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*Research article*

# Existence of nontrivial positive solutions for generalized quasilinear elliptic equations with critical exponent

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**Abstract:** In this paper, we are concerned with the existence of nontrivial positive solutions for the following generalized quasilinear elliptic equations with critical growth

$$-\text{div}(g^p(u)|\nabla u|^{p-2}\nabla u) + g^{p-1}(u)g'(u)|\nabla u|^p + V(x)|u|^{p-2}u = h(x,u), \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $1 < p < N$. Under some suitable conditions, we prove that the above equation has a nontrivial positive solution by variational methods. To some extent, our results improve and supplement some existing relevant results.

**Keywords:** generalized quasilinear elliptic equations; variational methods; nontrivial positive solutions
**Mathematics Subject Classification:** 35J20, 35J60, 35J62

## 1. Introduction

In this paper, we consider the existence of nontrivial positive solutions for the following generalized quasilinear elliptic equations

$$-\text{div}(g^p(u)|\nabla u|^{p-2}\nabla u) + g^{p-1}(u)g'(u)|\nabla u|^p + V(x)|u|^{p-2}u = h(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $1 < p < N$, $g : \mathbb{R} \to \mathbb{R}^+$ is an even differential function with $g'(t) \geq 0$ for all $t \geq 0$ and $g(0) = 1$, $V : \mathbb{R}^N \to \mathbb{R}$ and $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Notice that if we take $p = 2$ and $g^2(u) = 1 + \frac{[(l(u^2))']^2}{2}$, where $l$ is a given real function, then (1.1) turns into

$$-\triangle u + V(x)u - \triangle(l(u^2))l'(u^2)u = h(x,u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Equation (1.2) is related to the existence of solitary wave solutions for quasilinear Schrödinger equations

$$iz_t = -\triangle z + W(x)z - h(x,z) - \triangle(l(|z|^2))l'(|z|^2)z, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $l : \mathbb{R} \to \mathbb{R}$ are suitable functions. The form of (1.3) has many applications in physics. For instance, the case $l(s) = s$ was used to model the time evolution of the condensate wave function in superfluid film [16, 17]. In the case of $l(s) = \sqrt{1+s}$, Eq (1.3) was used as a model of the self-channeling of a high-power ultrashort laser in matter [3, 29]. For more details on the physical background, we can refer to [2, 4] and references therein.

Putting $z(t,x) = \exp(-iEt)u(x)$ in (1.3), where $E \in \mathbb{R}$ and $u$ is a real function, we obtain a corresponding equation of elliptic type (1.2).

If we take $p = 2$, then (1.1) turns into

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

In recent years, many researchers have studied (1.4) under various hypotheses on the potential and nonlinearity, for example [6-9, 24, 28, 38].

If we set $p = 2, g^2(u) = 1 + 2u^2$, i.e., $l(s) = s$, we can get the superfluid film equation in plasma physics

$$-\triangle u + V(x)u - \triangle(u^2)u = h(x,u), \quad x \in \mathbb{R}^N. \tag{1.5}$$

Equation (1.5) has been extensively studied, see [5, 20, 27, 30, 31].

If we take $p = 2, g^2(u) = 1 + \frac{u^2}{2(1+u^2)}$, i.e., $l(s) = \sqrt{1+s}$, (1.2) derives the following equation

$$-\triangle u + V(x)u - [\triangle(1+u^2)^{\frac{1}{2}}]\frac{u}{2(1+u^2)^{\frac{1}{2}}} = h(x,u), \quad x \in \mathbb{R}^N, \tag{1.6}$$

which models of the self-channeling of a high-power ultrashort laser in matter. For (1.6), there are many papers studying the existence of solutions, see [10, 13, 33] and references therein.

Furthermore, if we set $g^p(u) = 1 + 2^{p-1}u^p$ in (1.1), then we get the following quasilinear elliptic equations

$$-\triangle_p(u) + V(x)|u|^{p-2}u - \triangle_p(u^2)u = h(x,u), \quad u \in W^{1,p}(\mathbb{R}^N), \tag{1.7}$$

where $\triangle_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator with $1 < p \leq N$. In [11], where $h(x,u) = h(u)$, the authors constructed infinitely many nodal solutions for (1.7) under suitable assumptions.

We point out that the related semilinear elliptic equations with the asymptotically periodic condition have been extensively researched, see [12, 19, 22, 23, 25, 34–37] and their references.

Especially, in [12], Lins and Silva considered the following asymptotically periodic $p$-laplacian equations

$$\begin{cases} -\triangle_p u + V(x)u^{p-1} = K(x)u^{p^*-1} + g(x,u), x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), u \geq 0. \end{cases} \tag{1.8}$$

Assume that the potential $V$ satisfies

$(V_0)$ there exist a constant $a_0 > 0$ and a function $\bar{V} \in C(\mathbb{R}^N, \mathbb{R})$, 1-periodic in $x_i$, $1 \leq i \leq N$, such that $\bar{V}(x) \geq V(x) \geq a_0 > 0$ and $\bar{V} - V \in \mathcal{K}$, where

$$\mathcal{K} := \{h \in C(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N) : \forall \varepsilon > 0, \text{meas}\{x \in \mathbb{R}^N : |h(x)| \geq \varepsilon\} < +\infty\},$$

and the asymptotically periodic of $g$ at infinity was assumed to the following condition

($g_0$) there exist a constant $p < q_1 < p^*$ and functions $h \in \mathcal{K}$, $g_0 \in C(\mathbb{R}^N, \mathbb{R})$, 1-periodic in $x_i$, $1 \le i \le N$, such that

$$|g(x, s) - g_0(x, s)| \le h(x)|s|^{q_1 - 1}, \text{for all } (x, s) \in \mathbb{R}^N \times [0, +\infty).$$

For the other conditions on $g$, please see [12].

In recent paper [36], Xue and Tang studied the following quasilinear Schrodinger equation

$$-\triangle u + V(x)u - \triangle(u^2)u = K(x)|u|^{22^*-2}u + g(x, u), x \in \mathbb{R}^N, \tag{1.9}$$

they proposed reformative conditions, which unify the asymptotic process of the potential and nonlinear term at infinity, see the below condition ($V_1$) and (i) of ($f_5$). It is easy to see that this conditions contains more elements than those in [12]. To the best of our knowledge, there is no work concerning with the unified asymptotic process of the potential and nonlinear term at infinity for general quasilinear elliptic equations.

Motivated by above papers, under the asymptotically periodic conditions, we establish the existence of a nontrivial positive solution for Eq (1.1) with critical nonlinearity. We assume $h(x, u) = f(x, u) + K(x)g(u)|G(u)|^{p^*-2}G(u)$. Equation (1.1) can be rewritten in the following form:

$$\begin{aligned} &-\text{div}(g^p(u)|\nabla u|^{p-2}\nabla u) + g^{p-1}(u)g'(u)|\nabla u|^p + V(x)|u|^{p-2}u \\ &\quad = f(x, u) + K(x)g(u)|G(u)|^{p^*-2}G(u), \quad x \in \mathbb{R}^N, \end{aligned} \tag{1.10}$$

where $p^* = \frac{Np}{N-p}$ for $N \ge 3$, $G(t) = \int_0^t g(\tau)d\tau$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous function.

