{4.24}$$

uniformly in i. Also, we have

$$\int_0^T h(\zeta^*(t), \vartheta^*(t), t)\mathrm{d}t = \lim_{j\to\infty}(\frac{T}{2})\sum_{l=0}^{M_j} w_l h(\zeta^*(t_l), \vartheta^*(t_l), t_l).$$

Therefore

$$\int_0^T h(\zeta^*(t),\vartheta^*(t),t)\mathrm{d}t = (\frac{T}{2})\lim_{j\to\infty}\Big(\sum_{l=0}^{M_j} w_l h(\bar{\zeta}_l^{**},\bar{\vartheta}_l^{**},t_l) + \sum_{i=0}^{M_j} w_i\Big(h(\zeta^*(t_l),\vartheta^*(t_l),t_l) - h(\bar{\zeta}_l^{**},\bar{\vartheta}_l^{**},t_l)\Big)\Big).$$

According to the relation $\sum_{i=0}^{M} w_i = 2$ *and also the uniform convergence of* (4.23) *and* (4.24)*, and by* (4.10)*, we have*

$$\begin{aligned}&\lim_{j\to\infty}\Big\|\sum_{l=0}^{M_j} w_l\Big(h(\zeta^*(t_l),\vartheta^*(t_l),t_i) - h(\bar{\zeta}_l^{**},\bar{\vartheta}_l^{**},t_l)\Big)\Big\|_\infty \\ &\leq \lim_{j\to\infty} N_2\sum_{i=0}^{M_j} w_l\Big(\big\|\zeta^*(t_l) - \bar{\zeta}_l^{**}\big\|_\infty + \big\|(\vartheta^*(t_l) - \bar{\vartheta}_l^{**}\big\|_\infty\Big) = 0,\end{aligned}$$

where N_2 *is the Lipschitz constant of function* h*. Thus*

$$\int_0^T h(\zeta^*(t),\vartheta^*(t),t)\mathrm{d}t = \lim_{j\to\infty}\frac{T}{2}\sum_{l=0}^{M_j} w_l h(\bar{\zeta}_l^{**},\bar{\vartheta}_l^{**},t_l). \tag{4.25}$$

Also, we have

$$\lim_{j\to\infty}\Lambda(\bar{\zeta}_{M_j}^{**},t_{M_j}) = \Lambda(\zeta^*(T),T). \tag{4.26}$$

Step 3. *Suppose* $(\zeta^{**}(\cdot),\vartheta^{**}(\cdot))$ *is an optimal solution of problems* (2.6)–(2.9) *with the feature that* $\zeta^{**}(\cdot)\in W_m^{n,\infty}$ *for* $n\geq 2$*. Pursuant to Theorem* (4.1)*, there exists a sequence of feasible solution,* $\{\hat{\zeta}_M(\cdot),\hat{\vartheta}_M(\cdot)\}_{M=M_1}^{\infty}$*, of problems* (2.6)–(2.9) *that converges uniformly to* $(\zeta^{**}(\cdot),\vartheta^{**}(\cdot))$*. Now, by optimality of* $(\zeta^{**}(\cdot),\vartheta^{**}(\cdot))$ *and* $(\bar{\zeta}_i^{**},\bar{\vartheta}_i^{**})$*, we have*

$$J(\zeta^{**}(\cdot),\vartheta^{**}(\cdot)) \leq J(\zeta^*(\cdot),\vartheta^*(\cdot)) = \lim_{j\to\infty}\bar{J}_{M_j}(\bar{\zeta}^{**},\bar{\vartheta}^{**}) \leq \lim_{i\to\infty}\bar{J}_{M_j}(\hat{\zeta},\hat{\vartheta}), \tag{4.27}$$

where

$$\hat{\zeta} = (\hat{\zeta}_M(t_0),\hat{\zeta}_M(t_1),...,\hat{\zeta}_M(t_{M_j})),\quad \hat{\vartheta} = (\hat{\vartheta}_M(t_0),\hat{\vartheta}_M(t_1),...,\hat{\vartheta}_M(t_{M_j}))$$

Using similar reasoning as in Step 2, it is easy to offer that

$$J(\zeta^{**}(\cdot),\vartheta^{**}(\cdot)) = \lim_{j\to\infty}\bar{J}_{M_j}(\hat{\zeta},\hat{\vartheta}). \tag{4.28}$$

