Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(6): 9627–9647.
DOI: 10.3934/math.2022535
Received: 23 November 2021
Revised: 28 February 2022
Accepted: 02 March 2022
Published: 15 March 2022

Research article

Commutators of multilinear θ-type generalized fractional integrals on non-homogeneous metric measure spaces

Xiangxing Tao and Jiahui Wang*

Department of Mathematics, School of Science, Zhejiang University of Science and Technology, Hangzhou 310023, China

* **Correspondence:** Email: wangjh0121@163.com; Tel: +8618868181648.

Abstract: Let $\mathcal{I}_{\alpha,m}$ be the multilinear θ-type generalized fractional integrals and $\vec{b}_\sigma$ be the vector with each $b_{\sigma_i} \in \widetilde{\mathrm{RBMO}}(\mu)$. The boundedness for $\mathcal{I}_{\alpha,m}$ and the iterated multi-commutators $\mathcal{I}_{\alpha,m,\vec{b}_\sigma}$ on Lebesgue spaces over non-homogeneous spaces are showed in this paper.

Keywords: multilinear θ-type generalized fractional integrals; iterated commutators; non-homogeneous measures; boundedness; Lebesgue spaces
Mathematics Subject Classification: 42B35, 47B37, 28A35

1. Introduction

In the last decade, the study on non-homogeneous metric measure spaces, which include both spaces of homogeneous type and non-doubling measure spaces as special cases, has attracted much attention, see for example [1–9]. The purpose of the paper is to establish the boundedness of the multilinear θ-type generalized fractional integrals and the iterated commutators of them on Lebesgue spaces over non-homogeneous spaces.

In this paper, we always assume that $(\mathcal{X}, d, \mu)$ is the non-homogeneous metric measure space in the sense of T. Hytönen [1], i.e., the metric measure space $(\mathcal{X}, d, \mu)$ have the geometrically doubling and upper doubling properties, which can be defined as follows:

Definition 1.1. *A metric measure space $(\mathcal{X}, d)$ is called geometrically doubling if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.*

Definition 1.2. *A metric measure space $(\mathcal{X}, d, \mu)$ is said to be upper doubling if μ is a Borel measure on $\mathcal{X}$ and there exists a dominating function*

$$\lambda : \mathcal{X} \times \mathbb{R}_+ \to \mathbb{R}_+$$

and a constant C_λ such that

$$r \to \lambda(x,r) \text{ is non-decreasing},$$

$$\mu(B(x,r)) \le \lambda(x,r) \le C_\lambda \lambda(x,\frac{r}{2}), \tag{1.1}$$

for all $x \in \mathcal{X}$ and $r > 0$.

We note that we can also assume that, for all $x, y \in \mathcal{X}$ with $d(x,y) \le r$,

$$\lambda(x,r) \le C_\lambda \lambda(y,r). \tag{1.2}$$

In fact, T. Hytönen, D. Yang and D. Yang [3] showed that there exists another function $\widetilde{\lambda} \le \lambda$ such that, for all $x, y \in \mathcal{X}$ with $d(x,y) \le r$, the inequality (1.2) holds for $\widetilde{\lambda}$.

Let $L^\infty_b(\mu)$ be the space of all $L^\infty(\mu)$ functions with bounded support. For $\beta \in (0,1)$, $f \in L^\infty_b(\mu)$ and $x \in \mathcal{X}$, the fractional integral $I_\beta f(x)$ is defined by

$$I_\beta f(x) := \int_{\mathcal{X}} \frac{f(y)d\mu(y)}{[\lambda(y,d(x,y))]^{1-\beta}}. \tag{1.3}$$

In case $\mathcal{X} = \mathbb{R}^d$ with Lebesgue measure, $I_\beta f$ can be rewritten as classical fractional integral, which is bounded from L^t to L^s for $1 < t < d/\beta$ and $1/s = 1/t - \beta/d$. In non-homogeneous metric measure space, we need an additional assumption for μ to get the similar boundedness for $I_\beta f$.

Definition 1.3. *Let $\epsilon \in (0,\infty)$. A dominating function λ is said to satisfy the ϵ-weak reverse doubling condition if, for all $r \in (0, 2diam(\mathcal{X}))$ and $a \in (1, 2diam(\mathcal{X})/r)$, there exists a number $C(a) \in [1,\infty)$, depending only on a and $\mathcal{X}$, such that, for all $x \in \mathcal{X}$,*

$$\lambda(x,ar) \ge C(a)\lambda(x,r)$$

and moreover,

$$\sum_{k=1}^{\infty} \frac{1}{[C(a^k)]^\epsilon} < \infty.$$

Lemma 1.1. *[2] Assume that $\mu(\mathcal{X}) = \infty$. Let $0 < \beta < 1, 1 < t < 1/\beta$ and $1/s = 1/t - \beta$. If λ satisfies the ϵ-weak reverse doubling condition for some $\epsilon \in (0, \inf\{\beta, 1-\beta, 1/s\})$, then I_β, defined by* (1.3)*, is bounded from $L^t(\mu)$ to $L^s(\mu)$.*

Suppose that θ is a non-negative nondecreasing function on $(0,+\infty)$ satisfying, for $n \in (0,\infty)$,

$$\int_0^1 \frac{\theta(t)}{t} \left|\log t\right|^n dt < \infty.$$

In this paper, we consider the following multilinear θ-type generalized fractional integral.

Definition 1.4. *Let $\alpha \in (0,m)$. We call K_α is a multilinear θ-type generalized fractional integral kernel, if*

$$K_\alpha \in L^1_{loc}\left(\mathcal{X}^{m+1} \backslash \{(x, y_1, \ldots, y_i, \ldots, y_m) : x = y_i, 1 \le i \le m\}\right)$$

and

(i) there exists a positive constant C_{K_α} depending on K_α such that

$$|K_\alpha(x, y_1, \ldots, y_m)| \le \frac{C_{K_\alpha}}{\left[\sum_{i=1}^m \lambda(x, d(x, y_i))\right]^{m-\alpha}}, \tag{1.4}$$

for all $(x, y_1, \ldots, y_i, \ldots, y_m) \in \mathcal{X}^{m+1}$ with $x \neq y_i$ for some i;

(ii) there exists $\theta \in (0, 1]$ and a positive constant C_{K_α} depending on K_α such that

$$|K_\alpha(x, y_1, \ldots, y_m) - K_\alpha(x', y_1, \ldots, y_m)| \le C_{K_\alpha}\theta\left(\frac{d(x, x')}{\sum_{i=1}^m d(x, y_i)}\right)\frac{1}{\left[\sum_{i=1}^m \lambda(x, d(x, y_i))\right]^{m-\alpha}} \tag{1.5}$$

for all $(x, y_1, \ldots, y_m) \in \mathcal{X}^{m+1}$ with $C_{K_\alpha}d(x, x') \le \max_{1\le j\le m} d(x, y_j)$, and for every j,

$$\begin{aligned}&\left|K_\alpha(x, y_1, \ldots, y_j, \ldots, y_m) - K_\alpha(x, y_1, \ldots, y_j', \ldots, y_m)\right| \\ &\le C_{K_\alpha}\theta\left(\frac{d(y_j, y_j')}{\sum_{i=1}^m d(x, y_i)}\right)\frac{1}{\left[\sum_{i=1}^m \lambda(x, d(x, y_i))\right]^{m-\alpha}}\end{aligned} \tag{1.6}$$

provided $C_{K_\alpha}d(y_j, y_j') \le \max_{1\le j\le m} d(x, y_j)$.

For any $\vec{f} = (f_1, \ldots, f_m)$, $f_i \in L_b^\infty(\mu), i = 1, 2, \ldots, m$ and $x \notin \cap_{i=1}^m \operatorname{supp} f_i$, the multilinear θ-type generalized fractional integral $\mathcal{I}_{\alpha,m}$ is defined by

$$\mathcal{I}_{\alpha,m}\vec{f}(x) = \int_{\mathcal{X}^m} K_\alpha(x, y_1, \ldots, y_m)\prod_{i=1}^m f_i(y_i)d\mu(y_i)$$

with kernel K_α satisfying (1.4)–(1.6).

The first theorem of this paper is the boundedness for the multilinear θ-type generalized fractional integral on Lebesgue spaces.

Theorem 1.2. *Assume that $\mu(\mathcal{X}) = \infty$. Let $0 < \beta < 1, 1 < t < 1/\beta, 1/s = 1/t - \beta, 0 < \alpha < m, 1 < p_1, \ldots, p_m < \infty$ and $0 < 1/q = 1/p_1 + \cdots + 1/p_m - \alpha < 1$. Suppose λ satisfies the ϵ-weak reverse doubling condition for some $\epsilon \in (0, \inf\{\beta, 1-\beta, 1/s\})$. Then the multilinear θ-type generalized fractional integral $\mathcal{I}_{\alpha,m}$ defined by Definition 1.4 is bounded from $L^{p_1}(\mu) \times \cdots \times L^{p_m}(\mu)$ to $L^q(\mu)$, that is, there exists a constant $C > 0$ such that*

$$\left\|\mathcal{I}_{\alpha,m}\vec{f}\right\|_{L^q(\mu)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mu)}.$$

In order to consider the commutators of the multilinear θ-type generalized fractional integral, we need to give the following concepts.

We call $B \subset \mathcal{X}$ a (η, β)-doubling ball if $\mu(\eta B) \le \beta\mu(B)$ for $\eta, \beta \in (1, +\infty)$. In [1], T. Hytönen pointed out that if a metric measure space $(\mathcal{X}, d, \mu)$ is upper doubling and $\eta, \beta \in (1, \infty)$ with $\beta > (C_\lambda)^{\log_2 \eta} =: \eta^\nu$, then for any ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{N}$ such that $\eta^j B$ is (η, β)-doubling. Meanwhile, let $(\mathcal{X}, d)$ be geometrically doubling and $\beta > \eta^{n_0}$ with $n_0 := \log_2 N_0$ and μ is a Borel measure on $\mathcal{X}$ which is finite on bounded sets, then for μ-a.e. $x \in \mathcal{X}$, there exist arbitrarily small (η, β)-doubling balls centered at x. In fact, their radius may be chosen to be of the form $\eta^{-j}r$, $j \in \mathbb{N}$, for any preassigned number $r > 0$.

Throughout this paper, for any $\eta \in (1, \infty)$ and ball B, the smallest (η, β_η)-doubling ball of the form $\eta^j B$ with $j \in \mathbb{Z}_+$ is denoted by $\widetilde{B}^\eta$, where

$$\beta_\eta := \max\left\{\eta^{3n_0}, \eta^{3\nu}\right\} + 30^{n_0} + 30^\nu. \tag{1.7}$$

In what follows, by a doubling ball we mean a $(6, \beta_6)$-doubling ball and simply denote $\widetilde{B}^6$ by $\widetilde{B}$.