We observe that the energy functional associate with (1.10) is given by

$$\mathcal{I}(u) = \frac{1}{p}\int_{\mathbb{R}^N} g^p(u)|\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} F(x, u)dx - \frac{1}{p^*}\int_{\mathbb{R}^N} K(x)|G(u)|^{p^*}dx,$$

where $F(x, u) = \int_0^u f(x, \tau)d\tau$. However, $\mathcal{I}$ may be not well defined in $W^{1,p}(\mathbb{R}^N)$ because of the term $\int_{\mathbb{R}^N} g^p(u)|\nabla u|^p dx$. To overcome this difficulty, we make use of a change of variable constructed by [32],

$$v = G(u) = \int_0^u g(t)dt.$$

Then we obtain the following functional

$$\mathcal{J}(v) = \frac{1}{p}\int_{\mathbb{R}^N} \left[|\nabla v|^p + V(x)|G^{-1}(v)|^p\right]dx - \int_{\mathbb{R}^N} F(x, G^{-1}(v))dx - \frac{1}{p^*}\int_{\mathbb{R}^N} K(x)|v|^{p^*}dx.$$

Since $g$ is a nondecreasing positive function, we can get $|G^{-1}(v)| \le \frac{1}{g(0)}|v|$. From this and our hypotheses, it is clear that $\mathcal{J}$ is well defined in $W^{1,p}(\mathbb{R}^N)$ and $\mathcal{J} \in C^1$.

If $u$ is said to be a weak solution for Eq (1.10), then it should satisfy

$$\begin{aligned} &\int_{\mathbb{R}^N} \Big[g^p(u)|\nabla u|^{p-2}\nabla u \nabla\psi + g^{p-1}(u)g'(u)|\nabla u|^p\psi + V(x)|u|^{p-2}u\psi - f(x, u)\psi \\ &\quad - K(x)g(u)|G(u)|^{p^*-2}G(u)\psi\Big]dx = 0, \text{ for all } \psi \in C_0^\infty(\mathbb{R}^N). \end{aligned} \tag{1.11}$$

Let $\psi = \frac{1}{g(u)}\varphi$, we know that (1.11) is equivalent to

$$\langle \mathcal{J}'(v), \varphi\rangle = \int_{\mathbb{R}^N} \Big[|\nabla v|^{p-2}\nabla v\nabla\varphi + V(x)\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))}\varphi - \frac{f(x,G^{-1}(v))}{g(G^{-1}(v))}\varphi - K(x)|v|^{p^*-2}v\varphi\Big]dx = 0,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$.

Therefore, in order to obtain a nontrivial solution of (1.1), it suffices to study the following equations

$$-\Delta_p v + V(x)\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))} - \frac{f(x,G^{-1}(v))}{g(G^{-1}(v))} - K(x)|v|^{p^*-2}v = 0.$$

Obviously, if $v$ is a nontrivial critical point of the functional $\mathcal{J}$, then $u = G^{-1}(v)$ is a nontrivial critical point of the functional $\mathcal{I}$, i.e., $u = G^{-1}(v)$ is a nontrivial solution of equation (1.1).

In the asymptotically periodic potential case, the functional $\mathcal{J}$ loses the $\mathbb{Z}^N$-translation invariance due to the asymptotically periodic potential. For this reason, many effective methods applied in periodic problems become invalid in asymptotically periodic problems. In this paper, we adopt some tricks to overcome the difficulties caused by the dropping of periodicity of $V(x)$.

Before stating our results, we introduce some hypotheses on the potential $V, K$:

$(V_1)$ $0 < V_{min} \le V(x) \le V_0(x) \in L^\infty(\mathbb{R}^N)$ and $V(x) - V_0(x) \in \mathcal{F}_0$, where

$$\mathcal{F}_0 := \Big\{k(x) : \forall\varepsilon > 0, \lim_{|y|\to\infty} \text{meas}\,\{x \in B_1(y) : |k(x)| \ge \varepsilon\} = 0\Big\},$$

$V_{min}$ is a positive constant and $V_0(x)$ is $1-$periodic in $x_i$, $1 \le i \le N$.

$(K_1)$ The function $K \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in $x_i$, $1 \le i \le N$ and there exists a point $x_0 \in \mathbb{R}^N$ such that

(i) $K(x) \ge \inf_{x\in\mathbb{R}^N} K(x) > 0$, for all $x \in \mathbb{R}^N$;

(ii) $K(x) = \|K\|_\infty + O(|x - x_0|^{\frac{N-p}{p-1}})$, as $x \to x_0$.

Moreover, the nonlinear term $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ should satisfy the following assumptions:

$(f_0)$ $f(x,t) = 0, t \le 0$ uniformly for $x \in \mathbb{R}^N$;

$(f_1)$ $\lim_{t\to0^+} \frac{f(x,t)}{g(t)|G(t)|^{p-1}} = 0$ uniformly for $x \in \mathbb{R}^N$;

$(f_2)$ $\lim_{t\to+\infty} \frac{f(x,t)}{g(t)|G(t)|^{p^*-1}} = 0$ uniformly for $x \in \mathbb{R}^N$;

$(f_3)$ $\frac{f(x,G^{-1}(t))t}{g(G^{-1}(t))} - pF(x,G^{-1}(t)) \ge \frac{f(x,G^{-1}(ts))ts}{g(G^{-1}(ts))} - pF(x,G^{-1}(ts))$ for all $t \in \mathbb{R}^+$ and $s \in [0,1]$;

$(f_4)$ there exists an open bounded set $\Omega \subset \mathbb{R}^N$, containing $x_0$ given by $(K_1)$ such that

$$\lim_{t\to+\infty}\frac{F(x,t)}{|G(t)|^{\mu}}=+\infty \text{ uniformly for } x\in\Omega,\ \text{where } \mu=p^*-\frac{p}{p-1},\ \text{if } N<p^2;$$

$$\lim_{t\to+\infty}\frac{F(x,t)}{|G(t)|^{p}\log|G(t)|}=+\infty \text{ uniformly for } x\in\Omega,\ \text{if } N=p^2;$$

$$\lim_{t\to+\infty}\frac{F(x,t)}{|G(t)|^{p}}=+\infty \text{ uniformly for } x\in\Omega,\ \text{if } N>p^2,\ \text{where } F(x,t)=\int_0^t f(x,\tau)d\tau;$$

($f_5$) There exists a periodic function $f_0\in C(\mathbb{R}^N\times\mathbb{R}^+,\mathbb{R}^+)$, which is $1-$periodic in $x_i, 1\le i\le N$, such that

(i) $f(x,t)\ge f_0(x,t)$ for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}^+$ and $f(x,t)-f_0(x,t)\in\mathcal{F}$, where

$$\mathcal{F}:=\Big\{k(x,t):\forall\varepsilon>0,\ \lim_{|y|\to\infty}\text{meas}\{x\in B_1(y):|k(x,t)|\ge\varepsilon\}=0 \text{ uniformly for } |t| \text{ bound}\Big\};$$

(ii) $$\frac{f_0(x,G^{-1}(t))t}{g(G^{-1}(t))}-pF_0(x,G^{-1}(t))\ge\frac{f_0(x,G^{-1}(ts))ts}{g(G^{-1}(ts))}-pF_0(x,G^{-1}(ts)) \text{ for all } t\in\mathbb{R}^+ \text{ and } s\in[0,1],$$

where $F_0(x,t)=\int_0^t f_0(x,\tau)d\tau.$

In the asymptotically periodic case, we establish the following theorem.

**Theorem 1.1** Assume that $(V_1),(K_1)$ and $(f_1)-(f_5)$ hold. Then Eq (1.1) has a nontrivial positive solution.

In the special case: $V=V_0, f=f_0$, we can get a nontrivial positive solution for the periodic equation from Theorem 1.1. Indeed, considering the periodic equation

$$\begin{aligned}&-\text{div}(g^p(u)|\nabla u|^{p-2}\nabla u)+g^{p-1}(u)g'(u)|\nabla u|^p+V_0(x)|u|^{p-2}u\\&=f_0(x,u)+K(x)g(u)|G(u)|^{p^*-2}G(u),\end{aligned}\tag{1.12}$$

under the hypothesis:

($V_2$) The function $V_0(x)$ is 1-periodic in $x_i, 1\le i\le N$ and there exists a constant $V_{min}>0$ such that

$$0<V_{min}\le V_0(x)\in L^\infty(\mathbb{R}^N),\ \text{ for all } x\in\mathbb{R}^N.$$

In the periodic case, we obtain the following theorem.

**Theorem 1.2** Assume that $(V_2),(K_1)$ hold, $f_0$ satisfies $(f_1)-(f_4)$. Then Eq (1.12) has a nontrivial positive solution.