Inasmuch as $(\hat{\zeta}_M(\cdot),\hat{\vartheta}_M(\cdot))_{M=M_1}^{\infty}$ *converge uniformly to* $(\zeta^{**}(\cdot),\vartheta^{**}(\cdot))$*, Eqs* (4.27) *and* (4.28) *present that*

$$J(\zeta^{**}(\cdot),\vartheta^{**}(\cdot)) = J(\zeta^*(\cdot),\vartheta^*(\cdot)).$$

It is equivalent to saying that $(\zeta^*(\cdot),\vartheta^*(\cdot))$ *is an optimal solution that achieves the optimal cost. By Steps 1–3,* $(\zeta^*(\cdot),\vartheta^*(\cdot))$ *is an optimal solution to the OC problems* (2.6)–(2.9).

□

5. Numerical simulation

To evaluate the performance and the superiority of the proposed method, we present some numerical examples and compare the results with those of others. Here, we employ the FMINCON command in MATLAB software to find the solutions of the corresponding NLP problems (3.7)–(3.12). All calculations are run on a Core i3 PC Laptop with 2.5 GHz of CPU and 4 GB RAM.

Example 5.1. *In this example, we solve the TDOC problem related to harmonic oscillator described in [36, 37]. The problem is as follows*

$$\text{Minimize} \quad J = 5\zeta_1^2(2) + \frac{1}{2}\int_0^2 \vartheta^2(t)\mathrm{d}t,$$

subject to

$$\begin{aligned}
&\dot{\zeta}_1(t) = \zeta_2(t), \quad 0 \le t \le 2,\\
&\dot{\zeta}_2(t) = -\zeta_1(t) - \zeta_2(t-1) + \vartheta(t), \quad 0 \le t \le 2,\\
&\zeta_2(t) = 0, \quad -1 \le t \le 0,\\
&\zeta_1(0) = 10.
\end{aligned}$$

The optimal solutions to this problem are

$$J^* = 3.399118,$$

$$\vartheta^*(t) = \begin{cases} \alpha \sin(2-t) + (\frac{\alpha}{2})(1-t)\sin(t-1), & 0 \le t \le 1,\\ \alpha \sin(2-t), & 1 \le t \le 2, \end{cases}$$

where $\alpha \approx 2.5599$*. The corresponding NLP problems* (3.7)–(3.12) *can be written as follow*

$$\text{Minimize} \quad J = 5\zeta_1^2(2) + \frac{1}{2}\sum_{i=0}^{M} w_i \bar{\vartheta}_i^2, \tag{5.1}$$

subject to

$$\sum_{j=0}^{M} \bar{\zeta}_{1_j} D_{ij} - \bar{\zeta}_{2_i} = 0, \quad i = 0, 1, \ldots, M, \tag{5.2}$$

$$\sum_{j=0}^{M} \bar{\zeta}_{2_j} D_{ij} - (-\bar{\zeta}_{1_i} + \bar{\vartheta}_i) = 0, \quad i = 0, 1, \ldots, l_{\mu_1}, \tag{5.3}$$

$$\sum_{j=0}^{M} \bar{\zeta}_{2_j} D_{ij} - (-\bar{\zeta}_{1_i} - \sum_{j=0}^{M} \bar{\zeta}_{2_j} L_j(t_i - 1) + \bar{\vartheta}_i) = 0, \quad i = l_{\mu_1} + 1, \ldots, M, \tag{5.4}$$

$$\zeta_1(0) = 10. \tag{5.5}$$

We solve the problems (5.1)–(5.5)*. In Table 1, the results are given for objective functional J. In Figures 1 and 2 are illustrated the approximate solutions for control and states variables. Furthermore, we compare our results with those of other methods. The CPU time of our method for* $M = 8$ *is* 1.9187 *second.*

Table 1. Approximate optimal values of objective functional J and $|J - J^*|$ for Example 1.

