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Research article

Symmetry solutions and conservation laws of a new generalized 2D Bogoyavlensky-Konopelchenko equation of plasma physics

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Abstract: In physics as well as mathematics, nonlinear partial differential equations are known as veritable tools in describing many diverse physical systems, ranging from gravitation, mechanics, fluid dynamics to plasma physics. In consequence, we analytically examine a two-dimensional generalized Bogoyavlensky-Konopelchenko equation in plasma physics in this paper. Firstly, the technique of Lie symmetry analysis of differential equations is used to find its symmetries and perform symmetry reductions to obtain ordinary differential equations which are solved to secure possible analytic solutions of the underlying equation. Then we use Kudryashov's and (G'/G)-expansion methods to acquire analytic solutions of the equation. As a result, solutions found in the process include exponential, elliptic, algebraic, hyperbolic and trigonometric functions which are highly important due to their various applications in mathematic and theoretical physics. Moreover, the obtained solutions are represented in diagrams. Conclusively, we construct conservation laws of the underlying equation through the use of multiplier approach. We state here that the results secured for the equation understudy are new and highly useful.

Keywords: two-dimensional generalized Bogoyavlensky-Konopelchenko equation; Lie point symmetries; analytic solutions; conservation laws
Mathematics Subject Classification: 35L65, 35B06

1. Introduction

Plasma physics simply refers to the study of a state of matter consisting of charged particles. Plasmas are usually created by heating a gas until the electrons become detached from their parent atom or molecule. In addition, plasma can be generated artificially when a neutral gas is heated or subjected to a strong electromagnetic field. The presence of free charged particles makes plasma electrically conductive with the dynamics of individual particles and macroscopic plasma motion governed by collective electromagnetic fields [1].

Nonlinear partial differential equations (NPDE) in the fields of mathematics and physics play numerous important roles in theoretical sciences. They are the most fundamental models essential in studying nonlinear phenomena. Such phenomena occur in plasma physics, oceanography, aerospace industry, meteorology, nonlinear mechanics, biology, population ecology, fluid mechanics to mention a few. We have seen in [2] that the authors studied a generalized advection-diffusion equation which is a NPDE in fluid mechanics, characterizing the motion of buoyancy propelled plume in a bent-on absorptive medium. Moreover, in [3], a generalized Korteweg-de Vries-Zakharov-Kuznetsov equation was studied. This equation delineates mixtures of warm adiabatic fluid, hot isothermal as well as cold immobile background species applicable in fluid dynamics. Furthermore, the authors in [4] considered a NPDE where they explored important inclined magneto-hydrodynamic flow of an upper-convected Maxwell liquid through a leaky stretched plate. In addition, heat transfer phenomenon was studied with heat generation and absorption effect. The reader can access more examples of NPDEs in [5–16].

In order to really understand these physical phenomena it is of immense importance to solve NPDEs which govern these aforementioned phenomena. However, there is no general systematic theory that can be applied to NPDEs so that their analytic solutions can be obtained. Nevertheless, in recent times scientists have developed effective techniques to obtain viable analytical solutions to NPDEs, such as inverse scattering transform [16], simple equation method [17], Bäcklund transformation [18], F-expansion technique [19], extended simplest equation method [20], Hirota technique [21], Lie symmetry analysis [22–27], bifurcation technique [28, 29], the (G'/G)-expansion method [30], Darboux transformation [31], sine-Gordon equation expansion technique [32], Kudryashov's method [33], and so on.

The (2+1)-dimensional Bogoyavlensky-Konopelchenko (BK) equation given as

$$u_{tx} + 6\alpha u_x u_{xx} + 3\beta u_x u_{xy} + 3\beta u_y u_{xx} + \alpha u_{xxxx} + \beta u_{xxxy} = 0, \tag{1.1}$$

where parameters α and β are constants, is a special case of the KdV equation in [34] which was introduced as a (2+1)-dimensional version of the KdV and it is described as an interaction of a long wave propagation along x-axis and a Riemann wave propagation along the y-axis [35]. In addition to that, few particular properties of the equation have been explored. The authors in [36] provided a Darboux transformation for the BK equation and the obtained transformation was used to construct a family of solutions of this equation. In [37], with 3β replaced by 4β and $u_y = v_x$ in (1.1), the authors integrated the result once to get

$$u_t + \alpha u_{xxx} + \beta v_{xxx} + 3\alpha u_x^2 + 4\beta u_x v_x = 0, \quad u_y - v_x = 0. \tag{1.2}$$

Further, they utilized Lie group theoretic approach to obtain solutions of the system of Eq (1.2). They also engaged the concept of nonlinear self-adjointness of differential equations in conjunction with formal Lagrangian of (1.2) for constructing nonlocal conservation laws of the system. In addition, various applications of BK equation (1.1) were highlighted in [37]. Further investigations on certain particular cases of (1.1) were also carried out in [38, 39].

In [40], the 2D generalized BK equation that reads

$$u_{tx} + k_1 u_{xxxx} + k_2 u_{xxxy} + \frac{2k_1k_3}{k_2} u_x u_{xx} + k_3(u_x u_y)_x + \gamma_1 u_{xx} + \gamma_2 u_{xy} + \gamma_3 u_{yy} = 0 \tag{1.3}$$

was studied and lump-type and lump solutions were constructed by invoking the Hirota bilinear method. Liu et al. [41] applied the Lie group analysis together with (G'/G)-expansion and power series methods and obtained some analytic solutions of (1.3).

Yang et al. [42] recently examined a generalized combined fourth-order soliton equation expressed as

$$\alpha(6u_xu_{xx} + u_{xxxx}) + \beta[3(u_xu_t)_x + u_{xxxt}] + \gamma[3(u_xu_y)_x + u_{xxxy}] \\ + \delta_1u_{yt} + \delta_2u_{xx} + \delta_3u_{xt} + \delta_4u_{xy} + \delta_5u_{yy} + \delta_6u_{tt} = 0, \quad (1.4)$$

with constant parameters α, β and γ which are not all zero, whereas all constant coefficients δ_i, $1 \leq i \leq 6$, are arbitrary. It was observed that Eq (1.4) comprises three fourth-order terms and second-order terms that consequently generalizes the standard Kadomtsev-Petviashvili equation. Soliton equations are known to have applications in plasma physics and other nonlinear sciences such as fluid mechanics, atomic physics, biophysics, nonlinear optics, classical and quantum fields theories.

Assuming $\alpha = 0, \beta = 1, \gamma = 0$ and $\delta_1 = \delta_2 = 1$, $\delta_3 = \delta_4 = \delta_5 = \delta_6 = 0$, the authors gain an integrable (1+2)-dimensional extension of the Hirota-Satsuma equation commonly referred to as the Hirota-Satsuma-Ito equation in two dimensions [43] given as

$$u_{ty} + u_{xx} + 3(u_xu_t)_x + u_{xxxt} = 0 \quad (1.5)$$

that satisfies the Hirota three-soliton condition and also admits a Hirota bilinear structure under logarithmic transformation presented in the form

$$u = 2(\ln f)_x, \text{ where } \left(D_x^3D_t + D_yD_t + D_x^2\right) f \cdot f = 0, \quad (1.6)$$

whose lump solutions have been calculated in [44]. On taking parameters $\alpha = 1, \beta = 0, \gamma = 0$ along with $\delta_1 = \delta_4 = \delta_6 = 0$ whereas $\delta_2 = \delta_3 = \delta_5 = 1$, they eventually came up with a two dimensional equation [42]:

$$u_{tx} + 6u_xu_{xx} + u_{xxxx} + u_{xxxy} + 3\left(u_xu_y\right)_x + u_{xx} + u_{yy} = 0, \quad (1.7)$$

which is called a two-dimensional generalized Bogoyavlensky-Konopelchenko (2D-gBK) equation. We notice that if one takes $\alpha = \beta = 1$ in Eq (1.1) with the introduction of two new terms u_{xx} and u_{yy}, the new generalized version (1.7) is achieved.

In consequence, we investigate explicit solutions of the new two-dimensional generalized Bogoyavlensky-Konopelchenko equation (1.7) of plasma physics in this study. In order to achieve that, we present the paper in the subsequent format. In Section 2, we employ Lie symmetry analysis to carry out the symmetry reductions of the equation. In addition, direct integration method will be employed in order to gain some analytic solutions of the equation by solving the resulting ordinary differential equations (ODEs) from the reduction process. We achieve more analytic solutions of (1.7) via the conventional (G'/G)-expansion method as well as Kudryashov's technique. In addition, by choosing suitable parametric values, we depict the dynamics of the solutions via 3-D, 2-D as well as contour plots. Section 3 presents the conservation laws for 2D-gBK equation (1.7) through the multiplier method and in Section 4, we give the concluding remarks.

