Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(6): 9716–9730.
DOI: 10.3934/math.2022541
Received: 06 December 2021
Revised: 05 March 2022
Accepted: 07 March 2022
Published: 16 March 2022

Research article

Type 2 degenerate modified poly-Bernoulli polynomials arising from the degenerate poly-exponential functions

Dojin Kim[1], **Patcharee Wongsason**[2] **and Jongkyum Kwon**[3,*]

[1] Department of Mathematics, Dongguk University, Seoul 04620, Republic of Korea
[2] Department of Mathematics, Statistics and Computers, Faculty of Sciences, Ubon Ratchathani University, Ubon Ratchathani, Thailand
[3] Department of Mathematics Education and ERI, Gyeongsang National University, Jinju 52828, Republic of Korea

* **Correspondence:** Email: mathkjk26@gnu.ac.kr.

Abstract: We present a new type of degenerate poly-Bernoulli polynomials and numbers by modifying the polyexponential function in terms of the degenerate exponential functions and degenerate logarithm functions. Also, we introduce a new variation of the degenerate unipoly-Bernoulli polynomials by the similar modification. Based on these polynomials, we investigate some properties, new identities, and their relations to the known special functions and numbers such as the degenerate type 2-Bernoulli polynomials, the type 2 degenerate Euler polynomials, the degenerate Bernoulli polynomials and numbers, the degenerate Stirling numbers of the first kind, and λ-falling factorial sequence. In addition, we compute some of the proposed polynomials and present their zeros and behaviors for different variables in specific cases.

Keywords: degenerate polyexponential functions; degenerate poly-Bernoulli polynomials; degenerate unipoly functions
Mathematics Subject Classification: 05A19, 11B73, 11B83

1. Introduction

The degenerate types of Bernoulli and Euler polynomials are initially introduced by Carlitz [3, 4] with arithmetic and combinatorial results. Recently, these degenerate version of special functions and numbers have been intensively studied by many authors introducing various types of polynomials, functions and numbers (see [1, 5–8, 10–12, 15, 17] for which a rich variety of literature is available and references therein for further details). These special polynomials and numbers have been significant roles not only in combinatorial and arithmetic problems, but also various branches of problems arising in mathematics, theoretical physics, and engineering in order to new means of mathematical approaches.

In particular, a lot of research focusing on type 2 degenerate polynomials has been conducted: for example, the type 2 degenerate central Fubini polynomials [16], the type 2 degenerate poly-Bernoulli numbers and polynomials [11], the type 2 degenerate Euler and Bernoulli polynomials [6], the type 2 poly-ApostolBernoulli polynomials [9], the degenerate poly-type 2-Bernoulli Polynomials [1], the type 2 degenerate Euler and Bernoulli polynomials [6], a modification of the type 2 degenerate poly-Bernoulli polynomials [13], and the partially degenerate polyexponential-Bernoulli polynomials of the second kind [2].

Following by these studies, we introduce a new type of degenerate poly-Bernoulli polynomials and unipoly-Bernoulli polynomials called the type 2 degenerate modified poly-Bernoulli polynomials and the type 2 degenerate modified unipoly-Bernoulli polynomials attached to polynomials, respectively. We investigate their useful properties as well as their relations to express the proposed polynomials in terms of existing types or known functions and numbers. Further, we calculate some of the introduced polynomials and present their behaviors as well as their zeros for different variables in specific cases.

We first recall several definitions in order to introduce our new type of poly-Bernoulli polynomials.

The Bernoulli polynomials (see [10, 11] for detail) are given by the generating function

$$\frac{t}{e^t-1}e^{tx}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad(|t|<1), \tag{1.1}$$

and for case $x=0$, $B_n := B_n(0)$ are called the Bernoulli numbers.

The polyexponential functions are introduced in [8, 11] for $k\in\mathbb{Z}$, which are defined by

$$Ei_k(t)=\sum_{n=1}^{\infty}\frac{t^n}{(n-1)!n^k}\quad(|t|<1). \tag{1.2}$$

For example, in case $k=1$, it satisfies that $Ei_1(t)=e^t-1$, so that $Ei_1\left(\log(t+1)\right)=t$. For $k\le 0$, $Ei_k(t)$ are expressed as a product of polynomials with the exponential function such that

$$\begin{aligned}&Ei_0(t)=te^t, && Ei_{-1}(t)=t(t+1)e^t,\\ &Ei_{-2}(t)=t(t^2+3t+1)e^t, && Ei_{-3}(t)=t(t^3+6t^2+7t+1)e^t,\cdots,\end{aligned} \tag{1.3}$$

which are obtained from the recurrence relation $\frac{d}{dt}Ei_k(t)=\frac{1}{t}Ei_{k-1}(t)$ (see [11] for detail).

As a degenerate type of the polyexponential functions in (1.2), the author [11] considered the modified degenerate polyexponential functions for $k\in\mathbb{N}$ given by

$$Ei_{k,\lambda}(t)=\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}t^n}{(n-1)!n^k}\quad(|t|<1), \tag{1.4}$$

where $(t)_{n,\lambda}$ is the λ-falling factorial sequence [11] defined by $(t)_{n,\lambda}=t(t-\lambda)(t-2\lambda)\cdots(t-(n-1)\lambda)$ for $n\ge 1$ and $(t)_{0,\lambda}=1$. It is well known that $Ei_{1,\lambda}(t)=e_\lambda(t)-1$ and $Ei_{1,\lambda}\left(\log_\lambda(1+t)\right)=t$.