Method	*presented method*	*method [38]*	*method [36]*	*method [37]*	*method [39]*
J	**3.39911**366	3.3208	3.399070	3.384700	3.3991
	$M = 8$	$N = 2, M = 2$	$N = 2, M = 9$		
$\lvert J - J^* \rvert$	4.3×10^{-6}	7.8×10^{-2}	4.8×10^{-5}	1.4×10^{-2}	$1.8 * 10^{-5}$

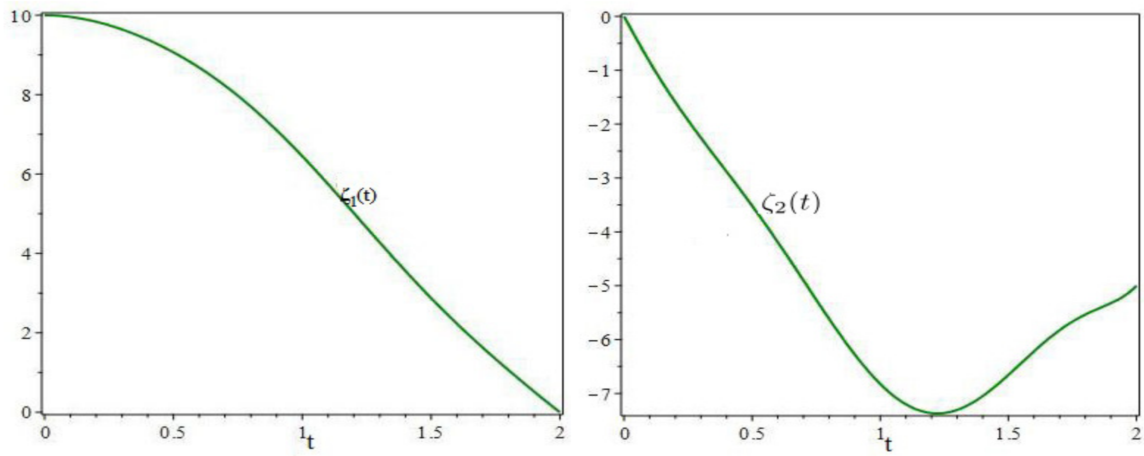

Figure 1. The obtained approximate for optimal state variable $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$ with $M = 8$, for Example 1.

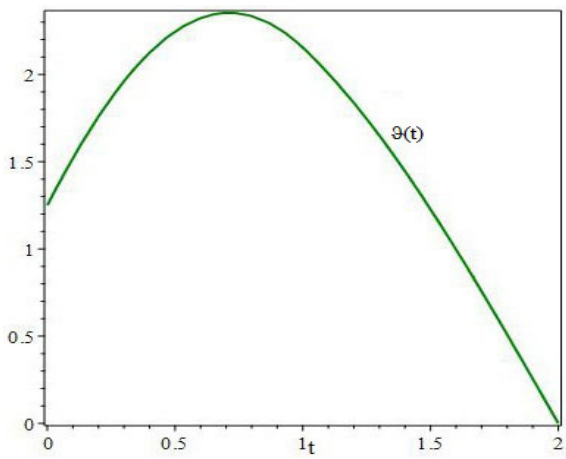

Figure 2. The obtained approximate for optimal control variable $\vartheta(\cdot)$ with $M = 8$, for Example 1.

Now, we add the following inequality constraints to the above problem

$$\zeta_1(2) \geq 2, \quad \zeta_2(2) \leq -5.$$

Again, we solve the problem with these constraints. The results of the proposed method and those of several other methods are presented in Table 2.

Table 2. Approximate optimal values of objective functional J for Example 1.

Method	***presented method***	***method [38]***	***method [40]***
J	30.434982	31.2759	31.303
	$M = 8$	$M = 2, N = 2$	

Example 5.2. *Consider the following TDOC problem [41]*

$$\text{Minimize} \quad J = \frac{1}{8}\int_0^1 (\zeta^2(t) + \vartheta^2(t))\mathrm{d}t,$$

subject to

$$\dot{\zeta}(t) = 0.25(\zeta(t) + \vartheta(t - \frac{0.1}{4}) + \vartheta(t)), \quad 0 \leq t \leq 1,$$
$$\vartheta(t) = 0, \quad -\frac{0.1}{4} \leq t \leq 0,$$
$$\zeta(0) = 1.$$

In Table 3, we are reported the results of J with different values of M, also we compare our results with the those of method [41]. The obtained approximations of optimal state and control variables are shown in Figure 3.

Table 3. Approximate optimal values of objective functional J with different values of M for Example 2.