2. Symmetry analysis and analytic solutions of (1.7)

In this section we in the first place compute the Lie point symmetries of Eq (1.7) and thereafter engage them to generate analytic solutions.

2.1. Lie point symmetries of (1.7)

A one-parameter Lie group of symmetry transformations associated with the infinitesimal generators related to (gbk) can be presented as

$$\begin{aligned}\bar{t} &= t + \epsilon\, \xi^1\,(t, x, y, u) + O(\epsilon^2),\\ \bar{x} &= x + \epsilon\, \xi^2\,(t, x, y, u) + O(\epsilon^2),\\ \bar{y} &= y + \epsilon\, \xi^3\,(t, x, y, u) + O(\epsilon^2),\\ \bar{u} &= u + \epsilon\, \phi\,(t, x, y, u) + O(\epsilon^2).\end{aligned} \tag{2.1}$$

We calculate symmetry group of 2D-gBK equation (1.7) using the vector field

$$R = \xi^1(t, x, y, u)\frac{\partial}{\partial t} + \xi^2(t, x, y, u)\frac{\partial}{\partial x} + \xi^3(t, x, y, u)\frac{\partial}{\partial y} + \phi(t, x, y, u)\frac{\partial}{\partial u}, \tag{2.2}$$

where $\xi^i, i = 1, 2, 3$ and ϕ are functions depending on t, x, y and u. We recall that (2.2) is a Lie point symmetry of Eq (1.7) if

$$R^{[4]}(u_{tx} + 6u_x u_{xx} + u_{xxxx} + u_{xxxy} + 3\left(u_x u_y\right)_x + u_{xx} + u_{yy})|_{Q=0} = 0, \tag{2.3}$$

where $Q = u_{tx} + 6u_x u_{xx} + u_{xxxx} + u_{xxxy} + 3\left(u_x u_y\right)_x + u_{xx} + u_{yy}$. Here, $R^{[4]}$ denotes the fourth prolongation of R defined by

$$R^{[4]} = R + \eta^t \partial_{u_t} + \eta^x \partial_{u_x} + \eta^y \partial_{u_y} + \eta^{tx} \partial_{u_{tx}} + \eta^{xx} \partial_{u_{xx}} + \eta^{yy} \partial_{u_{yy}} + \eta^{xxxx} \partial_{u_{xxxx}} + \eta^{xxxy} \partial_{u_{xxxy}}, \tag{2.4}$$

where coefficient functions η^t, η^x, η^y, η^{xt}, η^{xx}, η^{xy}, η^{yy}, η^{xxxx} and η^{xxxy} can be calculated from [22–24].

Writing out the expanded form of the determining equation (2.3), splitting over various derivatives of u and with the help of Mathematica, we achieve the system of linear partial differential equations (PDEs):

$$\begin{aligned}&\xi^3_x = 0,\ \ \xi^1_x = 0,\ \ \xi^1_y = 0,\ \ \xi^2_u = 0,\ \ \xi^1_u = 0,\ \ \xi^3_u = 0,\\ &\xi^1_{tt} + 5\xi^2_{yy} = 0,\ \ \xi^1_t + 5\phi_u = 0,\ \ 5\xi^2_x - \xi^1_t = 0,\\ &5\xi^2_y - 2\xi^1_t = 0,\ \ 5\xi^3_y - 3\xi^1_t = 0,\ \ 3\phi_{xx} - \xi^3_{yy} = 0,\ \ \xi^3_t - 3\phi_x + 2\xi^2_y = 0,\\ &4\xi^1_t - 5\xi^2_t + 30\phi_x + 15\phi_y = 0,\ \ \phi_{tx} + \phi_{xx} + \phi_{xxxx} + \phi_{xxxy} + \phi_{yy} = 0.\end{aligned}$$

The solution of the above system of PDEs is

$$\begin{aligned}&\xi^1 = A_1 + A_2 t,\ \ \xi^2 = F(t) + \frac{1}{5}A_2(x + 2y),\ \ \xi^3 = A_4 - \frac{4}{5}A_2 t + 3A_3 t + \frac{3}{5}A_2 y,\\ &\eta = G(t) - \frac{1}{5}A_2 u + A_3 x - \frac{4}{15}A_2 y - 2A_3 y + \frac{1}{3}yF'(t),\end{aligned}$$

where A_1–A_3 are arbitrary constants and $F(t)$, $G(t)$ are arbitrary functions of t. Consequently, we secure the Lie point symmetries of (1.7) given as

$$\begin{aligned}&R_1 = \frac{\partial}{\partial t},\quad R_2 = \frac{\partial}{\partial y},\quad R_3 = 3F(t)\frac{\partial}{\partial x} + yF'(t)\frac{\partial}{\partial u},\quad R_4 = 3t\frac{\partial}{\partial y} + (x - 2y)\frac{\partial}{\partial u},\\ &R_5 = G(t)\frac{\partial}{\partial u},\quad R_6 = 15t\frac{\partial}{\partial t} + (3x + 6y)\frac{\partial}{\partial x} + (9y - 12t)\frac{\partial}{\partial y} - (4y + 3u)\frac{\partial}{\partial u}.\end{aligned} \tag{2.5}$$

2.2. Lie group transformations associated to (2.5)

We contemplate the exponentiation of the vector fields (2.5) by computing the flow or one parameter group generated by (2.5) via the Lie equations [22, 23]:

$$\begin{aligned} \frac{d\bar{t}}{d\epsilon} &= \xi^1(\bar{t},\bar{x},\bar{y},\bar{u}), \quad \bar{t}|_{\epsilon=0} = t,\\ \frac{d\bar{x}}{d\epsilon} &= \xi^2(\bar{t},\bar{x},\bar{y},\bar{u}), \quad \bar{x}|_{\epsilon=0} = x,\\ \frac{d\bar{y}}{d\epsilon} &= \xi^3(\bar{t},\bar{x},\bar{y},\bar{u}), \quad \bar{y}|_{\epsilon=0} = y,\\ \frac{d\bar{u}}{d\epsilon} &= \phi(\bar{t},\bar{x},\bar{y},\bar{u}), \quad \bar{u}|_{\epsilon=0} = u. \end{aligned}$$

Therefore, by taking $F(t) = G(t) = t$ in (2.5), one computes a one parameter transformation group of 2D-gBK (1.7). Thus, we present the result in the subsequent theorem.

Theorem 2.1. *Let $T^i_\epsilon(t,x,y,u), i = 1,2,3,\dots,6$ be transformations group of one parameter generated by vectors $R_1, R_2, R_3\dots, R_6$ in (2.5), then, for each of the vectors, we have accordingly*

$$\begin{aligned} T^1_\epsilon &: (\tilde{t},\tilde{x},\tilde{y},\tilde{u}) \longrightarrow (t+\epsilon_1, x, y, u),\\ T^2_\epsilon &: (\tilde{t},\tilde{x},\tilde{y},\tilde{u}) \longrightarrow (t, x, y+\epsilon_2, u),\\ T^3_\epsilon &: (\tilde{t},\tilde{x},\tilde{y},\tilde{u}) \longrightarrow (t, 3\epsilon_3 t + x, y, \epsilon_3 y + u),\\ T^4_\epsilon &: (\tilde{t},\tilde{x},\tilde{y},\tilde{u}) \longrightarrow (t, x, 3\epsilon_4 t + y, u + (x-2y)\epsilon_4 - 3\epsilon_4^2 t),\\ T^5_\epsilon &: (\tilde{t},\tilde{x},\tilde{y},\tilde{u}) \longrightarrow (t, x, y, \epsilon_5 t + u),\\ T^6_\epsilon &: \tilde{t},\tilde{x},\tilde{y},\tilde{u}) \longrightarrow \Big(te^{15\epsilon_6}, (2e^{9\epsilon_6} - e^{3\epsilon_6} - e^{15\epsilon_6})t + xe^{3\epsilon_6} + (e^{9\epsilon_6} - e^{3\epsilon_6})y,\\ &\qquad (2e^{9\epsilon_6} - 2e^{15\epsilon_6})t + ye^{9\epsilon_6}, \frac{1}{9}\left[(4e^{18\epsilon_6} - 6e^{12\epsilon_6} + 2)t + (3 - 3e^{12\epsilon_6})y + 9u\right]e^{-3\epsilon_6}\Big), \end{aligned}$$

where $\epsilon \in \mathbb{R}$ is regarded as the group parameter.