**Remark 1.3** Compared with the results obtained by [12, 21, 23, 36], our results are new and different due to the following some facts:

(i) Compared with Eq (1.8) and Eq (1.9), Eq (1.10) is more general. In our results, there is no need to assume $f(x,t)\in C^1(\mathbb{R}^N,\mathbb{R})$. To some extent, our results extends the results of the work [12, 23, 36, 38].

(ii) We choose condition ($f_3$) to be weaker than Ambrosetti-Rabinowitz type condition (see [21]).

(iii) The aim of ($f_3$) is to ensure that the Cerami sequence is bounded, which is different from the conditions of [12].

The rest of this paper is organized as follows: in Section 2, we present some preliminary lemmas. We will prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively.

## 2. Some preliminary lemmas

In this section, we present some useful lemmas.

Let us recall some basic notions. $W := W^{1,p}(\mathbb{R}^N)$ is the usual Sobolev space endowed with the norm $\|u\|_W = \left(\int_{\mathbb{R}^N}(|\nabla u|^p + |u|^p)dx\right)^{\frac{1}{p}}$, we denote by $L^s(\mathbb{R}^N)$ the usual Lebesgue space endowed with the norm $\|u\|_s = \left(\int_{\mathbb{R}^N}|u|^s dx\right)^{\frac{1}{s}}, \forall s \in [1, +\infty)$ and let $C$ denote positive constants. Next, we define the following working space

$$X := \{u \in L^{p^*}(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N), \int_{\mathbb{R}^N} V(x)|u|^p dx < \infty\}$$

endowed with the norm $\|u\| = \left(\int_{\mathbb{R}^N}(|\nabla u|^p + V(x)|u|^p)dx\right)^{\frac{1}{p}}$. According to [22], it is easy to verify that the norms $\|\cdot\|$ and $\|\cdot\|_W$ are equivalent under the assumption $(V_1)$.

Next, we summarize some properties of $g, G$ and $G^{-1}$.

**Lemma 2.1 [32]** The functions $g, G$ and $G^{-1}$ satisfy the following properties:

(i) the functions $G(\cdot)$ and $G^{-1}(\cdot)$ are strictly increasing and odd;

(ii) $G(s) \le g(s)s$ for all $s \ge 0$; $G(s) \ge g(s)s$ for all $s \le 0$;

(iii) $g(G^{-1}(s)) \ge g(0) = 1$ for all $s \in \mathbb{R}$;

(iv) $\dfrac{G^{-1}(s)}{s}$ is decreasing on $(0, +\infty)$ and increasing on $(-\infty, 0)$;

(v) $|G^{-1}(s)| \le \dfrac{1}{g(0)}|s| = |s|$ for all $s \in \mathbb{R}$;

(vi) $\dfrac{|G^{-1}(s)|}{g(G^{-1}(s))} \le \dfrac{1}{g^2(0)}|s| = |s|$ for all $s \in \mathbb{R}$;

(vii) $\dfrac{G^{-1}(s)s}{g(G^{-1}(s))} \le |G^{-1}(s)|^2$ for all $s \in \mathbb{R}$;

(viii) $\lim\limits_{|s|\to 0} \dfrac{G^{-1}(s)}{s} = \dfrac{1}{g(0)} = 1$ and

$$\lim_{|s|\to\infty} \frac{G^{-1}(s)}{s} = \begin{cases} \frac{1}{g(\infty)}, & \text{if } g \text{ is bounded,} \\ 0, & \text{if } g \text{ is unbounded.} \end{cases}$$

Denote

$$\bar{f}(x, s) = V(x)|s|^{p-2}s - V(x)\frac{|G^{-1}(s)|^{p-2}G^{-1}(s)}{g(G^{-1}(s))} + \frac{f(x, G^{-1}(s))}{g(G^{-1}(s))}. \tag{2.1}$$

Then

$$\bar{F}(x, s) = \int_0^s \bar{f}(x, \tau)d\tau = \frac{1}{p}V(x)|s|^p - \frac{1}{p}V(x)|G^{-1}(s)|^p + F(x, G^{-1}(s)). \tag{2.2}$$

Therefore,

$$\mathcal{J}(v) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v|^p + V(x)|v|^p)dx - \int_{\mathbb{R}^N}\bar{F}(x, v)dx - \frac{1}{p^*}\int_{\mathbb{R}^N}K(x)|v|^{p^*}dx. \tag{2.3}$$

**Lemma 2.2** The functions $\bar{f}(x,s)$ and $\bar{F}(x,s)$ satisfy the following properties under $(f_1)-(f_3),(f_5)$:
(i) $\lim_{s\to0^+}\frac{\bar{f}(x,s)}{|s|^{p-1}}$=0 and $\lim_{s\to0^+}\frac{\bar{F}(x,s)}{|s|^{p}}=0$ uniformly in $x\in\mathbb{R}^N$;
(ii) $\lim_{s\to+\infty}\frac{\bar{f}(x,s)}{|s|^{p^*-1}}$=0 and $\lim_{s\to+\infty}\frac{\bar{F}(x,s)}{|s|^{p^*}}=0$ uniformly in $x\in\mathbb{R}^N$;
(iii) $t\bar{f}(x,t)-p\bar{F}(x,t)\ge st\bar{f}(x,st)-p\bar{F}(x,st)$ for all $t\in\mathbb{R}^+$ and $s\in[0,1]$;
(iv) $\bar{F}(x,s)\ge0$ for all $(x,s)\in\mathbb{R}^N\times\mathbb{R}^+$.

**Proof** From $(f_1)$ and Lemma 2.1-(8), we have

$$\lim_{s\to0^+}\frac{\bar{f}(x,s)}{s^{p-1}}=\lim_{s\to0^+}\Big[V(x)(1-(\frac{G^{-1}(s)}{s})^{p-1}\cdot\frac{1}{g(G^{-1}(s))})\Big]+\lim_{s\to0^+}\frac{f(x,G^{-1}(s))}{s^{p-1}g(G^{-1}(s))}=0,$$

uniformly in $x\in\mathbb{R}^N$. Moreover, by $(f_2)$ and Lemma 2.1-(8), one has

$$\lim_{s\to+\infty}\frac{\bar{f}(x,s)}{s^{p^*-1}}=-\lim_{s\to+\infty}\Big[V(x)(\frac{G^{-1}(s)}{s})^{p-1}\cdot\frac{1}{g(G^{-1}(s))}\cdot\frac{1}{s^{p^*-p}}\Big]+\lim_{s\to+\infty}\frac{f(x,G^{-1}(s))}{s^{p^*-1}g(G^{-1}(s))}=0,$$

uniformly in $x\in\mathbb{R}^N$. Then using the L'Hospital rule, we obtain

$$\lim_{s\to0^+}\frac{\bar{F}(x,s)}{|s|^p}=0\text{ and }\lim_{s\to+\infty}\frac{\bar{F}(x,s)}{|s|^{p^*}}=0,$$

uniformly in $x\in\mathbb{R}^N$. Hence, (i) and (ii) hold.

Let $\mathcal{H}(x,t)=t\bar{f}(x,t)-p\bar{F}(x,t)$ for $(x,t)\in(\mathbb{R}^N,\mathbb{R}^+)$.

We claim that $\mathcal{H}(x,t)$ is an increasing function with respect to $t$.

By (2.1) and (2.2), we directly calculate

$$\begin{aligned}\mathcal{H}(x,t)&=t\bar{f}(x,t)-p\bar{F}(x,t)\\&=V(x)|G^{-1}(t)|^{p-2}\Big[G^{-1}(t)^2-\frac{G^{-1}(t)t}{g(G^{-1}(t))}\Big]+\frac{f(x,G^{-1}(t))t}{g(G^{-1}(t))}-pF(x,G^{-1}(t)).\end{aligned}$$

Next, set $\eta(t)=G^{-1}(t)^2-\frac{G^{-1}(t)t}{g(G^{-1}(t))}$ for any $t\in\mathbb{R}^+$. To compete the claim, combining with $(f_3)$ and Lemma 2.1-(1), We only need to prove that $\eta(t)$ is an increasing function on $\mathbb{R}^+$. Please see literature [26], for the reader's convenience, we give a brief proof.

