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Research article

Complete convergence and complete integral convergence of partial sums for moving average process under sub-linear expectations

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Abstract: In this paper, we establish the complete convergence and complete integral convergence of partial sums for moving average process based on independent random variables under the sub-linear expectations. The results in the paper extend some convergence properties of moving average process under independent assumption from probability space to the sub-linear expectation space.

Keywords: moving average process; complete integral convergence; complete convergence; independent random variables; sub-linear expectation
Mathematics Subject Classification: 60F15

1. Introduction

Limit theory is one of the important branches of probability theory. In recent years, the limitation of classical limit theory which only applies to some deterministic models has been gradually highlighted. In order to explain the uncertain phenomena in practical problems and to solve nonlinear probability and distribution problems, many scholars began to study limit theory under sub-linear expectation. Peng [1–3] was the first scholar to put forward the definition of sub-linear expectation space, and he also constructed the relevant theoretical system. Sub-linear expectation space provides a more flexible framework for solving sub-linear probability problems. On the basis of Peng's research, Zhang [4–6] obtained a series of important conclusions such as strong law of large numbers, exponential inequality, Rosenthal's inequality and so on under sub-linear expectations, which laid a foundation for the limit theory research in sub-linear expectation space.

Complete convergence and complete moment convergence are two important parts of probability limit theory. The concept of complete convergence was first proposed by Hsu [7]. As a more general form of convergence, complete moment convergence was put forward later than complete convergence. Chow [8] first proposed complete γ-order moment convergence. In probability space, the study of complete convergence and complete moment convergence becomes mature relatively. Sung [9] proved the complete convergence for arrays of rowwise independent random, Wu [10, 11] obtained the complete convergence theorem of negatively dependent sequences and arrays of rowwise negatively dependent random variables respectively, Bao et al. [12] expanded the range of random variables and established the complete convergence of weighted sums of arrays of rowwise END random variables and so on. Complete integral convergence in sub-linear expectation space corresponds to complete moment convergence in probability space. In recent ten years, the results of complete convergence and complete integral convergence in sub-linear expectations are increasingly abundant, such as Deng and Wang [13], Li and Wu [14] obtained complete convergence and complete integral convergence for extended independent random variables respectively, Lu et al. [15] proved complete f-moment convergence for END variables, Zhong and Wu [16] extended the complete integral convergence of weighted sum of END random variables, Xu and Cheng [17] got convergence theorems for sum of i.i.d random variables, Ding [18] obtained a general form of precise asymptotics for complete convergence, and Zhang [19] discussed sufficient and necessary conditions for convergence of independent random variables, he also obtained the Lévy and Kolmogorov maximum inequalities under sub-linear expectations.

There are a lot of conclusions about convergence properties of moving average process after the notion of it was defined. For the study on limit theory of moving average process in classical probability space, Ibragimov [20] first established the central limit theorem in 1962, and then Burton and Dehling [21] obtained the large deviation theorem. Subsequently, Li et al. [22] proved that the result of complete convergence of moving average process. Zhang [23] studied the complete convergence of moving average process under dependent conditions. With the attention of scholars in this field, many research results have been obtained. For example, Yang et al. [24] proved complete convergence of moving average process based on AANA sequence, Yang and Hu [25] obtained complete moment convergence of moving average process generated by PNQD random variables, Qiu and Chen [26] investigated convergence of moving average process under END assumptions, and so on. Recently, Tao et al. [27] studied complete convergence of moving average process generated by WOD random variable sequence, Song and Zhu [28] studied complete convergence and complete moment convergence of the maximum partial sum of moving average process based on ρ^--mixing random variable sequence. Based on the above study, there are some research results on linear process in sub-linear expectation space. For example, Liu and Zhang [29–31] got the central limit theorem for linear processes generated by IID random variables, and investigated the law of the iterated logarithm and large deviation principle for linear process based on stationary independent random variables.

In this paper, we aim to extend some convergence properties of Song and Zhu [28] from probability space to sub-linear space and get corresponding results.

2. Preliminaries

We use the framework and notations of Peng [1–3]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq c(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \ \forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}_n,$$

for some $c > 0,\ m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables. In this case we denote $X \in \mathcal{H}$.

Definition 2.1. A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: For all $X, Y \in \mathcal{H}$, we have

(1) Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}X \geq \hat{\mathbb{E}}Y$;

(2) Constant preserving: $\hat{\mathbb{E}}c = c$;

(3) Sub-additivity: $\hat{\mathbb{E}}(X + Y) \leq \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$; whenever $\hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(4) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda\hat{\mathbb{E}}X, \lambda \geq 0$.

Here $\bar{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space.
Given a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}X := -\hat{\mathbb{E}}(-X),\ \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that for all $X, Y \in \mathcal{H}$,

$$\hat{\varepsilon}X \leq \hat{\mathbb{E}}X,\ \ \hat{\mathbb{E}}(X + c) = \hat{\mathbb{E}}X + c,\ \ |\hat{\mathbb{E}}(X - Y)| \leq \hat{\mathbb{E}}|X - Y| \textit{ and } \hat{\mathbb{E}}(X - Y) \geq \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y.$$

If $\hat{\mathbb{E}}Y = \hat{\varepsilon}Y$, then $\hat{\mathbb{E}}(X + aY) = \hat{\mathbb{E}}X + a\hat{\mathbb{E}}Y$ for any $a \in \mathbb{R}$. Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0,\ V(\Omega) = 1 \text{ and } V(A) \leq V(B) \text{ for } \forall A \subseteq B, A, B \in \mathcal{G}.$$

It is called sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. In the sub-linear space $(\Omega, \mathcal{H}, \hat{E})$, we denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\hat{\mathbb{E}}\xi; I(A) \leq \xi, \xi \in \mathcal{H}\},\ \mathcal{V}(A) := 1 - \mathbb{V}(A^c),\ \forall A \in \mathcal{F},$$

where A^c is the complement set of A. By definition of $\mathbb{V}$ and $\mathcal{V}$, it is obvious that $\mathbb{V}$ is sub-additive, and

$$\mathcal{V} \leq \mathbb{V},\ \forall A \in \mathcal{F}.$$

If $I(A) \in \mathcal{H}$, then $\mathbb{V}(A) = \hat{\mathbb{E}}\left(I\left(A\right)\right),\ \mathcal{V}(A) = \hat{\varepsilon}\left(I\left(A\right)\right)$.

If $f \leq I(A) \leq g, f, g \in \mathcal{H}$, then $\hat{\mathbb{E}}f \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}g, \quad \hat{\varepsilon}f \leq \mathcal{V}(A) \leq \hat{\varepsilon}g$.

Definition 2.2. We define the Choquet integrals/expecations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V = \int_0^\infty V(X \geq t)\mathrm{d}t + \int_{-\infty}^0 \left[V(X \geq t) - 1\right]\mathrm{d}t,$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$ respectively.

Definition 2.3. In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \dots, Y_n), Y_i \in \mathcal{H}$ is said to be independent to another random vector $X = (X_1, \dots, X_m), X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}$ if

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}[\varphi(x, Y)]\Big|_{x=X}\right], \forall \varphi \in C_{l,\,\mathrm{Lip}}\left(\mathbb{R}_m \times \mathbb{R}_n\right),$$

whenever $\bar{\varphi}(x) := \hat{\mathbb{E}}[|\varphi(x, Y)|] < \infty$ for all x and $\hat{\mathbb{E}}[|\bar{\varphi}(X)|] < \infty$.

Next, we introduce a definition that extends the concept of stationary sequence to sub-linear expectation spaces.

Definition 2.4. $\{Y_n, n \in \mathbb{N}\}$ is said to be a sequence of stationary random variables on the $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, if for any a function $\phi_n \in C_{l,Lip}(\mathbb{R}^n) : \mathbb{R}^n \to \mathbb{R}$, then

$$\hat{\mathbb{E}}\left[\phi_n\left(Y_1, Y_2, \dots, Y_n\right)\right] = \hat{\mathbb{E}}\left[\phi_n\left(Y_{1+k}, Y_{2+k}, \dots, Y_{n+k}\right)\right], \quad \forall n \geq 1, k \in \mathbb{N}$$

Definition 2.5. In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be stochastically dominated by a random variable X if there exists a positive constant c such that

$$\hat{\mathbb{E}}\left(f\left(|X_n|\right)\right) \leqslant c\hat{\mathbb{E}}(f(|X|)), \forall 0 \leqslant f \in C_{l,\mathrm{Lip}}(\mathbb{R}).$$

Definition 2.6. (i) A sub-linear expectation $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ is called to be countably sub-additive if

$$\hat{\mathbb{E}}(X) \leq \sum_{n=1}^{\infty} \hat{\mathbb{E}}\left(X_n\right), \quad \text{where} \quad X \leq \sum_{n=1}^{\infty} X_n, \quad X, X_n \in \mathcal{H}, \quad X \geq 0, X_n \geq 0, \quad n \geq 1.$$

(ii) A function $V : \mathcal{F} \to [0, 1]$ is called to be countably sub-additive if

$$V\left\{\bigcup_{n=1}^{\infty} A_n\right\} \leq \sum_{n=1}^{\infty} V\left(A_n\right), \quad \forall A_n \in \mathcal{F}.$$

In this paper, the symbol c stands for a generic positive constant which may differ from one place to another. Let $a_n \ll b_n$ denote that there exists a constant $c > 0$ such that $a_n \leq cb_n$ for sufficiently large n, and $I(\cdot)$ denote an indicator function.

