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Research article

Determinantal inequalities for block Hadamard product and Khatri-Rao product of positive definite matrices

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Abstract: In this paper, we first give an alternative proof for a result of Liu et al. in [Math. Inequal. Appl. 20 (2017) 537–542]. Then we present two inequalities for the block Hadamard product and the Khatri-Rao product respectively. The inequalities obtained extend the result of Liu et al.

Keywords: determinantal inequality; block Hadamard product; Khatri-Rao product; positive definite matrix
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1. Introduction

Let $\mathbb{M}_n$ be the set of $n \times n$ complex matrices. If X is positive semidefinite, we put $X \geq 0$. For two Hermitian matrices $X, Y \in \mathbb{M}_n, X \geq Y$ means $X - Y$ is positive semidefinite. If X is positive definite, we put $X > 0$.

The Hadamard product of $A, B \in \mathbb{M}_n$ is denoted by $A \circ B$, and the Hadamard product of $A_1, \ldots, A_m \in \mathbb{M}_n$ is denoted by $\prod_{i=1}^{m} \circ A_i$. The Kronecker product of A and B is denoted by $A \otimes B = (a_{ij}B)$. If $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{M}_n$ with A_{11} nonsingular, then the Schur complement of A_{11} in A is defined as $A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$. A well known property of the Schur complement is

$$\det A = \det A_{11} \det(A/A_{11}). \tag{1.1}$$

The set of all complex matrices partitioned as $p \times p$ blocks with each block $q \times q$ is denoted by $\mathbb{M}_p(\mathbb{M}_q)$. Let $\mathbf{A} = (A_{ij}), \mathbf{B} = (B_{ij}) \in \mathbb{M}_p(\mathbb{M}_q)$. The block Hadamard product of $\mathbf{A}$ and $\mathbf{B}$ is given by $\mathbf{A} \square \mathbf{B} := (A_{ij}B_{ij})$, where $A_{ij}B_{ij}$ denotes the usual matrix product of A_{ij} and B_{ij}. If every block of $\mathbf{A}$ commutes with corresponding block of $\mathbf{B}$, we say that $\mathbf{A}, \mathbf{B}$ block commute. The Khatri-Rao product of $\mathbf{A}$ and $\mathbf{B}$ is given by $\mathbf{A} * \mathbf{B} := (A_{ij} \otimes B_{ij})$. We denote by $\mathbf{I_p} \in \mathbb{M}_p(\mathbb{M}_q)$ the $pq \times pq$ identity matrix, which is partitioned according to the block structure of the matrices of $\mathbb{M}_p(\mathbb{M}_q)$. Clearly, when $q = 1$, that is, A and B are $p \times p$ matrices with complex entries, the block Hadamard product and the Khatri-Rao product coincide with the Hadamard product. These matrix products have been used in many fields, such as matrix analysis, statistical analysis, communication and information theory, etc. For more information, we refer to [3–6, 9, 10].

Let $A_i \in \mathbb{M}_n, i = 1, \dots, m$, be positive definite matrices whose diagonal blocks are n_j-square matrices $A_i^{(j)}, j = 1, \dots, k$ (so $n_1 + \cdots + n_k = n$). Denote $\sum_{i=1}^m$ or $\prod_{i=1}^m \circ$ by $\star$, then

$$\det\left(\star A_i\right) \leq \det\left(\star\, (A_i^{(1)})\right) \cdots \det\left(\star\, (A_i^{(k)})\right)$$

follows directly from Fischer's inequality [2, p. 506].

By making use of a result of Lin [7], Choi [1] proved the following inquality,

$$\det\left(\sum_{i=1}^m A_i^{-1}\right) \geq \det\left(\sum_{i=1}^m (A_i^{(1)})^{-1}\right) \cdots \det\left(\sum_{i=1}^m (A_i^{(k)})^{-1}\right). \tag{1.2}$$

Later, Liu et al. [8] gave a new proof of Choi's inequality(1.2), and they also obtained the following theorem.