For $\lambda\in\mathbb{R}\backslash\{0\}$, the author in [3, 4] introduced the degenerate Bernoulli polynomials and the degenerate Bernoulli numbers, which are respectively defined by

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}ß_{n,\lambda}(x)\frac{t^n}{n!},\quad \frac{t}{e_\lambda(t)-1}=\sum_{n=0}^{\infty}ß_{n,\lambda}\frac{t^n}{n!}, \tag{1.5}$$

where the degenerate exponential functions are given by

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = e_\lambda^1(t). \tag{1.6}$$

As a variant of degenerate Bernoulli polynomials, the author in [1] considered the degenerate poly-type 2-Bernoulli polynomials $B_{n,\lambda}^{(k)}$, which are defined by the generating function

$$\frac{Li_k(1 - e^{-t})}{e_\lambda(t) - e_\lambda^{-1}(t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}(x) \frac{t^n}{n!},$$

where $Li_k(t)$ for $k \in \mathbb{Z}$ are the polylogarithm functions [5] defined by

$$Li_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k} \quad (|t| < 1).$$

For the case $k = 1$ of $B_{n,\lambda}^{(k)}$, $B_{n,\lambda}^{(1)}(x) := \beta_{n,\lambda}(x)$ are called the degenerate type 2-Bernoulli polynomials, which satisfy

$$\frac{t}{e_\lambda(t) - e_\lambda^{-1}(t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}, \tag{1.7}$$

since the identity holds $Li_1(1 - e^{-t}) = t$ for $k = 1$.

In [10], another variant of the degenerate Bernoulli polynomials, called the degenerate Bernoulli polynomials of the second kind, is introduced by modifying the generating function such as

$$\frac{t}{\log_\lambda(1 + t)} (1 + t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!}. \tag{1.8}$$

Here the degenerate logarithm function $\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1)$ is the compositional inverse of $e_\lambda(t)$, i.e., $\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t$.

Further, applying the polyexponential function (1.2) to the degenerate Bernoulli polynomials, the type 2 degenerate poly-Bernoulli polynomials [11] are introduced by the following formula

$$\frac{Ei_{k,\lambda}(\log_\lambda(1 + t))}{e_\lambda(t) - 1} e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}, \tag{1.9}$$

in which, for case $x = 0$, $B_{n,\lambda}^{(k)} := B_{n,\lambda}^{(k)}(0)$ are called the type 2 degenerate poly-Bernoulli numbers.

As a different variation of (1.8) using the polyexponential function, the type 2 degenerate poly-Bernoulli polynomials [14] of the second kind are introduced by the generating function

$$\frac{Ei_{k,\lambda}(\log_\lambda(1 + t))}{\log_\lambda(1 + t)} (1 + t)^x = \sum_{j=0}^{\infty} Pb_{j,\lambda}^{(k)}(x) \frac{t^j}{j!}, \quad k \in \mathbb{Z}. \tag{1.10}$$

In further research, the type 2 degenerate Euler polynomials are introduced by the author in [6] based on the following generating function

$$\frac{2}{e_\lambda^{\frac{1}{2}}(t) + e_\lambda^{-\frac{1}{2}}(t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x) \frac{t^n}{n!}, \tag{1.11}$$

and $\mathcal{E}_{n,\lambda} := \mathcal{E}_{n,\lambda}(0)$ are called the type 2 degenerate Euler numbers when $x = 0$.

2. Type 2 degenerate modified poly-Bernoulli polynomials

In this section, motivated by the previous mentioned works about variant types of degenerate poly-Bernoulli polynomials, we introduce a new type of degenerate poly-Bernoulli polynomials and discuss their properties.

Definition 2.1. *Let us define the type 2 degenerate modified poly-Bernoulli polynomials $\mathscr{B}_{n,\lambda}^{[k]}(x)$ for $k \in \mathbb{Z}$ by the generating function given by*

$$\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t) - e_\lambda^{-1}(t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathscr{B}_{n,\lambda}^{[k]}(x) \frac{t^n}{n!},$$

where $Ei_{k,\lambda}$ are the degenerate modified polyexponential function given in (1.4). *Especially, when $x = 0$, $\mathscr{B}_{n,\lambda}^{[k]} := \mathscr{B}_{n,\lambda}^{[k]}(0)$ are called the type 2 degenerate modified poly-Bernoulli numbers.*

Remark 2.1. *Since $Ei_{1,\lambda}(\log_\lambda(1+t)) = t$ for $k = 1$, one can see that $\mathscr{B}_{n,\lambda}^{[1]}(x)$ satisfy*

$$\frac{t}{e_\lambda(t) - e_\lambda^{-1}(t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathscr{B}_{n,\lambda}^{[1]}(x) \frac{t^n}{n!},$$

which provides the same generating function listed in (1.7)*, so that $\mathscr{B}_{n,\lambda}^{[k]}(x)$ represent the degenerate type 2-Bernoulli polynomials $\beta_{n,\lambda}(x)$ for $k = 1$. Thus, the type 2 degenerate modified poly-Bernoulli polynomials $\mathscr{B}_{n,\lambda}^{[k]}(x)$ is a generalization of the degenerate type 2-Bernoulli polynomials $\beta_{n,\lambda}(x)$, while the type 2 degenerate poly-Bernoulli polynomials* (1.10) *of the second kind is a generalization of the degenerate Bernoulli polynomials of the second kind.*

First, we can express the type 2 degenerate modified poly-Bernoulli polynomials in terms of the degenerate Stirling numbers of the first kind and the degenerate type 2-Bernoulli polynomials (1.7).

Theorem 2.1. *For $n \geq 0$ and $\lambda \in \mathbb{R}\backslash\{0\}$, the following identity holds:*

$$\mathscr{B}_{n,\lambda}^{[k]}(x) = \sum_{m=0}^{n} \frac{1}{m+1} \binom{n}{m} \sum_{\ell=1}^{m+1} \frac{S_{1,\lambda}(m+1,\ell)}{\ell^{k-1}} (1)_{\ell,\lambda} \beta_{n-m,\lambda}(x),$$

where $S_{1,\lambda}(n,m)$ are the degenerate Stirling numbers of the first kind defined by

$$(t)_{n,\lambda} = \sum_{m=0}^{n} S_{1,\lambda}(n,m) t^m.$$

Proof. We first note (see [11] for detail) that the degenerate polyexponential functions $Ei_{k,\lambda}(t)$ satisfy that

$$
\begin{aligned}
Ei_{k,\lambda}(\log_\lambda(1+t)) &= \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda}(\log_\lambda(1+t))^m}{(m-1)!m^k} \\
&= \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda}}{m^{k-1}} \frac{(\log_\lambda(1+t))^m}{m!} \\
&= \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda}}{m^{k-1}} \sum_{n=m}^{\infty} S_{1,\lambda}(n,m)\frac{t^n}{n!} \qquad (2.1)\\
&= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \frac{(1)_{m,\lambda}}{m^{k-1}} S_{1,\lambda}(n,m) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{t}{n+1} \left(\sum_{\ell=1}^{n+1} S_{1,\lambda}(n+1,\ell)\frac{(1)_{\ell,\lambda}}{\ell^{k-1}} \right) \frac{t^n}{n!}.
\end{aligned}
$$

