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Research article

Instability of standing waves for a quasi-linear Schrödinger equation in the critical case

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Abstract: We consider the following quasi-linear Schrödinger equation.

$$i\frac{\partial\psi}{\partial t} + \triangle\psi + \psi\triangle|\psi|^2 + |\psi|^{p-1}\psi = 0, x \in \mathbb{R}^D, D \geq 1, \tag{Q}$$

where $\psi : \mathbb{R}^+ \times \mathbb{R}^D \to \mathbb{C}$ is the wave function, $p = 3 + \frac{4}{D}$. It is known that the set of standing waves is stable for $1 < p < 3 + \frac{4}{D}$ and it is strongly unstable for $3 + \frac{4}{D} < p < \frac{3D+2}{D-2}$. In this paper, we prove that the standing waves are strongly unstable for $p = 3 + \frac{4}{D}$. Moreover, a property on the set of the ground states of (Q) is investigated.

Keywords: quasi-linear Schrödinger equation; ground states; instability
Mathematics Subject Classification: 35F25, 35Q55

1. Introduction

This note is concerned with the quasi-linear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} + \triangle\psi + \psi\triangle|\psi|^2 + |\psi|^{p-1}\psi = 0, x \in \mathbb{R}^D, D \geq 1, \tag{1.1}$$

where $\psi : \mathbb{R}^+ \times \mathbb{R}^D \to \mathbb{C}$ is the wave function, $\triangle$ is the Laplacian operator on $\mathbb{R}^D$, $1 < p < 22^* - 1$ ($2^* = \infty$, if $D = 1, 2$; $2^* = \frac{2D}{D-2}$ if $D \geq 3$). Quasi-linear equations of the form (1.1) come from a superfluid film equation in plasma physics, which was introduced in [3, 9].

Due to the focusing nature of the power nonlinearity $|\psi|^{p-1}\psi$ in (1.1), there exists a standing wave solution given by

$$\psi(t, x) = e^{i\omega t}u(x),$$

where u is a nontrivial solution of the semi-linear elliptic problem

$$\begin{cases} -\triangle u - u\triangle |u|^2 + \omega u = |u|^{p-1}u \text{ in } \mathbb{R}^D, \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{1.2}$$

The stability and instability of standing waves of (1.1) has been studied in [7]. Similar to [7], we introduce the following notations.

Notation.

- $\|\cdot\|_q$ and $\|\cdot\|_{H^k}$ denote the norms in $L^q := L^q(\mathbb{R}^D)$ and Sobolev space $H^k(\mathbb{R}^D) = W^{1,k}(\mathbb{R}^D)$, respectively.
- $E(a)$ denotes the integer part of a.
- The natural working space for (1.2) is $X_{\mathbb{C}}$, defined by

$$X_{\mathbb{C}} := \{u \in H^1(\mathbb{R}^D) : \int |u|^2 |\nabla |u||^2 < \infty\}.$$

The local and global well-posedness of the Cauchy problem to (1.1) have been studied by Poppenberg in [11] for smooth initial data, precisely belonging to the space H^∞. On the local well-posedness for the Cauchy problem to (1.1) in H^k, we make the following assumptions.

Assumption (A1). Let $D \geq 1$, $s = 2E(\frac{D}{2}) + 2$. Then, the Cauchy problem for (1.1) is locally well-posed in $H^s(\mathbb{R}^D)$, that is, for any $\psi_0 \in H^{s+2}(\mathbb{R}^D)$ there exists a positive T and a unique solution $\psi(t)$ of (1.1) with $\psi(0,x) = \psi_0(x)$ satisfying

$$\psi(t) \in L^\infty(0,T; H^{s+2}(\mathbb{R}^D)) \cap C([0,T]; H^s(\mathbb{R}^D)).$$

Moreover, it has the conservation laws of the mass

$$\|\psi(t,\cdot)\|_2 = \|\psi_0\|_2 \tag{1.3}$$

and the energy

$$\mathcal{E}(\psi(t)) = \mathcal{E}(\psi(0)) \tag{1.4}$$

for every $t \in [0,T)$, where

$$\mathcal{E}(\psi) := \frac{1}{2}\int |\nabla \psi|^2 + \frac{1}{4}\int |\nabla |\psi|^2|^2 - \frac{1}{p+1}\int |\psi|^{p+1}. \tag{1.5}$$

Remark 1.1. In view of Theorem 1.1 in [7], the assumption (A1) is verified provided that $p \in (1, 22^* - 1)$ is an odd integer or $p \in (4E(\frac{D}{2}) + 9, 22^* - 1)$. We also refer the readers to [5] for some other results on the local well-posedness for the Cauchy problem to (1.1).