Theorem 1. *Let $A_i \in \mathbb{M}_n, i = 1, \dots, m$, be positive definite whose diagonal blocks are n_j-square matrices $A_i^{(j)}$ for $j = 1, \dots, k$ (so $n_1 + \cdots + n_k = n$). Then*

$$\det\left(\prod_{i=1}^m \circ A_i^{-1}\right) \geq \det\left(\prod_{i=1}^m \circ (A_i^{(1)})^{-1}\right) \cdots \det\left(\prod_{i=1}^m \circ (A_i^{(k)})^{-1}\right).$$

In this paper, we give an alternative proof of Theorem 1, this is done in Section 2. In Section 3, we present the following inequality for the block Hadamard product. Clearly, when $q = 1$, Theorem 2 reduces to Theorem 1.

Theorem 2. *Let $\mathbf{A}_i \in \mathbb{M}_p(\mathbb{M}_q)$, $i = 1, \dots, m$, partition $\mathbf{A}_i$ with diagonal blocks $\mathbf{A}_i^{(j)} \in \mathbb{M}_{p_j}(\mathbb{M}_q)$, $j = 1, \dots, t$ (so $p_1 + \cdots + p_t = p$). If $\mathbf{A}_i, i = 1, \dots, m$, are positive definite and block commute, then*

$$\det\left(\prod_{i=1}^m \square \mathbf{A}_i^{-1}\right) \geq \det\left(\prod_{i=1}^m \square (\mathbf{A}_i^{(1)})^{-1}\right) \cdots \det\left(\prod_{i=1}^m \square (\mathbf{A}_i^{(t)})^{-1}\right).$$

The result for the Khatri-Rao product is given in Section 4.

2. Alternative proof of Theorem 1

We list some lemmas which are important for our proof.

Lemma 1. *[2, Corollary 7.7.4] If $A, B \in \mathbb{M}_n$ such that $A \geq B > 0$, then* $\det A \geq \det B$.

Lemma 2. *[2, Theorem 7.5.3] Let $A, B \in \mathbb{M}_n$ be positive semidefinite. Then $A \circ B \geq 0$.*

Lemma 3. *[11, Theorem 7.13] Let $A \in \mathbb{M}_n$ be positive definite. Partition A as $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with A_{11} square. Let A^{-1} be conformally partitioned as A, then*

(1) $(A_{ii})^{-1} \leq (A^{-1})_{ii}, i = 1, 2$;

(2) $A^{-1}/(A^{-1})_{11} = (A_{22})^{-1}$.

Lemma 4. *[2, p.504] Let $A, B \in \mathbb{M}_n$ be positive definite. Partition A, B as $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ with $A_{11}, B_{11} \in \mathbb{M}_k$, $1 \leq k < n$, then*

$$(A \circ B)/(A_{11} \circ B_{11}) \geq (A/A_{11}) \circ (B/B_{11}).$$

By Lemma 4 and induction, we have

Lemma 5. *Let $A_i \in \mathbb{M}_n$, i=1,...,m, be positive definite and conformally partitioned, and $A_i^{(1)}$ be the $(1, 1)$ block of A_i. Then*

$$\Big(\prod_{i=1}^{m} \circ A_i\Big)/\Big(\prod_{i=1}^{m} \circ A_i^{(1)}\Big) \geq \prod_{i=1}^{m} \circ (A_i/A_i^{(1)}).$$

Now we are ready to present.