From (1.7), (2.1) and Definition 2.1, we can see that

$$
\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{B}_{n,\lambda}^{[k]}(x)\frac{t^n}{n!} &= \frac{e_\lambda^x(t)}{e_\lambda(t)-e_\lambda^{-1}(t)} Ei_{k,\lambda}(\log_\lambda(1+t)) \\
&= \left(\sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{1}{m+1} \left(\sum_{\ell=1}^{m+1} S_{1,\lambda}(m+1,\ell)\frac{(1)_{\ell,\lambda}}{\ell^{k-1}} \right) \frac{t^m}{m!} \right) \qquad (2.2)\\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{1}{m+1}\binom{n}{m} \sum_{\ell=1}^{m+1} \frac{S_{1,\lambda}(m+1,\ell)}{\ell^{k-1}} (1)_{\ell,\lambda}\beta_{n-m,\lambda}(x) \right) \frac{t^n}{n!}.
\end{aligned}
$$

By comparing the coefficients on the both sides of (2.2), one can conclude the desired result.

Remark 2.2. *(1) One can see that the type 2 degenerate poly-Bernoulli polynomials* (1.10) *of the second kind are expressed in terms of the degenerate Stirling numbers of the first kind and the degenerate Bernoulli polynomials of the second kind with the same combinatorial coefficients (see [14, Theorem 2.1]) as a similar expression.*

(2) From $\frac{d}{dt}Ei_{k,\lambda}(\log_\lambda(1+t)) = \frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}Ei_{k-1,\lambda}(\log_\lambda(1+t))$ *for* $k \geq 2$ *(see [11] for detail),* $Ei_{k,\lambda}(\log_\lambda(1+t))$ *can be expressed as the multiple integrals of* $Ei_{1,\lambda}(\log_\lambda(1+t))$ *as follows:*

$$
Ei_{k,\lambda}(\log_\lambda(1+t)) = \int_0^t \frac{(1+t_1)^{\lambda-1}}{\log_\lambda(1+t_1)} \int_0^{t_1} \frac{(1+t_2)^{\lambda-1}}{\log_\lambda(1+t_2)} \cdots \int_0^{t_{k-2}} \frac{(1+t_{k-1})^{\lambda-1}}{\log_\lambda(1+t_{k-1})} t_{k-1}dt_{k-1}\cdots dt_2dt_1,
$$

in which $k-1$ *times of integrals are performed by the product with* $\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}$ *each time.*

Next, using the property of Remark 2.1, we have the following result.

Theorem 2.2. *For* $n \geq 0,\ k \in \mathbb{Z}$, *and* $\lambda \in \mathbb{R}\backslash\{0\}$, $\mathcal{B}_{n,\lambda}^{[k]}(x)$ *satisfy the following identity:*

$$
\begin{aligned}
\mathcal{B}_{n,\lambda}^{[k]}(x) = \sum_{m=0}^{n}\binom{n}{m} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1,\cdots,m_{k-1}} \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1} \\
\times \frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}\beta_{n-m,\lambda}(x). \qquad (2.3)
\end{aligned}
$$

Proof. By making use of Definition 2.1, (1.7), (1.8) and Remark 2.1, we can verify that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}^{[k]}(x)\frac{t^n}{n!} &= \frac{e_\lambda^x(t)}{e_\lambda(t)-e_\lambda^{-1}(t)}Ei_{k,\lambda}\left(\log_\lambda(1+t)\right)\\
&= \frac{e_\lambda^x(t)}{e_\lambda(t)-e_\lambda^{-1}(t)}\left(\int_0^t\frac{(1+t_1)^{\lambda-1}}{\log_\lambda(1+t_1)}\cdots\int_0^{t_{k-2}}\frac{(1+t_{k-1})^{\lambda-1}}{\log_\lambda(1+t_{k-1})}t_{k-1}dt_{k-1}\cdots dt_1\right)\\
&= \frac{te_\lambda^x(t)}{e_\lambda(t)-e_\lambda^{-1}(t)}\left(\sum_{m=0}^{\infty}\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\cdots,m_{k-1}}\frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\right.\\
&\qquad\left.\times\frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}\frac{t^m}{m!}\right)\\
&= \left(\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\cdots,m_{k-1}}\frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\right.\\
&\qquad\left.\times\frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}\frac{t^m}{m!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\cdots,m_{k-1}}\frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\right.\\
&\qquad\left.\times\frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}\beta_{n-m,\lambda}(x)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.4}$$

Hence, the conclusion is established by comparing coefficients on both sides.

We next give the expression for $\mathcal{B}_{n,\lambda}^{[k]}(x)$ as a sum of the products of the type 2 degenerate modified poly-Bernoulli numbers and λ-falling factorial sequence.

Theorem 2.3. *Let k be any integer. Then the following identity holds true for all $n \geq 0$,*

$$\mathcal{B}_{n,\lambda}^{[k]}(x) = \sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_{m,\lambda}^{[k]}(x)_{n-m,\lambda}. \tag{2.5}$$

Proof. By considering $\mathcal{B}_{n,\lambda}^{[k]}(x)$ to be the product of $\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)-e_\lambda^{-1}(t)}$ and $e_\lambda^x(t)$ and using the identity of $e_\lambda^x(t) = \sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}$, the desired result for $\mathcal{B}_{n,\lambda}^{[k]}(x)$ is obtained by the binomial convolution of the sequences $\left\{\mathcal{B}_{n,\lambda}^{[k]}\right\}_{n=0}^{\infty}$ and $\{(x)_{n,\lambda}\}_{n=0}^{\infty}$.