It is known that a solution $u \in X_{\mathbb{C}}$ of (1.2) is essentially a critical point of the variational functional S_ω: $X_{\mathbb{C}} \to \mathbb{R}$, defined by

$$S_\omega(v) = \frac{1}{2}\int |\nabla v|^2 + \frac{1}{4}\int |\nabla |v|^2|^2 + \frac{\omega}{2}\int |v|^2 - \frac{1}{p+1}\int |v|^{p+1}. \tag{1.6}$$

The ground states of (1.2) is defined as follows.

Definition 1.1. Let $\mathcal{X}_\omega$ be the set of the solutions of (1.2); namely,

$$\mathcal{X}_\omega = \{u \in X_{\mathbb{C}} : S'_\omega(u) = 0, u \neq 0\},$$

and let $\mathcal{G}_\omega$ be the set of the ground states of (1.2); that is

$$\mathcal{G}_\omega = \{v \in \mathcal{X}_\omega : S_\omega(v) \leq S_\omega(u), \forall u \in \mathcal{X}_\omega\}.$$

Remark 1.2. The existence of ground states for (1.2) was proved in [6, 7, 10, 13]. It is shown that $\mathcal{G}_\omega$ is not empty for any $\omega > 0$ and $1 < p < 22^* - 1$. When ω is large enough, the uniqueness of the ground state is studied in [1].

Under the assumption (A1), it is proved in [7] that the standing wave $e^{i\omega t}Q_\omega(x)$ with $Q_\omega \in \mathcal{G}_\omega$ is strongly unstable by blowup for $3 + \frac{4}{D} < p < 22^* - 1$ and stable in some sense for $1 < p < 3 + \frac{4}{D}$ (See Theorem 3.1 for detail). Hence, the power $p = 3 + \frac{4}{D}$ is the critical exponent of the nonlinearity for the stability and instability of standing waves. As we can see, for the critical case $p = 3 + 4/D$, it is still unknown whether or not the standing wave of Eq (1.1) is stable.

The main objective of this note is to prove the instability of the standing wave for Eq (1.1) with $p = 3 + 4/D$. Moreover, our argument can give a simpler proof of the instability result for $p > 3 + 4/D$, which has been established in [7]. More precisely, we establish and prove the following theorem.

Theorem 1.2. *Assume (A1). Let $\omega > 0$, $s = 2E(\frac{D}{2}) + 2$, $3 + \frac{4}{D} \leq p < 22^* - 1$ and $Q_\omega \in \mathcal{G}_\omega$. Then the standing wave $e^{i\omega t}Q_\omega(x)$ of equation (1.1) is strongly unstable by blowup. More precisely, for all $\varepsilon > 0$, there exists $\psi_0 \in H^{s+2}(\mathbb{R}^D)$ such that $\|\psi_0 - Q_\omega\|_{H^1} < \varepsilon$ and the solution $\psi(t)$ of (1.1) with $\psi(0) = \psi_0$ satisfies*

$$\lim_{t \to T_{\psi_0}} \|\psi(t)\|_{H^1} = \infty \text{ with } T_{\psi_0} < \infty.$$

Remark 1.3. The assumptions of Theorem 1.2 hold for

$$p = 7 \text{ or } p \geq 9 \text{ when } D = 1, \qquad p = 5, 7, 9 \text{ or } p \geq 13 \text{ if } D = 2,$$
$$p = 5, 7, 9 \text{ if } D = 3 \qquad \text{and } p = 5 \text{ if } D = 4.$$

We note that for the case $p = 3 + \frac{4}{D}$, the assumptions of Theorem 1.2 hold if the space dimension is limited to $D = 1, 2$.

As in many previous works [2, 8, 12], the arguments of Theorem 1.2 is based upon a minimization problem related to the ground states which are used to define appropriate invariant sets and to further derive the blow-up solutions with initial data near the ground states. In [7], to prove the instability of the standing waves of Eq (1.1) with $p \in (3 + \frac{4}{D}, 22^* - 1)$ the authors introduce the minimization problem

$$\inf\{S_\omega(v) : P(v) = 0, N(v) \leq 0\}, \tag{1.7}$$

where the functionals $P(\cdot)$ and $N(\cdot)$ are defined by (2.4) and (2.5) respectively in the next section. Then, it is proved for $p > 3 + \frac{4}{D}$, the minimization problem (1.7) is equivalent to the minimizing problem

$$\inf\{S_\omega(v) : N(v) = 0\}, \tag{1.8}$$

which has been solved in [10]. However, such an argument fails when $p = 3 + \frac{4}{D}$. In the present paper, to overcome this difficulty, we replace the minimization problem (1.7) with

$$\inf\{S_\omega(v) : P(v) = 0\}. \tag{1.9}$$

By establishing the equivalence between the minimization problems (1.9) and (1.8), we manage to prove the instability of the standing waves of Eq (1.1) with $p \geq 3 + \frac{4}{D}$.

2. Proof of Theorem 1.2

In view of the result of [7], we have the following properties on the ground states of (1.2).