By Lemma 2.1-(2) and $g'(t)\ge0$ for all $t\ge0$, one has

$$G(t)\Big[\frac{g(t)-g'(t)t}{g^2(t)}\Big]\le t,$$

which deduces that

$$\frac{G(t)}{g(t)}\Big(\frac{t}{g(t)}\Big)'\le\frac{t}{g(t)},$$

for all $t\ge0$. Set $\xi=G(t)$. Then

$$G(t)\frac{d}{d\xi}\Big(\frac{t}{g(t)}\Big)\le\frac{t}{g(t)},$$

and thus

$$\xi\Big[\frac{G^{-1}(\xi)}{g(G^{-1}(\xi))}\Big]'\le\frac{G^{-1}(\xi)}{g(G^{-1}(\xi))},$$

for all $\xi \geq 0$. Therefore,

$$\eta'(t) = \frac{G^{-1}(t)}{g(G^{-1}(t))} - \Big[\frac{G^{-1}(t)}{g(G^{-1}(t))}\Big]' t \geq 0,$$

for all $t \geq 0$. It follows that $\eta(t)$ is increasing with respect to $t \geq 0$. Thus, $\eta(st) \leq \eta(t)$ for all $s \in [0, 1]$ and $t \geq 0$, and then

$$G^{-1}(st)^2 - \frac{G^{-1}(st)st}{g(G^{-1}(st))} \leq G^{-1}(t)^2 - \frac{G^{-1}(t)t}{g(G^{-1}(t))},$$

for all $s \in [0, 1]$ and $t \geq 0$. So (iii) holds. Moreover, Lemma 2.1-(5) and $(f_5)-(i)$ imply that $\bar{F}(x, s) \geq 0$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}^+$. This completes the proof. □

**Lemma 2.3** Assume that $(V_1), (K_1)$ and $(f_1) - (f_2)$ are satisfied. Then the functional $\mathcal{J}$ satisfies the following mountain pass geometry structure:

(i) there exist positive constants $\rho$ and $b$ such that $\mathcal{J}(v) \geq b$ for $\|v\| = \rho$;

(ii) there exists a function $v_0 \in X$ such that $\|v_0\| > \rho$ and $\mathcal{J}(v_0) < 0$.

**Proof** By $(f_1), (f_2)$, Lemma 2.2-(1), (2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ and $q \in (p, p^*)$ such that

$$\bar{F}(x, s) \leq \varepsilon(|s|^p + |s|^{p^*}) + C_\varepsilon |s|^q, \tag{2.4}$$

for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}^+$. Therefore, by (2.3) and (2.4), we have

$$\begin{aligned}
\mathcal{J}(v) &= \frac{1}{p}\int_{\mathbb{R}^N}\Big[|\nabla v|^p + V(x)|v|^p\Big]dx - \int_{\mathbb{R}^N}\bar{F}(x, v)dx - \frac{1}{p^*}\int_{\mathbb{R}^N}K(x)|v|^{p^*}dx \\
&\geq \frac{1}{p}\int_{\mathbb{R}^N}\Big[|\nabla v|^p + V(x)|v|^p\Big]dx - \varepsilon\int_{\mathbb{R}^N}|v|^p dx - \varepsilon\int_{\mathbb{R}^N}|v|^{p^*}dx \\
&\quad - C_\varepsilon\int_{\mathbb{R}^N}|v|^q dx - \frac{\|K\|_\infty}{p^*}\int_{\mathbb{R}^N}|v|^{p^*}dx \\
&= \frac{1}{p}\|v\|^p - \varepsilon\int_{\mathbb{R}^N}|v|^p dx - (\varepsilon + \frac{\|K\|_\infty}{p^*})\int_{\mathbb{R}^N}|v|^{p^*}dx - C_\varepsilon\int_{\mathbb{R}^N}|v|^q dx \\
&\geq (\frac{1}{p} - \varepsilon C)\|v\|^p - C\|v\|^{p^*} - C\|v\|^q,
\end{aligned}$$

where $\varepsilon$ is small enough, thus (i) is proved because $p < q < p^*$.

It follows from (2.3) that Lemma 2.2-(4), for any fixed $v \in X$ with $v \geq 0$ and $v \not\equiv 0$, we obtain

$$\begin{aligned}
\mathcal{J}(tv) &= \frac{t^p}{p}\int_{\mathbb{R}^N}\Big[|\nabla v|^p + V(x)|v|^p\Big]dx - \int_{\mathbb{R}^N}\bar{F}(x, tv)dx - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N}K(x)|v|^{p^*}dx \\
&\leq \frac{t^p}{p}\|v\|^p - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N}K(x)|v|^{p^*}dx.
\end{aligned}$$

Obviously, $\mathcal{J}(tv) \to -\infty$ as $t \to +\infty$. Thus there exists a $t_0 > 0$ large enough such that $\mathcal{J}(t_0 v) < 0$ with $\|t_0 v\| > \rho$. Hence, we take $v_0 = t_0 v$, (ii) is proved. □

**Lemma 2.4** Assume that $(V_1), (K_1)$ and $(f_1)-(f_3)$ hold, then there exists a bounded Cerami sequence $\{v_n\} \subset X$ for $\mathcal{J}$.

**Proof** From Lemma 2.3, we know that $\mathcal{J}$ satisfies the mountain pass geometry structure. By the mountain pass theorem (see [1]), there exists a Cerami sequence $\{v_n\} \subset X$ such that

$$\mathcal{J}(v_n) \to c \text{ and } (1 + \|v_n\|)\|\mathcal{J}'(v_n)\|_* \to 0,$$

where

$$c := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}\mathcal{J}(\gamma(t)),\quad \Gamma := \{\gamma\in C([0,1],X) : \gamma(0)=0, \mathcal{J}(\gamma(1))<0\}.$$

As in [14], we can take a sequence $\{t_n\}\subset[0,1]$ satisfying

$$\mathcal{J}(t_nv_n) := \max_{t\in[0,1]}\mathcal{J}(tv_n). \tag{2.5}$$

We claim that $\{\mathcal{J}(t_nv_n)\}$ is bounded from above.

Indeed, without loss of the generality, we may assume that $t_n\in(0,1)$ for all $n\in\mathbb{N}$. Thus, by Lemma 2.2-(3), we have

$$\begin{aligned}\mathcal{J}(t_nv_n)-\frac{1}{p}\langle\mathcal{J}'(t_nv_n),t_nv_n\rangle &= \int_{\mathbb{R}^N}\Big[\frac{1}{p}t_nv_n\bar{f}(x,t_nv_n)-\bar{F}(x,t_nv_n)\Big]dx+(\frac{1}{p}-\frac{1}{p^*})t_n^{p^*}\int_{\mathbb{R}^N}K(x)|v_n|^{p^*}dx\\ &\le \int_{\mathbb{R}^N}\Big[\frac{1}{p}v_n\bar{f}(x,v_n)-\bar{F}(x,v_n)\Big]dx+(\frac{1}{p}-\frac{1}{p^*})\int_{\mathbb{R}^N}K(x)|v_n|^{p^*}dx\\ &= \mathcal{J}(v_n)-\frac{1}{p}\langle\mathcal{J}'(v_n),v_n\rangle\ = c+o_n(1),\end{aligned}$$

which implies that $\{\mathcal{J}(t_nv_n)\}$ is bounded from above.