Let us present the recurrence relation for $\mathcal{B}_{n,\lambda}^{[k]}(x)$ from the derivations of the expressions for $Ei_{k,\lambda}\left(\log_\lambda(1+t)\right)$.

Theorem 2.4. *For $n \geq 1$ and $k \in \mathbb{Z}$, the following relation holds:*

$$\mathcal{B}_{n,\lambda}^{[k]}(1) - \mathcal{B}_{n,\lambda}^{[k]}(-1) = \sum_{m=1}^{n}\frac{(1)_{m,\lambda}S_{1,\lambda}(n,m)}{m^{k-1}}. \tag{2.6}$$

Proof. We first note from [10] that

$$Ei_{k,\lambda}\left(\log_\lambda(1+t)\right)=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(1)_{m,\lambda}S_{1,\lambda}(n,m)}{m^{k-1}}\right)\frac{t^n}{n!}. \tag{2.7}$$

On the other hand, in the virtue of the result of Theorem 2.3, it follows that

$$\begin{aligned}Ei_{k,\lambda}\left(\log_\lambda(1+t)\right)&=\left(e_\lambda(t)-e_\lambda^{-1}(t)\right)\sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}^{[k]}\frac{t^n}{n!}\\&=\sum_{m=0}^{\infty}\left((1)_{m,\lambda}-(-1)_{m,\lambda}\right)\frac{t^m}{m!}\sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}^{[k]}\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\left(\mathcal{B}_{m,\lambda}^{[k]}(1)_{n-m,\lambda}-\mathcal{B}_{m,\lambda}^{[k]}(-1)_{n-m,\lambda}\right)\right)\frac{t^n}{n!}\\&=\sum_{n=1}^{\infty}\left(\mathcal{B}_{n,\lambda}^{[k]}(1)-\mathcal{B}_{n,\lambda}^{[k]}(-1)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

Therefore, we can show the desired relation by comparing coefficients of (2.7) and (2.8).

Also, $\mathcal{B}_{n,\lambda}^{[k]}(x)$ can be expressed in terms of the type 2 degenerate poly-Bernoulli numbers and the type 2 degenerate Euler polynomials.

Theorem 2.5. *For $n\geq 0$ and $k\in\mathbb{Z}$, $\mathcal{B}_{n,\lambda}^{[k]}(x)$ can be expressed as the product of $B_{n,\lambda}^{(k)}$ and $\mathcal{E}_{n,\lambda}(x)$ such as*

$$\mathcal{B}_{n,\lambda}^{[k]}(x)=\frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}B_{m,\lambda}^{(k)}\mathcal{E}_{n-m,\lambda}\left(x+\frac{1}{2}\right). \tag{2.9}$$

Proof. With the help of (1.9) and (1.11), one can have

$$\begin{aligned}\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^x(t)&=\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)+1)(e_\lambda(t)-1)}e_\lambda^{x+1}(t)\\&=\frac{1}{2}\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)-1}\frac{2}{e_\lambda^{\frac{1}{2}}(t)+e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^{x+\frac{1}{2}}(t)\\&=\frac{1}{2}\left(\sum_{n=0}^{\infty}B_{n,\lambda}^{(k)}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}\left(x+\frac{1}{2}\right)\frac{t^m}{m!}\right)\\&=\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_{m,\lambda}^{(k)}\mathcal{E}_{n-m,\lambda}\left(x+\frac{1}{2}\right)\right)\frac{t^n}{n!}.\end{aligned}$$

Thus, the previous identity gives the identity (2.9) by the comparing coefficients.

We close this section with an expression for $\mathcal{B}_{n,\lambda}^{[k]}(x)$ in terms of $\mathcal{B}_{n,\lambda}^{[1]}$ and $S_{1,\lambda}(n,m)$.

Theorem 2.6. *For $n\geq 0$ and $k\in\mathbb{Z}$, the following identity holds*

$$\mathcal{B}_{n,\lambda}^{[k]}(x)=\sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_{n-m,\lambda}^{[1]}(x)\frac{1}{m+1}\sum_{\ell=0}^{m}\frac{(1)_{\ell+1,\lambda}S_{1,\lambda}(m+1,\ell+1)}{(\ell+1)^{k-1}}.$$

Proof. Note that from (2.1)

$$\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{t}=\sum_{m=0}^{\infty}\frac{1}{m+1}\left(\sum_{\ell=0}^{m}S_{1,\lambda}(m+1,\ell+1)\frac{(1)_{\ell+1,\lambda}}{(\ell+1)^{k-1}}\right)\frac{t^m}{m!}.$$

From the previous note and Remark 2.1, the followings are established:

$$\begin{aligned}\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^x(t)&=\frac{Ei_{k,\lambda}(\log_\lambda(1+t))}{t}\frac{te_\lambda^x(t)}{e_\lambda(t)-e_\lambda^{-1}(t)}\\&=\left(\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{\ell=1}^{n+1}\frac{(1)_{\ell,\lambda}S_{1,\lambda}(n+1,\ell)}{\ell^{k-1}}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\mathcal{B}_{m,\lambda}^{[1]}(x)\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\frac{\mathcal{B}_{n-m,\lambda}^{[1]}(x)}{m+1}\sum_{\ell=0}^{m}\frac{(1)_{\ell+1,\lambda}S_{1,\lambda}(m+1,\ell+1)}{(\ell+1)^{k-1}}\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, the assertion is obtained by comparing coefficients.

3. Type 2 degenerate modified unipoly-Bernoulli polynomials attached to polynomials

In this section, we consider a new type of the degenerate unipoly-Bernoulli polynomials attached to p, where p is any arithmetic function defined on $\mathbb{N}$. The unipoly function attached to p is recently introduced in [8], which is defined by

$$u_k(x|p)=\sum_{n=1}^{\infty}\frac{p(n)}{n^k}x^n,\quad k\in\mathbb{Z}.$$

Applying the degenerate type of unipoly function attached to p in the degenerate poly-Bernoulli polynomials of type 2, we introduce a new type of degenerate unipoly polynomials.