Theorem 2.2. *Hence, suppose $u(t,x,y) = \Theta(t,x,y)$ satisfies the 2D-gBK (1.7), in the same vein, the functions given in the structure*

$$\begin{aligned} u^1(t,x,y) &= \Theta(t-\epsilon_1, x, y, z),\\ u^2(t,x,y) &= \Theta(t, x, y-\epsilon_2, z, u),\\ u^3(t,x,y) &= \Theta(t, x-3\epsilon_3 t, y) - \epsilon_3 y,\\ u^4(t,x,y) &= \Theta(t, x, 3\epsilon_4 t + y) - (x-2y)\epsilon_4 + 3\epsilon_4^2 t,\\ u^5(t,x,y) &= \Theta(t,x,y) - \epsilon_5 t,\\ u^6(t,x,y) &= \Theta\Big[te^{15\epsilon_6}, (2e^{9\epsilon_6} - e^{3\epsilon_6} - e^{15\epsilon_6})t + xe^{3\epsilon_6} + (e^{9\epsilon_6} - e^{3\epsilon_6})y,\\ &\qquad (2e^{9\epsilon_6} - 2e^{15\epsilon_6})t + ye^{9\epsilon_6}\Big] - \frac{2}{3}te^{12\epsilon_6} + \frac{4}{9}te^{18\epsilon_6} + \frac{2}{9}t - \frac{1}{3}ye^{12\epsilon_6} + \frac{1}{3}y \end{aligned}$$

will do, where $u^i(t,x,y) = T^\epsilon_i \cdot \Theta(t,x,y), i = 1,2,3,\dots,6$ with $\epsilon << 1$ regarded as any positive real number.

2.3. Symmetry reduction of 2D-gBK equation (1.7)

In this subsection, we utilize symmetries (2.5) with a view to reduce Eq (1.7) to ordinary differential equations and thereafter obtain the analytic solutions of Eq (1.7) by solving the respective ODEs.

Case 1. Invariant solutions via R_1–R_3

Taking $F(t) = 1/3$, we linearly combine translational symmetries R_1–R_3 as $R = bR_1 + cR_2 + aR_3$ with nonzero constant parameters a, b and c. Subsequently utilizing the combination reduces 2D-gBK equation (1.7) to a PDE with two independent variables. Thus, solution to the characteristic equation associated with the symmetry R leaves us with invariants

$$r = ct - ay, \quad s = cx - by, \quad \theta = u. \tag{2.6}$$

Now treating θ above as the new dependent variable as well as r, s as new independent variables, (1.7) then transforms into the PDE:

$$\begin{aligned}&c^2\theta_{rs} + 6c^3\theta_s\theta_{ss} + c^4\theta_{ssss} - 3ac^2\theta_s\theta_{sr} - 6bc^2\theta_s\theta_{ss} - 3ac^2\theta_r\theta_{ss} - ac^3\theta_{sssr}\\ &- bc^3\theta_{ssss} + c^2\theta_{ss} + a^2\theta_{rr} + 2ab\theta_{sr} + b^2\theta_{ss} = 0.\end{aligned} \tag{2.7}$$

We now utilize the Lie point symmetries of (2.7) in a bid to transform it to an ODE. From (2.7), we achieve three translation symmetries:

$$Q_1 = \frac{\partial}{\partial r}, \quad Q_2 = \frac{\partial}{\partial s}, \quad Q_3 = \frac{\partial}{\partial \theta}.$$

The linear combination $Q = Q_1 + \omega Q_2$ ($\omega \neq 0$ being an arbitrary constant) leads to two invariants:

$$z = s - \omega r, \quad \theta = \Theta, \tag{2.8}$$

that secures group-invariant solution $\Theta = \Theta(z)$. Thus, on using these invariants, (2.7) is transformed into the fourth-order nonlinear ODE:

$$\begin{aligned}&(c^2 - \omega c^2 + a^2\omega^2 - 2ba\omega + b^2)\Theta''(z) - 6(\beta bc^2 - \beta c^2 a\omega - c^3)\Theta'(z)\Theta''(z)\\ &+ (c^3 a\omega + c^4 - bc^3)\Theta''''(z) = 0,\end{aligned}$$

which we rewrite in a simple structure as

$$A\Theta''(z) - B\Theta'(z)\Theta''(z) + C\Theta''''(z) = 0, \tag{2.9}$$

where $A = c^2 - \omega c^2 + a^2\omega^2 - 2ba\omega + b^2$, $B = 6(bc^2 - c^2 a\omega - c^3)$, $C = c^3 a\omega + c^4 - bc^3$ and $z = cx + (a\omega - b)y - c\omega t$.

2.4. Some analytic solutions of 2D-gBK equation (1.7)

In this section, we seek travelling wave solutions of the 2D-gBK equation (1.7).

A. Elliptic function solution of (1.7)

On integrating equation (2.9) once, we accomplish a third-order ODE:

$$A\Theta'(z) - \frac{1}{2}B\Theta'^2(z) + C\Theta'''(z) + C_1 = 0, \tag{2.10}$$

where C_1 is a constant of integration. Multiplying Eq (2.10) by $\Theta''(z)$, integrating once and simplifying the resulting equation, we have the second-order nonlinear ODE:

$$\frac{1}{2}A\Theta'(z)^2 - \frac{1}{6}B\Theta'(z)^3 + \frac{1}{2}C\Theta''(z)^2 + C_1\Theta'(z) + C_2 = 0,$$

where C_2 is a constant of integration. The above equation can be rewritten as

$$\Theta''(z)^2 = \frac{B}{3C}\Theta'(z)^3 - \frac{A}{C}\Theta'(z)^2 - \frac{2C_1}{C}\Theta'(z) - \frac{2C_2}{C}. \tag{2.11}$$

Letting $U(z) = \Theta'(z)$, Eq (2.11) becomes

$$U'(z)^2 = \frac{B}{3C}U(z)^3 - \frac{A}{C}U(z)^2 - \frac{2C_1}{C}U(z) - \frac{2C_2}{C}. \tag{2.12}$$

Suppose that the cubic equation

$$U(z)^3 - \frac{3A}{B}U(z)^2 - \frac{6C_1}{B}U(z) - \frac{6C_2}{B} = 0 \tag{2.13}$$

has real roots c_1–c_3 such that $c_1 > c_2 > c_3$, then Eq (2.12) can be written as

$$U'(z)^2 = \frac{B}{3C}(U(z) - c_1)(U(z) - c_2)(U(z) - c_3), \tag{2.14}$$

whose solution with regards to Jacobi elliptic function [45, 46] is

$$U(z) = c_2 + (c_1 - c_2)\,\mathrm{cn}^2\left\{\sqrt{\frac{B(c_1 - c_2)}{12C}}z, \Delta^2\right\},\ \Delta^2 = \frac{c_1 - c_2}{c_1 - c_3}, \tag{2.15}$$

with (cn) being the elliptic cosine function. Integration of (2.15) and reverting to the original variables secures a solution of 2D-gBK equation (1.7) as

$$\begin{aligned} u(t, x, y) = &\sqrt{\frac{12C(c_1 - c_2)^2}{B(c_1 - c_3)\Delta^8}}\left\{\mathrm{EllipticE}\left[\mathrm{sn}\left(\frac{B(c_1 - c_3)}{12C}z, \Delta^2\right), \Delta^2\right]\right\} \\ &+ \left\{c_2 - (c_1 - c_2)\frac{1 - \Delta^4}{\Delta^4}\right\}z + C_3, \end{aligned} \tag{2.16}$$

with $z = cx + (a\omega - b)y - c\omega t$ and C_3 a constant of integration. We note that (2.16) is a general solution of (1.7), where EllipticE$[p; q]$ is the incomplete elliptic integral [46, 47] expressed as

$$\mathrm{EllipticE}[p; q] = \int_0^p \sqrt{\frac{1 - q^2r^2}{1 - r^2}}dr.$$

We present wave profile of periodic solution (2.16) in Figure 1 with 3D, contour and 2D plots with parametric values $a = -4$, $b = 0.2$, $c = -0.1$, $\omega = 0.1$, $c_1 = 100$, $c_2 = 50.05$, $c_3 = -60$, $B = 10$, $C = 70$, where $t = 1$ and $-10 \le x, y \le 10$.

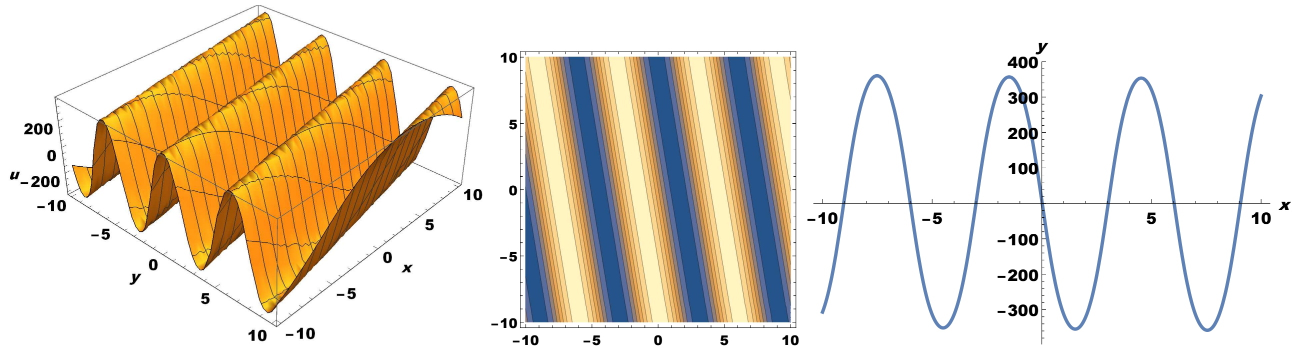

Figure 1. Elliptic solution wave profile of (2.16) at $t = 1$.