Now, we prove that $\{v_n\}$ is bounded in $X$. Assume by contradiction that $\{v_n\}$ is unbounded, then up to a subsequence, we may assume that $\|v_n\|\to+\infty$. Set $w_n=\frac{v_n}{\|v_n\|}$. Clearly, $w_n$ is bounded in $X$ and $\|w_n\|=1$. Then, there exists $w\in X$ such that $w_n\rightharpoonup w$ in $X$. Set $\Lambda=\{x\in\mathbb{R}^N : w(x)\neq0\}$. If $\text{meas}(\Lambda)>0$, the Fatou lemma and Lemma 2.2-(4) implies

$$\begin{aligned}0=\limsup_{n\to\infty}\frac{\mathcal{J}(v_n)}{\|v_n\|^p} &= \frac{1}{p}-\liminf_{n\to\infty}\Big[\int_{\mathbb{R}^N}\frac{\bar{F}(x,v_n)}{\|v_n\|^p}dx+\frac{1}{p^*}\int_\Lambda K(x)|w_n|^p|v_n|^{p^*-p}dx\Big]\\ &\le \frac{1}{p}-\frac{\inf_{x\in\mathbb{R}^N}K(x)}{p^*}\liminf_{n\to\infty}\int_\Lambda|w_n|^p|v_n|^{p^*-p}dx\to-\infty,\end{aligned}$$

which is a contradiction. Thus $w=0$. For any $B>0$, by $\|v_n\|\to+\infty$ and (2.5), we have

$$\mathcal{J}(t_nv_n)\ge\mathcal{J}(\frac{B}{\|v_n\|}v_n)=\mathcal{J}(Bw_n)=\frac{B^p}{p}-\int_{\mathbb{R}^N}\bar{F}(x,Bw_n)dx-\frac{B^{p^*}}{p^*}\int_{\mathbb{R}^N}K(x)|w_n|^{p^*}dx.$$

By $(f_1),(f_2)$, Lemma 2.2-(1), (2), for $\varepsilon>0$, there exists $C_\varepsilon>0$ and $q\in(p,p^*)$ such that

$$\bar{f}(x,s)s\le\varepsilon(|s|^p+|s|^{p^*})+C_\varepsilon|s|^q, \tag{2.6}$$

for all $(x,s)\in\mathbb{R}^N\times\mathbb{R}^+$. Hence, by (2.6), we get

$$\inf_{x\in\mathbb{R}^N}K(x)\int_{\mathbb{R}^N}|w_n|^{p^*}dx\le\int_{\mathbb{R}^N}K(x)|w_n|^{p^*}dx=\frac{1}{\|v_n\|^{p^*-p}}-\frac{1}{\|v_n\|^{p^*}}\int_{\mathbb{R}^N}\bar{f}(x,v_n)v_ndx+o_n(1)\to0,$$

as $n\to\infty$, namely, $\int_{\mathbb{R}^N}|w_n|^{p^*}dx\to0$ as $n\to\infty$. Then by interpolation inequality, we have $\int_{\mathbb{R}^N}|w_n|^qdx\to0$ as $n\to\infty$. Moreover, from (2.4), we have

$$\Big|\int_{\mathbb{R}^N}\bar{F}(x,Bw_n)dx\Big|\le\varepsilon B^p\int_{\mathbb{R}^N}|w_n|^pdx+\varepsilon B^{p^*}\int_{\mathbb{R}^N}|w_n|^{p^*}dx+C_\varepsilon B^q\int_{\mathbb{R}^N}|w_n|^qdx.$$

By the arbitrariness of $\varepsilon$, we can get $\int_{\mathbb{R}^N} \bar{F}(x, Bw_n)dx \to 0$ as $n \to \infty$. Consequently,

$$\liminf_{n\to\infty} \mathcal{J}(t_n v_n) \geq \frac{B^p}{p}, \ \forall B > 0.$$

This contradicts the fact that $\mathcal{J}(t_n v_n)$ is bounded above. Therefore, $\{v_n\}$ is bounded in $X$. The proof of Lemma 2.4 is complete □

Next, we do an estimate on $c$ and follow the approach presented in [12]. Given $\varepsilon > 0$, we consider the function

$$U_\varepsilon(x) = \frac{\left[N\left(\frac{N-p}{p-1}\right)^{p-1}\varepsilon\right]^{\frac{N-p}{p^2}}}{(\varepsilon + |x - x_0|^{\frac{p}{p-1}})^{\frac{N-p}{p}}},$$

which is a solution of the following equation

$$-\triangle_p u = |u|^{p^*-2}u, \ \text{in } \mathbb{R}^N$$

and

$$S := \inf_{u\in D^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p dx}{(\int_{\mathbb{R}^N} |u|^{p^*} dx)^{\frac{p}{p^*}}}.$$

can be achieved at $U_\varepsilon$.

Let $\phi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ be a cut-off function such that $\phi \equiv 1$ in $B_1(0)$, $\phi \equiv 0$ in $\mathbb{R}^N\setminus B_2(0)$. Define

$$u_\varepsilon = \phi U_\varepsilon, \ \ v_\varepsilon = \frac{u_\varepsilon}{\left(\int_{\mathbb{R}^N} K(x)|u|_\varepsilon^{p^*} dx\right)^{\frac{1}{p^*}}}, \tag{2.7}$$

then by a direct computation, there exist positive constants $l_1, l_2$ and $\varepsilon_0$ such that

$$l_1 < \int_{\mathbb{R}^N} K(x)|u|_\varepsilon^{p^*} dx < l_2, \ \text{for all } 0 < \varepsilon < \varepsilon_0, \tag{2.8}$$

$$\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^p \leq \|K\|_\infty^{\frac{p-N}{N}} S + O(\varepsilon^{\frac{N-p}{p}}), \ \text{as } \varepsilon \to 0^+, \tag{2.9}$$

and as $\varepsilon \to 0$, we have

$$\int_{\mathbb{R}^N} |v_\varepsilon|^p dx = \begin{cases} O(\varepsilon^{\frac{N-p}{p}}), & \text{if } N < p^2, \\ O(\varepsilon^{p-1}|\log\varepsilon|), & \text{if } N = p^2, \\ O(\varepsilon^{p-1}), & \text{if } N > p^2. \end{cases} \tag{2.10}$$

**Lemma 2.5** Assume that $(V_1), (K_1)$ and $(f_1) - (f_2), (f_4)$ are satisfied. Then $c < \frac{1}{N}\|K\|_\infty^{\frac{p-N}{p}} S^{\frac{N}{p}}$.

**Proof** For $t > 0$, $v_\varepsilon$ defined by above (2.7), we have

$$\mathcal{J}(tv_\varepsilon) = \frac{1}{p}\int_{\mathbb{R}^N} \left[t^p|\nabla v_\varepsilon|^p + V(x)|G^{-1}(tv_\varepsilon)|^p\right]dx - \int_{\mathbb{R}^N} F(x, G^{-1}(tv_\varepsilon))dx - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N} K(x)|v_\varepsilon|^{p^*}dx. \tag{2.11}$$

Lemma 2.3 implies that there exists $t_\varepsilon > 0$ such that $\mathcal{J}(t_\varepsilon v_\varepsilon) = \max_{t\geq 0} \mathcal{J}(tv_\varepsilon)$.

We claim that there exist $T_1, T_2 > 0$ such that $T_1 \leq t_\varepsilon \leq T_2$ for $\varepsilon$ small enough.

Indeed, if $t_\varepsilon \to 0$ as $\varepsilon \to 0$, we have $0 < \mathcal{J}(t_\varepsilon v_\varepsilon) \to \mathcal{J}(0) = 0$, which is a contradiction.