To prove our results,we need the following lemmas.

Lemma 2.1. *(Zhang [6]) (i) Markov inequality:* $\forall X \in \mathcal{H}$,

$$\mathbb{V}(|X| \geq x) \leq \frac{\hat{\mathbb{E}}(|X|^p)}{x^p}, \ \forall x > 0, p > 0,$$

(ii) Hölder inequality: $\forall X, Y \in \mathcal{H}, p, q > 1$ *satisfying* $p^{-1} + q^{-1} = 1$,

$$\hat{\mathbb{E}}(|XY|) \leq \left(\hat{\mathbb{E}}\left(|X|^p\right)\right)^{1/p} \left(\hat{\mathbb{E}}\left(|Y|^q\right)\right)^{1/q}.$$

Lemma 2.2. *(Zhong and Wu [16]) Suppose* $X \in \mathcal{H}, p > 0, \alpha > 0$, *and* $l(x)$ *is a slowly varying function.*

(i) Then for any $c > 0$,

$$C_{\mathbb{V}}\left[|X|^{\alpha p} l\left(|X|^p\right)\right] < \infty \Leftrightarrow \sum_{n=1}^{\infty} n^{\alpha-1} l(n) \mathbb{V}\left(|X| > cn^{1/p}\right) < \infty,$$

taking $l(x)=1$ *and* $l(x)=\log(1+|X|)$ *respectively, we can get that* $\forall c>0$,

$$C_{\mathbb{V}}\left(|X|^{\alpha p}\right)<\infty \Leftrightarrow \sum_{n=1}^{\infty} n^{\alpha-1}\mathbb{V}\left(|X|>cn^{1/p}\right)<\infty, \tag{2.1}$$

$$C_{\mathbb{V}}\left[|X|^{\alpha p}\log(1+|X|)\right]<\infty \Leftrightarrow \sum_{n=1}^{\infty} n^{\alpha-1}\log(1+n)\,\mathbb{V}\left(|X|>cn^{1/p}\right)<\infty. \tag{2.2}$$

(ii) If $C_{\mathbb{V}}\left[|X|^{\alpha p} l\left(|X|^{p}\right)\right]<\infty$, *then for any* $\theta>1$ *and* $c>0$,

$$\sum_{k=1}^{\infty}\theta^{k\alpha} l\left(\theta^{k}\right)\mathbb{V}\left(|X|>c\theta^{k/p}\right)<\infty,$$

taking $l(x)=1$, $C_{\mathbb{V}}\left[|X|^{\alpha p}\right]<\infty$ *implies*

$$\sum_{k=1}^{\infty}\theta^{k\alpha}\mathbb{V}\left(|X|>c\theta^{k/p}\right)<\infty, \tag{2.3}$$

and taking $l(x)=\log(1+|X|)$, $C_{\mathbb{V}}\left[|X|^{\alpha p}\log(1+|X|^{p})\right]<\infty$ *implies that*

$$\sum_{k=1}^{\infty}\theta^{k\alpha}\log(1+\theta^{k})\mathbb{V}\left(|X|>c\theta^{k/p}\right)<\infty. \tag{2.4}$$

Lemma 2.3. *(Zhang [5]) Suppose that* X_k *is independent to* $(X_{k+1},\ldots,X_n)$ *for each* $k=1,\ldots,n-1$, *and* $\hat{\mathbb{E}}X_n\leq 0$ *for* $n\geq 1$. *Then*

$$\hat{\mathbb{E}}\left[(S_n^{+})^{r}\right]\leq C_r\left\{\sum_{k=1}^{n}\hat{\mathbb{E}}\left[|X_k|^{r}\right]+\left(\sum_{k=1}^{n}\hat{\mathbb{E}}\left[|X_k|^{2}\right]\right)^{r/2}\right\},\ \text{for } r>2, \tag{2.5}$$

here C_r *is a positive constant depending only on* r.

Lemma 2.4. *(Xu and Cheng [17]) Let* Y *be a random variable under sub-linear expectation space* $(\Omega,\mathcal{H},\hat{\mathbb{E}})$. *Then for any* $\alpha>0,\gamma>0$ *and* $\beta<-1$,

$$\int_{1}^{\infty}u^{\beta}\hat{\mathbb{E}}\left[|Y|^{\alpha}I\left(|Y|\leq u^{\gamma}\right)\right]\mathrm{d}u\leq cC_{\mathbb{V}}\left(|Y|^{(\beta+1)/\gamma+\alpha}\right),$$
$$\int_{1}^{\infty}u^{\beta}\log(u)\,\hat{\mathbb{E}}\left[|Y|^{\alpha}I\left(|Y|\leq u^{\gamma}\right)\right]\mathrm{d}u\leq cC_{\mathbb{V}}\left(|Y|^{(\beta+1)/\gamma+\alpha}\log(1+|Y|)\right).$$

Lemma 2.5. *(Zhang [5]) If* $\hat{\mathbb{E}}$ *is countably sub-additive, then for* $X\in\mathcal{H}$,

$$\hat{\mathbb{E}}\left(|X|\right)\leq C_{\mathbb{V}}\left(|X|\right).$$

3. Main results

Theorem 3.1. *Let* $1 \le p < 2, \alpha > 1$. *Assume* $\{a_i, -\infty < i < \infty\}$ *is an absolutely summable sequence of real numbers. Suppose that* $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \ge 1\}$ *is a moving average process generated by a sequence* $\left\{Y_j, -\infty < j < \infty\right\}$ *of stationary independent random variables which is stochastically dominated by a random variable* Y. *If* $\hat{\mathbb{E}}$ *is countably sub-additive,* $C_{\mathbb{V}}\left(|Y|^{\alpha p}\right) < \infty$ *and* $\hat{\mathbb{E}} Y_j = \hat{\varepsilon} Y_j = 0$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha-2} \mathbb{V}\left\{|S_n| > \varepsilon n^{1/p}\right\} < \infty. \tag{3.1}$$

Theorem 3.2. *Assume that the conditions of Theorem 3.1 are satisfied. Let* $r > 0$, *and*

$$\begin{cases} C_{\mathbb{V}}\left(|Y|^{\alpha p}\right) < \infty, & \text{if } r < \alpha p, \\ C_{\mathbb{V}}\left(|Y|^{\alpha p} \log(1+|Y|)\right) < \infty, & \text{if } r = \alpha p, \\ C_{\mathbb{V}}\left(|Y|^{r}\right) < \infty, & \text{if } r > \alpha p, \end{cases}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha-2-r/p} C_{\mathbb{V}}\left\{|S_n| - \varepsilon n^{1/p}\right\}_{+}^{r} < \infty. \tag{3.2}$$

4. Proof of major results

4.1. Proof of Theorem 3.1

For fixed $n \ge 1$ and for $-\infty < j < +\infty$, denote

$$\begin{aligned} Y_j' &:= -n^{1/p} I\left(Y_j < -n^{1/p}\right) + Y_j I\left(\left|Y_j\right| \le n^{1/p}\right) + n^{1/p} I\left(Y_j > n^{1/p}\right), \\ Y_j'' &:= Y_j - Y_j' = \left(Y_j + n^{1/p}\right) I\left(Y_j < -n^{1/p}\right) + \left(Y_j - n^{1/p}\right) I\left(Y_j > n^{1/p}\right). \end{aligned}$$

Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} (Y_j' + Y_j'').$$

Thus,

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha-2} \mathbb{V}\left\{|S_n| > \varepsilon n^{1/p}\right\} \\ =& \sum_{n=1}^{\infty} n^{\alpha-2} \mathbb{V}\left(\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} (Y_j' + Y_j'')\right| > \varepsilon n^{1/p}\right) \\ \le& \sum_{n=1}^{\infty} n^{\alpha-2} \mathbb{V}\left(\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j''\right| > \frac{\varepsilon n^{1/p}}{2}\right) + \sum_{n=1}^{\infty} n^{\alpha-2} \mathbb{V}\left(\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j'\right| > \frac{\varepsilon n^{1/p}}{2}\right) \\ :=& I_1 + I_2. \end{aligned}$$

Let $2^{-\frac{1}{\alpha}} < \mu < 1$, $g(y) \in C_{l,Lip}(\mathbb{R})$, such that $0 \le g(y) \le 1$ for all y and $g(y) = 1$ if $|y| \le \mu$, $g(y) = 0$ if $|y| > 1$. And $g(y)$ is a decreasing function for $y \ge 0$. Then,

$$I(|y| \le \mu) \le g(|y|) \le I(|y| \le 1), \quad I(|y| > 1) \le 1 - g(|y|) \le I(|y| > \mu). \tag{4.1}$$