For $\eta > 1$ and any two balls $B \subset Q \subset \mathcal{X}$, let

$$\widetilde{K}_{B,Q}^{(\eta)} := 1 + \sum_{k=1}^{N_{B,Q}^{(\eta)}} \frac{\mu(\eta^k B)}{\lambda(c_B, \eta^k r_B)},$$

where $N_{B,Q}^{(\eta)}$ represents the smallest integer k satisfying $\eta^k r_B \geq r_Q$, c_B and r_B represent the center and radius of the ball B, respectively. One always denote $N_{B,Q}^{(6)}$ by $N_{B,Q}$.

Then we review the concept of the space $\widetilde{\mathrm{RBMO}}\,(\mu)$.

Definition 1.5. *Let $\rho > 1$ and $\gamma \geq 1$. We say that a $L^1_{loc}(\mu)$-function $b \in \widetilde{RBMO}_{\rho,\gamma}(\mu)$ provided there exists a constant $C > 0$ and, for any ball $B \subset \mathcal{X}$, a number b_B such that*

$$\frac{1}{\mu(\rho B)} \int_B |b(x) - b_B|\, d\mu(x) \leq C \tag{1.8}$$

and, for two balls B and Q such that $B \subset Q$,

$$\left|b_B - b_Q\right| \leq C\left[\widetilde{K}_{B,Q}^{(\rho)}\right]^\gamma. \tag{1.9}$$

The smallest constant C satisfying both (1.8) *and* (1.9) *is called the $\widetilde{RBMO}_{\rho,\gamma}(\mu)$ norm of b and denoted by $\|b\|_{\widetilde{RBMO}_{\rho,\gamma}(\mu)}$.*

Remark 1. *X. Fu, D. Yang and D. Yang [10] pointed that the norms for different choice of $\rho > 1$ and $\gamma \geq 1$ are equivalent. In what follows, we denote $\widetilde{RBMO}_{\rho,\gamma}(\mu)$ simply by $\widetilde{RBMO}\,(\mu)$. Obviously, $\widetilde{RBMO}\,(\mu) \subset RBMO(\mu)$ (the space regularized BMO(μ) which was introduced by X. Tolsa [11]).*

For $1 \leq r \leq m < \infty$, let C_r^m be the cluster of all finite subsets $\sigma = \{\sigma_1, \ldots, \sigma_r\}$ of $\{1, \ldots, m\}$. For arbitrary $\sigma \in C_r^m$, let $\vec{b}_\sigma = \{b_{\sigma_1}, \ldots, b_{\sigma_r}\}$, where $b_{\sigma_i}, i = 1, \ldots, m$ are $\widetilde{\mathrm{RBMO}}\,(\mu)$-functions. We define the iterated commutators, generated by the multilinear θ-type generalized fractional integral $\mathcal{I}_{\alpha,m}$ and $\widetilde{\mathrm{RBMO}}\,(\mu)$-functions $b_{\sigma_i}, \sigma_i \in \sigma$,

$$\begin{aligned}
&\mathcal{I}_{\alpha,m,\vec{b}_\sigma}\vec{f}(x) =: [b_{\sigma_r}, \cdots, [b_{\sigma_1}, \mathcal{I}_{\alpha,m}]]\,\vec{f}(x) \\
&= \int_{\mathcal{X}^m}\left[\prod_{\sigma_i\in\sigma}(b_{\sigma_i}(x) - b_{\sigma_i}(y_i))\right] K_\alpha(x, y_1, \ldots, y_m)\left(\prod_{j=1}^m f_j\left(y_j\right) d\mu\left(y_j\right)\right).
\end{aligned} \tag{1.10}$$

The second theorem is the boundedness for the iterated commutators generated by the multilinear θ-type generalized fractional integral operator and $\widetilde{\mathrm{RBMO}}\,(\mu)$-functions on Lebesgue spaces.

Theorem 1.3. *Let* $0 < \beta < 1, 1 < t < 1/\beta, 1/s = 1/t - \beta, 0 < \alpha < m, 1 < p_1, \ldots, p_m < \infty$ *and* $0 < 1/q = 1/p_1 + \cdots + 1/p_m - \alpha < 1$. *Suppose* λ *satisfies the* ϵ*-weak reverse doubling condition for some* $\epsilon \in (0, \inf\{\beta, 1-\beta, 1/s\})$ *and* $b_{\sigma_i} \in \widetilde{RBMO}(\mu), \sigma_i \in \sigma$. *Then the iterated commutators* $\mathcal{I}_{\alpha,m,\vec{b}_\sigma}$ *of the multilinear* θ*-type generalized fractional integral with* $\widetilde{RBMO}(\mu)$*-functions, defined by* (1.10)*, is bounded from* $L^{p_1}(\mu) \times \cdots \times L^{p_m}(\mu)$ *to* $L^q(\mu)$*, that is, there exists a constant* $C > 0$ *such that*

$$\left\|\mathcal{I}_{\alpha,m,\vec{b}_\sigma}\vec{f}\right\|_{L^q(\mu)} \leq C\left(\prod_{\sigma_i\in\sigma}\left\|b_{\sigma_i}\right\|_{\widetilde{RBMO}(\mu)}\right)\left(\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\mu)}\right).$$

Remark 2. *If* $\mu(\mathcal{X}) < \infty$*, the boundedness for the multilinear* θ*-type generalized fractional integral and its iterated multi-commutators with* $\widetilde{RBMO}(\mu)$*-functions on Lebesgue spaces also holds as long as* I_β*, defined by* (1.3)*, is bounded from* $L^t(\mu)$ *to* $L^s(\mu)$.

Throughout this paper, the letter r and m stand for the dimensions of $\vec{b}_\sigma$ and $\vec{f}$, respectively. C always denotes a positive constant independent of the main parameters involved, but may different in different currents.

This paper is organized as follows. In Section 2, we present some necessary lemmas being used in the proof of the theorems. In Section 3, we establish the boundedness of the multilinear θ-type generalized fractional integral $\mathcal{I}_{\alpha,m}$ and its the iterated commutators $\mathcal{I}_{\alpha,m,\vec{b}_\sigma}$ with $\widetilde{\mathrm{RBMO}}(\mu)$-functions on Lebesgue spaces.

2. Preliminaries

In order to prove the theorems, some necessary lemmas are presented in this section. At the beginning, we introduce the fractional coefficient $\widetilde{K}^{\alpha}_{B,Q}$. For $\alpha \in [0,1)$ and any two balls B and Q such that $B := B(c_B, r_B) \subset Q$, $\widetilde{K}^{\alpha}_{B,Q}$ is defined by

$$\widetilde{K}^{\alpha}_{B,Q} := 1 + \sum_{k=1}^{N_{B,Q}}\left[\frac{\mu(6^kB)}{\lambda(c_B, 6^kr_B)}\right]^{1-\alpha},$$

where $N_{B,Q}$ is the smallest integer k satisfying $6^kr_B \geq r_Q$. $\widetilde{K}^{0}_{B,Q}$ is denoted by $\widetilde{K}_{B,Q}$.

Now we can introduce the sharp maximal operator associated with $\widetilde{K}^{\alpha}_{B,Q}$.

$$M^{\sharp,(\alpha)}f(x) = \sup_{B\ni x}\frac{1}{\mu(6B)}\int_B\left|f(y) - m_{\widetilde{B}}(f)\right|d\mu(y) + \sup_{(B,Q)\in\Delta_x}\frac{\left|m_B(f) - m_Q(f)\right|}{\widetilde{K}^{\alpha}_{B,Q}},$$

where $\Delta_x := \{(B,Q) : x \in B \subset Q$ and B, Q are $(6,\beta_6)$-doubling balls$\}$ and $m_E(f)$ represents the mean value of the function $f \in L^1_{\mathrm{loc}}(\mu)$ over any measurable set E, namely, $m_E(f) := \frac{1}{\mu(E)}\int_E f(x)d\mu(x)$.

Then we recall some results about some maximal operators. The non-centered doubling maximal operator is defined by

$$Nf(x) = \sup_{B\ni x,\ B \text{ doubling}}\frac{1}{\mu(B)}\int_B|f(y)|\,d\mu(y). \tag{2.1}$$

Notice that for every $f \in L^1_{\text{loc}}(\mu)$ and μ-a.e. $x \in \mathcal{X}$, it holds true that the non-centered doubling maximal operator $Nf(x)$ satisfies $|f(x)| \le Nf(x)$.

Another non-centered maximal operator

$$M^{(\alpha)}_{t,(\rho)}f(x) = \sup_{B \ni x} \left\{ \frac{1}{[\mu(\rho B)]^{1-\alpha t}} \int_B |f(y)|^t d\mu(y) \right\}^{1/t} \tag{2.2}$$

for $\rho > 1, 0 \le \alpha < 1$ and $t \ge 1$. When $\alpha = 0$, we simply write $M^{(0)}_{t,(\rho)}f(x)$ as $M_{t,(\rho)}f(x)$, and when $t = 1, \alpha = 0$, $M^{(0)}_{1,(\rho)}f(x)$ is denoted by $M_{(\rho)}f(x)$.

Lemma 2.1. *[12] For every $p > 1$ and $\rho > 1$, there exists a constant $C_1 > 0$ such that, for all $f \in L^p(\mu)$, the non-centered maximal operator $M_{t,(\rho)}f$ satisfies*

$$\left\| M_{t,(\rho)}f \right\|_{L^p(\mu)} \le C_1 \|f\|_{L^p(\mu)} .$$

Let $0 < \alpha < 1, 1 < t < p < \frac{1}{\alpha}, \rho \ge 5$ and $\frac{1}{q} = \frac{1}{p} - \alpha$, there exists a constant $C_2 > 0$ such that, for all $f \in L^p(\mu)$, the non-centered maximal operator $M^{(\alpha)}_{t,(\rho)}f$, defined by (2.2), satisfies

$$\left\| M^{(\alpha)}_{t,(\rho)}f \right\|_{L^q(\mu)} \le C_2 \|f\|_{L^p(\mu)}.$$

Lemma 2.2. *[2] For $f \in L^1_{loc}(\mu)$, $\int_{\mathcal{X}} f(x)d\mu(x) = 0$ if $\mu(\mathcal{X}) < \infty$. Assume $0 < \alpha < 1$ and $\inf\{1, Nf\} \in L^p(\mu), 1 < p < \infty$, then there exists a constant $C > 0$, independing of f, such that*

$$\|Nf\|_{L^p(\mu)} \le C \left\| M^{\sharp,(\alpha)}f \right\|_{L^p(\mu)} .$$

Then we focus on the equivalent characterization of the space $\widetilde{\mathrm{RBMO}}(\mu)$.

Lemma 2.3. *[13] Let $\rho \in (1, \infty)$ and β_6 be as in (1.7). For $b \in L^1_{loc}(\mu)$, the following statements are equivalent:*

(i) $b \in \widetilde{RBMO}(\mu)$;

(ii) there exists a constant $C > 0$ such that, for all balls B,

$$\frac{1}{\mu(\rho B)} \int_B \left| b(x) - m_{\widetilde{B}}(b) \right| d\mu(x) \le C,$$

and for all $(6, \beta_6)$-doubling balls $B \subset Q$,

$$\left| m_B(b) - m_Q(b) \right| \le C\widetilde{K}_{B,Q}.$$

Moreover, the infimum of the above constant C is equivalent to $\|b\|_{\widetilde{RBMO}(\mu)}$.