Proof of Theorem 1. For all $i = 1, \ldots, m$, as A_i are positive definite, A_i^{-1} are all positive definite. Partition A_i^{-1} as A_i for $i = 1, \ldots, m$, the diagonal blocks of A_i^{-1} are n_j-square matrices $(A_i^{-1})^{(j)}$ for $j = 1, \ldots, p$. Using mathematical induction on k , we may assume $k = 2$. By Lemma 5 and Lemma 1, we get

$$\det\left(\Big(\prod_{i=1}^{m} \circ A_i^{-1}\Big)/\Big(\prod_{i=1}^{m} \circ (A_i^{-1})^{(1)}\Big)\right) \geq \det\left(\prod_{i=1}^{m} \circ \Big(A_i^{-1}/(A_i^{-1})^{(1)}\Big)\right). \tag{2.1}$$

Then we have

$$\begin{aligned}
\det\left(\prod_{i=1}^{m} \circ A_i^{-1}\right) &= \det\left(\prod_{i=1}^{m} \circ (A_i^{-1})^{(1)}\right) \det\left(\Big(\prod_{i=1}^{m} \circ A_i^{-1}\Big)/\Big(\prod_{i=1}^{m} \circ (A_i^{-1})^{(1)}\Big)\right) \\
&\geq \det\left(\prod_{i=1}^{m} \circ (A_i^{-1})^{(1)}\right) \det\left(\prod_{i=1}^{m} \circ \Big(A_i^{-1}/(A_i^{-1})^{(1)}\Big)\right) \\
&\geq \det\left(\prod_{i=1}^{m} \circ (A_i^{(1)})^{-1}\right) \det\left(\prod_{i=1}^{m} \circ \Big(A_i^{-1}/(A_i^{-1})^{(1)}\Big)\right) \\
&= \det\left(\prod_{i=1}^{m} \circ (A_i^{(1)})^{-1}\right) \det\left(\prod_{i=1}^{m} \circ (A_i^{(2)})^{-1}\right).
\end{aligned}$$

where the first equality above is by (1.1); the first inequality is by (2.1); the second inequality is by Lemma 3(1); the last equality is due to Lemma 3(2). □

3. Proof of Theorem 2

In order to prove Theorem 2, we need to show the following lemmas.

Lemma 6. *[4, Corollary 3.3] Let* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_p(\mathbb{M}_q)$*. If* $\mathbf{A}, \mathbf{B}$ *are positive semidefinite and block commute, then* $\mathbf{A} \square \mathbf{B} \geq 0$.

Lemma 7. *[4, Lemma 2.4] Let* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_p(\mathbb{M}_q)$*. If* $\mathbf{B}$ *is invertible, then* $\mathbf{A}, \mathbf{B}$ *block commute if and only if* $\mathbf{A}, \mathbf{B}^{-1}$ *block commute.*

Lemma 8. *Let* $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^* & A_{22} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^* & \mathbf{B}_{22} \end{pmatrix} \in \mathbb{M}_p(\mathbb{M}_q)$ *with* $\mathbf{A}_{11}, \mathbf{B}_{11} \in \mathbb{M}_h(\mathbb{M}_q), h < p$*. If* $\mathbf{A}, \mathbf{B}$ *are positive definite and block commute, then*

$$(\mathbf{A} \square \mathbf{B})/(\mathbf{A}_{11} \square \mathbf{B}_{11}) \geq (\mathbf{A}/\mathbf{A}_{11}) \square (\mathbf{B}/\mathbf{B}_{11}).$$

Proof. Let

$$\mathbf{E} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^* & \mathbf{A}_{12}^* \mathbf{A}_{11}^{-1} \mathbf{A}_{12} \end{pmatrix},$$

and

$$\mathbf{F} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^* & \mathbf{B}_{12}^* \mathbf{B}_{11}^{-1} \mathbf{B}_{12} \end{pmatrix}.$$

Then $\mathbf{E}$ and $\mathbf{F}$ are positive semidefinite and block commute.