Let $g_j(y) \in C_{l,Lip}(\mathbb{R})$, $j \ge 1$ such that $0 \le g_j(y) \le 1$ for all y and $g_j\left(\frac{y}{2^{j/\alpha}}\right) = 1$ if $2^{(j-1)/\alpha} < |y| \le 2^{j/\alpha}$, $g_j\left(\frac{y}{2^{j/\alpha}}\right) = 0$ if $|y| \le \mu 2^{(j-1)/\alpha}$ or $|y| > (1+\mu)2^{j/\alpha}$. Then,

$$I\left(2^{\frac{j-1}{\alpha}} < |Y| \le 2^{\frac{j}{\alpha}}\right) \le g_j\left(\frac{|Y|}{2^{j/\alpha}}\right) \le I\left(\mu 2^{\frac{j-1}{\alpha}} < |Y| \le (1+\mu)2^{\frac{j}{\alpha}}\right), \tag{4.2}$$

$$|Y|^q g\left(\frac{|Y|}{2^{k/\alpha}}\right) \le 1 + \sum_{j=1}^{k} |Y|^q g_j\left(\frac{|Y|}{2^{j/\alpha}}\right), \forall q > 0, \tag{4.3}$$

$$|Y|^q \left(1 - g\left(\frac{|Y|}{2^{k/\alpha}}\right)\right) \le \sum_{j=k}^{\infty} |Y|^q g_j\left(\frac{|Y|}{2^{j/\alpha}}\right), \forall q > 0. \tag{4.4}$$

For I_1, noting $\left|Y_j''\right| \le \left|Y_j\right| I\left(\left|Y_j\right| > n^{1/p}\right)$, $g(y)$ is a decreasing function for $y \ge 0$ and $\sum\limits_{i=-\infty}^{\infty} |a_i| < \infty$, then by (4.1) and countably sub-additive of $\hat{\mathbb{E}}$: $\hat{\mathbb{E}}\left(\sum\limits_{n=1}^{\infty} X_n\right) \le \sum\limits_{n=1}^{\infty} \hat{\mathbb{E}} X_n$ for $X_n \ge 0$, we have

$$\begin{aligned}
I_1 &\le c \sum_{n=1}^{\infty} n^{\alpha-2-1/p} \hat{\mathbb{E}} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j'' \right| \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-2-1/p} \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} \hat{\mathbb{E}} \left|Y_j''\right| \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-2-1/p} \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} \hat{\mathbb{E}} |Y_j| \left(1 - g\left(\frac{|Y_j|}{n^{1/p}}\right)\right) \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-1-1/p} \sum_{i=-\infty}^{\infty} |a_i| \hat{\mathbb{E}} |Y| \left(1 - g\left(\frac{|Y|}{n^{1/p}}\right)\right) \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-1-1/p} \hat{\mathbb{E}} |Y| \left(1 - g\left(\frac{|Y|}{n^{1/p}}\right)\right) \\
&= c \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} n^{\alpha-1-1/p} \hat{\mathbb{E}} |Y| \left(1 - g\left(\frac{|Y|}{n^{1/p}}\right)\right) \\
&\le c \sum_{k=1}^{\infty} 2^{k(\alpha-1/p)} \hat{\mathbb{E}} |Y| \left(1 - g\left(\frac{|Y|}{2^{k/p}}\right)\right).
\end{aligned}$$

Thus, combined with (4.4), countably sub-additive of $\hat{\mathbb{E}}$ and (2.3), we get

$$I_1 \le c \sum_{k=1}^{\infty} 2^{k(\alpha-1/p)} \hat{\mathbb{E}} \left(\sum_{j=k}^{\infty} |Y| g_j\left(\frac{|Y|}{2^{j/p}}\right) \right)$$

$$
\begin{aligned}
&\le c\sum_{k=1}^{\infty}2^{k(\alpha-1/p)}\sum_{j=k}^{\infty}\hat{\mathbb{E}}\,|Y|\,g_j\left(\frac{|Y|}{2^{j/p}}\right)\\
&\le c\sum_{j=1}^{\infty}\hat{\mathbb{E}}|Y|g_j\left(\frac{|Y|}{2^{j/p}}\right)\sum_{k=1}^{j}2^{k(\alpha-1/p)}\\
&\le c\sum_{j=1}^{\infty}2^{j\alpha}\hat{\mathbb{E}}\left(g_j\left(\frac{|Y|}{2^{j/p}}\right)\right)\\
&\le c\sum_{j=1}^{\infty}2^{j\alpha}\mathbb{V}\left(|Y|>\mu 2^{-1/p}2^{j/p}\right)\\
&<\infty.
\end{aligned}
$$

In the following, we prove $I_2<\infty$. First, we prove that

$$
n^{-1/p}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}Y_j'\right|\to 0\quad \text{as } n\to\infty.
$$

Since $\sum\limits_{i=-\infty}^{\infty}|a_i|<\infty$, $\hat{\mathbb{E}}Y_j=0$ and $|\hat{\mathbb{E}}X-\hat{\mathbb{E}}Y|\le\hat{\mathbb{E}}|X-Y|$, noting $\alpha>1$ and $\alpha p>1$. By countably sub-additive of $\hat{\mathbb{E}}$ and Lemma 2.4, we know $\hat{\mathbb{E}}\left(|Y|^{\alpha p}\right)\le C_{\mathbb{V}}\left(|Y|^{\alpha p}\right)<\infty$, together with (4.1), we have

$$
\begin{aligned}
n^{-1/p}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}Y_j'\right| &\le n^{-1/p}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\left|\hat{\mathbb{E}}Y_j-\hat{\mathbb{E}}Y_j'\right|\\
&\le n^{-1/p}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y_j-Y_j'\right|\\
&\le n^{1-1/p}\hat{\mathbb{E}}|Y|\left(1-g\left(\frac{|Y|}{n^{1/p}}\right)\right)\\
&\le n^{1-1/p}\frac{\hat{\mathbb{E}}|Y||Y|^{\alpha p-1}}{n^{(\alpha p-1)/p}}\left(1-g\left(\frac{|Y|}{n^{1/p}}\right)\right)\\
&\le cn^{1-\alpha}\hat{\mathbb{E}}|Y|^{\alpha p}\to 0\quad \text{as } n\to\infty.
\end{aligned}
$$

Therefore, for all n large enough, we obtain

$$
\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}Y_j'\right|<\frac{\varepsilon n^{1/p}}{4}. \tag{4.5}
$$

By (4.5), Markov inequality, Hölder inequality: $\hat{\mathbb{E}}(|XY|)\le\left(\hat{\mathbb{E}}\left(|X|^p\right)\right)^{1/p}\left(\hat{\mathbb{E}}\left(|Y|^q\right)\right)^{1/q}$ for $p,q>1$ satisfying $\frac{1}{p}+\frac{1}{q}=1$ and C_r inequality, taking $v>\max\left\{\alpha p,2,\frac{2p(\alpha-1)}{2-p}\right\}$, we have that

$$
I_2\le\sum_{n=1}^{\infty}n^{\alpha-2}\mathbb{V}\left(\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}\left(Y_j'-\hat{\mathbb{E}}Y_j'\right)\right|>\frac{\varepsilon n^{1/p}}{4}\right)
$$

$$
\begin{aligned}
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-v/p}\hat{\mathbb{E}}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n}\left(Y'_j-\hat{\mathbb{E}}Y'_j\right)\right|^v\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-v/p}\hat{\mathbb{E}}\left\{\sum_{i=-\infty}^{\infty} |a_i| \left|\sum_{j=i+1}^{i+n}\left(Y'_j-\hat{\mathbb{E}}Y'_j\right)\right|\right\}^v\\
&= c\sum_{n=1}^{\infty} n^{\alpha-2-v/p}\hat{\mathbb{E}}\left\{\sum_{i=-\infty}^{\infty} |a_i|^{1-1/v} \left(|a_i|^{1/v}\left|\sum_{j=i+1}^{i+n}\left(Y'_j-\hat{\mathbb{E}}Y'_j\right)\right|\right)\right\}^v\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-v/p}\hat{\mathbb{E}}\left\{\left(\sum_{i=-\infty}^{\infty}\left(|a_i|^{\frac{v-1}{v}}\right)^{\frac{v}{v-1}}\right)^{\frac{v-1}{v}}\left(\sum_{i=-\infty}^{\infty}\left(|a_i|^{\frac{1}{v}}\right)^v\left|\sum_{j=i+1}^{i+n}\left(Y'_j-\hat{\mathbb{E}}Y'_j\right)\right|^v\right)^{\frac{1}{v}}\right\}^v\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-v/p}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{v-1}\left(\sum_{i=-\infty}^{\infty}|a_i|\,\hat{\mathbb{E}}\left|\sum_{j=i+1}^{i+n}\left(Y'_j-\hat{\mathbb{E}}Y'_j\right)\right|^v\right)\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-v/p}\sum_{i=-\infty}^{\infty}|a_i|\,\hat{\mathbb{E}}\left(\left|\left(\sum_{j=i+1}^{i+n}\left(Y'_j-\hat{\mathbb{E}}Y'_j\right)\right)^+ + \left(\sum_{j=i+1}^{i+n}\left(Y'_j-\hat{\mathbb{E}}Y'_j\right)\right)^-\right|^v\right)\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-v/p}\sum_{i=-\infty}^{\infty}|a_i|\left\{\hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y'_j-\hat{\mathbb{E}}Y'_j\right)\right)^+\right)^v + \hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y'_j-\hat{\mathbb{E}}Y'_j\right)\right)^-\right)^v\right\}.
\end{aligned}
$$