However, contemplating a special case of (2.9) with $B = 0$, we integrate the equation twice and so we have

$$C\Theta''(z) + A\Theta(z) + K_1 z + K_2 = 0, \tag{2.17}$$

where K_1 and K_2 are integration constants. Solving the second-order linear ODE (2.17) and reverting to the basic variables, we achieve the trigonometric solution of 2D-gBK equation (1.7) as

$$\begin{aligned} u(t,x,y) = {} & A_1 \sin\left(\frac{\sqrt{a^2\omega^2 - \omega\left(2ab + c^2\right) + b^2 + c^2}\, z}{\sqrt{c^3(a\omega - b + c)}}\right) + A_2 \cos\left(\frac{\sqrt{a^2\omega^2 - \omega\left(2ab + c^2\right) + b^2 + c^2}\, z}{\sqrt{c^3(a\omega - b + c)}}\right) \\ & - \frac{K_1 z + K_2}{a^2\omega^2 - \omega\left(2ab + c^2\right) + b^2 + c^2}, \end{aligned} \tag{2.18}$$

with A_1 and A_2 as the integration constants as well as $z = cx + (a\omega - b)y - c\omega t$. We depict the wave dynamics of periodic solution (2.18) in Figure 2 via 3D, contour and 2D plots with dissimilar parametric values $a = 1$, $b = 0.2$, $c = -0.1$, $\omega = 0.1$, $A_1 = 20$, $A_2 = -2$, $K_1 = 1$, $K_2 = 10$, where $t = 2$ and $-10 \leq x, y \leq 10$.

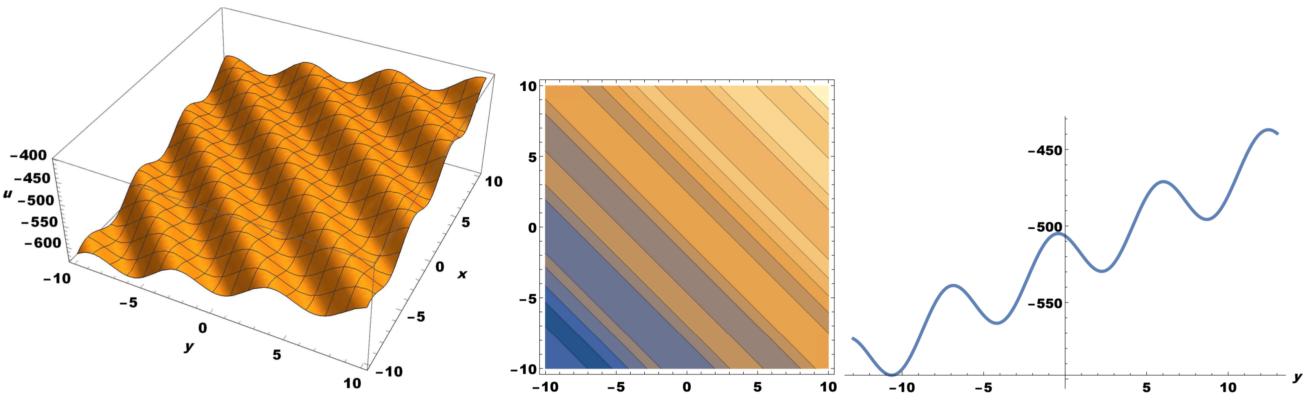

Figure 2. Wave profile of the trigonometric function solution (2.18) at $t = 2$.

B. Weierstrass elliptic solution of 2D-gBK equation (1.7)

We further explore Weierstrass elliptic function solution of (1.7), which is a technique often involved in getting general exact solutions to NPDEs [47, 48]. In order to accomplish this, we use the transformation

$$U(z) = \mathbf{W}(z) + \frac{A}{B} \tag{2.19}$$

and transform the nonlinear ordinary differential equation (NODE) (2.12) to

$$\mathbf{W}_{\xi}^2 = 4\mathbf{W}^3 - g_2\mathbf{W} - g_3, \quad \xi = \sqrt{\frac{B}{12C}}\, z, \tag{2.20}$$

with the invariants g_2 and g_3 given by

$$g_2 = -\frac{12A^2}{B^2} - \frac{24C_1}{B} \quad \text{and} \quad g_3 = -\frac{8A^3}{B^3} - \frac{24AC_1}{B^2} - \frac{24C_2}{B}.$$

Thus, we have the solution of NODE (2.12) as

$$U(z) = \frac{A}{B} + \wp\left(\sqrt{\frac{1}{12C}}(z - z_0); g_2; g_3\right), \tag{2.21}$$

where $\wp$ denotes the Weierstrass elliptic function [46]. In consequence, integration of (2.21) and reverting to the basic variables gives the solution of 2D-gBK equation (1.7) as

$$u(t, x, y) = \frac{A}{B}(z - z_0) - \sqrt{\frac{12B}{C}}\,\zeta\left[\sqrt{\frac{B}{12C}}(z - z_0); g_2, g_3\right], \tag{2.22}$$

with arbitrary constant z_0, $z = cx + (a\omega - b)y - c\omega t$ and ζ being the Weierstrass zeta function [46]. We give wave profile of Weierstrass function solution (2.22) in Figure 3 with 3D, contour and 2D plots using parameter values $a = 1$, $b = 0.2$, $c = -0.1$, $\omega = 0.1$, $A = 10$, $B = -2$, $z_0 = 0$, $C = 1$, $C_1 = 1$, $C_2 = 10$, where $t = 2$ and $-10 \le x, y \le 10$.

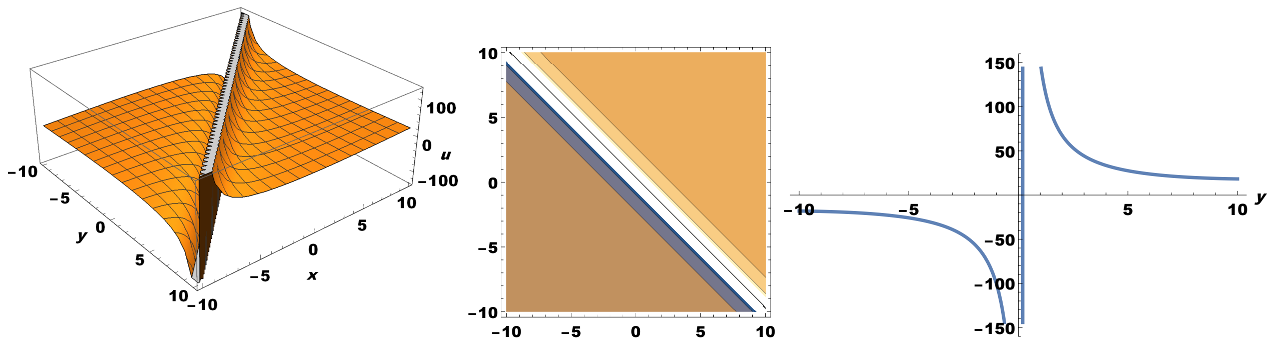

Figure 3. Wave profile of (2.22) at $t = 2$ of the Weierstrass zeta function solution.