On the other hand, from (2.11) and $(f_4)$, one has

$$\begin{aligned}
0 < \mathcal{J}(t_\varepsilon v_\varepsilon) &= \frac{1}{p}\int_{\mathbb{R}^N}\Big[t_\varepsilon^p|\nabla v_\varepsilon|^p + V(x)|G^{-1}(t_\varepsilon v_\varepsilon)|^p\Big]dx - \int_{\mathbb{R}^N} F(x, G^{-1}(t_\varepsilon v_\varepsilon))dx - \frac{t_\varepsilon^{p^*}}{p^*}\int_{\mathbb{R}^N} K(x)|v_\varepsilon|^{p^*}dx \\
&\le \frac{t_\varepsilon^p}{p}\int_{\mathbb{R}^N}\Big[|\nabla v_\varepsilon|^p + V(x)|v_\varepsilon|^p\Big]dx - \frac{t_\varepsilon^{p^*}}{p^*}\int_{\mathbb{R}^N} K(x)|v_\varepsilon|^{p^*}dx \\
&\to -\infty,
\end{aligned}$$

as $t_\varepsilon \to +\infty$, a contradiction, which implies that the claim holds.

To complete the proof, it suffices to show that $\mathcal{J}(t_\varepsilon v_\varepsilon) < \frac{1}{N}\|K\|_\infty^{\frac{p-N}{p}} S^{\frac{N}{p}}$. Therefore,

$$\begin{aligned}
\mathcal{J}(t_\varepsilon v_\varepsilon) &= \frac{1}{p}\int_{\mathbb{R}^N}\Big[t_\varepsilon^p|\nabla v_\varepsilon|^p + V(x)|G^{-1}(t_\varepsilon v_\varepsilon)|^p\Big]dx - \int_{\mathbb{R}^N} F(x, G^{-1}(t_\varepsilon v_\varepsilon))dx - \frac{t_\varepsilon^{p^*}}{p^*}\int_{\mathbb{R}^N} K(x)|v_\varepsilon|^{p^*}dx \\
&\le \frac{t_\varepsilon^p}{p}\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^p dx + \frac{t_\varepsilon^p}{p}\int_{\mathbb{R}^N} V(x)|v_\varepsilon|^p dx - \int_{\mathbb{R}^N} F(x, G^{-1}(t_\varepsilon v_\varepsilon))dx - \frac{t_\varepsilon^{p^*}}{p^*} \\
&\le \frac{1}{N}\Big(\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^p dx\Big)^{\frac{N}{p}} + C\int_{\mathbb{R}^N}|v_\varepsilon|^p dx - \int_{\mathbb{R}^N} F(x, G^{-1}(t_\varepsilon v_\varepsilon))dx,
\end{aligned}$$

for some constant $C = \frac{T_2^p\|V_0\|_\infty}{p} > 0$.

Indeed, for $t > 0$, define $l(t) = \frac{t^p}{p}\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^p dx - \frac{t^{p^*}}{p^*}$, we have that $t_0 = \Big(\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^p dx\Big)^{\frac{1}{p^*-p}}$ is a maximum point of $l$ and $l(t_0) = \frac{1}{N}\Big(\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^p dx\Big)^{\frac{N}{p}}$. Applying the inequality

$$(a+b)^\kappa \le a^\kappa + \kappa(a+b)^{\kappa-1}b, \quad a, b \ge 0,\ \kappa \ge 1. \tag{2.12}$$

By (2.9) and (2.12), we have

$$\begin{aligned}
\mathcal{J}(t_\varepsilon v_\varepsilon) &\le \frac{1}{N}\Big(\|K\|_\infty^{\frac{p-N}{N}} S + O(\varepsilon^{\frac{N-p}{p}})\Big)^{\frac{N}{p}} + C\int_{\mathbb{R}^N}|v_\varepsilon|^p dx - \int_{\mathbb{R}^N} F(x, G^{-1}(t_\varepsilon v_\varepsilon))dx \\
&\le \frac{1}{N}\|K\|_\infty^{\frac{p-N}{p}} S^{\frac{N}{p}} + C\int_{\mathbb{R}^N}|v_\varepsilon|^p dx - \int_{\mathbb{R}^N} F(x, G^{-1}(t_\varepsilon v_\varepsilon))dx + O(\varepsilon^{\frac{N-p}{p}}).
\end{aligned}\tag{2.13}$$

Now consider

$$r(\varepsilon) = \begin{cases} \varepsilon^{\frac{N-p}{p}}, & \text{if } N < p^2, \\ \varepsilon^{p-1}|\log\varepsilon|, & \text{if } N = p^2, \\ \varepsilon^{p-1}, & \text{if } N > p^2. \end{cases}\tag{2.14}$$

From (2.13) and (2.14), we have

$$\mathcal{J}(t_\varepsilon v_\varepsilon) \le \frac{1}{N}\|K\|_\infty^{\frac{p-N}{p}} S^{\frac{N}{p}} + r(\varepsilon)\Big[C - \frac{\int_{\mathbb{R}^N} F(x, G^{-1}(t_\varepsilon v_\varepsilon))dx}{r(\varepsilon)}\Big].$$

Next we claim that

$$\lim_{\varepsilon\to 0^+}\frac{\int_{\mathbb{R}^N} F(x, G^{-1}(t_\varepsilon v_\varepsilon))dx}{r(\varepsilon)} = +\infty. \tag{2.15}$$

It follows $(f_4)$ that any $A>0$, there exists $R=R_A>0$ such that for all $(x,s)\in\Omega\times[R_A,+\infty)$,

$$F(x,G^{-1}(s))\geq\begin{cases}A|s|^{\mu}, & \text{if } N<p^2,\\ A|s|^p\log|s|, & \text{if } N=p^2,\\ A|s|^p, & \text{if } N>p^2,\end{cases}\tag{2.16}$$

where $\mu=p^*-\frac{p}{p-1}$. Now consider the function $\eta_\varepsilon:[0,+\infty)\to\mathbb{R}$ defined by

$$\eta_\varepsilon(r)=\frac{\varepsilon^{\frac{N-p}{p^2}}}{(\varepsilon+r^{\frac{p}{p-1}})^{\frac{N-p}{p}}}.$$

Since $\phi\equiv1$ in $B_1(0)$, due to (2.8), we choose a constant $C>0$ such that $v_\varepsilon(x)\geq C\eta_\varepsilon(|x|)$ for $|x|<1$. Note that $\eta_\varepsilon$ is decreasing and $G^{-1}$ is increasing, there exists a positive constant $C$ such that, for $|x|<\varepsilon^{\frac{p-1}{p}}$,

$$G^{-1}(t_\varepsilon v_\varepsilon)\geq G^{-1}(T_1C\eta_\varepsilon(|x|))\geq G^{-1}(T_1C\eta_\varepsilon(\varepsilon^{\frac{p-1}{p}}))\geq G^{-1}(C\varepsilon^{\frac{(N-p)(1-p)}{p^2}}).$$

Then we can choose $\varepsilon_1>0$ such that

$$C\varepsilon^{\frac{(N-p)(1-p)}{p^2}}\geq1,\ \ G^{-1}(t_\varepsilon v_\varepsilon)\geq G^{-1}(C\varepsilon^{\frac{(N-p)(1-p)}{p^2}})\geq R,\tag{2.17}$$

for $|x|<\varepsilon^{\frac{p-1}{p}}$, $0<\varepsilon<\varepsilon_1$. It follows from (2.16) and (2.17) that

$$F(x,G^{-1}(s))\geq\begin{cases}CA\varepsilon^{\frac{(N-p)(1-p)\mu}{p^2}}, & \text{if } N<p^2,\\ CA\varepsilon^{\frac{(N-p)(1-p)}{p}}\log\varepsilon, & \text{if } N=p^2,\\ CA\varepsilon^{\frac{(N-p)(1-p)}{p}}, & \text{if } N>p^2,\end{cases}\tag{2.18}$$

for $|x|<\varepsilon^{\frac{p-1}{p}}$, $0<\varepsilon<\varepsilon_1$.