Because of the fact that $\left\{Y'_j-\hat{\mathbb{E}}Y'_j, -\infty<j<\infty\right\}$ is independent, by (2.5) and C_r inequality, together with $\left|\hat{\mathbb{E}}Y'_j\right|^v \le \hat{\mathbb{E}}\left|Y'_j\right|^v$, for all i and n, we have

$$
\begin{aligned}
\hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y'_j-\hat{\mathbb{E}}Y'_j\right)\right)^+\right)^v &\le c\left\{\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y'_j-\hat{\mathbb{E}}Y'_j\right|^v + \left(\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y'_j-\hat{\mathbb{E}}Y'_j\right|^2\right)^{v/2}\right\}\\
&\le c\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y'_j-\hat{\mathbb{E}}Y'_j\right|^v + c\left(\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y'_j-\hat{\mathbb{E}}Y'_j\right|^2\right)^{v/2}\\
&\le c\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y'_j\right|^v + c\left(\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y'_j\right|^2\right)^{v/2}. \qquad (4.6)
\end{aligned}
$$

By $\hat{\mathbb{E}}Y_j = \hat{\varepsilon}Y_j = 0$, it shall be noted that $\left\{-Y_j, -\infty<j<\infty\right\}$ also satisfies the conditions of Theorem 3.1. Considering $-Y'_j$ instead of Y'_j in (4.6), then by $(x+y)^- \le x^- + |y|$ and C_r inequality, for all i and n, we can obtain

$$
\begin{aligned}
&\hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y'_j-\hat{\mathbb{E}}Y'_j\right)\right)^-\right)^v\\
=&\hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y'_j-\hat{\varepsilon}Y'_j+\hat{\varepsilon}Y'_j-\hat{\mathbb{E}}Y'_j\right)\right)^-\right)^v
\end{aligned}
$$

$$
\begin{aligned}
&\leq \hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y_j'-\hat{\varepsilon}Y_j'\right)\right)^{-}+\left|\sum_{j=i+1}^{i+n}\left(\hat{\varepsilon}Y_j'-\hat{\mathbb{E}}Y_j'\right)\right|\right)^{v}\\
&\leq c\left(\hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y_j'-\hat{\varepsilon}Y_j'\right)\right)^{-}\right)^{v}+\hat{\mathbb{E}}\left|\sum_{j=i+1}^{i+n}\left(\hat{\varepsilon}Y_j'-\hat{\mathbb{E}}Y_j'\right)\right|^{v}\right)\\
&\leq c\hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y_j'-\hat{\varepsilon}Y_j'\right)\right)^{-}\right)^{v}+c\hat{\mathbb{E}}\left(\sum_{j=i+1}^{i+n}\left|\hat{\varepsilon}Y_j'-\hat{\mathbb{E}}Y_j'\right|\right)^{v}\\
&\leq c\hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y_j'+\hat{\mathbb{E}}\left(-Y_j'\right)\right)\right)^{-}\right)^{v}+c\left(\sum_{j=i+1}^{i+n}\left|\hat{\varepsilon}Y_j'\right|\right)^{v}+c\left(\sum_{j=i+1}^{i+n}\left|\hat{\mathbb{E}}Y_j'\right|\right)^{v}\\
&\leq c\hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(-Y_j'-\hat{\mathbb{E}}\left(-Y_j'\right)\right)\right)^{+}\right)^{v}+c\left(\sum_{j=i+1}^{i+n}\left|-\hat{\mathbb{E}}\left(-Y_j'\right)\right|\right)^{v}+c\left(\sum_{j=i+1}^{i+n}\left|\hat{\mathbb{E}}Y_j'\right|\right)^{v}\\
&\leq c\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left(\left|\left(-Y_j'\right)\right|^{v}\right)+c\left(\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|\left(-Y_j'\right)\right|^{2}\right)^{v/2}+c\left(\sum_{j=i+1}^{i+n}\left|\hat{\mathbb{E}}\left(-Y_j'\right)\right|\right)^{v}+c\left(\sum_{j=i+1}^{i+n}\left|\hat{\mathbb{E}}Y_j'\right|\right)^{v}\\
&\leq c\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y_j'\right|^{v}+c\left(\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y_j'\right|^{2}\right)^{v/2}+c\left(\sum_{j=i+1}^{i+n}\left|\hat{\mathbb{E}}\left(\pm Y_j'\right)\right|\right)^{v}. \qquad (4.7)
\end{aligned}
$$

By (4.6) and (4.7), we have

$$
\begin{aligned}
I_2 \leq& c\sum_{n=1}^{\infty}n^{\alpha-2-v/p}\sum_{i=-\infty}^{\infty}|a_i|\left\{\hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y_j'-\hat{\mathbb{E}}Y_j'\right)\right)^{+}\right)^{v}+\hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y_j'-\hat{\mathbb{E}}Y_j'\right)\right)^{-}\right)^{v}\right\}\\
\leq& c\sum_{n=1}^{\infty}n^{\alpha-2-v/p}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y_j'\right|^{v}+c\sum_{n=1}^{\infty}n^{\alpha-2-v/p}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y_j'\right|^{2}\right)^{v/2}\\
&+c\sum_{n=1}^{\infty}n^{\alpha-2-v/p}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\left|\hat{\mathbb{E}}\left(\pm Y_j'\right)\right|\right)^{v}\\
:=& I_{21}+I_{22}+I_{23}.
\end{aligned}
$$

For I_{21}, by (2.1), (4.2), (4.3) and $v > \alpha p$, we have

$$
\begin{aligned}
I_{21} &\leq c\sum_{n=1}^{\infty}n^{\alpha-2-v/p}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\left[n^{v/p}\hat{\mathbb{E}}\left(1-g\left(\frac{|Y_j|}{n^{1/p}}\right)\right)+\hat{\mathbb{E}}\left|Y_j\right|^{v}g\left(\frac{\mu\left|Y_j\right|}{n^{1/p}}\right)\right]\\
&\leq c\sum_{n=1}^{\infty}n^{\alpha-1-v/p}\left[n^{v/p}\hat{\mathbb{E}}\left(1-g\left(\frac{|Y|}{n^{1/p}}\right)\right)+\hat{\mathbb{E}}|Y|^{v}g\left(\frac{\mu|Y|}{n^{1/p}}\right)\right]\\
&\leq c\sum_{n=1}^{\infty}n^{\alpha-1}\mathbb{V}\left(|Y|>\mu n^{1/p}\right)+c\sum_{n=1}^{\infty}n^{\alpha-1-v/p}\hat{\mathbb{E}}|Y|^{v}g\left(\frac{\mu|Y|}{n^{1/p}}\right)\\
&\ll \sum_{n=1}^{\infty}n^{\alpha-1-v/p}\hat{\mathbb{E}}|Y|^{v}g\left(\frac{\mu|Y|}{n^{1/p}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} n^{\alpha-1-v/p} \hat{\mathbb{E}}|Y|^v g\left(\frac{\mu|Y|}{n^{1/p}}\right) \\
&\le \sum_{k=1}^{\infty} 2^{k(\alpha-v/p)} \hat{\mathbb{E}}|Y|^v g\left(\frac{\mu|Y|}{2^{(k+1)/p}}\right) \\
&\le \sum_{k=1}^{\infty} 2^{k(\alpha-v/p)} \hat{\mathbb{E}}\left(1 + \sum_{j=1}^{k} |Y|^v g_j\left(\frac{\mu|Y|}{2^{(j+1)/p}}\right)\right) \\
&\le \sum_{k=1}^{\infty} 2^{k(\alpha-v/p)} + \sum_{k=1}^{\infty} 2^{k(\alpha-v/p)} \sum_{j=1}^{k} \hat{\mathbb{E}}\left(|Y|^v g_j\left(\frac{\mu|Y|}{2^{(j+1)/p}}\right)\right) \\
&\ll \sum_{j=1}^{\infty} 2^{jv/p} \mathbb{V}\left(|Y| > 2^{j/p}\right) \sum_{k=j}^{\infty} 2^{k(\alpha-v/p)} \\
&\le \sum_{j=1}^{\infty} 2^{j\alpha} \mathbb{V}\left(|Y| > 2^{j/p}\right) \\
&< \infty. \qquad (4.8)
\end{aligned}
$$

For I_{22}, we consider the following two cases.