2.4.1. Solution of (1.7) by Kudrayshov's approach

This part of the study furnishes the solution of 2D-gBK equation (1.7) through the use of Kudryashov's approach [33]. This technique is one of the most prominent way to obtain closed-form solutions of NPDEs. Having reduced Eq (1.7) to the NODE (2.9), we assume the solution of (2.9) as

$$\Theta(z) = \sum_{n=0}^{N} B_n Q^n(z), \tag{2.23}$$

with $Q(z)$ satisfying the first-order NODE

$$Q'(z) = Q^2(z) - Q(z). \tag{2.24}$$

We observe that the solution of (2.24) is

$$Q(z) = \frac{1}{1 + \exp(z)}. \tag{2.25}$$

The balancing procedure for NODE (2.9) produces $N = 1$. Hence, from (2.23), we have

$$\Theta(z) = B_0 + B_1 Q(z). \tag{2.26}$$

Now substituting (2.26) into (2.9) and using (2.24), we gain a long determining equation and splitting on powers of $Q(z)$, we get algebraic equations for the coefficients B_0 and B_1 as

$$\begin{aligned}
Q(z)^5 :\ & 2aB_1c^3\omega + aB_1^2c^2\omega - 2bB_1c^3 - bB_1^2c^2 + 2B_1c^4 + B_1^2c^3 = 0,\\
Q(z)^4 :\ & 2bB_1c^3 - 2aB_1c^3\omega - aB_1^2c^2\omega + bB_1^2c^2 - 2B_1c^4 - B_1^2c^3 = 0,\\
Q(z)^3 :\ & a^2B_1\omega^2 - 2abB_1\omega + 25aB_1c^3\omega + 12aB_1^2c^2\omega + b^2B_1 - 25bB_1c^3\\
& - 12bB_1^2c^2 + 25B_1c^4 + 12B_1^2c^3 - B_1c^2\omega + B_1c^2 = 0,\\
Q(z)^2 :\ & 2abB_1\omega - a^2B_1\omega^2 - 5aB_1c^3\omega - 2aB_1^2c^2\omega - b^2B_1 + 5bB_1c^3\\
& + 2bB_1^2c^2 - 5B_1c^4 - 2B_1^2c^3 + B_1c^2\omega - B_1c^2 = 0,\\
Q(z) :\ & a^2B_1\omega^2 - 2abB_1\omega + aB_1c^3\omega + b^2B_1 - bB_1c^3 + B_1c^4 - B_1c^2\omega\\
& + B_1c^2 = 0.
\end{aligned} \tag{2.27}$$

The solution of the above system gives

$$B_0 = 0, \quad B_1 = -2c, \quad a = \frac{2b\omega - c^3\omega \mp \sqrt{c^2\omega^2\left(c^4 - 4c^2 + 4\omega - 4\right)}}{2\omega^2}. \tag{2.28}$$

Hence, the solution of 2D-gBK equation (1.7) associated with (2.28) is given as

$$u(t, x, y) = \frac{-2c}{1 + \exp\{cx + (a\omega - b)y - c\omega t\}}. \tag{2.29}$$

The wave profile of solution (2.29) is shown in Figure 4 with 3D, contour and 2D plots using parameter values $a = 1$, $b = -0.2$, $c = 20$, $\omega = 0.05$, $B_0 = 0$ with $t = 7$ and $-6 \le x, y \le 6$.

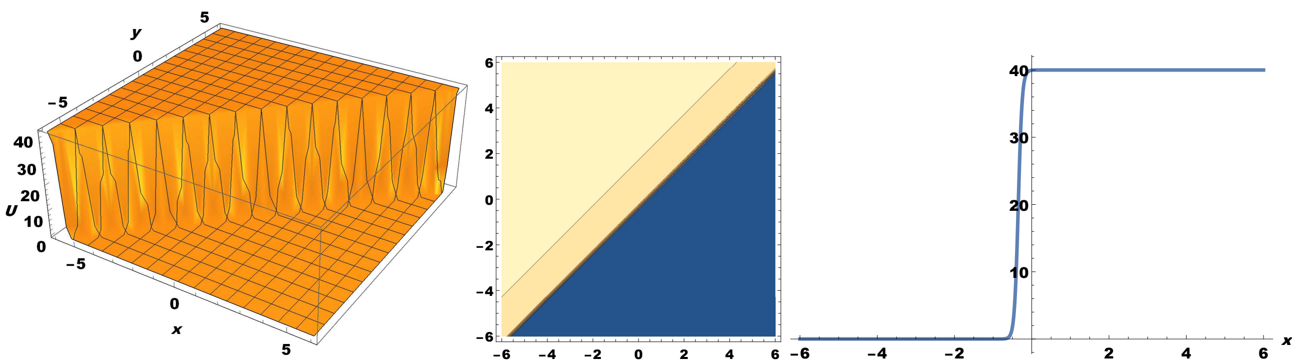

Figure 4. The wave profile of solution (2.29) at $t = 7$.

2.4.2. Solution of (1.7) through (G'/G)-expansion technique

We reckon the (G'/G)-expansion technique [30] in the construction of analytic solutions of 2D-gBK equation (1.7) and so we contemplate a solution structured as

$$\Theta(z) = \sum_{j=0}^{M} B_j \left(\frac{Q'(z)}{Q(z)}\right)^j, \tag{2.30}$$

where $Q(z)$ satisfies

$$Q''(z) + \lambda Q'(z) + \mu Q(z) = 0 \tag{2.31}$$

with λ and μ taken as constants. Here, $B_0, \ldots, B_M$ are parameters to be determined. Utilization of balancing procedure for (2.9) produces $M = 1$ and as a result, the solution of (1.7) assumes the form

$$\Theta(z) = B_0 + B_1 \left(\frac{Q'(z)}{Q(z)}\right). \tag{2.32}$$

Substituting the value of $\Theta(z)$ from (2.32) into (2.9) and using (2.31) and following the steps earlier adopted, leads to an algebraic equation in B_0 and B_1, which splits over various powers of $Q(z)$ to give the system of algebraic equations whose solution is secured as

$$B_0 = 0, \quad B_1 = 2c, \quad a = \frac{16b\omega - B_1^3\lambda^2\omega \pm \sqrt{\Omega_0 + 64B_1^2\omega^3 - 64B_1^2\omega^2} + 4B_1^3\mu\omega}{16\omega^2},$$

where $\Omega_0 = B_1^6\lambda^4\omega^2 - 8B_1^6\lambda^2\mu\omega^2 - 16B_1^4\lambda^2\omega^2 + 16B_1^6\mu^2\omega^2 + 64B_1^4\mu\omega^2$. Thus, we have three types of solutions of the 2D-gBK equation (1.7) given as follows:

When $\lambda^2 - 4\mu > 0$, we gain the hyperbolic function solution

$$u(t,x,y) = B_0 + B_1\left(\Delta_1 \frac{A_1 \sinh(\Delta_1 z) + A_2 \cosh(\Delta_1 z)}{A_1 \cosh(\Delta_1 z) + A_2 \sinh(\Delta_1 z)} - \frac{\lambda}{2}\right), \tag{2.33}$$

with $z = cx + (a\omega - b)y - c\omega t$, $\Delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$ together with A_1, A_2 being arbitrary constants. The wave profile of solution (2.33) is shown in Figure 5 with 3D, contour and 2D plots using parameter values $a = 3$, $b = 0.5$, $c = 10$, $\omega = -0.1$, $B_0 = 0$, $\lambda = -0.971$, $\mu = 10$, $A_1 = 5$, $A_2 = 1$, where $t = 10$ and $-10 \leq x, y \leq 10$.

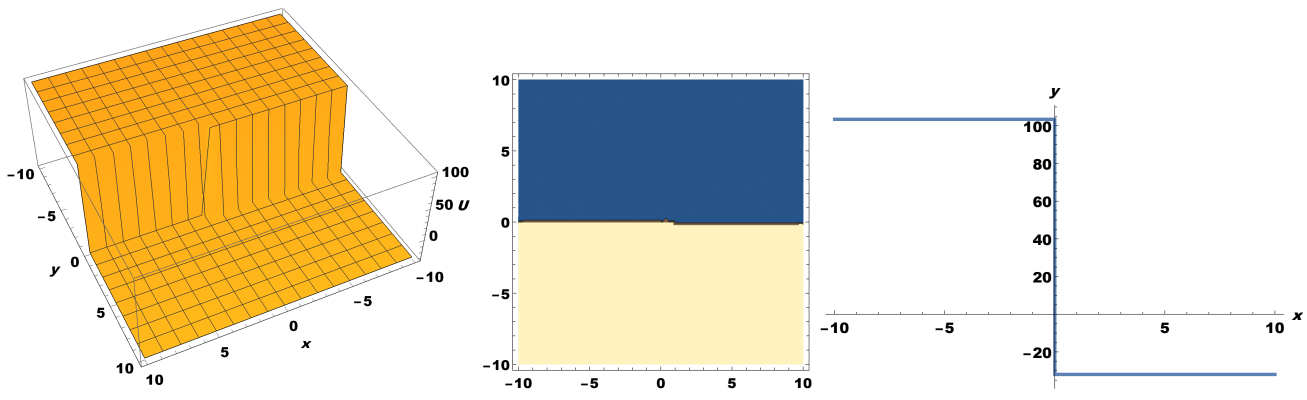

Figure 5. The wave profile of solution (2.33) at $t = 10$.