Proposition 2.1. [Theorem 1.3 in [7]] *For* $1 < p < 22^* - 1$ *and* $\omega > 0$, $\mathcal{G}_\omega$ *is nonempty and any* $u \in \mathcal{G}_\omega$ *is of the form*

$$u(x) = e^{i\theta}|u(x)|, \quad x \in \mathbb{R}^D,$$

for some $\theta \in \mathbb{S}^1$. *In particular, the elements of* $\mathcal{G}_\omega$ *are unique up to a constant complex phase, real-valued and non-negative. Furthermore, any real nonnegative ground state* $u \in \mathcal{G}_\omega$ *satisfies the following properties.*

(i) $u > 0$, *in* $\mathbb{R}^D$,
(ii) u *is a radially symmetric decreasing function with respect to some point,*
(iii) $u \in C^2(\mathbb{R}^D)$,
(iv) *for all* $\alpha \in \mathbb{N}^D$ *with* $|\alpha| \leq 2$, *there exists* $(c_\alpha, \delta_\alpha) \in (\mathbb{R}_+^*)^2$ *such that*

$$|D^\alpha u(x)| \leq C_\alpha e^{-\delta_\alpha |x|}, \text{ for all } x \in \mathbb{R}^D.$$

To prove that a solution blows up in finite time, we need the following virial identities for the solution of Eq (1.1).

Lemma 2.1. (Lemma 3.2 in [7]) *Let* $\psi_0 \in H^1$, $|x|^2\psi_0 \in L^2$ *and* $\psi(t)$ *be the solution of Eq (1.1) with* $\psi(0) = \psi_0$. *Then the function* $J(t) := \int |x|^2|\psi|^2$ *is* C^2 *and*

$$J'(t) = 4\Im \int (x \cdot \nabla\psi)\bar{\psi},$$

$$J''(t) = 8P(\psi), \tag{2.1}$$

where $P(u)$ *is defined by (2.4).*

Similar to [7], we know that the solution u of Eq (1.2) satisfies two identities.

$$\int |\nabla u|^2 + \omega \int |u|^2 + 4 \int |u|^2|\nabla|u||^2 - \int |u|^{p+1} = 0, \tag{2.2}$$

$$\frac{D-2}{D}\left(\frac{1}{2}\int |\nabla u|^2 + \int |u|^2|\nabla|u||^2\right) + \frac{\omega}{2}\int |u|^2 - \frac{1}{p+1}\int |u|^{p+1} = 0. \tag{2.3}$$

The above identities lead to the following lemma.

Lemma 2.2. *For any solution* u *of Eq (1.2), we know*

$$P(u) = 0 \quad \text{and} \quad N(u) = 0,$$

where $P(u)$ *and* $N(u)$ *are defined by*

$$P(v) := \int |\nabla v|^2 + (D+2)\int |v|^2|\nabla|v||^2 - \frac{D(p-1)}{2(p+1)}\int |v|^{p+1}, \tag{2.4}$$

$$N(v) := \int |\nabla v|^2 + \omega \int |v|^2 + 4\int |v|^2|\nabla|v||^2 - \int |v|^{p+1}. \tag{2.5}$$

Let

$$\begin{aligned}\mathcal{N} &= \{v \in X_{\mathbb{C}}; N(v) = 0, v \neq 0\},\\ \mathcal{M} &= \{v \in X_{\mathbb{C}}; P(v) = 0, v \neq 0\},\end{aligned}$$

and then we consider the minimization problems

$$d_{\mathcal{N}} = \inf_{v\in\mathcal{N}} S_{\omega}(v), \tag{2.6}$$

$$d_{\mathcal{M}} = \inf_{v\in\mathcal{M}} S_{\omega}(v). \tag{2.7}$$

It is clear that

$$d_{\mathcal{M}} = \inf_{v\in\mathcal{M}} S^{+}_{\omega}(v),$$

where the functional S^{+}_{ω} is defined by

$$\begin{aligned}S^{+}_{\omega}(v) &= S_{\omega}(v) - \tfrac{2}{D(p-1)}P(v)\\ &= \tfrac{D(p-1)-4}{2D(p-1)}\int |\nabla v|^2 + \tfrac{\omega}{2}\int |v|^2 + \tfrac{D(p-3)-4}{D(p-1)}\int |v|^2|\nabla|v||^2.\end{aligned}$$

We introduce the following minimizing problem

$$d_{\mathcal{M}'} = \inf\{S^{+}_{\omega}(v) : P(v) \le 0, v \in X_{\mathbb{C}}\}. \tag{2.8}$$

From Lemma 3.4 in [7], we have the variational characterization of the ground state solutions of Eq (1.2).

Lemma 2.3. *For $3 \le p < 22^{*} - 1$, it holds that the set of minimizer of problem (2.6) is exactly the set $\mathcal{G}_{\omega}$.*

Now we investigate the behavior of the functionals $S_{\omega}(u)$ and $P(u)$ under the scaling.