By Lemma 6, we get that

$$\mathbf{E} \square \mathbf{F} = \begin{pmatrix} \mathbf{A}_{11} \square \mathbf{B}_{11} & \mathbf{A}_{12} \square \mathbf{B}_{12} \\ \mathbf{A}_{12}^* \square \mathbf{B}_{12}^* & (\mathbf{A}_{12}^* \mathbf{A}_{11}^{-1} \mathbf{A}_{12}) \square (\mathbf{B}_{12}^* \mathbf{B}_{11}^{-1} \mathbf{B}_{12}) \end{pmatrix}$$

is positive semidefinite, thus

$$(\mathbf{A}_{12}^* \mathbf{A}_{11}^{-1} \mathbf{A}_{12}) \square (\mathbf{B}_{12}^* \mathbf{B}_{11}^{-1} \mathbf{B}_{12}) \geq (\mathbf{A}_{12}^* \square \mathbf{B}_{12}^*)(\mathbf{A}_{11} \square \mathbf{B}_{11})^{-1}(\mathbf{A}_{12} \square \mathbf{B}_{12}),$$

that is

$$(\mathbf{A}_{22} - \mathbf{A}/\mathbf{A}_{11}) \square (\mathbf{B}_{22} - \mathbf{B}/\mathbf{B}_{11}) \geq \mathbf{A}_{22} \square \mathbf{B}_{22} - (\mathbf{A} \square \mathbf{B})/(\mathbf{A}_{11} \square \mathbf{B}_{11}),$$

which implies

$$(\mathbf{A} \square \mathbf{B})/(\mathbf{A}_{11} \square \mathbf{B}_{11}) \geq \mathbf{A}_{22} \square (\mathbf{B}/\mathbf{B}_{11}) + (\mathbf{A}/\mathbf{A}_{11}) \square \mathbf{B}_{22} - (\mathbf{A}/\mathbf{A}_{11}) \square (\mathbf{B}/\mathbf{B}_{11}).$$

It follows from

$$\mathbf{A}_{22} \geq \mathbf{A}/\mathbf{A}_{11}, \quad \mathbf{B}_{22} \geq \mathbf{B}/\mathbf{B}_{11},$$

that

$$\begin{aligned} (\mathbf{A} \square \mathbf{B})/(\mathbf{A}_{11} \square \mathbf{B}_{11}) &\geq \mathbf{A}_{22} \square (\mathbf{B}/\mathbf{B}_{11}) \\ &\geq (\mathbf{A}/\mathbf{A}_{11}) \square (\mathbf{B}/\mathbf{B}_{11}). \end{aligned}$$

This completes the proof. □

Lemma 9. *Let* $\mathbf{A}_i = \begin{pmatrix} (\mathbf{A}_i)_{11} & (\mathbf{A}_i)_{12} \\ (\mathbf{A}_i)_{12}^* & (\mathbf{A}_i)_{22} \end{pmatrix} \in \mathbb{M}_p(\mathbb{M}_q)$ *with* $(\mathbf{A}_i)_{11} \in \mathbb{M}_h(\mathbb{M}_q), h < p, i = 1, \dots, m.$ *If* $\mathbf{A}_i, i = 1, \dots, m$, *are positive definite and block commute, then*

$$\det\left(\prod_{i=1}^{m} \Box \mathbf{A}_i\right) \geq \det\left(\prod_{i=1}^{m} \Box (\mathbf{A}_i)_{11}\right) \det\left(\prod_{i=1}^{m} \Box (\mathbf{A}_i/(\mathbf{A}_i)_{11})\right).$$

Proof. By Lemma 8 and induction, we can get

$$\left(\prod_{i=1}^{m} \Box \mathbf{A}_i\right) / \left(\prod_{i=1}^{m} \Box (\mathbf{A}_i)_{11}\right) \geq \prod_{i=1}^{m} \Box (\mathbf{A}_i/(\mathbf{A}_i)_{11}).$$

Then

$$\begin{aligned} \det\left(\prod_{i=1}^{m} \Box \mathbf{A}_i\right) &= \det\left(\prod_{i=1}^{m} \Box (\mathbf{A}_i)_{11}\right) \det\left(\left(\prod_{i=1}^{m} \Box \mathbf{A}_i\right) / \left(\prod_{i=1}^{m} \Box (\mathbf{A}_i)_{11}\right)\right) \\ &\geq \det\left(\prod_{i=1}^{m} \Box (\mathbf{A}_i)_{11}\right) \det\left(\prod_{i=1}^{m} \Box (\mathbf{A}_i/(\mathbf{A}_i)_{11})\right). \end{aligned}$$

□

Now we give the proof of Theorem 2.