If $1 < \alpha p < 2$, noting $\hat{\mathbb{E}}\left|Y_j'\right| \le \hat{\mathbb{E}}\left|Y_j\right|$, $\sum_{i=-\infty}^{\infty} |a_i| < \infty$ and $\hat{\mathbb{E}}\left(|Y|^{\alpha p}\right) \le C_{\mathbb{V}}\left(|Y|^{\alpha p}\right) < \infty$, by $v > 2$, $\alpha - 2 - (\alpha - 1)v/2 < -1$, we have

$$
\begin{aligned}
I_{22} &\le c \sum_{n=1}^{\infty} n^{\alpha-2-v/p} \sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} n^{(2-\alpha p)/p} \hat{\mathbb{E}}\left|Y_j'\right|^{\alpha p}\right)^{v/2} \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-2-v/p} \sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} n^{(2-\alpha p)/p} \hat{\mathbb{E}}\left|Y_j\right|^{\alpha p}\right)^{v/2} \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-2-v/p+v/2} \left(n^{(2-\alpha p)/p} \hat{\mathbb{E}}|Y|^{\alpha p}\right)^{v/2} \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-2-(\alpha-1)v/2} \\
&< \infty. \qquad (4.9)
\end{aligned}
$$

If $\alpha p \ge 2$, it follows by $v > \frac{2p(\alpha-1)}{2-p}$, $\alpha - 2 - v/p + v/2 < -1$, we have

$$
\begin{aligned}
I_{22} &\le c \sum_{n=1}^{\infty} n^{\alpha-2-v/p} \sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} \hat{\mathbb{E}}\left|Y_j\right|^2\right)^{v/2} \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-2-v/p} \sum_{i=-\infty}^{\infty} |a_i| \left(n \hat{\mathbb{E}}|Y|^2\right)^{v/2} \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-2-v/p} \left(n \hat{\mathbb{E}}|Y|^2\right)^{v/2}
\end{aligned}
$$

$$\leq c\sum_{n=1}^{\infty} n^{\alpha-2-v/p+v/2}$$
$$< \infty. \tag{4.10}$$

To prove $I_{23} < \infty$, we first estimate $\sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} \left| \hat{\mathbb{E}}\left(\pm Y_j'\right) \right| \right)^v$. In view of $\hat{\mathbb{E}}\left(\pm Y_j\right) = 0$ and $\left|\hat{\mathbb{E}}X - \hat{\mathbb{E}}Y\right| \leq \hat{\mathbb{E}}|X-Y|$. Noting $\left|Y_j''\right| \leq \left|Y_j\right| I\left(\left|Y_j\right| > n^{1/p}\right)$, and by Lemma 2.4, we obtain

$$\begin{aligned}
\sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} \left| \hat{\mathbb{E}}\left(\pm Y_j'\right) \right| \right)^v &= \sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} \left| \hat{\mathbb{E}}\left(\pm Y_j\right) - \hat{\mathbb{E}}\left(\pm Y_j'\right) \right| \right)^v \\
&\leq \sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} \hat{\mathbb{E}} \left| \left(\pm Y_j\right) - \left(\pm Y_j'\right) \right| \right)^v \\
&= \sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} \hat{\mathbb{E}} \left| \left(\pm Y_j''\right) \right| \right)^v \\
&\leq \sum_{i=-\infty}^{\infty} |a_i| \left[\sum_{j=i+1}^{i+n} \hat{\mathbb{E}} \left|Y_j\right| \left(1 - g\left(\frac{\left|Y_j\right|}{n^{1/p}} \right) \right) \right]^v \\
&\leq c\sum_{i=-\infty}^{\infty} |a_i| \left[n\hat{\mathbb{E}}|Y| \left(1 - g\left(\frac{|Y|}{n^{1/p}} \right) \right) \right]^v \\
&\leq c\left[\frac{n\hat{\mathbb{E}}|Y||Y|^{\alpha p-1}}{n^{(\alpha p-1)/p}} \left(1 - g\left(\frac{|Y|}{n^{1/p}} \right) \right) \right]^v \\
&\ll n^{v-\alpha v+v/p}.
\end{aligned} \tag{4.11}$$

Noting $v > \max\left\{\alpha p, 2, \frac{2p(\alpha-1)}{2-p}\right\}$, such that $\alpha - 2 - (\alpha-1)v < -1$, by (4.11), we have

$$\begin{aligned}
I_{23} &= c\sum_{n=1}^{\infty} n^{\alpha-2-v/p} \sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} \left| \hat{\mathbb{E}}\left(\pm Y_j'\right) \right| \right)^v \\
&\leq c\sum_{n=1}^{\infty} n^{\alpha-2-v/p} \left(n^{v-\alpha v+v/p} \right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha-2-(\alpha-1)v} \\
&< \infty.
\end{aligned} \tag{4.12}$$

From (4.8)–(4.10) and (4.12), it follows that $I_2 < \infty$. Therefore (3.1) holds, the proof of Theorem 3.1 is completed.

4.2. Proof of Theorem 3.2

For $\forall\varepsilon > 0$, we have by Theorem 3.1 that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha-2-r/p} C_{\mathbb{V}}\left(|S_n| - \varepsilon n^{1/p}\right)_+^r \\
=&\sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_0^{\infty} \mathbb{V}\left(|S_n| - \varepsilon n^{1/p} > x^{1/r}\right) \mathrm{d}x \\
=&\sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_0^{n^{r/p}} \mathbb{V}\left(|S_n| > \varepsilon n^{1/p} + x^{1/r}\right) \mathrm{d}x + \sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} \mathbb{V}\left(|S_n| > \varepsilon n^{1/p} + x^{1/r}\right) \mathrm{d}x \\
\leq&\sum_{n=1}^{\infty} n^{\alpha-2} \mathbb{V}\left(|S_n| > \varepsilon n^{1/p}\right) + \sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} \mathbb{V}\left(|S_n| > x^{1/r}\right) \mathrm{d}x \\
\leq& c\sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} \mathbb{V}\left(|S_n| > x^{1/r}\right) \mathrm{d}x.
\end{aligned}
$$

Hence, to prove (3.2), it suffices to show that

$$
H := \sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} \mathbb{V}\left(|S_n| > x^{1/r}\right) \mathrm{d}x < \infty. \tag{4.13}
$$

Let $x > n^{r/p}$, denote that

$$
\begin{aligned}
Y_j^{(1)} &:= -x^{1/r} I\left(Y_j < -x^{1/r}\right) + Y_j I\left(\left|Y_j\right| \leq x^{1/r}\right) + x^{1/r} I\left(Y_j > x^{1/r}\right), \\
Y_j^{(2)} &:= Y_j - Y_j^{(1)} = \left(Y_j + x^{1/r}\right) I\left(Y_j < -x^{1/r}\right) + \left(Y_j - x^{1/r}\right) I\left(Y_j > x^{1/r}\right).
\end{aligned} \tag{4.14}
$$

For H, note that $\sum\limits_{k=1}^{n} X_k = \sum\limits_{i=-\infty}^{\infty} a_i \sum\limits_{j=i+1}^{i+n} Y_j$, we have

$$
\begin{aligned}
H \leq& \sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} \mathbb{V}\left(\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j^{(2)}\right| > \frac{x^{1/r}}{2}\right) \mathrm{d}x + \sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} \mathbb{V}\left(\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j^{(1)}\right| > \frac{x^{1/r}}{2}\right) \mathrm{d}x \\
\leq& \sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} \mathbb{V}\left(\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j^{(2)}\right| > \frac{x^{1/r}}{2}\right) \mathrm{d}x \\
&+ \sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} \mathbb{V}\left(x^{-1/r}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \left(Y_j^{(1)} - \hat{\mathbb{E}} Y_j^{(1)}\right)\right| > \frac{1}{2} - x^{-1/r}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \hat{\mathbb{E}} Y_j^{(1)}\right|\right) \mathrm{d}x \\
:=& H_1 + H_2.
\end{aligned}
$$

For H_1, noting that $g(y)$ is decreasing in $y \geq 0$, by Markov inequality, (4.4), (4.2), (2.3) and (2.4), we get

$$
H_1 \leq c\sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} x^{-1/r} \hat{\mathbb{E}}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j^{(2)}\right| \mathrm{d}x
$$