Lemma 2.4. *[13] Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space. Then, for every $\rho \in (1, \infty)$ and $p \in [1, \infty)$, there exists a constant $C > 0$ such that, for all $b \in \widetilde{RBMO}(\mu)$ and balls B,*

$$\left(\frac{1}{\mu(\rho B)} \int_B |b(x) - b_B|^p \, d\mu(x) \right)^{1/p} \le C \|b\|_{\widetilde{RBMO}(\mu)}$$

where b_B is as in Definition 1.5.

Because $\widetilde{\mathrm{RBMO}}(\mu) \subset \mathrm{RBMO}(\mu)$, according to Lemma 3.11 in [14], Lemmas 14, 15 in [5] and Lemma 3.2 in [11], we can give the following three lemmas directly.

Lemma 2.5. *Let $b \in \widetilde{RBMO}(\mu)$, $q \in (0,\infty)$. For all $x \in \mathcal{X}$,*

$$b_q(x) := \begin{cases} b(x), & \text{if } |b(x)| \le q, \\ q\dfrac{b(x)}{|b(x)|}, & \text{if } |b(x)| > q, \end{cases}$$

then $b_q \in \widetilde{RBMO}(\mu)$ and there exists a constant $C > 0$ such that $\left\|b_q\right\|_{\widetilde{RBMO}(\mu)} \le C\|b\|_{\widetilde{RBMO}(\mu)}$.

Lemma 2.6. *For $1 < \rho < \infty$ and $1 \le p < \infty$, then $b \in \widetilde{RBMO}(\mu)$ if and only if for all balls $B \in \mathcal{X}$,*

$$\left(\frac{1}{\mu(\rho B)}\int_B \left|b_B - m_{\widetilde{B}}(b)\right|^p d\mu(x)\right)^{\frac{1}{p}} \le C\|b\|_{\widetilde{RBMO}(\mu)},$$

and for any two $(6,\beta_6)$-doubling balls $B \subset Q$,

$$\left|m_B(b) - m_Q(b)\right| \le C\widetilde{K}_{B,Q}\|b\|_{\widetilde{RBMO}(\mu)}.$$

Lemma 2.7. *For any $k \in \mathbb{N}_+$ and $b \in \widetilde{RBMO}(\mu)$,*

$$\left|m_{\widetilde{6^k\frac{6}{5}B}}(b) - m_{\widetilde{B}}(b)\right| \le Ck\|b\|_{\widetilde{RBMO}(\mu)}.$$

With Lemmas 2.4, 2.6 and 2.7 in mind, we can give the following lemma, which will be used frequently later.

Lemma 2.8. *If $b \in \widetilde{RBMO}(\mu)$, then*

(i) there exists a constant $C_1 > 0$ such that, for all balls $B \in \mathcal{X}$, $1 \le \tau < \rho < \infty$, $1 \le \eta < \infty$ and $1 \le p < \infty$,

$$\left(\frac{1}{\mu(\rho B)}\int_{\tau B} \left|b(x) - m_{\widetilde{\eta B}}(b)\right|^p d\mu(x)\right)^{\frac{1}{p}} \le C_1\|b\|_{\widetilde{RBMO}(\mu)}.$$

(ii) there exists a constant $C_2 > 0$ such that, for all balls $B \in \mathcal{X}$, $k \in \mathbb{N}_+$ and $1 \le p < \infty$,

$$\left(\frac{1}{\mu(6^k 6B)}\int_{6^k\frac{6}{5}B} \left|b(x) - m_{\widetilde{B}}(b)\right|^p d\mu(x)\right)^{\frac{1}{p}} \le C_2 k\|b\|_{\widetilde{RBMO}(\mu)}.$$

Proof. As

$$\begin{aligned}
&\left(\frac{1}{\mu(\rho B)}\int_{\tau B} \left|b(x) - m_{\widetilde{\eta B}}(b)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&\le \left(\frac{1}{\mu(\rho B)}\int_{\tau B} |b(x) - b_{\tau B}|^p d\mu(x)\right)^{\frac{1}{p}} + \left(\frac{1}{\mu(\rho B)}\int_{\tau B} \left|b_{\tau B} - m_{\widetilde{\tau B}}(b)\right|^p d\mu(x)\right)^{\frac{1}{p}}
\end{aligned}$$

$$+\left(\frac{1}{\mu(\rho B)}\int_{\tau B}\left|m_{\widetilde{\tau B}}(b)-m_{\widetilde{\eta B}}(b)\right|^{p}d\mu(x)\right)^{\frac{1}{p}}$$

and $b \in \widetilde{\mathrm{RBMO}}\,(\mu)$, applying Lemmas 2.4 and 2.6, the desired result (i) can be directly obtained.

Then using (i) and Lemma 2.7,

$$\left(\frac{1}{\mu(6^k 6B)}\int_{6^k\frac{6}{5}B}\left|b(x)-m_{\widetilde{B}}(b)\right|^{p}d\mu(x)\right)^{\frac{1}{p}}$$
$$\leq\left(\frac{1}{\mu(6^k 6B)}\int_{6^k\frac{6}{5}B}\left|b(x)-m_{\widetilde{6^k\frac{6}{5}B}}(b)\right|^{p}d\mu(x)\right)^{\frac{1}{p}}+\left(\frac{1}{\mu(6^k 6B)}\int_{6^k\frac{6}{5}B}\left|m_{\widetilde{6^k\frac{6}{5}B}}(b)-m_{\widetilde{B}}(b)\right|^{p}d\mu(x)\right)^{\frac{1}{p}}.$$

Therefore, the desired result (ii) can also be directly obtained. □

Below we present an important lemma which is crucial in the proof of Theorem 1.3.

Lemma 2.9. *Let* $0<\beta<1, 1<t<1/\beta, 1<t<q<\infty, 1/s=1/t-\beta, 0<\alpha<m, 1<p_1,\ldots,p_m<\infty$ *and suppose* λ *satisfies the* ϵ*-weak reverse doubling condition for some* $\epsilon\in(0,\inf\{\beta,1-\beta,1/s\})$ *and* $b_{\sigma_i}\in\widetilde{RBMO}(\mu), \sigma_i\in\sigma$*. Then there exists a constant* $C>0$ *such that for every* $x\in\mathcal{X}, f_i\in L^{p_i}(\mu), i=1,\ldots,m,$

$$M^{\sharp,(\alpha/m)}\left(\mathcal{I}_{\alpha,m,\vec{b}_\sigma}\vec{f}\right)(x)$$
$$\leq C\sum_{j=1}^{r}\sum_{\sigma(j)}\left(\prod_{\sigma_i\in\sigma(j)}\left\|b_{\sigma_i}\right\|_{\widetilde{RBMO}(\mu)}\right)M_{t,(6)}\left(\mathcal{I}_{\alpha,m,\vec{b}_{\sigma(j)'}}\vec{f}\right)(x)+C\left(\prod_{\sigma_i\in\sigma}\left\|b_{\sigma_i}\right\|_{\widetilde{RBMO}(\mu)}\right)\left(\prod_{l=1}^{m}M^{(\alpha/m)}_{p_l,(5)}f_l(x)\right), \tag{2.3}$$

where $\sigma(j)$ *are the sets consisting of any* j *elements in* σ *and* $\sigma(j)'=\sigma\backslash\sigma(j)$.

Proof. By Lemma 2.5 and a standard limit argument, without loss of generality, we may assume that $b_{\sigma_i},\sigma_i\in\sigma$ are bounded functions, that is, $b_{\sigma_i}\in L^\infty(\mu),\sigma_i\in\sigma$. Furthermore, taking $b_{\sigma_i}=b_i, i=1,\ldots,r$ is feasible. As $L^\infty_c(\mu)$ (the set of all $L^\infty(\mu)$ functions with compact support) is dense in $L^p(\mu)$ for $1<p<\infty$, by standard density arguments, it is enough to consider the case that $f_i\in L^\infty_c(\mu), i=1,\ldots,m$. Similarly to Theorem 9.1 in [11], let

$$h_B:=m_B\left(\mathcal{I}_{\alpha,m}\left(\left(m_{\widetilde{B}}(b_1)-b_1\right)f_1\chi_{\mathcal{X}\backslash\frac{6}{5}B},\ldots,\left(m_{\widetilde{B}}(b_r)-b_r\right)f_r\chi_{\mathcal{X}\backslash\frac{6}{5}B},f_{r+1}\chi_{\mathcal{X}\backslash\frac{6}{5}B},\ldots,f_m\chi_{\mathcal{X}\backslash\frac{6}{5}B}\right)\right),$$

and

$$h_Q:=m_Q\left(\mathcal{I}_{\alpha,m}\left(\left(m_Q(b_1)-b_1\right)f_1\chi_{\mathcal{X}\backslash\frac{6}{5}Q},\ldots,\left(m_Q(b_r)-b_r\right)f_r\chi_{\mathcal{X}\backslash\frac{6}{5}Q},f_{r+1}\chi_{\mathcal{X}\backslash\frac{6}{5}Q},\ldots,f_m\chi_{\mathcal{X}\backslash\frac{6}{5}Q}\right)\right).$$

In order to prove (2.3), suppose B is an arbitrary ball and Q is a doubling ball containing B, it is sufficient to show that, for every $x\in B$,

$$\begin{aligned}&\frac{1}{\mu(6B)}\int_B\left|[b_r,\cdots,[b_1,\mathcal{I}_{\alpha,m}]]\vec{f}(z)-h_B\right|d\mu(z)\\&\leq C\sum_{j=1}^{r}\left[\left(\prod_{i=r-j+1}^{r}\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)M_{t,(6)}\left(\left[b_{r-j},\cdots,[b_1,\mathcal{I}_{\alpha,m}]\right]\vec{f}\right)(x)\right]\\&\quad+C\left(\prod_{i=1}^{r}\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^{m}M^{(\alpha/m)}_{p_l,(5)}f_l(x)\right),\end{aligned}\tag{2.4}$$

and for $Q \supset B \ni x$,

$$\begin{aligned}\left|h_B - h_Q\right| \le & C \sum_{j=1}^{r-1} \sum_{p=0}^{r-j-1} \sum_{k=0}^{r-j-p} \left(\prod_{i=r-j-p-k+1}^{r} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) M_{t,(6)} \left(\left[b_{r-j-p-k}, \cdots, [b_1, \mathcal{I}_{\alpha,m}] \right] \vec{f} \right)(x) \\ & + C \left(\widetilde{K}_{B,Q}^{\alpha/m} \right)^{r+m} \left(\prod_{i=1}^{r} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) \left(\prod_{l=1}^{m} M_{p_l,(5)}^{(\alpha/m)} f_l (x) \right).\end{aligned} \tag{2.5}$$