Lemma 2.4. *Let $v \in H^1 \setminus \{0\}$, $\omega > 0$, $P(v) = 0$ and assume that $p \ge 3 + \frac{4}{D}$. Let $\lambda > 0$ and define $v^{\lambda} = \lambda^{\frac{D}{2}} v(\lambda x)$. Then it holds that*

$$S_{\omega}(v^{\lambda}) < S_{\omega}(v) \quad \textit{for } \lambda \in (0,1) \cup (1,\infty).$$

Proof. For the case $p > 3 + \frac{4}{D}$, the result is proved by Lemma 3.3 in [7].
Let $p = 3 + \frac{4}{D}$. Set $f(\lambda) = S_{\omega}(v^{\lambda})$. Direct computation leads to

$$\begin{aligned}f(\lambda) &= S_{\omega}(v^{\lambda})\\ &= \tfrac{\lambda^2}{2}\int |\nabla v|^2 + \lambda^{D+2}\left(\int |v|^2|\nabla|v||^2 - \tfrac{1}{4+\frac{4}{D}}\int |v|^{4+\frac{4}{D}}\right) + \tfrac{\omega}{2}\int |v|^2,\\ P(v^{\lambda}) &= \lambda^2\int |\nabla v|^2 + \lambda^{D+2}(D+2)\left(\int |v|^2|\nabla|v||^2 - \tfrac{1}{4+\frac{4}{D}}\int |v|^{4+\frac{4}{D}}\right),\\ f'(\lambda) &= \tfrac{\partial}{\partial\lambda}S_{\omega}(v^{\lambda})\\ &= \lambda\int |\nabla v|^2 + \lambda^{D+1}(D+2)\left(\int |v|^2|\nabla|v||^2 - \tfrac{1}{4+\frac{4}{D}}\int |v|^{4+\frac{4}{D}}\right) = \tfrac{1}{\lambda}P(v^{\lambda}),\\ f''(\lambda) &= \tfrac{\partial^2}{\partial\lambda^2}S_{\omega}(v^{\lambda})\\ &= \int |\nabla v|^2 + \lambda^{D}(D+2)(D+1)\left(\int |v|^2|\nabla|v||^2 - \tfrac{1}{4+\frac{4}{D}}\int |v|^{4+\frac{4}{D}}\right).\end{aligned}$$

The fact $P(v) = 0$ implies that

$$(D+2)\left(\int |v|^2|\nabla|v||^2 - \frac{1}{4+\frac{4}{D}}\int |v|^{4+\frac{4}{D}}\right) = -\int |\nabla v|^2 =: -\delta < 0.$$

Thus, $f''(\lambda)$ is positive for small values of λ, and tends to $-\infty$ as $\lambda \to +\infty$, and is strictly decreasing. Hence, there exists $\lambda_2 > 0$ such that

$$f''(\lambda_2) = 0 \text{ and } f''(\lambda)(\lambda_2 - \lambda) > 0 \text{ for } \lambda \neq \lambda_2. \tag{2.9}$$

Since f' is increasing for $\lambda < \lambda_2$ and $f'(0) = 0$, f' takes positive values at least for $\lambda < \lambda_2$. For $\lambda > \lambda_2$, f' decreases, and tends to $-\infty$. Thus, there exists $\lambda_1 > \lambda_2$ such that

$$f'(\lambda_1) = 0 \text{ and } f'(\lambda)(\lambda_1 - \lambda) > 0 \text{ for } \lambda \neq \lambda_1. \tag{2.10}$$

Therefore, λ_1 is the unique critical point of f on $(0, +\infty)$. Noticing $f'(\lambda) = \frac{1}{\lambda}P(v^\lambda)$ and $P(v) = 0$, we infer that $\lambda_1 = 1$, and $f(\lambda)$ attains maximum at $\lambda = 1$. □

Using Lemma 2.4, we get the following lemma.

Lemma 2.5. *For $\omega > 0$ and $p \geq 3 + \frac{4}{D}$, it holds that*

$$d_{\mathcal{N}} = d_{\mathcal{M}} := d. \tag{2.11}$$

Proof. We prove the result in two steps.
Step 1. $d_{\mathcal{M}} \leq d_{\mathcal{N}}$. Let $f \in H^1$ be a solution to (2.6). By Lemma 2.3, it is a ground state of (1.2); and by applying the Pohozaev identity (2.3) to a standing solution we immediately deduce $P(f) = 0$. By definition $N(f) = 0$, we have $f \in \mathcal{M}$. Hence $d_{\mathcal{M}} \leq d_{\mathcal{N}}$, since $S_\omega(f) = d_{\mathcal{N}}$.
Step 2. $d_{\mathcal{N}} \leq d_{\mathcal{M}}$. On the other hand, for any $h \in \mathcal{M}$, setting $h^\lambda = \lambda^{\frac{D}{2}} h(\lambda x)$, we have

$$N(h^\lambda) = \lambda^2 \int |\nabla h|^2 + \omega \int |h|^2 + 4\lambda^{D+2} \int |h|^2 |\nabla |h||^2 - \lambda^{\frac{D(p-1)}{2}} \int |h|^{p+1}. \tag{2.12}$$