$$
\begin{aligned}
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty} x^{-1/r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y_j^{(2)}\right|\mathrm{d}x\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty} x^{-1/r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}|Y_j|\left(1-g\left(\frac{|Y_j|}{x^{1/r}}\right)\right)\mathrm{d}x\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-1-r/p}\int_{n^{r/p}}^{\infty} x^{-1/r}\hat{\mathbb{E}}|Y|\left(1-g\left(\frac{|Y|}{x^{1/r}}\right)\right)\mathrm{d}x\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-1-r/p}\sum_{m=n}^{\infty}\int_{m^{r/p}}^{(m+1)^{r/p}} x^{-1/r}\hat{\mathbb{E}}|Y|\left(1-g\left(\frac{|Y|}{x^{1/r}}\right)\right)\mathrm{d}x\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-1-r/p}\sum_{m=n}^{\infty} m^{r/p-1/p-1}\hat{\mathbb{E}}|Y|\left(1-g\left(\frac{|Y|}{m^{1/p}}\right)\right)\\
&= c\sum_{m=1}^{\infty} m^{r/p-1/p-1}\hat{\mathbb{E}}|Y|\left(1-g\left(\frac{|Y|}{m^{1/p}}\right)\right)\sum_{n=1}^{m} n^{\alpha-1-r/p}\\
&\le\begin{cases}
c\sum\limits_{m=1}^{\infty} m^{\alpha-1-1/p}\hat{\mathbb{E}}|Y|\left(1-g\left(\frac{|Y|}{m^{1/p}}\right)\right), & \text{if } r<\alpha p,\\
c\sum\limits_{m=1}^{\infty} m^{\alpha-1-1/p}\hat{\mathbb{E}}|Y|\left(1-g\left(\frac{|Y|}{m^{1/p}}\right)\right)\log(1+m), & \text{if } r=\alpha p,\\
c\sum\limits_{m=1}^{\infty} m^{r/p-1-1/p}\hat{\mathbb{E}}|Y|\left(1-g\left(\frac{|Y|}{m^{1/p}}\right)\right), & \text{if } r>\alpha p,
\end{cases}\\
&=\begin{cases}
c\sum\limits_{k=0}^{\infty}\sum\limits_{m=2^k}^{2^{k+1}-1} m^{\alpha-1-1/p}\hat{\mathbb{E}}|Y|\left(1-g\left(\frac{|Y|}{m^{1/p}}\right)\right), & \text{if } r<\alpha p,\\
c\sum\limits_{k=0}^{\infty}\sum\limits_{m=2^k}^{2^{k+1}-1} m^{\alpha-1-1/p}\hat{\mathbb{E}}|Y|\left(1-g\left(\frac{|Y|}{m^{1/p}}\right)\right)\log(1+m), & \text{if } r=\alpha p,\\
c\sum\limits_{k=0}^{\infty}\sum\limits_{m=2^k}^{2^{k+1}-1} m^{r/p-1-1/p}\hat{\mathbb{E}}|Y|\left(1-g\left(\frac{|Y|}{m^{1/p}}\right)\right), & \text{if } r>\alpha p,
\end{cases}\\
&\le\begin{cases}
c\sum\limits_{k=1}^{\infty} 2^{k(\alpha-1/p)}\hat{\mathbb{E}}\left(\sum\limits_{j=k}^{\infty}|Y|\,g_j\left(\frac{|Y|}{2^{j/p}}\right)\right), & \text{if } r<\alpha p,\\
c\sum\limits_{k=1}^{\infty} 2^{k(\alpha-1/p)}\hat{\mathbb{E}}\left(\sum\limits_{j=k}^{\infty}|Y|\,g_j\left(\frac{|Y|}{2^{j/p}}\right)\right)\log\left(1+2^k\right), & \text{if } r=\alpha p,\\
c\sum\limits_{k=1}^{\infty} 2^{k(r-1)/p}\hat{\mathbb{E}}\left(\sum\limits_{j=k}^{\infty}|Y|\,g_j\left(\frac{|Y|}{2^{j/p}}\right)\right), & \text{if } r>\alpha p,
\end{cases}\\
&\le\begin{cases}
c\sum\limits_{j=1}^{\infty}\hat{\mathbb{E}}|Y|g_j\left(\frac{|Y|}{2^{j/p}}\right)\sum\limits_{k=1}^{j} 2^{k(\alpha-1/p)}, & \text{if } r<\alpha p,\\
c\sum\limits_{j=1}^{\infty}\hat{\mathbb{E}}|Y|g_j\left(\frac{|Y|}{2^{j/p}}\right)\sum\limits_{k=1}^{j} 2^{k(\alpha-1/p)}\log\left(1+2^k\right), & \text{if } r=\alpha p,\\
c\sum\limits_{j=1}^{\infty}\hat{\mathbb{E}}|Y|g_j\left(\frac{|Y|}{2^{j/p}}\right)\sum\limits_{k=1}^{j} 2^{k(r-1)/p}, & \text{if } r>\alpha p,
\end{cases}
\end{aligned}
$$

$$\leq \begin{cases} c\sum\limits_{j=1}^{\infty} 2^{j\alpha}\mathbb{V}\left(|Y|>\mu 2^{-1/p}2^{j/p}\right), & \text{if } r<\alpha p,\\ c\sum\limits_{j=1}^{\infty} 2^{j\alpha}\log\left(1+2^{j}\right)\mathbb{V}\left(|Y|>\mu 2^{-1/p}2^{j/p}\right), & \text{if } r=\alpha p,\\ c\sum\limits_{j=1}^{\infty} 2^{rj/p}\mathbb{V}\left(|Y|>\mu 2^{-1/p}2^{j/p}\right), & \text{if } r>\alpha p,\end{cases}$$

$$<\infty. \tag{4.15}$$

In the following, we prove that $H_2 < \infty$. First,we show that

$$\sup_{x\geq n^{r/p}} x^{-1/r}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \hat{\mathbb{E}}Y_j^{(1)}\right| \to 0 \text{ as } n\to\infty.$$

By (4.14) and (4.1), we can imply

$$\hat{\mathbb{E}}\left|Y_j^{(1)}\right| \ll \hat{\mathbb{E}}\left[|Y|g\left(\frac{\mu|Y|}{x^{1/r}}\right)\right] + x^{1/r}\hat{\mathbb{E}}\left[1-g\left(\frac{|Y|}{x^{1/r}}\right)\right] \leq \hat{\mathbb{E}}\left[|Y|g\left(\frac{\mu|Y|}{x^{1/r}}\right)\right] + x^{1/r}\mathbb{V}(|Y|>\mu x^{1/r}), \tag{4.16}$$

$$\hat{\mathbb{E}}\left|Y_j^{(2)}\right| \ll \hat{\mathbb{E}}|Y|\left[1-g\left(\frac{|Y|}{x^{1/r}}\right)\right]. \tag{4.17}$$

If $r \leq \alpha p$, by (2.1) and $\alpha > 1$, we can get

$$\sum_{n=1}^{\infty}\mathbb{V}\left(|Y|>\mu n^{1/p}\right) \leq \sum_{n=1}^{\infty} n^{\alpha-1}\mathbb{V}\left(|Y|>\mu n^{1/p}\right) < \infty,$$

and $\mathbb{V}\left(|Y|>\mu n^{1/p}\right)\downarrow$, so we get $n\mathbb{V}\left(|Y|>\mu n^{1/p}\right) \to 0$ as $n\to\infty$.

For $x \geq n^{r/p}$, note that $\sum\limits_{i=-\infty}^{\infty}|a_i| < \infty$, By (4.16), countably sub-additive of $\hat{\mathbb{E}}$ and Lemma 2.4, we have

$$\begin{aligned}\sup_{x\geq n^{r/p}} x^{-1/r}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \hat{\mathbb{E}}Y_j^{(1)}\right| &\leq \sup_{x\geq n^{r/p}} x^{-1/r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\left|\hat{\mathbb{E}}Y_j^{(1)}\right|\\ &\leq \sup_{x\geq n^{r/p}} x^{-1/r}n\left(\hat{\mathbb{E}}|Y|g\left(\frac{\mu|Y|}{x^{1/r}}\right)+x^{1/r}\mathbb{V}\left(|Y|>\mu x^{1/r}\right)\right)\\ &\leq n^{1-1/p}\hat{\mathbb{E}}|Y|g\left(\frac{\mu|Y|}{n^{1/p}}\right)+n\mathbb{V}\left(|Y|>\mu n^{1/p}\right)\\ &\leq cn^{1-\alpha}\hat{\mathbb{E}}|Y|^{\alpha p}\to 0 \quad \text{ as } n\to\infty.\end{aligned}$$

If $r > \alpha p$, since $\hat{\mathbb{E}}Y_j = 0$, by (4.17) and Lemma 2.4, we have

$$\begin{aligned}\sup_{x\geq n^{r/p}} x^{-1/r}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \hat{\mathbb{E}}Y_j^{(1)}\right| &\leq \sup_{x\geq n^{r/p}} x^{-1/r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\left|\hat{\mathbb{E}}Y_j-\hat{\mathbb{E}}Y_j^{(1)}\right|\\ &\leq n^{-1/p}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y_j^{(2)}\right|\end{aligned}$$

$$\leq n^{1-1/p}\hat{\mathbb{E}}|Y|\left(1-g\left(\frac{|Y|}{n^{1/p}}\right)\right)$$
$$\leq n^{1-1/p}\frac{\mathbb{E}|Y||Y|^{r-1}}{n^{(r-1)/p}}\left(1-g\left(\frac{|Y|}{n^{1/p}}\right)\right)$$
$$\leq cn^{1-r/p}\hat{\mathbb{E}}|Y|^{r}\to 0 \quad \text{as } n\to\infty.$$

It follows that for all n large enough, we get

$$\sup_{x\geq n^{r/p}} x^{-1/r}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}Y_j^{(1)}\right|<\frac{1}{4}, \tag{4.18}$$

which implies that

$$H_2\leq c\sum_{n=1}^{\infty}n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty}\mathbb{V}\left(\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}\left(Y_j^{(1)}-\hat{\mathbb{E}}Y_j^{(1)}\right)\right|>\frac{x^{1/r}}{4}\right)\mathrm{d}x.$$