Definition 3.1. *Let k be any integer. Then we define the type 2 degenerate modified unipoly-Bernoulli polynomials attached to polynomials p, which are given by*

$$\frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{e_\lambda(t)-e_\lambda^{-1}(t)}e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathcal{PB}_{n,\lambda,p}^{[k]}(x)\frac{t^n}{n!},$$

where $u_{k,\lambda}(x|p)$ is the degenerate unipoly function [8, 11] attached to any arithmetic function $p(n)$ defined by

$$u_{k,\lambda}(x|p)=\sum_{n=1}^{\infty}p(n)\frac{(1)_{n,\lambda}}{n^k}x^n.$$

In particular, when $x=0$, $\mathcal{PB}_{n,\lambda,p}^{[k]}=\mathcal{PB}_{n,\lambda,p}^{[k]}(0)$ are called the type 2 degenerate modified unipoly-Bernoulli numbers.

We note that if the attached arithmetic function satisfies $p(n)=\frac{1}{\Gamma(n)}$, $n\in\mathbb{N}$, it is seen that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{PB}_{n,\lambda,\frac{1}{\Gamma}}^{[k]}(x)\frac{t^n}{n!}&=\frac{e_\lambda^x(t)}{e_\lambda(t)-e_\lambda^{-1}(t)}u_{k,\lambda}\left(\log_\lambda(1+t)\Big|\frac{1}{\Gamma}\right)\\&=\frac{e_\lambda^x(t)}{e_\lambda(t)-e_\lambda^{-1}(t)}\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}(\log_\lambda(1+t))^n}{n^k(n-1)!}.\end{aligned}\tag{3.1}$$

Also, one can see that $\mathcal{PB}^{[1]}_{n,\lambda,\frac{1}{\Gamma}}(x) = \beta_{n,\lambda}(x)$ when $k = 1$ from (1.7) and the identity holds

$$\sum_{n=0}^{\infty} \mathcal{PB}^{[1]}_{n,\lambda,\frac{1}{\Gamma}}(x)\frac{t^n}{n!} = \frac{e^x_\lambda(t)}{e_\lambda(t) - e^{-1}_\lambda(t)} \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}\left(\log_\lambda(1+t)\right)^n}{n!} = \frac{te^x_\lambda(t)}{e_\lambda(t) - e^{-1}_\lambda(t)}.$$

For example, using the scientific calculator, we can see that the first five polynomials $\mathcal{PB}^{[1]}_{k,\lambda,\frac{1}{\Gamma}}(x),\ k = 0, \cdots, 4$, are estimated as

$$\begin{aligned}
&\mathcal{PB}^{[1]}_{0,\lambda,\frac{1}{\Gamma}}(x) = 0.5,\\
&\mathcal{PB}^{[1]}_{1,\lambda,\frac{1}{\Gamma}}(x) = 0.5x + 0.25\lambda,\\
&\mathcal{PB}^{[1]}_{2,\lambda,\frac{1}{\Gamma}}(x) = 0.5x^2 - 0.0833\lambda^2 - 0.1667,\\
&\mathcal{PB}^{[1]}_{3,\lambda,\frac{1}{\Gamma}}(x) = 0.5x^3 - 0.75\lambda x^2 - 0.5x + 0.125\lambda^3 + 0.25\lambda,\\
&\mathcal{PB}^{[1]}_{4,\lambda,\frac{1}{\Gamma}}(x) = 0.5x^4 - 2\lambda x^3 + 2\lambda^2x^2 - x^2 + 2\lambda x - 0.3167\lambda^4 - 0.667\lambda^2 + 0.2333.
\end{aligned}$$

First, we show that $\mathcal{PB}^{[k]}_{n,\lambda,p}(x)$ can be expressed in terms of $\mathcal{B}^{[1]}_{n,\lambda}(x)$ and the degenerate Stirling numbers of the first kind.

Theorem 3.1. *For $k \in \mathbb{Z}, n \geq 0, \lambda \in \mathbb{R}\backslash\{0\}$, the following identity is established:*

$$\mathcal{PB}^{[k]}_{n,\lambda,p}(x) = \sum_{m=0}^{n}\sum_{\ell=0}^{m}\binom{n}{m}\frac{(1)_{\ell+1,\lambda}p(\ell+1)(\ell+1)!}{(\ell+1)^k}\frac{S_{1,\lambda}(n+1,\ell+1)}{n+1}\mathcal{B}^{[1]}_{n-m,\lambda}(x).$$

Proof. It is well known that

$$\frac{te^x_\lambda(t)}{e_\lambda(t) - e^{-1}_\lambda(t)} = \sum_{n=0}^{\infty}\mathcal{B}^{[1]}_{n,\lambda}(x)\frac{t^n}{n!} \tag{3.2}$$

and

$$\begin{aligned}
\frac{1}{t}u_{k,\lambda}(\log_\lambda(1+t)|p) &= \frac{1}{t}\sum_{n=0}^{\infty}\frac{p(n+1)(1)_{n+1,\lambda}}{(n+1)^k}(\log_\lambda(1+t))^{n+1}\\
&= \sum_{n=0}^{\infty}C_{n,k,\lambda}\frac{t^n}{n!},
\end{aligned} \tag{3.3}$$

where

$$C_{n,k,\lambda} := \sum_{m=0}^{n}\frac{p(m+1)(1)_{m+1,\lambda}(m+1)!}{(m+1)^k}\frac{S_{1,\lambda}(n+1,m+1)}{n+1},$$

which can be obtained from $(\log_\lambda(1+t))^{n+1}/(n+1)! = \sum_{m=n+1}^{\infty} S_{1,\lambda}(m,n+1)\frac{t^m}{m!}$. Since the product of the left hand sides of Eqs (3.2) and (3.3) provides the generating function in Definition 3.1, we obtain the desired identity by the binomial convolution of $\left\{\mathcal{B}^{[1]}_{n,\lambda}(x)\right\}_{n=0}^{\infty}$ and $\{C_{n,k,\lambda}\}_{n=0}^{\infty}$.

We next show that $\mathcal{PB}^{[k]}_{n,\lambda,p}(x)$ is expressed as a sum of the products of the type 2 degenerate modified unipoly-Bernoulli numbers and λ-falling factorial sequence for x.