In fact, for any ball $B \ni x$,

$$\begin{aligned}& \frac{1}{\mu(6B)} \int_B \left| [b_r, \cdots, [b_1, \mathcal{I}_{\alpha,m}]] \vec{f} - m_{\widetilde{B}} \left([b_r, \cdots, [b_1, \mathcal{I}_{\alpha,m}]] \vec{f} \right) \right| d\mu \\ & \le \frac{1}{\mu(6B)} \int_B \left| [b_r, \cdots, [b_1, \mathcal{I}_{\alpha,m}]] \vec{f} - h_B \right| d\mu + \left| h_B - h_{\widetilde{B}} \right| + C \frac{1}{\mu(6\widetilde{B})} \int_{\widetilde{B}} \left| [b_r, \cdots, [b_1, \mathcal{I}_{\alpha,m}]] \vec{f} - h_{\widetilde{B}} \right| d\mu \\ & \le C \sum_{j=1}^{r} \sum_{p=0}^{r-j} \sum_{k=0}^{r-j-p} \left(\prod_{i=r-j-p-k+1}^{r} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) M_{t,(6)} \left(\left[b_{r-j-p-k}, \cdots, [b_1, \mathcal{I}_{\alpha,m}] \right] \vec{f} \right) \\ & \quad + C \left(\prod_{i=1}^{r} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) \left(\prod_{l=1}^{m} M_{p_l,(5)}^{(\alpha/m)} f_l \right).\end{aligned}$$

On the other hand, for all doubling balls $B \subset Q$ with $x \in B$ such that $\widetilde{K}_{B,Q} \le \widetilde{K}_{B,Q}^{\alpha/m} \le P_0$, where P_0 is the constant in Lemma 9.3 in [11], using (2.5), we have

$$\begin{aligned}\left|h_B - h_Q\right| \le & C \sum_{j=1}^{r-1} \sum_{p=0}^{r-j-1} \sum_{k=0}^{r-j-p} \left(\prod_{i=r-j-p-k+1}^{r} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) M_{t,(6)} \left(\left[b_{r-j-p-k}, \cdots, [b_1, \mathcal{I}_{\alpha,m}] \right] \vec{f} \right)(x) \\ & + C \widetilde{K}_{B,Q}^{\alpha/m} (P_0)^{r+m-1} \left(\prod_{i=1}^{r} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) \left(\prod_{l=1}^{m} M_{p_l,(5)}^{(\alpha/m)} f_l (x) \right).\end{aligned}$$

Therefore,

$$\begin{aligned}& \left| m_B \left([b_r, \cdots, [b_1, \mathcal{I}_{\alpha,m}]] \vec{f} \right) - m_Q \left([b_r, \cdots, [b_1, \mathcal{I}_{\alpha,m}]] \vec{f} \right) \right| \\ \le & \left| m_B \left([b_r, \cdots, [b_1, \mathcal{I}_{\alpha,m}]] \vec{f} \right) - h_B \right| + \left| h_B - h_Q \right| + \left| h_Q - m_Q \left([b_r, \cdots, [b_1, \mathcal{I}_{\alpha,m}]] \vec{f} \right) \right| \\ = & C \widetilde{K}_{B,Q}^{\alpha/m} \sum_{j=1}^{r} \sum_{p=0}^{r-j} \sum_{k=0}^{r-j-p} \left(\prod_{i=r-j-p-k+1}^{r} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) M_{t,(6)} \left(\left[b_{r-j-p-k}, \cdots, [b_1, \mathcal{I}_{\alpha,m}] \right] \vec{f} \right) \\ & + C \widetilde{K}_{B,Q}^{\alpha/m} \left(\prod_{i=1}^{r} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) \left(\prod_{l=1}^{m} M_{p_l,(5)}^{(\alpha/m)} f_l \right).\end{aligned}$$

Next we estimate (2.4) and (2.5). Consider (2.4) first. For some fixed balls $B \ni x$, as

$$\begin{aligned}& \mathcal{I}_{\alpha,m} \left((m_{\widetilde{B}}(b_1) - b_1) f_1, \ldots, (m_{\widetilde{B}}(b_r) - b_r) f_r, f_{r+1}, \ldots, f_m \right)(z) \\ & = \sum_{j=1}^{r} \sum_{1 \le k_1 < \cdots < k_j \le r} \left[\prod_{p=1}^{j} \left(m_{\widetilde{B}} \left(b_{k_p} \right) - b_{k_p}(z) \right) \right]\end{aligned}$$

$$\int_{\mathcal{X}^m} K(z,y_1,\dots,y_m)\left[\prod_{i\in\{1,\dots,r\}\setminus\{k_1,\dots,k_j\}}(b_i(z)-b_i(y_i))\right]\left(\prod_{l=1}^m f_l(y_l)d\mu(y_l)\right)+[b_r,\cdots,[b_1,\mathcal{I}_{\alpha,m}]]\,\vec{f}(z),$$

thus

$$\begin{aligned}
&\left(\frac{1}{\mu(6B)}\int_B\left|[b_r,\cdots,[b_1,\mathcal{I}_{\alpha,m}]]\,\vec{f}(z)-h_B\right|^\eta d\mu(z)\right)^{\frac{1}{\eta}}\\
&\le\sum_{j=1}^r\sum_{1\le k_1<\cdots<k_j\le r}\left(\frac{1}{\mu(6B)}\int_B\left|\left[\prod_{p=1}^j\left(m_{\widetilde{B}}\left(b_{k_p}\right)-b_{k_p}(z)\right)\right]\right.\right.\\
&\qquad\left.\left.\int_{\mathcal{X}^m}K(z,y_1,\dots,y_m)\left[\prod_{i\in\{1,\dots,r\}\setminus\{k_1,\dots,k_j\}}(b_i(z)-b_i(y_i))\right]\left(\prod_{l=1}^m f_l(y_l)d\mu(y_l)\right)\right|^\eta d\mu(z)\right)^{\frac{1}{\eta}}\\
&\quad+\left(\frac{1}{\mu(6B)}\int_B\left|\mathcal{I}_{\alpha,m}\left((m_{\widetilde{B}}(b_1)-b_1)f_1,\dots,(m_{\widetilde{B}}(b_r)-b_r)f_r,f_{r+1},\dots,f_m\right)(z)-h_B\right|^\eta d\mu(z)\right)^{\frac{1}{\eta}}\\
&=:C\sum_{j=1}^r I_{1,j}+I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_{1,j}&=\left(\frac{1}{\mu(6B)}\int_B\left|\left[\prod_{p=r-j+1}^r\left(m_{\widetilde{B}}\left(b_p\right)-b_p(z)\right)\right]\int_{\mathcal{X}^m}K(z,y_1,\dots,y_m)\left[\prod_{i=1}^{r-j}(b_i(z)-b_i(y_i))\right]\left(\prod_{l=1}^m f_l(y_l)d\mu(y_l)\right)\right|^\eta d\mu(z)\right)^{\frac{1}{\eta}}\\
&=\left(\frac{1}{\mu(6B)}\int_B\left|\left[\prod_{p=r-j+1}^r\left(m_{\widetilde{B}}\left(b_p\right)-b_p(z)\right)\right]\left[b_{r-j},\cdots,[b_1,\mathcal{I}_{\alpha,m}]\right]\vec{f}(z)\right|^\eta d\mu(z)\right)^{\frac{1}{\eta}}.
\end{aligned}$$

Estimate $I_{1,j}$ for each $j\in\{1,\dots,r\}$. Let $\frac{1}{t}+\sum_{p=r-j+1}^r\frac{1}{t_p}=\frac{1}{\eta}$. For a fixed ball B with $x\in B$, Hölder's inequality and (i) of Lemma 2.8 give that

$$\begin{aligned}
I_{1,j}&\le C\left[\prod_{p=r-j+1}^r\left(\frac{1}{\mu(6B)}\int_B\left|m_{\widetilde{B}}\left(b_p\right)-b_p(z)\right|^{t_p}d\mu(z)\right)^{\frac{1}{t_p}}\right]\left(\frac{1}{\mu(6B)}\int_B\left|\left[b_{r-j},\cdots,[b_1,\mathcal{I}_{\alpha,m}]\right]\vec{f}(z)\right|^t d\mu(z)\right)^{\frac{1}{t}}\\
&\le C\left(\prod_{p=r-j+1}^r\|b_p\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)M_{t,(6)}\left(\left[b_{r-j},\cdots,[b_1,\mathcal{I}_{\alpha,m}]\right]\vec{f}\right)(x).
\end{aligned}$$

Next, we turn to estimate I_2. For a fixed ball B with $x\in B$ and $f_i\in L_c^\infty(\mu)$, write $f_i=f_i(y)\chi_{\frac{6}{5}B}(y)+f_i(y)\chi_{\mathcal{X}\setminus\frac{6}{5}B}(y)=:f_{i,1}(y)+f_{i,2}(y)$, $i=1,\dots,m$. For each $j'\in\{0,\dots,m\}$,

$$\begin{aligned}
&\mathcal{I}_{\alpha,m}\left((m_{\widetilde{B}}(b_1)-b_1)f_1,\dots,(m_{\widetilde{B}}(b_r)-b_r)f_r,f_{r+1},\dots,f_m\right)(z)\\
&=\sum_{j'=0}^{m-1}\sum_{1\le k_1<\cdots<k_{j'}\le m}\int_{\mathcal{X}^m}K(z,y_1,\dots,y_m)\left[\prod_{i=1}^r(m_{\widetilde{B}}(b_i)-b_i(y_i))\right]\\
&\qquad\left(\prod_{l\in\{1,\dots,m\}\setminus\{k_1,\dots,k_{j'}\}}f_{l,1}(y_l)d\mu(y_l)\right)\left(\prod_{l=1}^{j'}f_{k_l,2}(y_l)d\mu(y_l)\right)
\end{aligned}$$

$$+ \mathcal{I}_{\alpha,m}\left((m_{\widetilde{B}}(b_1) - b_1) f_{1,2}, \ldots, (m_{\widetilde{B}}(b_r) - b_r) f_{r,2}, f_{r+1,2}, \ldots, f_{m,2}\right)(z)$$
$$=: C \sum_{j'=0}^{m-1} I'_{2,1,j'} + I'_{2,2},$$

where

$$I'_{2,1,j'} = \int_{\mathcal{X}^m} K(z, y_1, \ldots, y_m) \left[\prod_{i=1}^{r} (m_{\widetilde{B}}(b_i) - b_i(y_i))\right] \left(\prod_{l_1=1}^{r-j'_1} f_{l_1,1}(y_{l_1}) d\mu(y_{l_1})\right)$$
$$\left(\prod_{l_2=r-j'_1+1}^{r} f_{l_2,2}(y_{l_2}) d\mu(y_{l_2})\right) \left(\prod_{l_3=r+1}^{m-j'_2} f_{l_3,1}(y_{l_3}) d\mu(y_{l_3})\right) \left(\prod_{l_4=m-j'_2+1}^{m} f_{l_4,2}(y_{l_4}) d\mu(y_{l_4})\right),$$

and $j'_1 \in \{0, \ldots, r\}$, $j'_2 \in \{0, \ldots, m-r\}$ satisfy $j' = j'_1 + j'_2$.