Proof of Theorem 2. For all $i = 1, \dots, m$, as $\mathbf{A}_i$ are positive definite, $\mathbf{A}_i^{-1}$ are all positive definite. Partition $\mathbf{A}_i^{-1}$ as $\mathbf{A}_i$ for $i = 1, \dots, m$, then the diagonal blocks of $(\mathbf{A}_i^{-1})^{(j)} \in \mathbb{M}_{p_j}(\mathbb{M}_q)$ for $j = 1, \dots, t$. By Lemma 7, we get that $\mathbf{A}_i^{-1}$, $i = 1, \dots, m$, block commute. Using mathematical induction on t , we may assume $t = 2$. By Lemma 9, we get

$$\det\left(\prod_{i=1}^{m} \Box \mathbf{A}_i^{-1}\right) \geq \det\left(\prod_{i=1}^{m} \Box (\mathbf{A}_i^{-1})^{(1)}\right) \det\left(\prod_{i=1}^{m} \Box \left(\mathbf{A}_i^{-1}/(\mathbf{A}_i^{-1})^{(1)}\right)\right).$$

By Lemma 3, we have

$$\begin{aligned} \det\left(\prod_{i=1}^{m} \Box \mathbf{A}_i^{-1}\right) &\geq \det\left(\prod_{i=1}^{m} \Box (\mathbf{A}_i^{(1)})^{-1}\right) \det\left(\prod_{i=1}^{m} \Box \left(\mathbf{A}_i^{-1}/(\mathbf{A}_i^{-1})^{(1)}\right)\right) \\ &= \det\left(\prod_{i=1}^{m} \Box (\mathbf{A}_i^{(1)})^{-1}\right) \det\left(\prod_{i=1}^{m} \Box (\mathbf{A}_i^{(2)})^{-1}\right). \end{aligned}$$

This completes the proof. □

Corollary 1. *Let* $\mathbf{A} \in \mathbb{M}_p(\mathbb{M}_q)$ *be positive definite, partition* $\mathbf{A}$ *with diagonal blocks* $\mathbf{A}^{(j)} \in \mathbb{M}_{p_j}(\mathbb{M}_q)$, $j = 1, \dots, t$ *(so* $p_1 + \cdots + p_t = p$*). Then*

$$\det\left(\mathbf{A}^{-1} \Box \mathbf{I}_p\right) \geq \det\left((\mathbf{A}^{(1)})^{-1} \Box \mathbf{I}_{p_1}\right) \cdots \det\left((\mathbf{A}^{(t)})^{-1} \Box \mathbf{I}_{p_t}\right).$$

Corollary 2. *Let* $\mathbf{A} \in \mathbb{M}_p(\mathbb{M}_q)$ *be positive definite, partition* $\mathbf{A}$ *with diagonal blocks* $\mathbf{A}^{(j)} \in \mathbb{M}_{p_j}(\mathbb{M}_q)$, $j = 1, \dots, t$ *(so* $p_1 + \cdots + p_t = p$*). Let* $\mathbf{B}^{(j)} \in \mathbb{M}_{p_j}(\mathbb{M}_q)$, $j = 1, \dots, t$, *be positive semidefinite,* $\mathbf{B} = diag(\mathbf{B}^{(1)}, \dots, \mathbf{B}^{(t)})$, $\mathbf{A}$ *and* $\mathbf{B}$ *block commute. Then*

$$\det\left(\mathbf{A}^{-1} \Box \mathbf{B}\right) \geq \det\left((\mathbf{A}^{(1)})^{-1} \Box \mathbf{B}^{(1)}\right) \cdots \det\left((\mathbf{A}^{(t)})^{-1} \Box \mathbf{B}^{(t)}\right). \tag{3.1}$$