By Markov inequality, Hölder inequality, C_r inequality and (2.5), similar to the proof of (4.6) and (4.7), taking $v>\max\left\{\alpha p, r, 2, \frac{2p(\alpha-1)}{2-p}\right\}$, we have

$$\begin{aligned}
H_2\leq&c\sum_{n=1}^{\infty}n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty}x^{-v/r}\hat{\mathbb{E}}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}\left(Y_j^{(1)}-\hat{\mathbb{E}}Y_j^{(1)}\right)\right|^{v}\mathrm{d}x\\
\leq&c\sum_{n=1}^{\infty}n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty}x^{-v/r}\hat{\mathbb{E}}\left\{\sum_{i=-\infty}^{\infty}|a_i|\left|\sum_{j=i+1}^{i+n}\left(Y_j^{(1)}-\hat{\mathbb{E}}Y_j^{(1)}\right)\right|\right\}^{v}\mathrm{d}x\\
=&c\sum_{n=1}^{\infty}n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty}x^{-v/r}\hat{\mathbb{E}}\left\{\sum_{i=-\infty}^{\infty}|a_i|^{1-1/v}\left(|a_i|^{1/v}\left|\sum_{j=i+1}^{i+n}\left(Y_j^{(1)}-\hat{\mathbb{E}}Y_j^{(1)}\right)\right|\right)\right\}^{v}\mathrm{d}x\\
\leq&c\sum_{n=1}^{\infty}n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty}x^{-v/r}\hat{\mathbb{E}}\left\{\left(\sum_{i=-\infty}^{\infty}\left(|a_i|^{\frac{v-1}{v}}\right)^{\frac{v}{v-1}}\right)^{\frac{v-1}{v}}\left(\sum_{i=-\infty}^{\infty}\left(|a_i|^{\frac{1}{v}}\right)^{v}\left|\sum_{j=i+1}^{i+n}\left(Y_j^{(1)}-\hat{\mathbb{E}}Y_j^{(1)}\right)\right|^{v}\right)^{\frac{1}{v}}\right\}^{v}\mathrm{d}x\\
\leq&c\sum_{n=1}^{\infty}n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty}x^{-v/r}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{v-1}\left(\sum_{i=-\infty}^{\infty}|a_i|\hat{\mathbb{E}}\left|\sum_{j=i+1}^{i+n}\left(Y_j^{(1)}-\hat{\mathbb{E}}Y_j^{(1)}\right)\right|^{v}\right)\mathrm{d}x\\
\leq&c\sum_{n=1}^{\infty}n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty}x^{-v/r}\sum_{i=-\infty}^{\infty}|a_i|\hat{\mathbb{E}}\left(\left|\left(\sum_{j=i+1}^{i+n}\left(Y_j^{(1)}-\hat{\mathbb{E}}Y_j^{(1)}\right)\right)^{+}+\left(\sum_{j=i+1}^{i+n}\left(Y_j^{(1)}-\hat{\mathbb{E}}Y_j^{(1)}\right)\right)^{-}\right|^{v}\right)\mathrm{d}x\\
\leq&c\sum_{n=1}^{\infty}n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty}x^{-v/r}\sum_{i=-\infty}^{\infty}|a_i|\left\{\hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y_j^{(1)}-\hat{\mathbb{E}}Y_j^{(1)}\right)\right)^{+}\right)^{v}+\hat{\mathbb{E}}\left(\left(\sum_{j=i+1}^{i+n}\left(Y_j^{(1)}-\hat{\mathbb{E}}Y_j^{(1)}\right)\right)^{-}\right)^{v}\right\}\mathrm{d}x.\\
\leq&c\sum_{n=1}^{\infty}n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty}x^{-v/r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y_j^{(1)}\right|^{v}\mathrm{d}x+c\sum_{n=1}^{\infty}n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty}x^{-v/r}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}\left|Y_j^{(1)}\right|^{2}\right)^{v/2}\mathrm{d}x\\
&+c\sum_{n=1}^{\infty}n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty}x^{-v/r}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\left|\hat{\mathbb{E}}\left(\pm Y_j^{(1)}\right)\right|\right)^{v}\mathrm{d}x
\end{aligned}$$

$$:= H_{21} + H_{22} + H_{23}. \tag{4.19}$$

For H_{21}, from $\sum_{i=-\infty}^{\infty} |a_i| < \infty$ and (4.16), we get

$$\begin{aligned}
H_{21} &\le c \sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} x^{-v/r} \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} \left[x^{v/r} \hat{\mathbb{E}}\left(1 - g\left(\frac{|Y_j|}{x^{1/r}}\right)\right) + \hat{\mathbb{E}}\left|Y_j\right|^v g\left(\frac{\mu\left|Y_j\right|}{x^{1/r}}\right)\right] \mathrm{d}x \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-1-r/p} \int_{n^{r/p}}^{\infty} \sum_{i=-\infty}^{\infty} |a_i| \hat{\mathbb{E}}\left(1 - g\left(\frac{|Y|}{x^{1/r}}\right)\right) \mathrm{d}x + c \sum_{n=1}^{\infty} n^{\alpha-1-r/p} \int_{n^{r/p}}^{\infty} \sum_{i=-\infty}^{\infty} |a_i| \, x^{-v/r} \hat{\mathbb{E}}|Y|^v g\left(\frac{\mu|Y|}{x^{1/r}}\right) \mathrm{d}x \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-1-r/p} \int_{n^{r/p}}^{\infty} \hat{\mathbb{E}}\left(1 - g\left(\frac{|Y|}{x^{1/r}}\right)\right) \mathrm{d}x + c \sum_{n=1}^{\infty} n^{\alpha-1-r/p} \int_{n^{r/p}}^{\infty} x^{-v/r} \hat{\mathbb{E}}|Y|^v g\left(\frac{\mu|Y|}{x^{1/r}}\right) \mathrm{d}x \\
&:= H_{211} + H_{212}.
\end{aligned}$$

Noting that $g(y)$ is a decreasing function for $y \ge 0$, by (4.1), (2.1) and (2.2), we have

$$\begin{aligned}
H_{211} &\le c \sum_{n=1}^{\infty} n^{\alpha-1-r/p} \sum_{m=n}^{\infty} \int_{m^{r/p}}^{(m+1)^{r/p}} \hat{\mathbb{E}}\left(1 - g\left(\frac{|Y|}{x^{1/r}}\right)\right) \mathrm{d}x \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-1-r/p} \sum_{m=n}^{\infty} \hat{\mathbb{E}}\left(1 - g\left(\frac{|Y|}{m^{1/p}}\right)\right) \\
&\le c \sum_{m=1}^{\infty} \mathbb{V}\left(|Y| > \mu m^{1/p}\right) \sum_{n=1}^{m} n^{\alpha-1-r/p} \\
&\le \begin{cases} c \sum_{m=1}^{\infty} m^{\alpha-1-r/p} \mathbb{V}\left(|Y| > \mu m^{1/p}\right), & \text{if } r < \alpha p, \\ c \sum_{m=1}^{\infty} m^{\alpha-1-r/p} \log(1+m) \mathbb{V}\left(|Y| > \mu m^{1/p}\right), & \text{if } r = \alpha p, \\ c \sum_{m=1}^{\infty} m^{\alpha-1-r/p} \mathbb{V}\left(|Y| > \mu m^{1/p}\right), & \text{if } r > \alpha p, \end{cases} \\
&\le \begin{cases} c \sum_{m=1}^{\infty} m^{\alpha-1} \mathbb{V}\left(|Y| > \mu m^{1/p}\right), & \text{if } r < \alpha p, \\ c \sum_{m=1}^{\infty} m^{\alpha-1} \log(1+m) \mathbb{V}\left(|Y| > \mu m^{1/p}\right), & \text{if } r = \alpha p, \\ c \sum_{m=1}^{\infty} m^{-1+r/p} \mathbb{V}\left(|Y| > \mu m^{1/p}\right), & \text{if } r > \alpha p, \end{cases} \\
&< \infty.
\end{aligned} \tag{4.20}$$

From $v > \alpha p \vee r$, by Lemma 2.4, we have

$$\begin{aligned}
H_{212} &\le c \sum_{n=1}^{\infty} n^{\alpha-1-r/p} \sum_{m=n}^{\infty} \int_{m^{r/p}}^{(m+1)^{r/p}} x^{-v/r} \hat{\mathbb{E}}|Y|^v g\left(\frac{\mu|Y|}{x^{1/r}}\right) \mathrm{d}x \\
&\le c \sum_{n=1}^{\infty} n^{\alpha-1-r/p} \sum_{m=n}^{\infty} m^{r/p-v/p-1} \hat{\mathbb{E}}|Y|^v g\left(\frac{\mu|Y|}{m^{1/p}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\le c\sum_{m=1}^{\infty} m^{r/p-v/p-1}\hat{\mathbb{E}}|Y|^v g\left(\frac{\mu|Y|}{m^{1/p}}\right)\sum_{n=1}^{m} n^{\alpha-1-r/p}\\
&\le \begin{cases} c\sum_{m=1}^{\infty} m^{\alpha-v/p-1}\hat{\mathbb{E}}|Y|^v g\left(\frac{\mu|Y|}{m^{1/p}}\right), & \text{if } r<\alpha p,\\ c\sum_{m=1}^{\infty} m^{\alpha-v/p-1}\log m\hat{\mathbb{E}}|Y|^v g\left(\frac{\mu|Y|}{m^{1/p}}\right), & \text{if } r=\alpha p,\\ c\sum_{m=1}^{\infty} m^{r/p-v/p-1}\hat{\mathbb{E}}|Y|^v g\left(\frac{\mu|Y|}{m^{1/p}}\right), & \text{if } r>\alpha p,\end{cases}\\
&\le \begin{cases} \int_1^{\infty} x^{\alpha-v/p-1}\hat{\mathbb{E}}\left[|Y|^v I\left(|Y|\le cx^{1/p}\right)\right]\mathrm{d}x, & \text{if } r<\alpha p,\\ \int_1^{\infty} x^{\alpha-v/p-1}\log x\hat{\mathbb{E}}\left[|Y|^v I\left(|Y|\le cx^{1/p}\right)\right]\mathrm{d}x, & \text{if } r=\alpha p,\\ \int_1^{\infty} x^{r/p-v/p-1}\hat{\mathbb{E}}\left[|Y|^v I\left(|Y|\le cx^{1/p}\right)\right]\mathrm{d}x, & \text{if } r>\alpha p,\end{cases}\\
&\le \begin{cases} cC_{\mathbb{V}}\left(|Y|^{\alpha p}\right)<\infty, & \text{if } r<\alpha p,\\ cC_{\mathbb{V}}\left(|Y|^{\alpha p}\log(1+|Y|)\right)<\infty, & \text{if } r=\alpha p,\\ cC_{\mathbb{V}}\left(|Y|^{r}\right)<\infty, & \text{if } r>\alpha p.\end{cases}
\end{aligned}
\tag{4.21}
$$

Combing (4.20) and (4.21), we conclude $H_{21}<\infty$. Next, we prove $H_{22}<\infty$.