When $\lambda^2 - 4\mu < 0$, we achieve the trigonometric function solution

$$u(t,x,y) = B_0 + B_1\left(\Delta_2 \frac{A_2\cos(\Delta_2 z) - A_1\sin(\Delta_2 z)}{A_1\cos(\Delta_2 z) + A_2\sin(\Delta_2 z)} - \frac{\lambda}{2}\right), \tag{2.34}$$

with $z = cx + (a\omega - b)y - c\omega t$, $\Delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$ together with A_1 and A_2 are arbitrary constants. The wave profile of solution (2.34) is shown in Figure 6 with 3D, contour and 2D plots using parameter values $a = 1$, $b = 0.5$, $c = 0.3$, $\omega = 0.3$, $B_0 = 0$, $\lambda = -0.971$, $\mu = 2$, $A_1 = 5$, $A_2 = 1$ with $t = 10$ and $-10 \leq x, y \leq 10$.

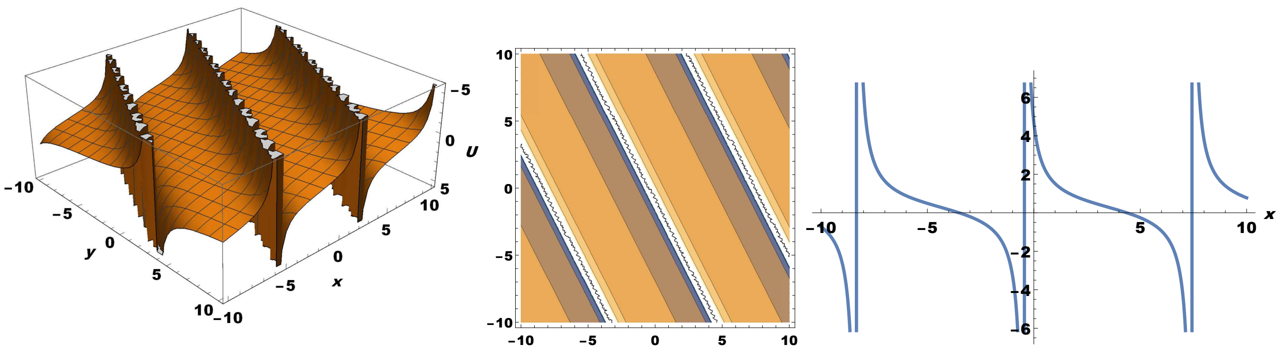

Figure 6. The wave profile of solution (2.34) at $t = 10$.

When $\lambda^2 - 4\mu = 0$, we gain the rational function solution

$$u(t,x,y) = B_0 + B_1\left(\frac{A_2}{A_1 + A_2 z} - \frac{\lambda}{2}\right), \tag{2.35}$$

with $z = cx + (a\omega - b)y - c\omega t$ and A_1, A_2 being arbitrary constants. We plot the graph of solution (2.35) in Figure 7 via 3D, contour and 2D plots using parametric values $a = 1$, $b = 1.01$, $c = 100$, $\omega = 0.1$, $B_0 = 10$, $\lambda = 10$, $A_1 = 3$, $A_2 = 10$, where $t = 2.4$ and $-5 \leq x, y \leq 5$.

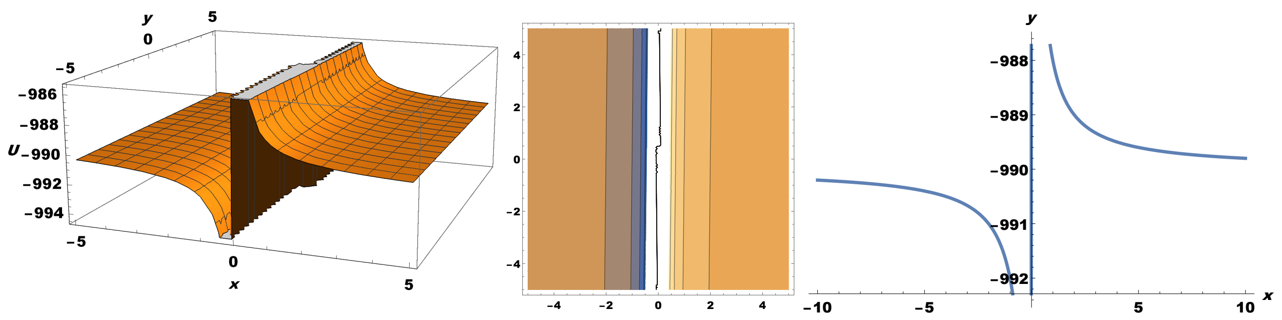

Figure 7. The wave profile of solution (2.35) at $t = 2.4$.

Case 2. Group-invariant solutions via R_4

Lagrange system associated with the symmetry $R_4 = 3t\partial/\partial y + (x - 2y)\partial/\partial u$ is

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{3t} = \frac{du}{(x-2y)}, \tag{2.36}$$

which leads to the three invariants $T = t,\ X = x,\ Q = u + (y^2/3t) - (xy/3t)$. Using these three invariants, the 2D-gBK equation (1.7) is reduced to

$$18TQ_XQ_{XX} + 3TQ_{TX} + 3TQ_{XX} + 3XQ_{XX} + 3Q_X + 3TQ_{XXXX} - 2 = 0. \tag{2.37}$$

Case 3. Group-invariant solutions via $R_1,\ R_2$ and R_5

We take $G(t) = 1$ and by combining the generators $R_1,\ R_2$ as well as R_5, we solve the characteristic equations corresponding to the combination and get the invariants $X = x$, $Y = y - t$ with group-invariant $u = Q(X, Y) + t$. With these invariants, the 2D-gBK equation (1.7) transforms to the NPDE

$$Q_{XX} + Q_{YY} - Q_{XY} + 3Q_XQ_{XY} + 3Q_YQ_{XX} + 6Q_XQ_{XX} + Q_{XXXX} + Q_{XXXY} = 0, \tag{2.38}$$

whose solution is given by

$$Q(X, Y) = 2A_2 \tanh\left[A_2X + A_2\left(\frac{1}{2} - \frac{1}{2}\sqrt{16A_2^4 - 24A_2^2 - 3} - 2A_2^2\right)Y + A_1\right] + A_3, \tag{2.39}$$

with arbitrary constants A_1–A_3. Thus, we achieve the hyperbolic solution of (1.7) as

$$\begin{aligned} u(t, x, y) = t + 2A_2 \tanh\Bigg[&\frac{1}{2}A_2(t - y)\sqrt{16A_2^4 - 24A_2^2 - 3} + \frac{1}{2}(4t - 4y)A_2^3 \\ &+ \frac{1}{2}(y + 2x - t)A_2 + A_1\Bigg] + A_3. \end{aligned} \tag{2.40}$$

The wave profile of solution (2.40) is shown in Figure 8 with 3D, contour and 2D plots using parameter values $A_1 = 70.1$, $A_2 = -30$, $A_3 = 0$, where $t = 0.5$ and $-10 \le x, y \le 10$.

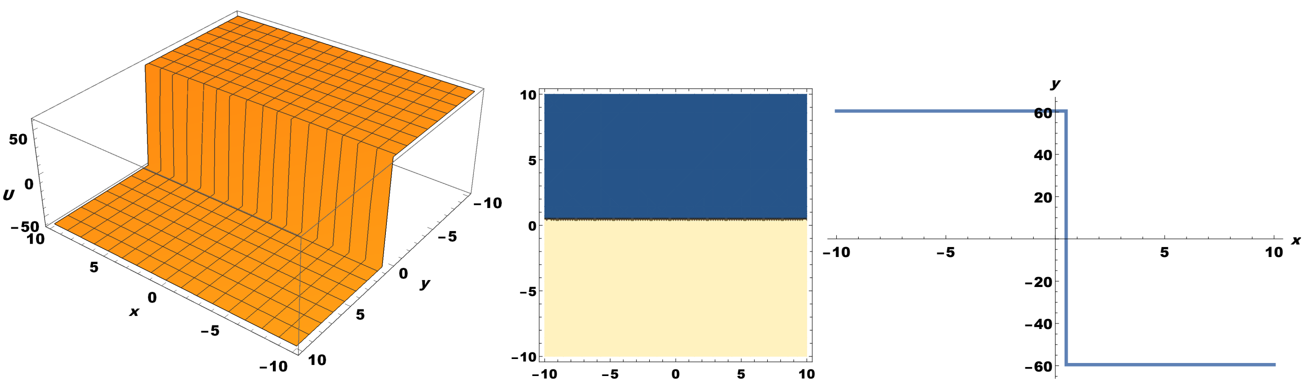

Figure 8. The wave profile of solution (2.40) at $t = 0.5$.