Using $(f_5)$-(i), one has

$$F(x,G^{-1}(s))+|s|^p\geq0,\ \ x\in\Omega,\ \ s\geq0.\tag{2.19}$$

Since $B_2(0)\subset\Omega$, by (2.18) and (2.19), for $0<\varepsilon<\varepsilon_1$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}F(x,G^{-1}(t_\varepsilon v_\varepsilon))dx&=\int_{B_{\varepsilon^{\frac{p-1}{p}}}}F(x,G^{-1}(t_\varepsilon v_\varepsilon))dx+\int_{\Omega\setminus B_{\varepsilon^{\frac{p-1}{p}}}}F(x,G^{-1}(t_\varepsilon v_\varepsilon))dx\\&\geq\int_{|x|<\varepsilon^{\frac{p-1}{p}}}F(x,G^{-1}(t_\varepsilon v_\varepsilon))dx-T_2^p\|v_\varepsilon\|_p^p,\end{aligned}\tag{2.20}$$

where

$$\int_{|x|<\varepsilon^{\frac{p-1}{p}}}F(x,G^{-1}(t_\varepsilon v_\varepsilon))dx\geq\begin{cases}CA\int_{|x|<\varepsilon^{\frac{p-1}{p}}}\varepsilon^{\frac{(N-p)(1-p)\mu}{p^2}}dx=CA\varepsilon^{\frac{N-p}{p}}, & \text{if } N<p^2,\\ CA\int_{|x|<\varepsilon^{\frac{p-1}{p}}}\varepsilon^{\frac{(N-p)(1-p)}{p}}\log\varepsilon dx=CA\varepsilon^{p-1}\log\varepsilon, & \text{if } N=p^2,\\ CA\int_{|x|<\varepsilon^{\frac{p-1}{p}}}\varepsilon^{\frac{(N-p)(1-p)}{p}}dx=CA\varepsilon^{p-1}, & \text{if } N>p^2.\end{cases}$$

Consequently, by (2.20), we obtain

$$\lim_{\varepsilon\to0^+}\frac{\int_{\mathbb{R}^N}F(x,G^{-1}(t_\varepsilon v_\varepsilon))dx}{r(\varepsilon)}\geq CA-T_2^p.\tag{2.21}$$

Choosing $A>0$ sufficiently large, (2.21) establishes (2.15). Lemma 2.5 is proved. □

## 3. The asymptotically period case

In this section, in order to overcome the difficulties caused by the loss of translation invariance due to the asymptotically periodic potential, we need to state the following technical convergence results. The detailed proofs can be found in [23, 36], where $p = 2$.

**Lemma 3.1** Assume that $(V_1)$, $(K_1)$, $(f_1)$ and (i) of $(f_5)$ hold. Suppose that $\{v_n\}$ is bounded in $X$, $\{y_n\} \subset \mathbb{Z}^N$ with $|y_n| \to +\infty$ and $v_n \to 0$ in $L^{\alpha}_{loc}(\mathbb{R}^N)$, for any $\alpha \in [p, p^*)$. Then up to a subsequence, one has

$$\int_{\mathbb{R}^N} \Big(V(x) - V_0(x)\Big)|G^{-1}(v_n)|^p dx \to o_n(1); \tag{3.1}$$

$$\int_{\mathbb{R}^N} (V(x) - V_0(x)) \frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)}{g(G^{-1}(v_n))} \varphi(x - y_n) dx \to o_n(1),\ \forall \varphi \in C_0^{\infty}(\mathbb{R}^N); \tag{3.2}$$

$$\int_{\mathbb{R}^N} \Big[F(x, G^{-1}(v_n)) - F_0(x, G^{-1}(v_n))\Big] dx \to o_n(1); \tag{3.3}$$

$$\int_{\mathbb{R}^N} \frac{f(x, G^{-1}(v_n)) - f_0(x, G^{-1}(v_n))}{g(G^{-1}(v_n))} \varphi(x - y_n) dx \to o_n(1),\ \forall \varphi \in C_0^{\infty}(\mathbb{R}^N). \tag{3.4}$$

**Proof of Theorem 1.1** Lemma 2.3 implies the existence of a Cerami sequence $\{v_n\} \subset X$. By Lemma 2.4, $\{v_n\}$ is bounded in $X$. Thus, there exists $v \in X$ such that $v_n \rightharpoonup v$ in $X$, $v_n \to v$ in $L^p_{loc}(\mathbb{R}^N)$, $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. For any $\varphi \in C_0^{\infty}(\mathbb{R}^N)$, one has

$$0 = \langle \mathcal{J}'(v_n), \varphi \rangle + o_n(1) = \langle \mathcal{J}'(v), \varphi \rangle,$$

that is, $v$ is a weak solution of Eq (1.10).

Now we prove that $v$ is nontrivial. By contradiction, we assume that $v = 0$. We divide the proof four steps.

Step 1: We claim that $\{v_n\} \subset X$ is also a Cerami sequence for the functional $\mathcal{J}_0 : X \to \mathbb{R}$, where

$$\mathcal{J}_0(v) = \frac{1}{p} \int_{\mathbb{R}^N} \Big[|\nabla v|^p + V_0(x)|G^{-1}(v)|^p\Big] dx - \int_{\mathbb{R}^N} F_0(x, G^{-1}(v)) dx - \frac{1}{p^*} \int_{\mathbb{R}^N} K(x)|v|^{p^*} dx.$$

From (3.1) and (3.3), we can deduce that

$$\begin{aligned} \Big|\mathcal{J}(v_n) - \mathcal{J}_0(v_n)\Big| &\le \frac{1}{p} \int_{\mathbb{R}^N} \Big|\Big(V(x) - V_0(x)\Big)|G^{-1}(v_n)|^p\Big| dx + \int_{\mathbb{R}^N} \Big|F(x, G^{-1}(v_n)) - F_0(x, G^{-1}(v_n))\Big| dx \\ &= o_n(1), \end{aligned} \tag{3.5}$$

and taking $\varphi \in X$ with $\|\varphi\| = 1$, by (3.2) and (3.4), we obtain that

$$\begin{aligned} \Big\|\langle \mathcal{J}'(v_n) - \mathcal{J}_0'(v_n)\Big\|_* \le \sup_{\varphi \in X, \|\varphi\|=1} \Big[ &\int_{\mathbb{R}^N} \Big|\Big(V(x) - V_0(x)\Big) \frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)}{g(G^{-1}(v_n))} \varphi\Big| dx \\ &+ \int_{\mathbb{R}^N} \Big|\frac{f(x, G^{-1}(v_n)) - f_0(x, G^{-1}(v_n))}{g(G^{-1}(v_n))} \varphi\Big| dx\Big] = o_n(1). \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we can get that $\{v_n\}$ is also a Cerami sequence for $\mathcal{J}_0$.

Step 2: We prove that $\{v_n\}$ is non-vanishing i.e.,

$$\beta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n|^p dx > 0. \tag{3.7}$$

If $\beta = 0$, the Lions lemma [18], we have $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (p, p^*)$.
Note that

$$o_n(1) = \langle \mathcal{J}'(v_n), v_n\rangle = \int_{\mathbb{R}^N} \Big[|\nabla v_n|^p + V(x)|v_n|^p\Big]dx - \int_{\mathbb{R}^N} \bar{f}(v_n, v)v_n dx - \int_{\mathbb{R}^N} K(x)|v_n|^{p^*} dx,$$

which combining with (2.6) leads to

$$\int_{\mathbb{R}^N} \Big[|\nabla v_n|^p + V(x)|v_n|^p\Big]dx - \int_{\mathbb{R}^N} K(x)|v_n|^{p^*} dx = o_n(1).$$

Therefore, there exists a constant $l \geq 0$ such that

$$\int_{\mathbb{R}^N} \Big[|\nabla v_n|^p + V(x)|v_n|^p\Big]dx \to l, \quad \int_{\mathbb{R}^N} K(x)|v_n|^{p^*} dx \to l.$$