Therefore,

$$I_2 = C\left(\frac{1}{\mu(6B)} \int_B \left|\sum_{j'=0}^{m-1} I'_{2,1,j'} + I'_{2,2} - h_B\right|^{\eta} d\mu(z)\right)^{\frac{1}{\eta}}$$
$$\leq C \sum_{j'=0}^{m-1} \left(\frac{1}{\mu(6B)} \int_B \left|I'_{2,1,j'}\right|^{\eta} d\mu(z)\right)^{\frac{1}{\eta}} + \left(\frac{1}{\mu(6B)} \int_B \left|I'_{2,2} - h_B\right|^{\eta} d\mu(z)\right)^{\frac{1}{\eta}}$$
$$=: C \sum_{j'=0}^{m-1} I_{2,1,j'} + I_{2,2}.$$

For $I_{2,1,j'}$, $j' \in \{0, \ldots, m-1\}$, using the condition (1.4), interval decomposition, Hölder's inequality, Lemmas 2.6–2.8, Eq (1.1) and Definition 1.3, we obtain

$$I_{2,1,j'} \leq C \left(\prod_{l_1=1}^{r-j'_1} \int_{\frac{6}{5}B} \left|b_{l_1}(y_{l_1}) - m_{\widetilde{B}}(b_{l_1})\right| \left|f_{l_1}(y_{l_1})\right| d\mu(y_{l_1})\right) \left(\prod_{l_3=r+1}^{m-j'_2} \int_{\frac{6}{5}B} \left|f_{l_3}(y_{l_3})\right| d\mu(y_{l_3})\right)$$
$$\left[\frac{1}{\mu(6B)} \int_B \left(\prod_{l_2=r-j'_1+1}^{r} \int_{\mathcal{X}\setminus\frac{6}{5}B} \frac{\left|b_{l_2}(y_{l_2}) - m_{\widetilde{B}}(b_{l_2})\right| \left|f_{l_2}(y_{l_2})\right| d\mu(y_{l_2})}{[\lambda(z, d(z, y_{k_2}))]^{(m-\alpha)/j'}}\right)^{\eta}\right.$$
$$\left.\left(\prod_{l_4=m-j'_2+1}^{m} \int_{\mathcal{X}\setminus\frac{6}{5}B} \frac{\left|f_{l_4}(y_{l_4})\right| d\mu(y_{l_4})}{[\lambda(z, d(z, y_{l_4}))]^{(m-\alpha)/j'}}\right)^{\eta} d\mu(z)\right]^{\frac{1}{\eta}}$$
$$\leq C (\mu(6B))^{\left(1-\frac{\alpha}{m}\right)(m-j')} \left(\frac{\mu(B)}{\mu(6B)}\right)^{\frac{1}{\eta}}$$
$$\left[\prod_{l_1=1}^{r-j'_1} \left(\frac{1}{\mu(6B)} \int_{\frac{6}{5}B} \left|b_{l_1}(y_{l_1}) - m_{\widetilde{B}}(b_{l_1})\right|^{p'_{l_1}} d\mu(y_{l_1})\right)^{\frac{1}{p'_{l_1}}} \left(\frac{1}{(\mu(6B))^{1-\alpha p_{l_1}/m}} \int_{\frac{6}{5}B} \left|f_{l_1}(y_{l_1})\right|^{p_{l_1}} d\mu(y_{l_1})\right)^{\frac{1}{p_{l_1}}}\right]$$
$$\left[\prod_{l_2=r-j'_1+1}^{r} \left(\sum_{k_2=1}^{\infty} \int_{6^{k_2}\frac{6}{5}B\setminus 6^{k_2-1}\frac{6}{5}B} \frac{\left|b_{l_2}(y_{l_2}) - m_{\widetilde{B}}(b_{l_2})\right| \left|f_{l_2}(y_{l_2})\right| d\mu(y_{l_2})}{[\lambda(c_B, d(c_B, y_{l_2}))]^{(m-\alpha)/j'}}\right)\right]$$

$$\left[\prod_{l_3=r+1}^{m-j_2'}\left(\frac{1}{(\mu(6B))^{1-\alpha p_{l_3}/m}}\int_{\frac{6}{5}B}\left|f_{l_3}(y_{l_3})\right|^{p_{l_3}}d\mu(y_{l_3})\right)^{\frac{1}{p_{l_3}}}\right]$$

$$\left[\prod_{l_4=m-j_2'+1}^{m}\left(\sum_{k_4=1}^{\infty}\int_{6^{k_4}\frac{6}{5}B\setminus 6^{k_4-1}\frac{6}{5}B}\frac{\left|f_{l_4}(y_{l_4})\right|d\mu(y_{l_4})}{[\lambda(c_B,d(c_B,y_{l_4}))]^{(m-\alpha)/j'}}\right)\right]$$

$$\leq C(\mu(6B))^{(1-\frac{\alpha}{m})(m-j')}\left(\prod_{l_1=1}^{r-j_1'}\left\|b_{l_1}\right\|_{\widetilde{\mathrm{RBMO}}(\mu)}M^{(\alpha/m)}_{p_{l_1},(5)}f_{l_1}(x)\right)$$

$$\left[\prod_{l_2=r-j_1'+1}^{r}\left(\sum_{k_2=1}^{\infty}\frac{\int_{6^{k_2}\frac{6}{5}B}\left|b_{l_2}(y_{l_2})-m_{\widetilde{B}}(b_{l_2})\right|\left|f_{l_2}(y_{l_2})\right|d\mu(y_{l_2})}{\left(\lambda\left(c_B,6^{k_2-1}\frac{6}{5}r_B\right)\right)^{(m-\alpha)/j'}}\right)\right]$$

$$\left(\prod_{l_3=r+1}^{m-j_2'}M^{(\alpha/m)}_{p_{l_3},(5)}f_{l_3}(x)\right)\left[\prod_{l_4=m-j_2'+1}^{m}\left(\sum_{k_4=1}^{\infty}\frac{\int_{6^{k_4}\frac{6}{5}B}\left|f_{l_4}(y_{l_4})\right|d\mu(y_{l_4})}{\left(\lambda\left(c_B,6^{k_4-1}\frac{6}{5}r_B\right)\right)^{(m-\alpha)/j'}}\right)\right]$$

$$\leq C(\mu(6B))^{(1-\frac{\alpha}{m})(m-j')}\left(\prod_{i=1}^{r}\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^{m}M^{(\alpha/m)}_{p_l,(5)}f_l(x)\right)$$

$$\left[\prod_{l_2=r-j_1'+1}^{r}\left(\sum_{k_2=1}^{\infty}\frac{k_2\left(\mu(6^{k_2}6B)\right)^{1-\alpha/m}}{\left(\lambda\left(c_B,6^{k_2-1}\frac{6}{5}r_B\right)\right)^{(m-\alpha)/j'}}\right)\right]\left[\prod_{l_4=m-j_2'+1}^{m}\left(\sum_{k_4=1}^{\infty}\frac{\left(\mu(6^{k_4}6B)\right)^{1-\alpha/m}}{\left(\lambda\left(c_B,6^{k_4-1}\frac{6}{5}r_B\right)\right)^{(m-\alpha)/j'}}\right)\right]$$

$$\leq C\left(\prod_{i=1}^{r}\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^{m}M^{(\alpha/m)}_{p_l,(5)}f_l(x)\right)$$

$$\left[\prod_{l_2=r-j_1'+1}^{r}\left(\sum_{k_2=1}^{\infty}\frac{k_2}{[C(6^{k_2})]^{[(m-\alpha)/j'-1+\alpha/m]}}\right)\right]\left[\prod_{l_4=m-j_2'+1}^{m}\left(\sum_{k_4=1}^{\infty}\frac{1}{[C(6^{k_4})]^{[(m-\alpha)/j'-1+\alpha/m]}}\right)\right]$$

$$\leq C\left(\prod_{i=1}^{r}\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^{m}M^{(\alpha/m)}_{p_l,(5)}f_l(x)\right).$$

For $I_{2,2}$, let $z, z_0 \in B$, write

$$I_{2,2}=\left(\frac{1}{\mu(6B)}\int_B\left|\frac{1}{\mu(B)}\int_B(\mathcal{I}_{\alpha,m}((m_{\widetilde{B}}(b_1)-b_1)f_{1,2},\ldots,(m_{\widetilde{B}}(b_r)-b_r)f_{r,2},f_{r+1,2},\ldots,f_{m,2})(z)\right.\right.$$

$$\left.\left.-\mathcal{I}_{\alpha,m}((m_{\widetilde{B}}(b_1)-b_1)f_{1,2},\ldots,(m_{\widetilde{B}}(b_r)-b_r)f_{r,2},f_{r+1,2},\ldots,f_{m,2})(z_0))d\mu(z_0)\right|^{\eta}d\mu(z)\right)^{\frac{1}{\eta}}$$

$$\leq\left[\frac{1}{\mu(6B)}\int_B\left(\frac{1}{\mu(B)}\int_B\left|\mathcal{I}_{\alpha,m}((m_{\widetilde{B}}(b_1)-b_1)f_{1,2},\ldots,(m_{\widetilde{B}}(b_r)-b_r)f_{r,2},f_{r+1,2},\ldots,f_{m,2})(z)\right.\right.\right.$$

$$\left.\left.\left.-\mathcal{I}_{\alpha,m}((m_{\widetilde{B}}(b_1)-b_1)f_{1,2},\ldots,(m_{\widetilde{B}}(b_r)-b_r)f_{r,2},f_{r+1,2},\ldots,f_{m,2})(z_0)\right|d\mu(z_0)\right)^{\eta}d\mu(z)\right]^{\frac{1}{\eta}}.$$

Thus, what we need to do is to give the following estimate which is following from the conditions (1.5) and (1.6), interval decomposition and properties of θ,

$$\left|\mathcal{I}_{\alpha,m}((m_{\widetilde{B}}(b_1)-b_1)f_{1,2},\ldots,(m_{\widetilde{B}}(b_r)-b_r)f_{r,2},f_{r+1,2},\ldots,f_{m,2})(z)\right.$$