Method	***J***
presented method	
$M = 3$	0.15175412
$M = 4$	0.14953759
$M = 5$	0.14858035
$M = 6$	0148359750
$M = 7$	0.14763888
method [41]	
$N = 4$	0.154268

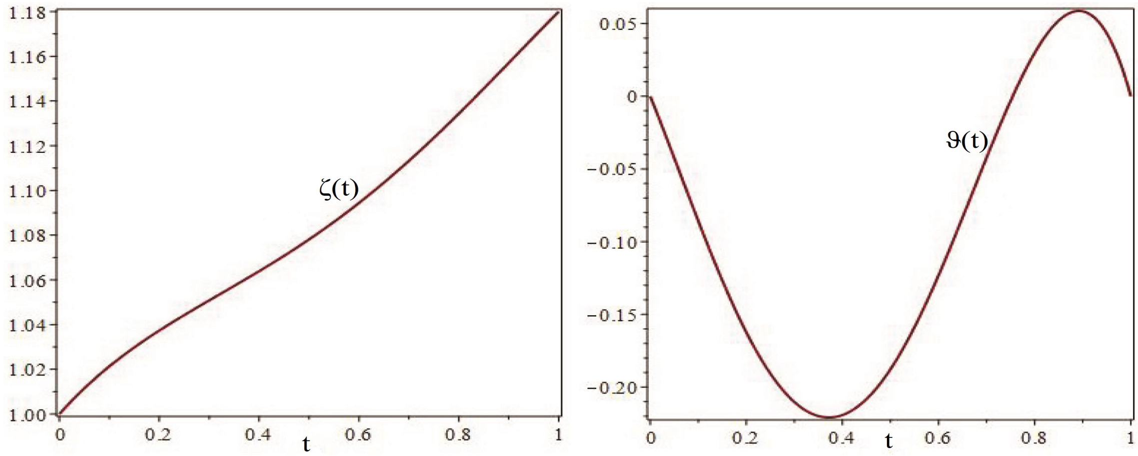

Figure 3. The obtained approximate for optimal state variable $\zeta(\cdot)$ and $\vartheta(\cdot)$ with $M = 4$, for Example 2.

Example 5.3. *Cosider the following TDOC problem [38].*

$$\textit{Minimize} \quad J = \frac{1}{2}\zeta^2(T) + \frac{1}{2}\int_0^T (\zeta^2(t) + \vartheta^2(t))\mathrm{d}t,$$

subject to

$$\dot{\zeta}(t) = \zeta(t)sin(\zeta(t)) + \zeta(t-1) + \vartheta(t), \quad 0 \le t \le T,$$
$$\zeta(t) = 10, \quad -1 \le t \le 0,$$

where $T = 2$. We use the corresponding NLP (3.7)–(3.12) *to solve this problem. The obtained approximate optimal values of J are given in Table 4. Compared with other works [38, 40, 42], the results show that the present method has better solutions.*

The obtained approximate optimal solutions are illustrated in Figure 4.

Table 4. Approximate optimal values of objective functional J with different values of M for Example 3.

Method	J
presented method	
$M = 4$	162.16538051
$M = 6$	161.70416416
$M = 8$	161.70813622
method [38]	
$N = 2, M = 2$	161.94
method [40]	
	162.001
method [42]	
	162.104