It is obvious that

$$N(h^\lambda) \to \omega \int |h|^2 > 0 \text{ as } \lambda \to 0. \tag{2.13}$$

On the other hand, for $p \geq 3 + \frac{4}{D}$, we claim that

$$N(h^\lambda) < 0 \text{ as } \lambda \to +\infty. \tag{2.14}$$

When $p > 3 + \frac{4}{D}$, one get $\frac{D(p-1)}{2} > D + 2$. Hence (2.14) follows from (2.12).
When $p = 3 + \frac{4}{D}$, (2.12) reduces to

$$N(h^\lambda) = \lambda^2 \int |\nabla h|^2 + \omega \int |h|^2 + \lambda^{D+2}(4 \int |h|^2 |\nabla |h||^2 - \int |h|^{4+\frac{4}{D}}). \tag{2.15}$$

As $P(h) = 0$, it follows from (2.4) with $p = 3 + \frac{4}{D}$ that

$$\int |h|^2 |\nabla |h||^2 - \frac{D}{4(D+1)} \int |h|^{p+1} = -\frac{1}{(D+2)} \int |\nabla h|^2 < 0,$$

from which we get that

$$4 \int |h|^2 |\nabla |h||^2 - \int |h|^{4+\frac{4}{D}} < 0.$$

Hence (2.14) follows from (2.15).
It follows from (2.13) and (2.14) that there exists $\hat{\lambda} \in (0, \infty)$ such that $N(h^{\hat{\lambda}}) = 0$ and $h^{\hat{\lambda}} \in \mathcal{N}$. Thus we get $S_\omega(h^{\hat{\lambda}}) \geq d_{\mathcal{N}}$. Using $P(h) = 0$, from Lemma 2.4, we obtain $S_\omega(h) \geq S_\omega(h^{\hat{\lambda}}) \geq d_{\mathcal{N}}$. Thus $S_\omega(h) \geq d_{\mathcal{N}}$ holds true for any $h \in \mathcal{M}$, which yields $d_{\mathcal{M}} \geq d_{\mathcal{N}}$. □

Lemma 2.6. *For $\omega > 0$ and $p \geq 3 + \frac{4}{D}$, it holds that*

$$d_{\mathcal{M}} = d_{\mathcal{M}'} = d. \tag{2.16}$$

Moreover, the minimum of (2.8) is attained at $v \in \mathcal{M}$.

Proof. It follows from the definitions of $d_{\mathcal{M}}$ and $d_{\mathcal{M}'}$ that $d_{\mathcal{M}} \geq d_{\mathcal{M}'}$. We need only to prove that $d_{\mathcal{M}} \leq d_{\mathcal{M}'}$. In fact, for any v satisfying $P(v) < 0$, we have

$$P(\lambda v) = \lambda^2 \int |\nabla v|^2 + (D+2)\lambda^4 \int |v|^2 |\nabla |v||^2 - \frac{D(p-1)}{2(p+1)} \lambda^{p+1} \int |v|^{p+1}.$$

The fact $p \geq 3 + \frac{4}{D}$ implies that $P(\lambda v) > 0$ for λ small sufficiently. Thus, there exists $\overline{\lambda} \in (0,1)$ such that $P(\overline{\lambda} v) = 0$, i.e., $\overline{\lambda} v \in \mathcal{M}$.

$$S^+_\omega(\overline{\lambda} v) = \frac{D(p-1)-4}{2D(p-1)} \overline{\lambda}^2 \int |\nabla v|^2 + \frac{\omega}{2} \overline{\lambda}^4 \int |v|^2 + \frac{D(p-3)-4}{D(p-1)} \overline{\lambda}^{p+1} \int |v|^2 |\nabla |v||^2.$$

Hence, $S^+_\omega(\overline{\lambda} v) < S^+_\omega(v)$. The conclusion of this lemma follows. □

Now, we prove the main theorem of this paper.

Proof of Theorem 1.2. We divide the proof into four steps.

Step 1. We construct a sequence of initial data $Q^\lambda_\omega(x)$ satisfying the properties.

$$S_\omega(Q^\lambda_\omega) < d, P(Q^\lambda_\omega) < 0.$$

Let $\varepsilon > 0$ be fixed and consider $Q^\lambda_\omega(x) = \lambda^{\frac{D}{2}} Q_\omega(\lambda x)$ for the ground state solution Q_ω. We get $\|Q^\lambda_\omega\|_2 = \|Q_\omega\|_2$. By the continuity of the mapping $\lambda \mapsto \lambda^{\frac{D}{2}} Q_\omega(\lambda x)$, if $\lambda > 1$ sufficiently closes to 1, it is clear that $\|Q^\lambda_\omega - Q_\omega\|_{H^1} \leq \varepsilon$. Moreover, using the facts $P(Q_\omega) = 0$, $S_\omega(Q_\omega) = d$ and $\lambda > 1$, we obtain from Lemma 2.4 that

$$S_\omega(Q^\lambda_\omega) < d, P(Q^\lambda_\omega) < 0. \tag{2.17}$$

Step 2. We prove that (2.17) is invariant under the flow of (1.1).