$$
\begin{aligned}
&-\mathcal{I}_{\alpha,m}\left((m_{\widetilde{B}}(b_1)-b_1)f_{1,2},\ldots,(m_{\widetilde{B}}(b_r)-b_r)f_{r,2},f_{r+1,2},\ldots,f_{m,2}\right)(z_0)\Big|\\
&\le C\int_{(\mathcal{X}\setminus\frac{6}{5}B)^m}\theta\left(\frac{d(z,z_0)}{\sum_{i=1}^m d(z,y_i)}\right)\frac{\prod_{i=1}^r\left|b_i(y_i)-m_{\widetilde{B}}(b_i)\right|}{\left[\sum_{i=1}^m\lambda(z,d(z,y_i))\right]^{m-\alpha}}\prod_{l=1}^m|f_l(y_l)|\,d\mu(y_l)\\
&\le C\sum_{k_1,\ldots,k_m=1}^{\infty}\int_{6^{k_1}\frac{6}{5}B\setminus6^{k_1-1}\frac{6}{5}B}\cdots\int_{6^{k_m}\frac{6}{5}B\setminus6^{k_m-1}\frac{6}{5}B}\theta\left(\frac{2r_B}{\sum_{i=1}^m d(c_B,y_i)}\right)\frac{\prod_{i=1}^r\left|b_i(y_i)-m_{\widetilde{B}}(b_i)\right|}{\left[\sum_{i=1}^m\lambda(c_B,d(c_B,y_i))\right]^{m-\alpha}}\left(\prod_{l=1}^m|f_l(y_l)|\right)\\
&\quad d\mu(y_m)\cdots d\mu(y_1)\\
&\le C\sum_{k_1,\ldots,k_m=1}^{\infty}\theta\left(\frac{2r_B}{\sum_{i=1}^m 6^{k_i-1}\frac{6}{5}r_B}\right)\left(\sum_{i=1}^m\lambda\left(c_B,6^{k_i-1}\frac{6}{5}r_B\right)\right)^{-m+\alpha}\\
&\quad\left(\prod_{l_1=1}^r\int_{6^{k_{l_1}}\frac{6}{5}B}\left|b_{l_1}(y_{l_1})-m_{\widetilde{B}}(b_{l_1})\right|\left|f_{l_1}(y_{l_1})\right|d\mu(y_{l_1})\right)\left(\prod_{l_2=r+1}^m\int_{6^{k_{l_2}}\frac{6}{5}B}\left|f_{l_2}(y_{l_2})\right|d\mu(y_{l_2})\right)\\
&\le C\sum_{k_{\max}=1}^{\infty}k_{\max}^{m-1}\theta\left(\frac{5}{6^{k_{\max}-1}3}\right)\left(\lambda\left(c_B,6^{k_{\max}-1}\frac{6}{5}r_B\right)\right)^{-m+\alpha}\left(\prod_{l_1=1}^r\int_{6^{k_{\max}}\frac{6}{5}B}\left|b_{l_1}(y_{l_1})-m_{\widetilde{B}}(b_{l_1})\right|\left|f_{l_1}(y_{l_1})\right|d\mu(y_{l_1})\right)\\
&\quad\left(\prod_{l_2=r+1}^m\int_{6^{k_{\max}}\frac{6}{5}B}\left|f_{l_2}(y_{l_2})\right|d\mu(y_{l_2})\right),
\end{aligned}
$$

where $k_{\max}=\max_{1\le i\le m}k_i$. Then Hölder's inequality, (ii) of Lemma 2.8, Definition 1.3 and properties of θ give that

$$
\begin{aligned}
&\Big|\mathcal{I}_{\alpha,m}\left((m_{\widetilde{B}}(b_1)-b_1)f_{1,2},\ldots,(m_{\widetilde{B}}(b_r)-b_r)f_{r,2},f_{r+1,2},\ldots,f_{m,2}\right)(z)\\
&\quad-\mathcal{I}_{\alpha,m}\left((m_{\widetilde{B}}(b_1)-b_1)f_{1,2},\ldots,(m_{\widetilde{B}}(b_r)-b_r)f_{r,2},f_{r+1,2},\ldots,f_{m,2}\right)(z_0)\Big|\\
&\le C\left(\prod_{i=1}^r\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^m M_{p_l,(5)}^{(\alpha/m)}f_l(x)\right)\sum_{k_{\max}=1}^{\infty}k_{\max}^{m+r-1}\theta\left((5/6)^{k_{\max}}\right)\\
&\le C\left(\prod_{i=1}^r\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^m M_{p_l,(5)}^{(\alpha/m)}f_l(x)\right)\int_0^1\frac{\theta(t)}{t}\left|\log t^{m+r-1}\right|dt\\
&\le C\left(\prod_{i=1}^r\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^m M_{p_l,(5)}^{(\alpha/m)}f_l(x)\right),
\end{aligned}
$$

where we use the following result

$$
\infty>\int_0^1\frac{\theta(t)}{t}\left|\log t^{m+r-1}\right|dt\ge\sum_{k=1}^{\infty}\int_{(\frac{5}{6})^k}^{(\frac{5}{6})^{k-1}}\frac{\theta\left((5/6)^k\right)}{(5/6)^{k-1}}\left|\log(5/6)^{k^{m+r-1}}\right|dt=C\sum_{k=1}^{\infty}k^{m+r-1}\theta\left((5/6)^k\right).
$$

Therefore,

$$
\begin{aligned}
I_{2,2}&\le C\left[\frac{1}{\mu(6B)}\int_B\left(\frac{1}{\mu(B)}\int_B\left|\left(\prod_{i=1}^r\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^m M_{p_l,(5)}^{(\alpha/m)}f_l(x)\right)\right|d\mu(z_0)\right)^{\eta}d\mu(z)\right]^{\frac{1}{\eta}}\\
&\le C\left(\prod_{i=1}^r\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^m M_{p_l,(5)}^{(\alpha/m)}f_l(x)\right).
\end{aligned}
$$

Thus,

$$I_2 \le C \sum_{j'=0}^{m-1} I_{2,1,j'} + I_{2,2} \le C \left(\prod_{i=1}^{r} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) \left(\prod_{l=1}^{m} M_{p_l,(5)}^{(\alpha/m)} f_l(x) \right).$$

Furthermore,

$$\begin{aligned}
&\left(\frac{1}{\mu(6B)} \int_B \left| [b_r, \cdots, [b_1, \mathcal{I}_{\alpha,m}]] \vec{f}(z) - h_B \right|^\eta d\mu(z) \right)^{\frac{1}{\eta}} \\
&\le C \sum_{j=1}^{r} \left[\left(\prod_{i=r-j+1}^{r} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) M_{t,(6)} \left([b_r, \cdots, [b_1, \mathcal{I}_{\alpha,m}]] \vec{f} \right)(x) \right] + C \left(\prod_{i=1}^{r} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) \left(\prod_{l=1}^{m} M_{p_l,(5)}^{(\alpha/m)} f_l(x) \right),
\end{aligned}$$

which completes the proof of (2.4).

Next, we turn to estimate (2.5). For all balls $Q \supset B \ni x$ with doubling balls Q, denote $N := N_{B,Q} + 1$, which means $6^N B \supset \frac{6}{5} Q$, then write

$$\begin{aligned}
&\left| h_B - h_Q \right| \\
&\le \Big| m_B \left(\mathcal{I}_{\alpha,m} \left((b_1 - m_{\widetilde{B}}(b_1)) f_1 \chi_{\mathcal{X} \setminus 6^N B}, \ldots, (b_r - m_{\widetilde{B}}(b_r)) f_r \chi_{\mathcal{X} \setminus 6^N B}, f_{r+1} \chi_{\mathcal{X} \setminus 6^N B}, \ldots, f_m \chi_{\mathcal{X} \setminus 6^N B} \right) \right) \\
&\qquad - m_Q \left(\mathcal{I}_{\alpha,m} \left((b_1 - m_{\widetilde{B}}(b_1)) f_1 \chi_{\mathcal{X} \setminus 6^N B}, \ldots, (b_r - m_{\widetilde{B}}(b_r)) f_r \chi_{\mathcal{X} \setminus 6^N B}, f_{r+1} \chi_{\mathcal{X} \setminus 6^N B}, \ldots, f_m \chi_{\mathcal{X} \setminus 6^N B} \right) \right) \Big| \\
&\quad + \Big| m_Q \left(\mathcal{I}_{\alpha,m} \left((b_1 - m_Q(b_1)) f_1 \chi_{\mathcal{X} \setminus 6^N B}, \ldots, (b_r - m_Q(b_r)) f_r \chi_{\mathcal{X} \setminus 6^N B}, f_{r+1} \chi_{\mathcal{X} \setminus 6^N B}, \ldots, f_m \chi_{\mathcal{X} \setminus 6^N B} \right) \right) \\
&\qquad - m_Q \left(\mathcal{I}_{\alpha,m} \left((b_1 - m_{\widetilde{B}}(b_1)) f_1 \chi_{\mathcal{X} \setminus 6^N B}, \ldots, (b_r - m_{\widetilde{B}}(b_r)) f_r \chi_{\mathcal{X} \setminus 6^N B}, f_{r+1} \chi_{\mathcal{X} \setminus 6^N B}, \ldots, f_m \chi_{\mathcal{X} \setminus 6^N B} \right) \right) \Big| \\
&\quad + C \sum_{j=1}^{m-1} \sum_{i_1=0}^{j-1} \Bigg| \frac{1}{\mu(B)} \int_B \int_{(\mathcal{X} \setminus \frac{6}{5} B)^{m-j}} \int_{(6^N B \setminus \frac{6}{5} B)^{i_1+1}} \int_{(\mathcal{X} \setminus 6^N B)^{j-i_1-1}} \\
&\qquad\qquad K(z, y_1, \ldots, y_m) \left[\prod_{i=1}^{r} (b_i(z) - m_{\widetilde{B}}(b_i)) \right] \left(\prod_{l=1}^{m} f_l(y_l) d\mu(y_l) \right) d\mu(z) \Bigg| \\
&\quad + C \sum_{j=1}^{m-1} \sum_{i_2=0}^{m-1} \sum_{i_3=1}^{m-j} \Bigg| \frac{1}{\mu(B)} \int_B \int_{(6^N B \setminus \frac{6}{5} B)^{m-j+i_2+1}} \int_{(\mathcal{X} \setminus 6^N B)^{j-i_2-i_3-1}} \\
&\qquad\qquad K(z, y_1, \ldots, y_m) \left[\prod_{i=1}^{r} (b_i(z) - m_{\widetilde{B}}(b_i)) \right] \left(\prod_{l=1}^{m} f_l(y_l) d\mu(y_l) \right) d\mu(z) \Bigg| \\
&\quad + C \sum_{j=1}^{m-1} \sum_{i_1=0}^{j-1} \Bigg| \frac{1}{\mu(Q)} \int_Q \int_{(\mathcal{X} \setminus \frac{6}{5} Q)^{m-j}} \int_{(6^N B \setminus \frac{6}{5} Q)^{i_1+1}} \int_{(\mathcal{X} \setminus 6^N B)^{j-i_1-1}} \\
&\qquad\qquad K(z, y_1, \ldots, y_m) \left[\prod_{i=1}^{r} (b_i(z) - m_{\widetilde{B}}(b_i)) \right] \left(\prod_{l=1}^{m} f_l(y_l) d\mu(y_l) \right) d\mu(z) \Bigg| \\
&\quad + C \sum_{j=1}^{m-1} \sum_{i_2=0}^{m-1} \sum_{i_3=1}^{m-j} \Bigg| \frac{1}{\mu(Q)} \int_Q \int_{(6^N B \setminus \frac{6}{5} Q)^{m-j+i_2+1}} \int_{(\mathcal{X} \setminus 6^N B)^{j-i_2-i_3-1}} \\
&\qquad\qquad K(z, y_1, \ldots, y_m) \left[\prod_{i=1}^{r} (b_i(z) - m_{\widetilde{B}}(b_i)) \right] \left(\prod_{l=1}^{m} f_l(y_l) d\mu(y_l) \right) d\mu(z) \Bigg|
\end{aligned}$$

$$=: J_1 + J_2 + J_3 + J_4 + J_5 + J_6.$$

In the above formula, the methods to estimate J_3, J_4, J_5 and J_6 are similar to that of J_2. Therefore, for the sake of simplicity, we only estimate the first two terms.