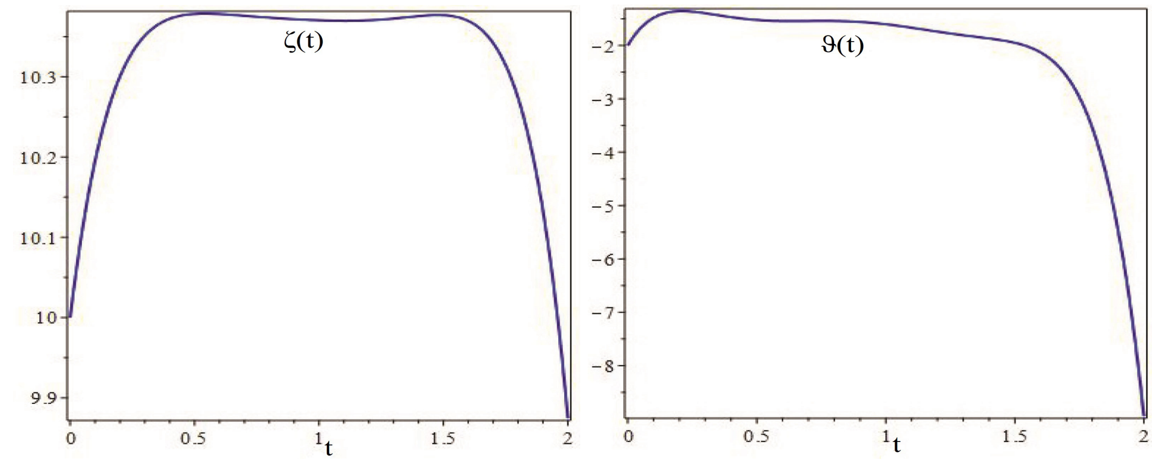

Figure 4. The obtained approximate for optimal state variable $\zeta(\cdot)$ and $\vartheta(\cdot)$ with $M = 6$, for Example 3.

Example 5.4. *Consider the following system*

$$\textit{Minimize} \quad J = \frac{1}{2}\int_0^2 (\zeta^2(t) + \vartheta^2(t))\mathrm{d}t,$$

subject to

$$\dot{\zeta}(t) = \zeta(t-\mu) + \vartheta(t), \quad 0 \leq t \leq 2,$$
$$\zeta(t) = 1, \quad -\mu \leq t \leq 0,$$

here, we have a delay in state and $\mu = 1$. We solve the problem with the method presented in this article, then compare those results with results [41]. Approximate values of J and $CPUtime$ are given in Table 5. The curves of obtained approximate solutions are illustrated in Figure 5.

Table 5. Approximate optimal values of objective functional J and $CPUtime$ with different values of M for Example 4.

Method	J	***CPUtime***(Sec)
presented method		
$M = 4$	1.70863830	1.719
$M = 5$	1.59019028	0.383
$M = 8$	1.65518595	0.651
$M = 10$	1.64349367	0.950
$M = 12$	1.63409908	1.010
method [41]		
$N = 20$	1.647453	4.358

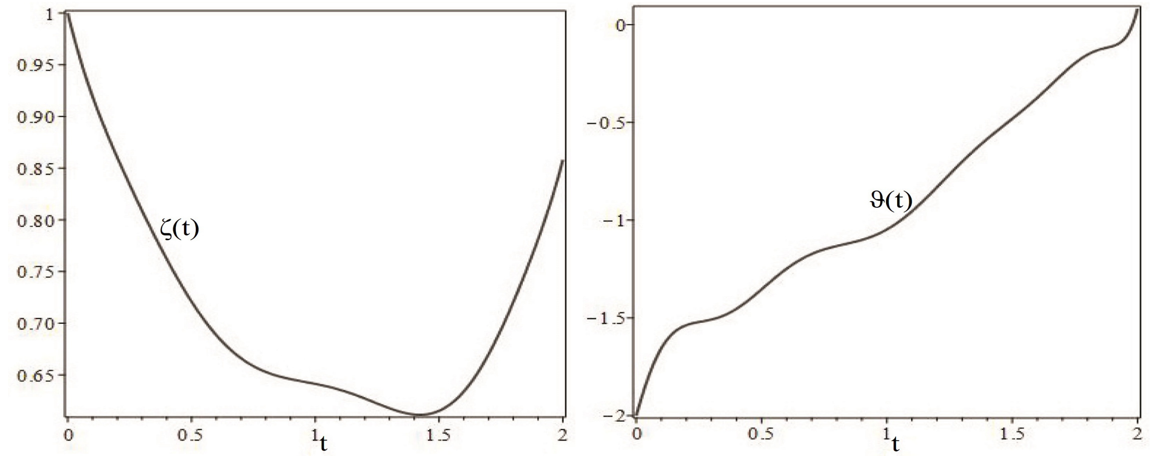

Figure 5. The obtained approximate for optimal state variable $\zeta(\cdot)$ and $\vartheta(\cdot)$ with $M = 10$, for Example 4.

6. Conclusions

In this paper, we introduced an extended spectral method based on the operational matrix of derivatives for solving the TDOC problems with delay on both control and state variables. One of the method's advantages is its complexity is much lower than the other methods utilized to solve the TDOC problems. Also, numerical results showed that our proposed method for TDOC problems has a higher convergence rate than other methods presented in [37–42].

For future works, we intend to extend the proposed method in this paper for solving the nonsmooth TDOC problems.

Conflict of interest

The authors declare that they have no conflict of interest.

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