Now fix a $\lambda > 1$ such that (2.17) is valid. Let $\psi^\lambda(t,x)$ be the solution of (1.1) with $\psi^\lambda(0) = Q^\lambda_\omega$. We claim that the properties described in (2.17) are invariant under the flow of (1.1). That is

$$S_\omega(\psi^\lambda(t)) < d, P(\psi^\lambda(t)) < 0, \text{ for all } t \in [0,T), \tag{2.18}$$

where $T \in (0, +\infty]$ is the maximal existence time. Using (1.3), (1.4) and (2.17), we get

$$S_\omega(\psi^\lambda(t)) = S_\omega(Q^\lambda_\omega) < d.$$

In turn, we infer that $P(\psi^\lambda(t)) \neq 0$ for all $t \in [0,T)$, otherwise if $P(\psi^\lambda(t)) = 0$ for some $t_0 \in [0,T)$, we would have $\psi^\lambda(t_0) \in \mathcal{M}$, yielding $S_\omega(\psi^\lambda(t)) \geq d$, which contradicts the first inequality of (2.18). Therefore, $P(\psi^\lambda(t)) < 0$ for all $t \in [0,T)$.

Step 3. We prove that $P(\psi^\lambda)$ stays negative and away from 0 for all $t \in [0,T)$.

It follows from Lemma 2.6 and the result of step 2 that

$$d \leq S^+_\omega(\psi^\lambda(t)) = S_\omega(\psi^\lambda(t)) - \frac{2}{D(p-1)} P(\psi^\lambda(t)).$$

Noticing

$$S_\omega(\psi^\lambda(t)) = S_\omega(\psi^\lambda(0)) = S_\omega(Q_\omega^\lambda),$$

we get

$$d \leq S_\omega^+(\psi^\lambda(t)) = S_\omega(Q_\omega^\lambda) - \frac{2}{D(p-1)}P(\psi^\lambda(t)). \tag{2.19}$$

It follows that

$$P(\psi^\lambda(t)) \leq \frac{D(p-1)}{2}(S_\omega(Q_\omega^\lambda) - d) := -\delta < 0. \tag{2.20}$$

Step 4. Conclusion

Since the ground sate of (1.2), $Q_\omega(x)$, has an exponential fall-off at infinity (Proposition 2.1 iv), it is clear that $|\cdot|\psi^\lambda(\cdot) \in L^2$.

By Lemma 2.1 and (2.20), we have

$$J(t) \leq J(0) + J'(0)t - 4\delta t^2,$$

where $J(t) = \int |x|^2|u|^2$. Thus we can find a T_0 such that

$$\lim_{t\to T_0} J(t) = 0.$$

Observing that

$$\int |u|^2 \leq C(\int |x|^2|u|^2)^{\frac{1}{2}}(\int |\nabla u|^2)^{\frac{1}{2}}$$

and C is independent of u, we obtain from the conservation of mass (1.3) that the solution blows up in finite time. Since it is clear that $Q_\omega^\lambda \to Q_\omega$ in H^1 when $\lambda \to 1$, Theorem 1.2 is proved. □

3. Remark on the stability of the standing wave

To study the stability of the standing waves of (1.1), Colin, Jeanjean and Squassina [7] introduce the following minimizing problem

$$m_\mu = \inf\{\mathcal{E}(v) : v \in H^1(\mathbb{R}^2), \|v\|^2_{L^2(\mathbb{R}^2)} = \mu\}. \tag{3.1}$$

The set of minimizers for (3.1) is denoted by Σ_μ:

$$\Sigma_\mu = \{v \in H^1(\mathbb{R}^2) : \mathcal{E}(v) = m_\mu, \|v\|^2_{L^2(\mathbb{R}^2)} = \mu\}.$$

Theorem 3.1. [Theorems 1.9 and 1.12 in [7]]
(1) Let $1 < p < 1 + \frac{4}{D}$. Then, the set Σ_μ is not empty for $\mu \in (0, \infty)$.
(2) Let $1 + \frac{4}{D} \leq p \leq 3 + \frac{4}{D}$. Then, the set Σ_μ is not empty if and only if $\mu \in [\mu(p, D), \infty)$, where $\mu(p, D)$ is defined by $\mu(p, D) = \inf\{\mu > 0 : m_\mu < 0\}$
(3) For $1 < p < 3 + \frac{4}{D}$, Σ_μ is stable in the sense that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\inf_{u\in\Sigma_\mu} \|\psi_0 - u\|_{H^1} < \delta \Rightarrow \sup_{t\in[0,T)} \inf_{u\in\Sigma_\mu} \|\psi(t,\cdot) - u\|_{H^1} < \varepsilon,$$

where $\psi(t,\cdot)$ is the solution of the Cauchy problem of (1.1) with initial condition $\psi_0 \in X_\mathbb{C} \cap H^s$, and T is the existence time for ψ.