For J_1, let $z, z_0 \in B$, write

$$\begin{aligned} J_1 = \bigg| \frac{1}{\mu(B)} \int_B \bigg[\frac{1}{\mu(Q)} \int_Q \Big(\mathcal{I}_{\alpha,m} \left((b_1 - m_{\widetilde{B}}(b_1)) f_1 \chi_{X \setminus 6^N B}, \ldots, (b_r - m_{\widetilde{B}}(b_r)) f_r \chi_{X \setminus 6^N B}, f_{r+1} \chi_{X \setminus 6^N B}, \ldots, \right. \\ \left. f_m \chi_{X \setminus 6^N B} \right)(z) - \mathcal{I}_{\alpha,m} \left((b_1 - m_{\widetilde{B}}(b_1)) f_1 \chi_{X \setminus 6^N B}, \ldots, (b_r - m_{\widetilde{B}}(b_r)) f_r \chi_{X \setminus 6^N B}, f_{r+1} \chi_{X \setminus 6^N B}, \ldots, \right. \\ \left. f_m \chi_{X \setminus 6^N B} \right)(z_0) \Big) d\mu(z_0) \bigg] d\mu(z) \bigg|. \end{aligned}$$

Similar to the estimate of $I_{2,2}$,

$$J_1 \leq C \left(\prod_{i=1}^{r} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) \left(\prod_{l=1}^{m} M_{p_l,(5)}^{(\alpha/m)} f_l(x) \right).$$

For J_2, first consider

$$\begin{aligned} & \mathcal{I}_{\alpha,m} \left((b_1 - m_Q(b_1)) f_1 \chi_{X \setminus 6^N B}, \ldots, (b_r - m_Q(b_r)) f_r \chi_{X \setminus 6^N B}, f_{r+1} \chi_{X \setminus 6^N B}, \ldots, f_m \chi_{X \setminus 6^N B} \right)(z) \\ & \quad - \mathcal{I}_{\alpha,m} \left((b_1 - m_{\widetilde{B}}(b_1)) f_1 \chi_{X \setminus 6^N B}, \ldots, (b_r - m_{\widetilde{B}}(b_r)) f_r \chi_{X \setminus 6^N B}, f_{r+1} \chi_{X \setminus 6^N B}, \ldots, f_m \chi_{X \setminus 6^N B} \right)(z) \\ & = \sum_{j=1}^{r-1} \sum_{1 \leq k_1 < \cdots < k_j \leq r} \sum_{p=0}^{r-j-1} \sum_{1 \leq n_1 < \cdots < n_p \leq r-j} \left[\prod_{i=1}^{j} (m_{\widetilde{B}}(b_{k_i}) - m_Q(b_{k_i})) \right] \left[\prod_{i=1}^{p} (m_Q(b_{n_i}) - m_{\widetilde{B}}(b_{n_i})) \right] \\ & \qquad \int_{(X \setminus 6^N B)^m} K(z, y_1, \ldots, y_m) \left[\prod_{i \in (\{1,\ldots,r\} \setminus \{k_1,\ldots,k_j\}) \setminus \{n_1,\ldots,n_p\}} (b_i(y_i) - m_Q(b_i)) \right] \left(\prod_{l=1}^{m} f_l(y_l) \, d\mu(y_l) \right) \\ & = C \sum_{j=1}^{r-1} \sum_{p=0}^{r-j-1} (-1)^p \left[\prod_{i=r-j-p+1}^{r} (m_{\widetilde{B}}(b_i) - m_Q(b_i)) \right] \mathcal{I}_{\alpha,m} \left((b_1 - m_Q(b_1)) f_1 \chi_{X \setminus 6^N B}, \ldots, \right. \\ & \qquad \left. \left(b_{r-j-p} - m_Q(b_{r-j-p}) \right) f_{r-j-p} \chi_{X \setminus 6^N B}, f_{r-j-p+1} \chi_{X \setminus 6^N B}, \ldots, f_r \chi_{X \setminus 6^N B} \right) \\ & =: C \sum_{j=1}^{r-1} \sum_{p=0}^{r-j-1} J_{2,j,p}. \end{aligned}$$

As

$$\begin{aligned} & \mathcal{I}_{\alpha,m} \left((b_1 - m_Q(b_1)) f_1 \chi_{X \setminus 6^N B}, \ldots, \left(b_{r-j-p} - m_Q(b_{r-j-p}) \right) f_{r-j-p} \chi_{X \setminus 6^N B}, f_{r-j-p+1} \chi_{X \setminus 6^N B}, \ldots, f_r \chi_{X \setminus 6^N B} \right) \\ = & \mathcal{I}_{\alpha,m} \left((b_1 - m_Q(b_1)) f_1, \ldots, \left(b_{r-j-p} - m_Q(b_{r-j-p}) \right) f_{r-j-p}, f_{r-j-p+1}, \ldots, f_r \right) \\ & - \mathcal{I}_{\alpha,m} \left((b_1 - m_Q(b_1)) f_1 \chi_{\frac{6}{5}Q}, \ldots, \left(b_{r-j-p} - m_Q(b_{r-j-p}) \right) f_{r-j-p} \chi_{\frac{6}{5}Q}, f_{r-j-p+1} \chi_{\frac{6}{5}Q}, \ldots, f_r \chi_{\frac{6}{5}Q} \right) \\ & - \sum_{i_1=1}^{r-j-p} \mathcal{I}_{\alpha,m} \Big((b_1 - m_Q(b_1)) f_1 \chi_{6^N B}, \ldots, (b_{i_1-1} - m_Q(b_{i_1-1})) f_{i_1-1} \chi_{6^N B}, (b_{i_1} - m_Q(b_{i_1})) f_{i_1} \chi_{X \setminus \frac{6}{5}Q}, \\ & \qquad (b_{i_1+1} - m_Q(b_{i_1+1})) f_{i_1+1} \chi_{6^N B}, \ldots, \left(b_{r-j-p} - m_Q(b_{r-j-p}) \right) f_{r-j-p} \chi_{6^N B}, \end{aligned}$$

$$f_{r-j-p+1}\chi_{6^N B}, \dots, f_m\chi_{6^N B}\Big)$$

$$- \sum_{i_2=r-j-p+1}^{r} \mathcal{I}_{\alpha,m}\Big((b_1 - m_Q(b_1))\, f_1\chi_{6^N B}, \dots, \left(b_{r-j-p} - m_Q(b_{r-j-p})\right) f_{r-j-p}\chi_{6^N B}, f_{r-j-p+1}\chi_{6^N B}, \dots,$$

$$f_{i_2-1}\chi_{6^N B}, f_{i_2}\chi_{\mathcal{X}\setminus\frac{6}{5}Q}, f_{i_2+1}\chi_{6^N B}, \dots, f_m\chi_{6^N B}\Big)$$

$$+ (m-1)\, \mathcal{I}_{\alpha,m}\Big((b_1 - m_Q(b_1))\, f_1\chi_{6^N B\setminus\frac{6}{5}Q}, \dots, \left(b_{r-j-p} - m_Q(b_{r-j-p})\right) f_{r-j-p}\chi_{6^N B\setminus\frac{6}{5}Q},$$

$$f_{r-j-p+1}\chi_{6^N B\setminus\frac{6}{5}Q}, \dots, f_r\chi_{6^N B\setminus\frac{6}{5}Q}\Big)$$

$$=: J_{2,1,j,p} + J_{2,2,j,p} + \sum_{i_1=1}^{r-j-p} J_{2,3,j,p,i_1} + \sum_{i_2=r-j-p+1}^{r} J_{2,4,j,p,i_2} + C J_{2,5,j,p},$$

according to Lemma 2.6,

$$\begin{aligned}
J_2 &= C \left| \frac{1}{\mu(Q)} \int_Q \left(\sum_{j=1}^{r-1} \sum_{p=0}^{r-j-1} J_{2,j,p} \right) d\mu(z) \right| \\
&\leq C \sum_{j=1}^{r-1} \sum_{p=0}^{r-j-1} \left| \frac{1}{\mu(Q)} \int_Q \left[\prod_{i=r-j-p+1}^{r} \left(m_{\widetilde{B}}(b_i) - m_Q(b_i)\right) \right] \Big(J_{2,1,j,p} + J_{2,2,j,p} \right. \\
&\qquad \left. + \sum_{i_1=1}^{r-j-p} J_{2,3,j,p,i_1} + \sum_{i_2=r-j-p+1}^{r} J_{2,4,j,p,i_2} + J_{2,5,j,p} \Big) d\mu(z) \right| \\
&\leq C \sum_{j=1}^{r-1} \sum_{p=0}^{r-j-1} \left(\prod_{i=r-j-p+1}^{r} \widetilde{K}_{B,Q} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) \Bigg(\left| m_Q\left(J_{2,1,j,p}\right) \right| + \left| m_Q\left(J_{2,2,j,p}\right) \right| \\
&\qquad + \sum_{i_1=1}^{r-j-p} \left| m_Q\left(J_{2,3,j,p,i_1}\right) \right| + \sum_{i_2=r-j-p+1}^{r} \left| m_Q\left(J_{2,4,j,p,i_2}\right) \right| + \left| m_Q\left(J_{2,5,j,p}\right) \right| \Bigg).
\end{aligned}$$

For $\left|m_Q\left(J_{2,1,j,p}\right)\right|$, using the property of μ, Hölder's inequality and (i) of Lemma 2.8, we obtain

$$\begin{aligned}
\left|m_Q\left(J_{2,1,j,p}\right)\right| &\leq \sum_{k=0}^{r-j-p} \sum_{1\leq k_1<\cdots<k_k\leq r-j-p} \frac{1}{\mu(Q)} \int_Q \left| \int_{\mathcal{X}^m} K(z, y_1, \dots, y_m) \left[\prod_{i=1}^{k} \left(b_{k_i}(z) - m_Q(b_{k_i})\right) \right] \right. \\
&\qquad \left. \left[\prod_{i\in\{1,\dots,r-j-p\}\setminus\{k_1,\dots,k_k\}} (b_i(y_i) - b_i(z)) \right] \left(\prod_{l=1}^{m} f_l(y_l) d\mu(y_l) \right) \right| d\mu(z) \\
&= C \sum_{k=0}^{r-j-p} \frac{1}{\mu(6Q)} \int_Q \left[\prod_{i=r-j-p-k+1}^{r-j-p} \left| b_i(z) - m_Q(b_i) \right| \right] \left| \left[b_{r-j-p-k}, \cdots, [b_1, \mathcal{I}_{\alpha,m}] \right] \vec{f}(z) \right| d\mu(z) \\
&\leq C \sum_{k=0}^{r-j-p} \left(\prod_{i=r-j-p-k+1}^{r-j-p} \|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)} \right) M_{t,(6)} \left(\left[b_{r-j-p-k}, \cdots, [b_1, \mathcal{I}_{\alpha,m}] \right] \vec{f} \right)(x).
\end{aligned}$$