If $1<\alpha p<2$, noting $\hat{\mathbb{E}}\left|Y_j^{(1)}\right|\le\hat{\mathbb{E}}\left|Y_j\right|$ and $\hat{\mathbb{E}}\left(|Y|^{\alpha p}\right)\le C_{\mathbb{V}}\left(|Y|^{\alpha p}\right)<\infty$, by $v>2$, $\alpha-2-\frac{v(\alpha-1)}{2}<-1$, we can obtain

$$
\begin{aligned}
H_{22}&\le c\sum_{n=1}^{\infty} n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty} x^{-v/r}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n} x^{\frac{2-\alpha p}{r}}\hat{\mathbb{E}}\left|Y_j^{(1)}\right|^{\alpha p}\right)^{v/2}\mathrm{d}x\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty} x^{-v/r}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n} x^{\frac{2-\alpha p}{r}}\hat{\mathbb{E}}\left|Y_j\right|^{\alpha p}\right)^{v/2}\mathrm{d}x\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty} x^{-v/r}\left(nx^{\frac{2-\alpha p}{r}}\hat{\mathbb{E}}|Y|^{\alpha p}\right)^{v/2}\mathrm{d}x\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-r/p+v/2}\int_{n^{r/p}}^{\infty} x^{\frac{-\alpha v p}{2r}}\mathrm{d}x\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-\frac{v(\alpha-1)}{2}}\\
&<\infty.
\end{aligned}
$$

If $\alpha p\ge 2$, by $v>\frac{2p(\alpha-1)}{2-p}$, $\alpha-2-v/p+v/2<-1$ and Lemma 2.5, we have

$$
\begin{aligned}
H_{22}&\le c\sum_{n=1}^{\infty} n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty} x^{-v/r}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\hat{\mathbb{E}}|Y_j|^2\right)^{v/2}\mathrm{d}x\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty} x^{-v/r}\sum_{i=-\infty}^{\infty}|a_i|\left(n\hat{\mathbb{E}}|Y|^2\right)^{v/2}\mathrm{d}x\\
&\le c\sum_{n=1}^{\infty} n^{\alpha-2-r/p}\int_{n^{r/p}}^{\infty} x^{-v/r}\left(n\hat{\mathbb{E}}|Y|^2\right)^{v/2}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leq c \sum_{n=1}^{\infty} n^{\alpha-2-r/p+v/2} \int_{n^{r/p}}^{\infty} x^{-v/r} \, \mathrm{d}x \\
&\leq c \sum_{n=1}^{\infty} n^{\alpha-2-v/p+v/2} \\
&< \infty.
\end{aligned}
$$

Finally, we show $H_{23} < \infty$. For $x > n^{r/p}$, noting $\sum\limits_{i=-\infty}^{\infty} |a_i| < \infty$, similar to the proof of (4.11), we get

$$
\begin{aligned}
\sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} \left| \hat{\mathbb{E}}\left(\pm Y_j^{(1)}\right) \right| \right)^v &= \sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} \left| \hat{\mathbb{E}}\left(\pm Y_j\right) - \hat{\mathbb{E}}\left(\pm Y_j^{(1)}\right) \right| \right)^v \\
&\leq \sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} \hat{\mathbb{E}} \left| \left(\pm Y_j\right) - \left(\pm Y_j^{(1)}\right) \right| \right)^v \\
&= \sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} \hat{\mathbb{E}} \left| \left(\pm Y_j^{(2)}\right) \right| \right)^v \\
&\leq \sum_{i=-\infty}^{\infty} |a_i| \left[\sum_{j=i+1}^{i+n} \hat{\mathbb{E}} \left| Y_j \right| \left(1 - g\left(\frac{|Y_j|}{x^{1/r}} \right) \right) \right]^v \\
&\leq c \left[n \hat{\mathbb{E}} |Y| \left(1 - g\left(\frac{|Y|}{n^{1/p}} \right) \right) \right]^v \\
&\leq \begin{cases} c \left[\frac{n\hat{\mathbb{E}}|Y||Y|^{\alpha p-1}}{n^{(\alpha p-1)/p}} \left(1 - g\left(\frac{|Y|}{n^{1/p}}\right)\right) \right]^v, & \text{if } r < \alpha p, \\ c \left[\frac{n\hat{\mathbb{E}}|Y||Y|^{\alpha p-1}}{n^{(\alpha p-1)/p}} \frac{\log(1+|Y|^p)}{\log(1+n)} \left(1 - g\left(\frac{|Y|}{n^{1/p}}\right)\right) \right]^v, & \text{if } r = \alpha p, \\ c \left[\frac{n\hat{\mathbb{E}}|Y||Y|^{r-1}}{n^{(r-1)/p}} \left(1 - g\left(\frac{|Y|}{n^{1/p}}\right)\right) \right]^v, & \text{if } r > \alpha p, \end{cases} \\
&\ll \begin{cases} n^{v-\alpha v+v/p}, & \text{if } r < \alpha p, \\ n^{v-\alpha v+v/p} \left(\log(1+n)\right)^{-v}, & \text{if } r = \alpha p, \\ n^{v-rv/p+v/p}, & \text{if } r > \alpha p. \end{cases}
\end{aligned} \tag{4.22}
$$

From (4.19) and (4.22), by $v > \max\left\{\alpha p, r, 2, \frac{2p(\alpha-1)}{2-p}\right\}$, such that $\alpha - 2 + v - \alpha v < -1$ and $\alpha - 2 + v - rv/p < -1$, we have that

$$
\begin{aligned}
H_{23} &\leq c \sum_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} x^{-v/r} \sum_{i=-\infty}^{\infty} |a_i| \left(\sum_{j=i+1}^{i+n} \left| \hat{\mathbb{E}}\left(\pm Y_j^{(1)}\right) \right| \right)^v \mathrm{d}x \\
&\leq \begin{cases} c \sum\limits_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{r^{r/p}}^{\infty} x^{-v/r} \left(n^{v-\alpha v+v/p}\right) \mathrm{d}x, & \text{if } r < \alpha p, \\ c \sum\limits_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} x^{-v/r} \left[n^{v-\alpha v+v/p} (\log(1+n))^{-v}\right] \mathrm{d}x, & \text{if } r = \alpha p, \\ c \sum\limits_{n=1}^{\infty} n^{\alpha-2-r/p} \int_{n^{r/p}}^{\infty} x^{-v/r} \left(n^{v-rv/p+v/p}\right) \mathrm{d}x, & \text{if } r > \alpha p, \end{cases}
\end{aligned}
$$

$$
\leq \begin{cases} c\sum\limits_{n=1}^{\infty} n^{\alpha-2-r/p+v+v/p-\alpha v}\int_{n^{r/p}}^{\infty} x^{-v/r}\,\mathrm{d}x, & \text{if } r<\alpha p,\\ c\sum\limits_{n=1}^{\infty} n^{\alpha-2-r/p+v+v/p-\alpha v}(\log(1+n))^{-v}\int_{n^{r/p}}^{\infty} x^{-v/r}\,\mathrm{d}x, & \text{if } r=\alpha p,\\ c\sum\limits_{n=1}^{\infty} n^{\alpha-2-r/p+v+v/p-rv/p}\int_{n^{r/p}}^{\infty} x^{-v/r}\,\mathrm{d}x, & \text{if } r>\alpha p, \end{cases}
$$

$$
\leq \begin{cases} c\sum\limits_{n=1}^{\infty} n^{\alpha-2+v-\alpha v}, & \text{if } r<\alpha p,\\ c\sum\limits_{n=1}^{\infty} n^{\alpha-2+v-\alpha v}(\log(1+n))^{-v}, & \text{if } r=\alpha p,\\ c\sum\limits_{n=1}^{\infty} n^{\alpha-2+v-rv/p}, & \text{if } r>\alpha p, \end{cases}
$$

$$
<\infty.
$$

It follows that $H_2 < \infty$, together with (4.15), we conclude that (4.13) holds. Therefore the proof of Theorem 3.2 is completed.

5. Conclusions

We have obtained the new results of complete convergence and complete integral convergence for moving average process under sub-linear expectations. Theorems of this paper are the extension of convergence properties of moving average process under the classical linear expectation space.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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