Proof. Assume that $\mathbf{B}$ is nonsingular, that is, $\mathbf{B}^{(j)}$ are all invertible for $j = 1, \ldots, t$. Then, (3.1) follows by Lemma 7 and Theorem 2. By a standard continuity argument, the statement is also true if $\mathbf{B}$ is singular. □

When $t = p$ in Theorem 2, we get the following.

Corollary 3. *Let $\mathbf{A}_i \in \mathbb{M}_p(\mathbb{M}_q), i = 1, \ldots, m$, with diagonal blocks q-square matrices $A_i^{(j)}$, $j = 1, \ldots, p$. If $\mathbf{A}_i, i = 1, \ldots, m$, are positive definite and block commute, then*

$$\det\left(\prod_{i=1}^{m} \square \mathbf{A}_i^{-1}\right) \geq \det\left(\prod_{i=1}^{m} (A_i^{(1)})^{-1}\right) \cdots \det\left(\prod_{i=1}^{m} (A_i^{(p)})^{-1}\right).$$

4. Results for the Khatri-Rao product

The following lemmas will be used in the main result of this section.

Lemma 10. *[9, Theorem 5] Let $\mathbf{A} \geq \mathbf{B} \geq 0, \mathbf{C} \geq \mathbf{D} \geq 0$, and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ be compatibly partitioned matrices. Then*

$$\mathbf{A} * \mathbf{C} \geq \mathbf{B} * \mathbf{D} \geq 0.$$

Lemma 11. *[5, Lemma 3.3] Let $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^* & \mathbf{A}_{22} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^* & \mathbf{B}_{22} \end{pmatrix} \in \mathbb{M}_p(\mathbb{M}_q)$ with $\mathbf{A}_{11}, \mathbf{B}_{11} \in \mathbb{M}_h(\mathbb{M}_q), h < p$. If $\mathbf{A}, \mathbf{B}$ are positive definite, then*

$$(\mathbf{A} * \mathbf{B})/(\mathbf{A}_{11} * \mathbf{B}_{11}) \geq (\mathbf{A}/\mathbf{A}_{11}) * (\mathbf{B}/\mathbf{B}_{11}).$$

By Lemma 11 and induction, we have

Lemma 12. *Let $\mathbf{A}_i = \begin{pmatrix} (\mathbf{A}_i)_{11} & (\mathbf{A}_i)_{12} \\ (\mathbf{A}_i)_{12}^* & (\mathbf{A}_i)_{22} \end{pmatrix} \in \mathbb{M}_p(\mathbb{M}_q)$ with $(\mathbf{A}_i)_{11} \in \mathbb{M}_h(\mathbb{M}_q), h < p, i = 1, \ldots, m$. If $\mathbf{A}_i, i = 1, \ldots, m$, are positive definite, then*

$$\det\left(\prod_{i=1}^{m} * \mathbf{A}_i\right) \geq \det\left(\prod_{i=1}^{m} * (\mathbf{A}_i)_{11}\right) \det\left(\prod_{i=1}^{m} * (\mathbf{A}_i/(\mathbf{A}_i)_{11})\right).$$

Now, we give the result for the Khatri-Rao product. Clearly, when $q = 1$, Theorem 3 reduces to Theorem 1.