Theorem 3.2. *Let $k \in \mathbb{Z}, n \geq 0, \lambda \in \mathbb{R}\backslash\{0\}$. Then we have the identity:*

$$\mathcal{PB}^{[k]}_{n,\lambda,p}(x) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{PB}^{[k]}_{n,\lambda,p}(x)_{n-m,\lambda}.$$

Proof. By making use of the identities

$$\frac{u_{k,\lambda}(\log_\lambda(1+t)|p)}{e_\lambda(t) - e_\lambda^{-1}(t)} = \sum_{n=0}^{\infty} \mathcal{PB}^{[k]}_{n,\lambda,p} \frac{t^n}{n!} \text{ and } e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!},$$

one can obtain the result from the binomial convolution of two identities.

Finally, we present the type 2 degenerate unipoly-Bernoulli numbers for $k = 1$ can be considered to the sum of the type 2 degenerate Euler polynomials (1.11) and the degenerate Bernoulli numbers (1.5).

Theorem 3.3. *Let $p(n) = \frac{1}{\Gamma(n)}$ for $n \geq 0, \lambda \in \mathbb{R}\backslash\{0\}$, we have*

$$\mathcal{PB}^{[1]}_{n,\lambda,p} = \frac{1}{2} \sum_{m=0}^{n} \binom{n}{m} \mathcal{E}_{n,\lambda}\left(\frac{1}{2}\right) ß_{n-m,\lambda}.$$

Proof. Recalling from Definition 3.1 for $k = 1$ that

$$\sum_{n=0}^{\infty} \mathcal{PB}^{[1]}_{n,\lambda,\frac{1}{\Gamma}}(x) \frac{t^n}{n!} = \frac{t e_\lambda^x(t)}{e_\lambda(t) - e_\lambda^{-1}(t)},$$

in which if $x = 0$, the right hand side of the previous equation provides that

$$\begin{aligned} \frac{t}{e_\lambda(t) - e_\lambda^{-1}(t)} &= \frac{1}{2} \frac{t}{e_\lambda(t) - 1} \frac{2e_\lambda^{\frac{1}{2}}(t)}{e_\lambda^{\frac{1}{2}}(t) + e_\lambda^{-\frac{1}{2}}(t)} \\ &= \frac{1}{2} \left(\sum_{n=0}^{\infty} ß_{n,\lambda} \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} \mathcal{E}_{m,\lambda}\left(\frac{1}{2}\right) \frac{t^m}{m!} \right) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \mathcal{E}_{n,\lambda}\left(\frac{1}{2}\right) ß_{n-m,\lambda} \right) \frac{t^n}{n!} \end{aligned}$$

with the help of the definitions (1.5) and (1.11). Hence, the identity follows by comparing coefficients.

4. Illustrative examples

In this section, we present several examples and some graphical representations of the type 2 degenerate modified poly-Bernoulli polynomials and the type 2 degenerate modified unipoly-Bernoulli polynomials attached to polynomials presented in the previous sections.

We first note that by using the recurrence relation $\frac{d}{dt}Ei_{k,\lambda}(t) = \frac{1}{t}Ei_{k-1,\lambda}(t)$ with the property of $\frac{d}{dt}e_\lambda(t) = \frac{1}{1+\lambda t}e_\lambda(t)$, $Ei_{k,\lambda}(t)$ for $k \le 1$ can be calculated as follows:

$$\begin{aligned} &Ei_{1,\lambda}(t) = e_\lambda(t) - 1, && Ei_{0,\lambda}(t) = \frac{t}{1+\lambda t}e_\lambda(t), \\ &Ei_{-1,\lambda}(t) = \frac{t(1+t)}{(1+\lambda t)^2}e_\lambda(t), && Ei_{-2,\lambda}(t) = \frac{t(t^2+3t-\lambda t+1)}{(1+\lambda t)^3}e_\lambda(t), \\ &Ei_{-3,\lambda}(t) = \frac{t(t^3+6t^2+\lambda^2t^2-4\lambda t^2-4\lambda t+7t+1)}{(1+\lambda t)^4}e_\lambda(t), && \cdots . \end{aligned} \tag{4.1}$$

Remark 4.1. *From* (1.3) *and* (4.1)*, we see that the polyexponential functions $Ei_k(t)$ are considered as the limit of the degenerate polyexponential functions $Ei_{k,\lambda}(t)$ as λ approaches to* 0*, i.e.,* $\lim_{\lambda\to 0} Ei_{k,\lambda}(t) = Ei_k(t)$.

In order to see some of the type 2 degenerate modified poly-Bernoulli polynomials, we calculate the first six polynomials of $\mathscr{B}^{[1]}_{n,\lambda}(x)$ for $k = 1$, which are listed below:

$$\begin{aligned} \mathscr{B}^{[1]}_{0,\lambda}(x) &= \frac{1}{2}, \\ \mathscr{B}^{[1]}_{1,\lambda}(x) &= \frac{\lambda}{4} + \frac{x}{2}, \\ \mathscr{B}^{[1]}_{2,\lambda}(x) &= -\frac{\lambda^2}{12} - \frac{1}{6} + \frac{x^2}{2}, \\ \mathscr{B}^{[1]}_{3,\lambda}(x) &= \frac{\lambda}{4} + \frac{\lambda^3}{8} - \frac{x}{2} - \frac{3\lambda x^2}{4} + \frac{x^3}{2}, \\ \mathscr{B}^{[1]}_{4,\lambda}(x) &= \frac{1}{60}(14 - 40\lambda^2 - 19\lambda^4) + 2\lambda x + (2\lambda^2 - 1)x^2 - 2\lambda x^3 + \frac{x^4}{2}, \\ \mathscr{B}^{[1]}_{5,\lambda}(x) &= \frac{1}{8}(-14\lambda + 20\lambda^3 + 9\lambda^5) + \frac{1}{6}(7 - 55\lambda^2)x + \frac{15}{2}(\lambda - \lambda^3)x^2 - \frac{5x^3}{3} + \frac{55\lambda^2x^3}{6} - \frac{15\lambda x^4}{4} + \frac{x^5}{2}. \end{aligned}$$

Further, following the presentation in [18], we investigate the zeros of $\mathscr{B}^{[1]}_{n,\lambda}(x), n = 1, 2, \cdots, 8$ for $\lambda = \frac{1}{4}$ in Table 1 and plot the zeros of $\mathscr{B}^{[1]}_{n,\lambda}(x) = 0$ for $n = 1, \cdots, 15$ when $\lambda = 1/4$ in Figures 1 and 2.