For any minimizer $u \in \Sigma_\mu$ of (3.1), there exists $\omega^* = \omega^*(\mu, u)$ such that u is a solution of (1.2) with $\omega = \omega^*$. As pointed by [7], to study the stability of $\mathcal{G}_\omega$ via Theorem 3.1, the relation between $\mathcal{G}_\omega$ and Σ_μ need to be investigated. Motivated by Theorem 1.3.1 in [14], we establish the following theorem.

Theorem 3.2. *Let $D \geq 3$. $u \in \Sigma_\mu$ be a solution of (1.2) with $\omega = \omega^*$, i.e. $\Sigma_\mu \cap \mathcal{X}_{\omega^*} \neq \emptyset$. Then it holds that $\mathcal{G}_{\omega^*} \subseteq \Sigma_\mu$.*

Proof. Let $u \in \Sigma_\mu \cap \mathcal{X}_{\omega^*}$. Define the functionals $T(\cdot)$ and $V(\cdot)$.

$$T(u) := \frac{1}{2}\int |\nabla u|^2 + \int |u|^2 |\nabla |u||^2. \tag{3.2}$$

$$V(u) := -\frac{\omega}{2}\int |u|^2 + \frac{1}{p+1}\int |u|^{p+1}. \tag{3.3}$$

It is clear that $S_\omega(u) = T(u) - V(u)$.
For any $\varphi \in \mathcal{G}_{\omega^*}$, we have $S_{\omega^*}(\varphi) \leq S_{\omega^*}(u)$. Noticing (2.3), we deduce that

$$\frac{2}{D}T(\varphi) \leq \frac{2}{D}T(u). \tag{3.4}$$

Setting $\widetilde{\varphi} = \varphi(\frac{x}{\lambda})$ with $\lambda = \frac{\sqrt{\mu}}{\|\varphi\|_{L^2}}$, we have $\|\widetilde{\varphi}\|_{L^2}^2 = \mu$. It follows that $\mathcal{E}(u) \leq \mathcal{E}(\widetilde{\varphi})$. Using the fact $S_{\omega^*}(v) = \mathcal{E}(v) + \frac{\omega^*}{2}\|v\|_{L^2}$, we get $S_{\omega^*}(u) \leq S_{\omega^*}(\widetilde{\varphi})$. Noticing

$$S_{\omega^*}(\widetilde{\varphi}) = T(\widetilde{\varphi}) - V(\widetilde{\varphi}),$$

$$T(\widetilde{\varphi}) = \lambda^{D-2}T(\varphi),$$

$$V(\widetilde{\varphi}) = \lambda^{D}V(\varphi),$$

$$V(\varphi) = \frac{D-2}{D}T(\varphi), \qquad \text{(by 2.3)}$$

we have

$$\frac{2}{D}T(u) \leq (\lambda^{D-2} - \lambda^{D}\frac{D-2}{D})T(\varphi). \tag{3.5}$$

It follows (3.4) and (3.5) that

$$\frac{2}{D} \leq \lambda^{D-2} - \lambda^{D}\frac{D-2}{D}.$$

That is

$$f(\lambda) := (D-2)\lambda^D - D\lambda^{D-2} + 2 \leq 0.$$

It is easy to check that $f(1) = 0$ and $f(\lambda) > 0$ for $\lambda \neq 1$. Thus $\lambda = 1$, which implies $\|\varphi\|_{L^2}^2 = \mu$ for all $u \in \mathcal{G}_{\omega^*}$ and $\mathcal{G}_{\omega^*} \subseteq \Sigma_\mu$. □

Remark 3.1. (1) Theorem 3.2 shows that the stable set Σ_μ consists of some ground state solutions of (1.2).
(2) If $\Sigma_\mu \cap \mathcal{X}_{\omega^*} \neq \emptyset$, Theorem 3.2 implies that $\mathcal{G}_{\omega^*} \subseteq \Sigma_\mu$. This shows that the elements of $\mathcal{G}_{\omega^*}$ share same L^2 norm. This conclusion is also implied by the uniqueness of the ground state of (1.2), which is showed in [1] under the condition that $\omega^{\frac{2}{p-1}} \geq c_0$ with c_0 being a constant depending only on p. (3) There are two questions remain open on the relation between Σ_μ and $\mathcal{G}_\omega$. Firstly, for different minimizer

in Σ_μ, we do not know if they have same Lagrange multiplier. Second, for any $\omega > 0$, it is unknown that if there exists μ such that $\mathcal{G}_\omega \subseteq \Sigma_\mu$.
(4) For the case $D = 2$, the same argument of Theorem 3.2 can show that if $u \in \Sigma_\mu \cap \mathcal{X}_{\omega^*}$ then $u \in \mathcal{G}_{\omega^*}$. However, we can not prove that the elements of $u \in \mathcal{G}_{\omega^*}$ have same L^2 norm.
(5) For Eq (1.2) without $-u\triangle|u|^2$, there are some classical results on the relation of Σ_μ and $\mathcal{G}_\omega$ [4].