Theorem 3. *Let $\mathbf{A}_i \in \mathbb{M}_p\ (\mathbb{M}_q)$, $i = 1, \ldots, m$, partition $\mathbf{A}_i$ with diagonal blocks $\mathbf{A}_i^{(j)} \in \mathbb{M}_{p_j}(\mathbb{M}_q)$, $j = 1, \ldots, t$ (so $p_1 + \cdots + p_t = p$). If $\mathbf{A}_i$, $i = 1, \ldots, m$, are positive definite, then*

$$\det\left(\prod_{i=1}^{m} * \mathbf{A}_i^{-1}\right) \geq \det\left(\prod_{i=1}^{m} * (\mathbf{A}_i^{(1)})^{-1}\right) \cdots \det\left(\prod_{i=1}^{m} * (\mathbf{A}_i^{(t)})^{-1}\right).$$

Proof. This follows analogous steps to the proof of Theorem 2. □

Corollary 4. *Let* $\mathbf{A} \in \mathbb{M}_p(\mathbb{M}_q)$ *be positive definite, partition* $\mathbf{A}$ *with diagonal blocks* $\mathbf{A}^{(j)} \in \mathbb{M}_{p_j}(\mathbb{M}_q)$*,* $j = 1, \ldots, t$ *(so* $p_1 + \cdots + p_t = p$*). Then*

$$\det\left(\mathbf{A}^{-1} * \mathbf{I}_{\boldsymbol{p}}\right) \geq \det\left((\mathbf{A}^{(1)})^{-1} * \mathbf{I}_{\boldsymbol{p}_1}\right) \cdots \det\left((\mathbf{A}^{(t)})^{-1} * \mathbf{I}_{\boldsymbol{p}_t}\right),$$

and

$$\det\left(\mathbf{I}_{\boldsymbol{p}} * \mathbf{A}^{-1}\right) \geq \det\left(\mathbf{I}_{\boldsymbol{p}_1} * (\mathbf{A}^{(1)})^{-1}\right) \cdots \det\left(\mathbf{I}_{\boldsymbol{p}_t} * (\mathbf{A}^{(t)})^{-1}\right).$$

Corollary 5. *Let* $\mathbf{A} \in \mathbb{M}_p(\mathbb{M}_q)$ *be positive definite, partition* $\mathbf{A}$ *with diagonal blocks* $\mathbf{A}^{(j)} \in \mathbb{M}_{p_j}(\mathbb{M}_q)$*,* $j = 1, \ldots, t$ *(so* $p_1 + \cdots + p_t = p$*). Let* $\mathbf{B}^{(j)} \in \mathbb{M}_{p_j}(\mathbb{M}_q)$*,* $j = 1, \ldots, t$*, be positive semidefinite,* $\mathbf{B} = diag(\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(t)})$*. Then*

$$\det\left(\mathbf{A}^{-1} * \mathbf{B}\right) \geq \det\left((\mathbf{A}^{(1)})^{-1} * \mathbf{B}^{(1)}\right) \cdots \det\left((\mathbf{A}^{(t)})^{-1} * \mathbf{B}^{(t)}\right),$$

and

$$\det\left(\mathbf{B} * \mathbf{A}^{-1}\right) \geq \det\left((\mathbf{B}^{(1)} * \mathbf{A}^{(1)})^{-1}\right) \cdots \det\left(\mathbf{B}^{(t)} * (\mathbf{A}^{(t)})^{-1}\right).$$

When $t = p$ in Theorem 3, we get the following.

Corollary 6. *Let* $\mathbf{A}_i \in \mathbb{M}_p(\mathbb{M}_q)$*,* $i = 1, \ldots, m$*, with diagonal blocks q-square matrices* $A_i^{(j)}$*,* $j = 1, \ldots, p$*. If* $\mathbf{A}_i$*,* $i = 1, \ldots, m$*, are positive definite, then*

$$\det\left(\prod_{i=1}^{m} * \mathbf{A}_i^{-1}\right) \geq \det\left(\prod_{i=1}^{m} \otimes (A_i^{(1)})^{-1}\right) \cdots \det\left(\prod_{i=1}^{m} \otimes (A_i^{(p)})^{-1}\right).$$

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Conflict of interest

We declare no conflict of interest.

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