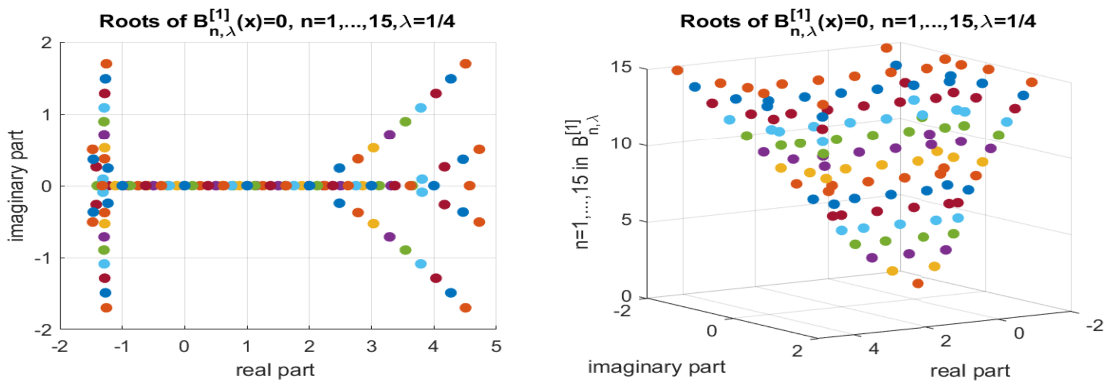

Figure 1. Stacks of all Roots of $\mathscr{B}^{[1]}_{n,\lambda}(x), n = 1, 2, \cdots, 15$ for $\lambda = \frac{1}{4}$.

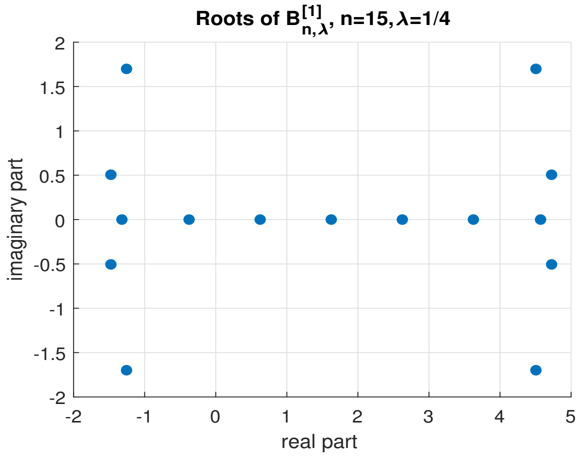

Figure 2. Roots of $\mathscr{B}^{[1]}_{n,\lambda}(x), n = 15, \lambda = \frac{1}{4}$ in $\mathbb{C}$.

Table 1. Approximate Roots of $\mathscr{B}^{[1]}_{n,\lambda}(x) = 0$ for $\lambda = 1/4$.

n	Real Roots	Complex Roots
$n = 1$	-0.12500	$-$
$n = 2$	$-0.5863,\ 0.5863$	$-$
$n = 3$	$0.1250,\ -0.8982,\ 1.1482$	$-$
$n = 4$	$-0.2750,\ -1.1099,\ 0.7750,\ 1.6099$	$-$
$n = 5$	$0.3750,\ -0.6285,\ 1.3785,$ $-1.2334,\ 1.9834$	$-$
$n = 6$	$-0.0076,\ 1.0076,\ -0.9906,$ $1.9906,\ -1.2289,\ 2.2289$	$-$
$n = 7$	$0.6250,\ -0.3755,\ 1.6255,$	$2.4803 \pm 0.2443i,\ -1.2303 \pm 0.2443i$
$n = 8$	$-0.7477,\ 2.2477,\ 0.2477,\ 1.2523,$	$-1.2853 \pm 0.3745i,\ 2.7853 \pm 0.3745i$

Also, we plot the behaviour of the type 2 degenerate modified unipoly-Bernoulli polynomials, $\mathcal{PB}^{[1]}_{6,\lambda,\frac{1}{\Gamma}}(x)$ and $\mathcal{PB}^{[1]}_{7,\lambda,\frac{1}{\Gamma}}(x)$ attached to polynomials $p = \frac{1}{\Gamma}$ in Figure 3 for various values of $\lambda = 30, 25, \cdots, 1/20$. The figures show the degenerate polynomials $\mathcal{PB}^{[1]}_{6,\lambda,\frac{1}{\Gamma}}(x)$ and $\mathcal{PB}^{[1]}_{7,\lambda,\frac{1}{\Gamma}}(x)$ approach to the type 2 Bernoulli polynomials $B^*_6(x)$ and $B^*_7(x)$, respectively, as λ is getting small, where $B^*_n(x)$ are defined by the generating function [6]

$$\frac{t}{e^t - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B^*_n(x) \frac{t^n}{n!}.$$