Besides, symmetries of (2.38) are found as

$$P_1 = \frac{\partial}{\partial X},\ P_2 = \frac{\partial}{\partial Y},\ P_3 = \frac{\partial}{\partial Q},\ P_4 = \left(\frac{1}{3}X + \frac{2}{3}Y\right)\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + \left(\frac{2}{3}X - 2Y - \frac{1}{3}Q\right)\frac{\partial}{\partial Q}.$$

Now, the symmetry P_1 furnishes the solution $Q(X,Y) = f(z)$, $z = Y$. So, Eq (2.38) gives the ODE $f''(z) = 0$. Hence, we have a solution of (1.7) as

$$u(t,x,y) = t + A_0(y - t) + A_1, \tag{2.41}$$

with A_0, A_1 as constants. Further, the symmetry P_2 yields $Q(X,Y) = f(z)$, $z = X$ and so Eq (2.38) reduces to

$$f''(z) + 6f'(z)f''(z) + f''''(z) = 0. \tag{2.42}$$

Integration of the above equation three times with respect to z gives

$$f'(z)^2 + 2f(z)^3 + f(z)^2 + 2A_0 f(z) + 2A_1 = 0, \tag{2.43}$$

and taking constants $A_0 = A_1 = 0$ and then integrating it results in the solution of (1.7) as

$$u(t,x,y) = t - \frac{1}{2}\left\{1 + \tan\left(\frac{1}{2}A_1 - \frac{1}{2}x\right)^2\right\}. \tag{2.44}$$

The wave profile of solution (2.44) is shown in Figure 9 with 3D, contour and 2D plots using parameter values $A_1 = 40$, $t = 3.5$ and $-10 \leq x \leq 10$.

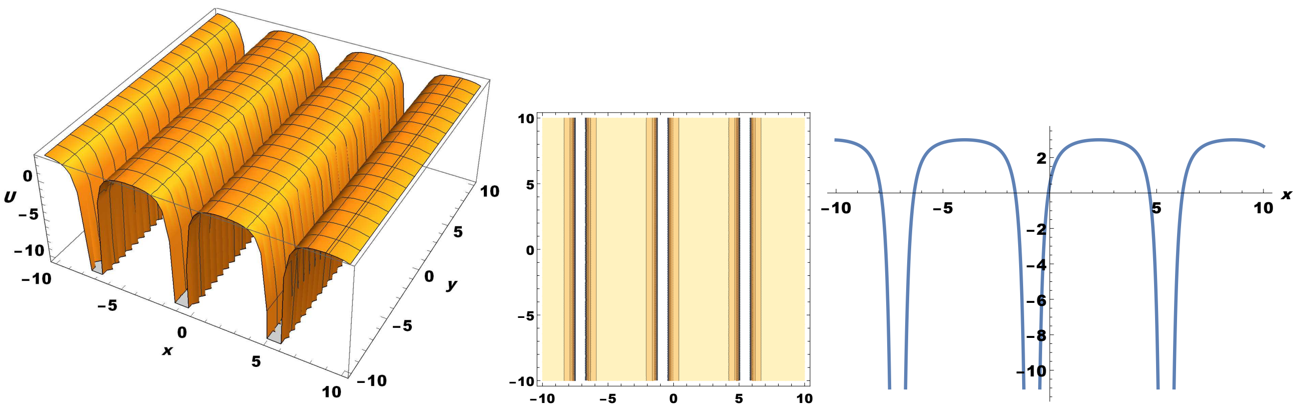

Figure 9. The wave profile of solution (2.44) at $t = 3.5$.

On combining P_1–P_3 as $P = c_0P_1 + c_1P_2 + c_2P_3$, we accomplish

$$Q(X,Y) = \frac{c_2}{c_0}X + f(z), \text{ where } z = c_0Y - c_1X. \tag{2.45}$$

Using the newly acquired invariants (2.45), Eq (2.38) transforms to the NODE:

$$c_0c_1^2f''(z) + 6c_1^2c_2f''(z) - 3c_0c_1c_2f''(z) + c_0^2c_1f''(z) + c_0^3f''(z) + 6c_0^2c_1^2f'(z)f''(z) \\ - 6c_0c_1^3f'(z)f''(z) + c_0c_1^4f''''(z) - c_0^2c_1^3f''''(z) = 0. \tag{2.46}$$

Engaging the Lie point symmetry P_4, we obtain

$$Q(X, Y) = X - 2Y + Y^{-1/3}f(z) \text{ with } z = Y^{-1/3}(X - Y), \tag{2.47}$$

and Eq (2.38) reduces to the NODE

$$6zf'(z) + z^2f''(z) - 18f'(z)^2 - 9f(z)f''(z) + 4f(z) - 18zf'(z)f''(z) - 12f'''(z) - 3zf''''(z) = 0. \tag{2.48}$$

Case 4. Group-invariant solutions via R_6

Lie point symmetry R_6 dissociates to the Lagrange system

$$\frac{dt}{15t} = \frac{dx}{3x + 6y} = \frac{dy}{9y - 12t} = \frac{du}{-(4y + 3u)},$$

which gives

$$u = t^{-1/5}Q(T, X) - \frac{2}{9}t - \frac{1}{3}y, \text{ with } T = (2t + y)t^{-3/5} \text{ and } X = (x - t - y)t^{-1/5}. \tag{2.49}$$

Substituting the expression of u in (1.7), we obtain the NPDE

$$5Q_{TT} - 3TQ_{TX} - XQ_{XX} - 2Q_X + 15Q_XQ_{TX} + 15Q_TQ_{XX} + 5Q_{TXXX} = 0, \tag{2.50}$$

which has two symmetries:

$$P_1 = \frac{\partial}{\partial Q}, \; P_2 = \frac{\partial}{\partial X} + \frac{1}{15}T\frac{\partial}{\partial Q}.$$

The symmetry P_2 gives $Q(X, Y) = f(z) + (1/15)TX$, $z = T$ and hence (2.50) reduces to

$$75f''(z) - 4z = 0.$$

Solving the above ODE and reverting to the basic variables gives the solution of (1.7) as

$$u(t, x, y) = \frac{1}{\sqrt[5]{t}}\left\{\frac{(2t + y)(x - t - y)}{15t^{4/5}} + \frac{2(2t + y)^3}{225t^{9/5}} + \frac{(2t + y)}{t^{3/5}}A_1 + A_2\right\} - \frac{2t}{9} - \frac{y}{3}, \tag{2.51}$$

where A_1 and A_2 are integration constants. The wave profile of solution (2.51) is shown in Figure 10 with 3D, contour and 2D plots using parameter values $A_1 = -0.3$, $A_2 = -50$ with $t = 1.1$ and $-10 \leq x, y \leq 10$.

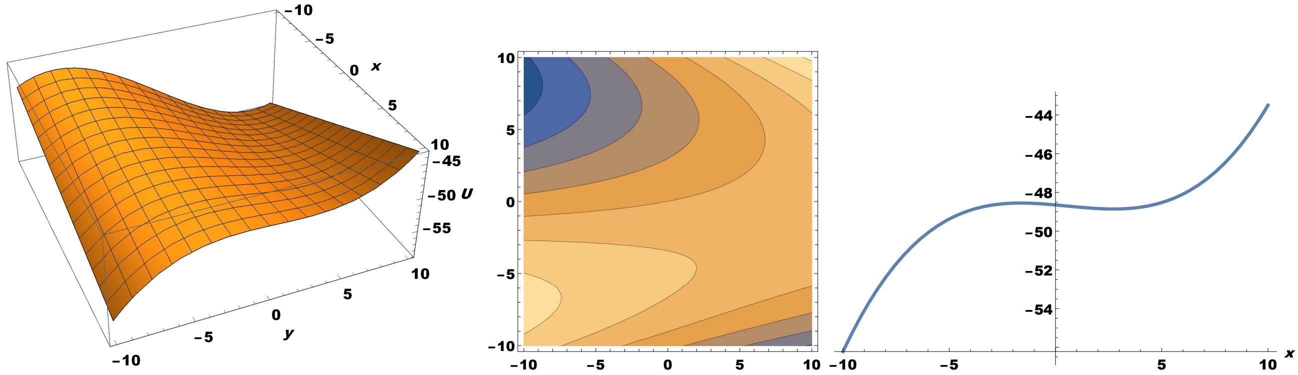

Figure 10. The wave profile of solution (2.51) at $t = 1.1$.

Next, we invoke the symmetry $P_1 + P_2$. This yields $Q(X, Y) = f(z) + X + (1/15)TX$, $z = T$. Consequently, we have the transformed version of (2.50) as

$$75f''(z) - 4z - 15 = 0.$$

Solving the above ODE and reverting to basic variables gives the solution of (1.7) as

$$u(t, x, y) = \frac{1}{\sqrt[5]{t}}\left\{\frac{(2t + y)(x - t - y)}{15t^{4/5}} + \frac{2(2t + y)^3}{225t^{9/5}} + \frac{(2t + y)^2}{10t^{6/5}} + \frac{(2t + y)}{t^{3/5}}A_1 + A_2\right\} - \frac{2t}{9} - \frac{y}{3}. \tag{2.52}$$

The wave profile of solution (2.52) is shown in Figure 11 with 3D, contour and 2D plots using parameter values $A_1 = -3.6$, $A_2 = 50$ with $t = 1.1$ and $-10 \leq x, y \leq 10$.