Obviously, $l > 0$. Otherwise, $\mathcal{J}(v_n) \to 0$ as $n \to \infty$, which contradicts with $c > 0$. Since

$$\begin{aligned} l &= \lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)|v_n|^{p^*} dx \leq \|K\|_\infty \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^{p^*} dx \\ &\leq \|K\|_\infty S^{-\frac{p^*}{p}} \lim_{n\to\infty} (\int_{\mathbb{R}^N} |\nabla v_n|^p dx)^{\frac{p^*}{p}} \leq \|K\|_\infty S^{-\frac{p^*}{p}} \lim_{n\to\infty} \|\nabla v_n\|^{p^*} \leq \|K\|_\infty S^{-\frac{p^*}{p}} l^{\frac{p^*}{p}}, \end{aligned}$$

that is, $l \geq \|K\|_\infty^{\frac{p-N}{p}} S^{\frac{N}{p}}$. Consequently, (2.4) implies that

$$\begin{aligned} c + o_n(1) = \mathcal{J}(v_n) &= \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|v_n|^p)dx - \int_{\mathbb{R}^N} \bar{F}(x, v_n)dx - \frac{1}{p^*} \int_{\mathbb{R}^N} K(x)|v_n|^{p^*} dx \\ &\to (\frac{1}{p} - \frac{1}{p^*})l = \frac{1}{N} l \geq \frac{1}{N} \|K\|_\infty^{\frac{p-N}{p}} S^{\frac{N}{p}}, \end{aligned}$$

as $n \to \infty$, which deduces that $c \geq \frac{1}{N}\|K\|_\infty^{\frac{p-N}{p}} S^{\frac{N}{p}}$, a contradiction.

Step 3: After a translation of $\{v_n\}$ denoted $\{w_n\}$, then $\{w_n\}$ converges weakly to a nonzero critical point of $\mathcal{J}_0$.

Choose $\{y_n\} \subset \mathbb{Z}^N$ such that $|y_n| \to +\infty$ as $n \to \infty$ and denote $w_n(x) = v_n(x + y_n)$. Then

$$\|w_n\| = \|v_n\|, \ \mathcal{J}_0(w_n) = \mathcal{J}_0(v_n), \ \mathcal{J}_0'(w_n) = \mathcal{J}_0'(v_n).$$

Thus, $\{w_n\}$ is a bounded $(PS)_{c_0}$ of $\mathcal{J}_0$, where $c_0$ is defined below. Going if necessary to a subsequence, we get that $w_n \rightharpoonup w$ in $X$ and $\mathcal{J}_0'(w) = 0$. So by Step 2 we get $w \neq 0$. Therefore, by $(f_5)$-(ii), Lemma

2.1-(8) and Fatou Lemma, we have

$$\begin{aligned}
c &= \liminf_{n\to\infty}\Big[\mathcal{J}_0(w_n) - \frac{1}{p}\langle \mathcal{J}_0'(w_n), w_n\rangle\Big] \\
&= \liminf_{n\to\infty}\frac{1}{p}\int_{\mathbb{R}^N} V_0(x)|G^{-1}(w_n)|^{p-2}\Big[(G^{-1}(w_n))^2 - \frac{G^{-1}(w_n)w_n}{g(G^{-1}(w_n))}\Big]dx \\
&\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^N}\Big[\frac{f_0(x,G^{-1}(w_n))w_n}{pg(G^{-1}(w_n))} - F_0(x,G^{-1}(w_n))\Big]dx + (\frac{1}{p}-\frac{1}{p^*})\liminf_{n\to\infty}\int_{\mathbb{R}^N} K(x)|w_n|^{p^*}dx \\
&\geq \frac{1}{p}\int_{\mathbb{R}^N} V_0(x)|G^{-1}(w)|^{p-2}\Big[(G^{-1}(w))^2 - \frac{G^{-1}(w)w}{g(G^{-1}(w))}\Big]dx \\
&\quad + \int_{\mathbb{R}^N}\Big[\frac{f_0(x,G^{-1}(w))w}{pg(G^{-1}(w))} - F_0(x,G^{-1}(w))\Big]dx + (\frac{1}{p}-\frac{1}{p^*})\int_{\mathbb{R}^N} K(x)|w|^{p^*}dx \\
&= \mathcal{J}_0(w) - \frac{1}{p}\langle \mathcal{J}_0'(w), w\rangle = \mathcal{J}_0(w),
\end{aligned}$$

which implies that $\mathcal{J}_0(w) \leq c$.

Step 4: We use $w$ to construct a path which allows us to get a contradiction with the definition of mountain pass level $c$.

Define the mountain pass level

$$c_0 := \inf_{\gamma\in\bar{\Gamma}}\sup_{t\in[0,1]}\mathcal{J}_0(\gamma(t)) > 0,$$

where $\bar{\Gamma} := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \mathcal{J}_0(\gamma(1)) < 0\}$. Applying similar arguments used in [15], we can construct a path $\gamma : [0,1] \to X$ such that

$$\begin{cases} \gamma(0) = 0, \;\; \mathcal{J}_0(\gamma(1)) < 0, \;\; w \in \gamma([0,1]), \\ \gamma(t)(x) > 0, \;\forall x \in \mathbb{R}^N, \; t \in [0,1], \\ \max_{t\in[0,1]}\mathcal{J}_0(\gamma(t)) = \mathcal{J}_0(w). \end{cases}$$

Then $c_0 \leq \max_{t\in[0,1]}\mathcal{J}_0(\gamma(t)) = \mathcal{J}_0(w)$. Due to the fact that $V(x) \leq V_0(x)$ but $V(x) \not\equiv V_0(x)$, we take the path $\gamma$ given by above and by $\gamma \in \bar{\Gamma} \subset \Gamma$, we have

$$c \leq \max_{t\in[0,1]}\mathcal{J}(\gamma(t)) = \mathcal{J}(\gamma(\bar{t})) < \mathcal{J}_0(\gamma(\bar{t})) \leq \max_{t\in[0,1]}\mathcal{J}_0(\gamma(t)) = \mathcal{J}_0(w) \leq c,$$

which is a contradiction. Consequently, $v$ is a nontrivial solution of Eq (1.10), then using the strong maximun principle, we obtain $v > 0$, namely, Eq (1.1) possesses a nontrivial positive solution $u = G^{-1}(v)$. This completes the proof of Theorem 1.1. □

## 4. The period case

In this section, we give the proof of Theorem 1.2.

**Proof of Theorem 1.2** By Lemma 2.3, there exists a Cerami sequence $\{v_n\} \subset X$ such that

$$\mathcal{J}_0(v_n) \to c \text{ and } (1 + \|v_n\|)\|\mathcal{J}_0'(v_n)\|_* \to 0.$$

Applying Lemma 2.4, the Cerami sequence $\{v_n\}$ is bounded in $X$. Similar to (3.7), it is easy to verify that $\{v_n\}$ is non-vanishing.

As in the step 3 of Theorem 1.1, set $w_n(x) = v_n(x + y_n)$. It is easy to know that $\|w_n\| = \|v_n\|$ and $\{w_n\}$ is bounded and non-vanishing. Going if necessary to a subsequence, we have

$$w_n \rightharpoonup w \neq 0 \text{ in } X,\ w_n \to w \text{ in } L^p_{loc}(\mathbb{R}^N).$$

Moreover, since $V_0(x)$, $K(x)$ and $f_0(x, u)$ are periodic on $X$, we see that $\{w_n\}$ is also a a Cerami sequence of $\mathcal{J}_0$. Then for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\langle \mathcal{J}_0'(w), \varphi\rangle = \lim_{n\to\infty} \langle \mathcal{J}_0'(w_n), \varphi\rangle.$$

That is $\mathcal{J}_0'(w) = 0$ and $w$ is a nontrivial solution to (1.12). By the strong maximun principle, we obtain $w > 0$. This completes the proof of Theorem 1.2. □

## 5. Conclusions

In [38], we discussed a class of generalized quasilinear Schrödinger equations with asymptotically periodic potential, where $p = 2$ and the nonlinear term is subcritical. In this current work, we have established the existence of nontrivial positive solutions for a class of generalized quasilinear elliptic equations with critical growth. In the next work, we will extend the study to the case of variable exponent $p = p(t)$.

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## Conflict of interest

The author declare no conflicts of interest in this paper.

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