(i) For $1 < p < 1 + \frac{4}{D}$, there exists one to one correspondence between $\mu > 0$ and $\omega > 0$ such that $\Sigma_\mu = \mathcal{G}_\omega$.
(ii) For $p = 1 + \frac{4}{D}$, there exists a constant μ_0 such that
  (a) $\|\varphi\|_{L^2}^2 = \mu_0$ for all $\varphi \in \mathcal{G}_\omega$ with $\omega \in (0, \infty)$,
  (b) Σ_μ is nonempty if and only if $\mu = \mu_0$,
  (c) $\Sigma_{\mu_0} = \bigcup\limits_{\omega \geq 0} \mathcal{G}_\omega$.

4. Conclusions

In this paper, we consider the strong instability of standing waves for the quasi-linear Schrödinger equation (1.1). In the critical case, i.e., $p = 3 + \frac{4}{D}$, we prove that the standing waves are strongly unstable by blow-up. This result is a complement to the result of Colin, Jeanjean and Squassina [7] where the instability of standing waves were studied in the supercritical case, i.e., $3 + \frac{4}{D} < p < \frac{3D+2}{D-2}$.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. S. Adachi, T. Watanabe, Uniqueness of the ground state solutions of quasilinear Schrödinger equations, *Nonlinear Anal.-Theor.*, **75** (2012), 819–833. https://doi.org/10.1016/j.na.2011.09.015
2. H. Berestycki, T. Cazenave, Instabilite des etats stationnaires dans les equations de Schrödinger et de Klein-Gordon non linearires, *C. R. Acad. Sci. Paris*, **293** (1981), 489–492.
3. L. Brizhik, A. Eremko, B. Piette, W. J. Zakrzewski, Electron self-trapping in a discrete two-dimensional lattice, *Physica D*, **159** (2001), 71–90. https://doi.org/10.1016/S0167-2789(01)00332-3
4. T. Cazenave, *Semilinear Schrödinger equations*, New York: New York University Courant Institute of Mathematical Sciences, 2003.
5. M. Colin, On the local well-posedness of quasilinear Schrödinger equations in arbitrary space dimension, *Commun. Part. Differ. Equ.*, **27** (2002), 325–354. https://doi.org/10.1081/PDE-120002789

6. M. Colin, L. Jeanjean, Solutions for a quasilinear Schrödinger equation: a dual approach, *Nonlinear Anal.-Theor.*, **56** (2004), 213–226. https://doi.org/10.1016/j.na.2003.09.008
7. M. Colin, L. Jeanjean, M. Squassina, Stability and instability results for standing waves of quasi–linear Schrödinger equations, *Nonlinearity*, **23** (2010), 1353–1385. https://doi.org/10.1088/0951-7715/23/6/006
8. B. H. Feng, R. P. Chen, Q. X. Wang, Instability of standing waves for the nonlinear Schrödinger–Poisson equation in the L^2– critical case, *J. Dyn. Differ. Equ.*, **32** (2020), 1357–1370. https://doi.org/10.1007/s10884-019-09779-6
9. S. Kurihura, Large amplitude quasi–solitons in superfluid film, *J. Phys. Soc. Jpn.*, **50** (1981), 3262–3267. https://doi.org/10.1143/JPSJ.50.3262
10. J. Q. Liu, Y. Q. Wang, Z. Q. Wang, Solutions for quasi-linear Schrödinger equations via the Nehari method, *Commun. Part. Differ. Equ.*, **29** (2004), 879–901. https://doi.org/10.1081/PDE-120037335
11. M. Poppenberg, On the local well posedness of quasi–linear Schrödinger equations in arbitrary space dimension, *J. Differ. Equ.*, **172** (2001), 83–115. https://doi.org/10.1006/jdeq.2000.3853
12. J. Zhang, Sharp threshold for blowup and global existence in nonlinear Schrödinger equations under a harmonic potential, *Commun. Part. Differ. Equ.*, **30** (2005), 1429–1443. https://doi.org/10.1080/03605300500299539
13. X. G. Zhang, J. Q. Jiang, Y. H. Wu, Y. J. Cui, The existence and nonexistence of entire large solutions for a quasilinear Schrödinger elliptic system by dual approach, *Appl. Math. Lett.*, **100** (2020), 106018. https://doi.org/10.1016/j.aml.2019.106018
14. X. Y. Zeng, *Variational problems arising in Bose-Einstein condensation*, Doctoral Thesis, Wuhan Institute of Physics and Mathematics Chinese Academy of Sciences, 2014.

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