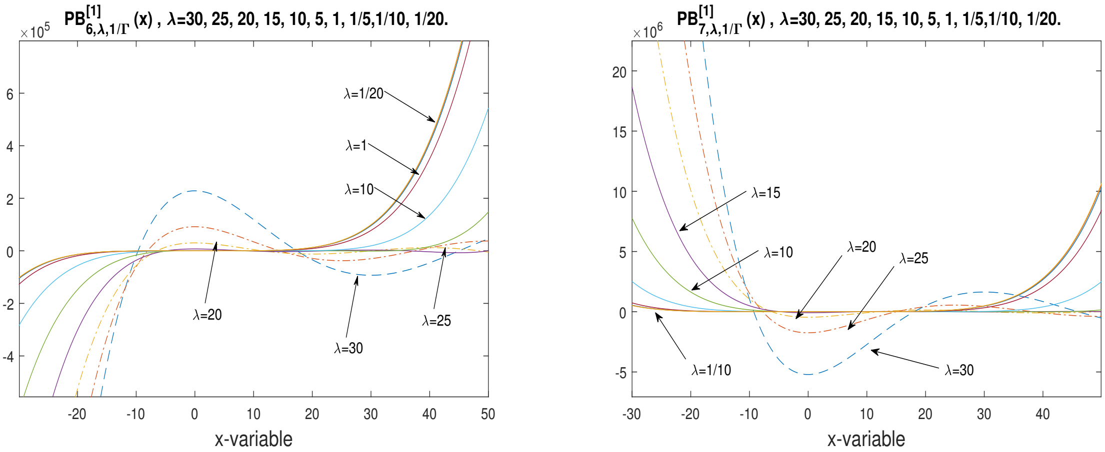

Figure 3. $\mathcal{PB}^{[1]}_{6,\lambda,\frac{1}{\Gamma}}(x)$ and $\mathcal{PB}^{[1]}_{7,\lambda,\frac{1}{\Gamma}}(x)$ for different λ.

5. Conclusions

Following recent research about type 2 degenerate polynomials, we studied the new types of degenerate poly-Bernoulli polynomials and degenerate unipoly-Bernoulli polynomials, which are obtained by modifying generating functions based on degenerate exponential functions and degenerate logarithm functions. We present the general properties of the polynomials including recursion relation and related identities in terms of well known special functions and numbers. Furthermore, in order to see their zeros and behaviors, we calculate some of the proposed polynomials for specific cases and display their zeros and polynomials for different variables. In general, the relation among special polynomials provides not only an important research topic but also useful identities in physics, science and engineering as well as in mathematics. Thus, it would be interested to find the relation among the variants of some special polynomials and their applications, which are one of our future work.

Acknowledgments

This work was supported by the Dongguk University Research Fund of 2020 and the corresponding author (Kwon) was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (No. 2020R1F1A1A0107156412).

The authors would like to thank the referees for their careful reading and helpful comments.

Conflict of interest

The authors declare no conflict of interest.

References

1. S. Araci, Degenerate poly-type 2-Bernoulli polynomials, *Math. Sci. Appl. E.*, **9** (2021), 1–8. https://doi.org/10.36753/mathenot.839111
2. S. Araci, A new class of Bernoulli polynomials attached to polyexponential functions and related identities, *Adv. Stud. Contemp. Math.*, **31** (2021), 195–204. https://dx.doi.org/10.17777/ascm2021.31.2.195
3. L. Carlitz, A degenerate Staudt-Clausen theorem, *Arch. Math.*, **7** (1956), 28–33. https://doi.org/10.1007/BF01900520
4. L. Carlitz, Degenerate Stirling, Bernoulli and Eulerian numbers, *Util. Math.*, **15** (1979), 51–88.
5. U. Duran, M. Acikgoz, S. Araci, Hermite based poly-Bernoulli polynomials with a q parameter, *Adv. Stud. Contemp. Math.*, **28** (2018), 285–296. https://dx.doi.org/10.17777/ascm2018.28.2.285
6. G. W. Jang, T. Kim, A note on type 2 degenerate Euler and Bernoulli polynomials, *Adv. Stud. Contemp. Math.*, **29** (2019), 147–159. Dhttps://dx.doi.org/10.17777/ascm2019.29.1.147
7. L. C. Jang, D. S. Kim, T. Kim, H. Lee, p-adic integral on Z_p associated with degenerate Bernoulli polynomials of the second kind, *Adv. Differ. Equ.*, **2020** (2020), 278. https://doi.org/10.1186/s13662-020-02746-2
8. D. S. Kim, T. Kim, A note on polyexponential and unipoly functions, *Russ. J. Math. Phys.*, **26** (2019), 40–49. https://doi.org/10.1134/S1061920819010047
9. J. Kwon, L. C. Jang, A note on the type 2 poly-Apostol-Bernoulli polynomials, *Adv. Stud. Contemp. Math.*, **30** (2020), 253–262. https://dx.doi.org/10.17777/ascm2020.30.2.253
10. T. Kim, D. S. Kim, A note on a new type degenerate Bernoulli numbers, *Russ. J. Math. Phys.*, **27** (2020), 227–235. https://doi.org/10.1134/S1061920820020090
11. T. Kim, D. S. Kim, J. Kwon, H. Lee, Degenerate polyexponential functions and type 2 degenerate poly-Bernoulli numbers and polynomials, *Adv. Differ. Equ.*, **2020** (2020), 168. https://doi.org/10.1186/s13662-020-02636-7
12. T. Kim, D. S. Kim, Degenerate polyexponential functions and degenerate Bell polynomials, *J. Math. Anal. Appl.*, **487** (2020), 124017. http://dx.doi.org/10.1016/j.jmaa.2020.124017
13. W. A. Khan, G. Muhiuddin, A. Muhyi, D. Al-Kadi, Analytical properties of type 2 degenerate poly-Bernoulli polynomials associated with their applications, *Adv. Differ. Equ.*, **2021** (2021), 420. http://dx.doi.org/10.1186/s13662-021-03575-7
14. W. A. Khan, A. Muhyi, R. Ali, K. A. H. Alzobydi, M. Singh, P. Agarwal, A new family of degenerate poly-Bernoulli polynomials of the second kind with its certain related properties, *AIMS Mathematics*, **6** (2021), 12680–12697. http://dx.doi.org/10.3934/math.2021731
15. S. Roman, *The umbral calculus*, Academic Press, 1984.
16. S. K. Sharma, W. A. Khan, S. Araci, S. S. Ahmed, New construction of type 2 degenerate central Fubini polynomials with their certain properties, *Adv. Differ. Equ.*, **2020** (2020), 587. https://doi.org/10.1186/s13662-020-03055-4

17. Y. Simsek, Identities on the Changhee numbers and Apostol-type Daehee polynomials, *Adv. Stud. Contemp. Math.*, **27** (2017), 199–212. http://dx.doi.org/10.17777/ascm2017.27.2.199

18. S. C. Woon, Analytic continuation of Bernoulli numbers, a new formula for the Riemann Zeta function, and the phenomenon of scattering of zeros, 1997, arXiv: physics/9705021.