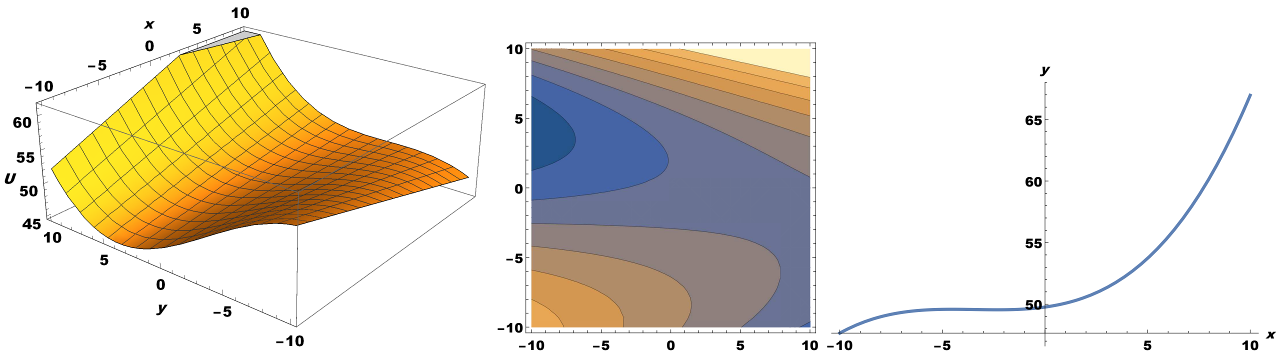

Figure 11. The wave profile of solution (2.52) at $t = 1.1$.

3. Conservation laws of (1.7)

In this section, we construct the conservation laws for 2D-gBK equation (1.7) by making use of the multiplier approach [22, 49, 50], but first we give some basic background of the method that we are adopting.

3.1. Fundamental operators and their relationship

Consider the n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$. The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \dots, \tag{3.1}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n, \tag{3.2}$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e., $u_{(1)} = \{u_i^\alpha\},\ \alpha = 1, ..., m,\ i = 1, ..., n.$ In the same vein $u_{(2)} = \{u_{ij}^\alpha\},\ \alpha = 1, ..., m,\ i, j = 1, ..., n$ and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$.

Now consider a kth-order system of PDEs:

$$G_\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0,\ \alpha = 1, 2, \dots, m. \tag{3.3}$$

The Euler-Lagrange operator, for every α, is presented as

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha},\ \ \alpha = 1, \dots, m. \tag{3.4}$$

An n-tuple $T = (T^1, T^2, \dots, T^n)$, such that

$$D_i T^i = 0 \tag{3.5}$$

holds for all solutions of (3.3) is known as the conserved vector of system (3.3).

The multiplier $\Omega_\alpha(x, u, u_{(1)}, \dots)$ of system (3.3) has the property that

$$D_i T^i = \Omega_\alpha G_\alpha \tag{3.6}$$

holds identically [22]. The determining equations for multipliers are obtained by taking the variational derivative of (3.6), namely

$$\frac{\delta}{\delta u^\alpha}(\Omega_\alpha G_\alpha) = 0. \tag{3.7}$$

The moment multipliers are generated from (3.7), the conserved vectors can be derived systematically using (3.6) as the determining equation.

3.2. Construction of conservation laws for (1.7)

Conservation laws of 2D-gBK equation (1.7) are derived by utilizing second-order multiplier $\Omega(t, x, y, u, u_t, u_x, u_y, u_{xx}, u_{xy})$, in Eq (3.7), where G is given as

$$G \equiv u_{tx} + 6u_x u_{xx} + u_{xxxx} + u_{xxxy} + 3\left(u_x u_y\right)_x + u_{xx} + u_{yy},$$

and the Euler operator $\delta/\delta u$ is expressed in this case as

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}-D_y\frac{\partial}{\partial u_y}+D_tD_x\frac{\partial}{\partial u_{tx}}+D_xD_y\frac{\partial}{\partial u_{xy}}$$
$$+D_x^2\frac{\partial}{\partial u_{xx}}+D_y^2\frac{\partial}{\partial u_{yy}}+D_x^4\frac{\partial}{\partial u_{xxxx}}+D_x^3D_y\frac{\partial}{\partial u_{xxxy}}.$$

Expansion of Eq (3.7) and splitting on diverse derivatives of dependent variable u gives

$$\begin{aligned}&\Omega_u=0,\ \ \Omega_x=0,\ \ \Omega_{yy}=0,\ \ \Omega_{yu_x}=0,\ \ \Omega_{u_xu_x}=0,\\&\Omega_{tu_x}=0,\ \ \Omega_{u_t}=0,\ \ \Omega_{u_{xx}}=0,\ \ \Omega_{u_{xy}}=0,\ \ \Omega_{u_y}=0.\end{aligned}\tag{3.8}$$

Solution to the above system of equations gives $\Omega(t,x,y,u,u_t,u_x,u_y,u_{xx},u_{xy})$ as

$$\Omega(t,x,y,u,u_t,u_x,u_y,u_{xx},u_{xy})=C_1u_x+f_1(t)y+f_2(t),\tag{3.9}$$

with C_1 being an arbitrary constant and $f_1(t)$, $f_2(t)$ being arbitrary functions of t. Using Eq (3.6), one obtains the following three conserved vectors of Eq (1.7) that correspond to the three multipliers u_x, $f_1(t)$ and $f_2(t)$:

Case 1. For the first multiplier $Q_1=u_x$, the corresponding conserved vector (T_1^t,T_1^x,T_1^y) is given by

$$\begin{aligned}T_1^t&=\frac{1}{2}u_x^2,\\T_1^x&=\frac{1}{2}u_x^2+2u_x^3-\frac{1}{2}u_{xx}^2-\frac{1}{2}u_{xx}u_{xy}+\frac{1}{2}u_xu_{xxy}+u_{xxx}u_x\\&\quad+\frac{1}{2}u_{xxx}u_y+\frac{1}{2}uu_{xxxy}+\frac{1}{2}uu_{yy}+uu_xu_{xy}+2u_yu_x^2,\\T_1^y&=\frac{1}{2}u_yu_x-uu_xu_{xx}-\frac{1}{2}uu_{xy}-\frac{1}{2}uu_{xxxx}.\end{aligned}$$

Case 2. For the second multiplier $Q_2=f_1(t)$, we obtain the corresponding conserved vector (T_2^t,T_2^x,T_2^y) as

$$\begin{aligned}T_2^t&=u_xf_1(t)y,\\T_2^x&=3yf_1(t)u_x^2+3yf_1(t)u_xu_y-yf_1'(t)u\\&\quad+yf_1(t)u_x+yf_1(t)u_{xxx}+yf_1(t)u_{xxy},\\T_2^y&=u_yf_1(t)y-uf_1(t).\end{aligned}$$

Case 3. Finally, for the third multiplier $Q_3=f_2(t)$, the corresponding conserved vector (T_3^t,T_3^x,T_3^y) is

$$\begin{aligned}T_3^t&=u_xf_2(t),\\T_3^x&=3u_x^2f_2(t)+3u_xu_yf_2(t)-uf_2'(t)+u_xf_2(t)\\&\quad+u_{xxx}f_2(t)+u_{xxy}f_2(t),\\T_3^y&=u_yf_2(t).\end{aligned}$$

Remark 3.1. *We notice that this method assists in the construction of conservation laws of (1.7) despite the fact that it possesses no variational principle [51]. Moreover, the presence of arbitrary functions in the multiplier indicates that 2D-gBK (1.7) has infinite number of conserved vectors.*

4. Conclusions

In this paper, we carried out a study on two-dimensional generalized Bogoyavlensky-Konopelchenko equation (1.7). We obtained solutions for Eq (1.7) with the use of Lie symmetry reductions, direct integration, Kudryashov's and (G'/G)-expansion techniques. We obtained solutions of (1.7) in the form of algebraic, rational, periodic, hyperbolic as well as trigonometric functions. Furthermore, we derived conservation laws of (1.7) by engaging the multiplier method were we obtained three local conserved vectors. We note here that the presence of the arbitrary functions $f_1(t)$ and $f_2(t)$ in the multipliers, tells us that one can generate unlimited number of conservation laws for the underlying equation.

Conflict of interest

The authors state no conflicts of interest.

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