For $\left|m_Q\left(J_{2,2,j,p}\right)\right|$, Hölder's inequality, Theorem 1.2, (i) of Lemma 2.8 and property of μ tell us

$$\begin{aligned}
\left|m_Q\left(J_{2,2,j,p}\right)\right| &\le \Big(\frac{1}{\mu(Q)}\int_Q \Big|\mathcal{I}_{\alpha,m}\Big((b_1-m_Q(b_1))\,f_1\chi_{\frac{6}{5}Q},\ldots,\left(b_{r-j-p}-m_Q(b_{r-j-p})\right)f_{r-j-p}\chi_{\frac{6}{5}Q},\\
&\qquad f_{r-j-p+1}\chi_{\frac{6}{5}Q},\ldots,f_r\chi_{\frac{6}{5}Q}\Big)(z)\Big|^s\,d\mu(z)\Big)^{\frac{1}{s}}\\
&\le C\left(\frac{1}{\mu(Q)}\right)^{\frac{1}{s}}\left(\prod_{l_1=1}^{r-j-p}\left\|(b_{l_1}-m_Q(b_{l_1}))\,f_{l_1}\chi_{\frac{6}{5}Q}\right\|_{L^{s_{l_1}}(\mu)}\right)\left(\prod_{l_2=r-j-p+1}^{m}\left\|f_{l_2}\chi_{\frac{6}{5}Q}\right\|_{L^{s_{l_2}}(\mu)}\right)\\
&\le C\left[\prod_{l_1=1}^{r-j-p}\left(\frac{1}{\mu(6Q)}\int_{\frac{6}{5}Q}\left|b_{l_1}(z)-m_Q(b_{l_1})\right|^{\frac{s_{l_1}p_{l_1}}{p_{l_1}-s_{l_1}}}d\mu(z)\right)^{\frac{1}{s_{l_1}}-\frac{1}{p_{l_1}}}\right.\\
&\qquad\left.\left(\frac{1}{(\mu(6Q))^{1-\alpha p_{l_1}/m}}\int_{\frac{6}{5}Q}\left|f_{l_1}(z)\right|^{p_{l_1}}d\mu(z)\right)^{\frac{1}{p_{l_1}}}\right]\\
&\qquad\left[\prod_{l_2=r-j-p+1}^{m}\left(\frac{1}{(\mu(6Q))^{1-\alpha p_{l_2}/m}}\int_{\frac{6}{5}Q}\left|f_{l_2}(z)\right|^{p_{l_2}}d\mu(z)\right)^{\frac{1}{p_{l_2}}}\right]\\
&\le C\left(\prod_{i=1}^{r-j-p}\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^{m}M^{(\alpha/m)}_{p_l,(5)}f_l(x)\right),
\end{aligned}$$

where $\frac{1}{s_1},\ldots,\frac{1}{s_m}\in(1,\infty)$ satisfy $0<\frac{1}{s}=\frac{1}{s_1}+\cdots+\frac{1}{s_m}-\alpha<1$.

For $\sum_{i_1=1}^{r-j-p}\left|m_Q\left(J_{2,3,j,p,i_1}\right)\right|$, we estimate $\left|J_{2,3,j,p,i_1}\right|$ first.

$$\begin{aligned}
\left|J_{2,3,j,p,i_1}\right| &\le C\int_{(6^NB)^{i_1-1}}\int_{\mathcal{X}\setminus\frac{6}{5}Q}\int_{(6^NB)^{m-i_1}}\frac{\prod_{k=1}^{r-j-p}\left|b_k(y_k)-m_Q(b_k)\right|}{\left[\sum_{k=1}^{m}\lambda(z,d(z,y_k))\right]^{m-\alpha}}\left(\prod_{l=1}^{m}|f_l(y_l)|\right)d\mu(y_m)\cdots d\mu(y_1)\\
&\le C\left(\prod_{l_1=\{1,\ldots,r-j-p\}\setminus\{i_1\}}\int_{6^NB}\left|b_{l_1}(y_{l_1})-m_Q(b_{l_1})\right|\left|f_{l_1}(y_{l_1})\right|d\mu(y_{l_1})\right)\\
&\qquad\left[\prod_{l_2=r-j-p+1}^{m}\left(\sum_{\gamma=1}^{N_{B,Q}}\int_{6^{\gamma+1}B\setminus6^{\gamma}B}\left|f_{l_2}(y_{l_2})\right|d\mu(y_{l_2})+\int_{6B}\left|f_{l_2}(y_{l_2})\right|d\mu(y_{l_2})\right)\right]\\
&\qquad\left(\sum_{k=1}^{\infty}\int_{6^k\frac{6}{5}Q\setminus6^{k-1}\frac{6}{5}Q}\frac{\left|b_i(y_i)-m_Q(b_i)\right||f_i(y_i)|}{(\lambda(c_Q,d(c_Q,y_i)))^{m-\alpha}}d\mu(y_i)\right)\\
&\le C\left(\prod_{l_1=1}^{r-j-p}\left\|b_{l_1}\right\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^{m}M^{(\alpha/m)}_{p_l,(5)}f_l(x)\right)\left(\widetilde{K}_{B,Q}\right)^{r-j-p-1}\left(\widetilde{K}^{\alpha/m}_{B,Q}\right)^{m-r+j+p}\\
&\qquad\left(\lambda\left(C_B,6^N5r_B\right)\right)^{(1-\frac{\alpha}{m})(m-1)}\sum_{k=1}^{\infty}\frac{k\left(\mu(6^k6Q)\right)^{1-\frac{\alpha}{m}}}{(\lambda(c_Q,6^{k-1}6r_Q/5))^{m-\alpha}}.
\end{aligned}$$

The properties of μ and λ give that

$$\left|J_{2,3,j,p,i_1}\right|\le C\left(\prod_{l_1=1}^{r-j-p}\left\|b_{l_1}\right\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^{m}M^{(\alpha/m)}_{p_l,(5)}f_l(x)\right)\left(\widetilde{K}_{B,Q}\right)^{r-j-p-1}\left(\widetilde{K}^{\alpha/m}_{B,Q}\right)^{m-r+j+p}$$

$$
\begin{aligned}
&\sum_{k=1}^{\infty} \frac{k\left(\mu(6^k 6Q)\right)^{1-\frac{\alpha}{m}} \left(\lambda\left(C_Q, 6^N 5 r_Q\right)\right)^{\left(1-\frac{\alpha}{m}\right)(m-1)}}{\left[C\left(6^{k-N} 6/5\right) \lambda\left(c_Q, 6^N 5 r_Q\right)\right]^{m-\alpha}} \\
\leq\, & C\left(\prod_{l_1=1}^{r-j-p} \left\|b_{l_1}\right\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^{m} M_{p_l,(5)}^{(\alpha/m)} f_l(x)\right)\left(\widetilde{K}_{B,Q}\right)^{r-j-p-1}\left(\widetilde{K}_{B,Q}^{\alpha/m}\right)^{m-r+j+p} \\
&\sum_{k=1}^{\infty} \frac{k\left(\mu(6^k 6Q)\right)^{1-\frac{\alpha}{m}}}{\left[C\left(6^{k-N} 6/5\right)\right]^{m-\alpha}\left(\lambda\left(c_Q, 6^N 5 r_Q\right)\right)^{1-\frac{\alpha}{m}}} \\
\leq\, & C\left(\widetilde{K}_{B,Q}\right)^{r-j-p-1}\left(\widetilde{K}_{B,Q}^{\alpha/m}\right)^{m-r+j+p}\left(\prod_{l_1=1}^{r-j-p} \left\|b_{l_1}\right\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^{m} M_{p_l,(5)}^{(\alpha/m)} f_l(x)\right).
\end{aligned}
$$

Therefore,

$$
\sum_{i_1=1}^{r-j-p}\left|m_Q\left(J_{2,3,j,p,i_1}\right)\right| \leq C\left(\widetilde{K}_{B,Q}\right)^{r-j-p-1}\left(\widetilde{K}_{B,Q}^{\alpha/m}\right)^{m-r+j+p}\left(\prod_{l_1=1}^{r-j-p}\left\|b_{l_1}\right\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^{m} M_{p_l,(5)}^{(\alpha/m)} f_l(x)\right).
$$

Similarly, the estimates of $\sum_{i_2=r-j-p+1}^{r}\left|m_Q\left(J_{2,4,j,p,i_2}\right)\right|$ and $\left|m_Q\left(J_{2,5,j,p}\right)\right|$ can be obtained. Therefore, we get

$$
\begin{aligned}
J_2 \leq\, & C\sum_{j=1}^{r-1}\sum_{p=0}^{r-j-1}\sum_{k=0}^{r-j-p}\left(\prod_{i=r-j-p-k+1}^{r}\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right) M_{t,(6)}\left(\left[b_{r-j-p-k},\cdots,\left[b_1,\mathcal{I}_{\alpha,m}\right]\right]\vec{f}\right)(x) \\
&+ C\left(\widetilde{K}_{B,Q}\right)^{r-1}\left(\widetilde{K}_{B,Q}^{\alpha/m}\right)^{m-1}\left(\prod_{i=1}^{r}\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^{m} M_{p_l,(5)}^{(\alpha/m)} f_l(x)\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left|h_B - h_Q\right| \leq\, & C\sum_{j=1}^{r-1}\sum_{p=0}^{r-j-1}\sum_{k=0}^{r-j-p}\left(\prod_{i=r-j-p-k+1}^{r}\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right) M_{t,(6)}\left(\left[b_{r-j-p-k},\cdots,\left[b_1,\mathcal{I}_{\alpha,m}\right]\right]\vec{f}\right)(x) \\
&+ C\left(\widetilde{K}_{B,Q}^{\alpha/m}\right)^{r+m}\left(\prod_{i=1}^{r}\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right)\left(\prod_{l=1}^{m} M_{p_l,(5)}^{(\alpha/m)} f_l(x)\right),
\end{aligned}
$$

which completes the proof of (2.5).

Thus, Lemma 2.9 is proved. □

3. Boundedness on Lebesgue spaces

In this section, we aim to establish the boundedness of the multilinear θ-type generalized fractional integral $\mathcal{I}_{\alpha,m}$ and its iterated commutators $\left[\vec{b}_\sigma, \mathcal{I}_{\alpha,m}\right]$ with $\widetilde{\mathrm{RBMO}}(\mu)$-functions on Lebesgue spaces.

Proof of Theorem 1.2. Assume $\alpha_1, \ldots, \alpha_m \in (0,1)$ satisfy $\alpha = \sum_{i=1}^{m}\alpha_i$ and $1 < p_i < \frac{1}{\alpha_i}, i = 1, \ldots, m$. For any $x \in \mathcal{X}$, by applying the (1.4), (1.5) and (1.6), the conditions of the kernel function, we have

$$
\left|\mathcal{I}_{\alpha,m}\vec{f}(x)\right| \leq \int_{\mathcal{X}^m} |K(x, y_1, \ldots, y_m)| \prod_{i=1}^{m} |f_i(y_i)|\, d\mu(y_i)
$$

$$
\begin{aligned}
&\leq C \int_{\mathcal{X}^m} \frac{\prod_{i=1}^m |f_i(y_i)|\, d\mu(y_i)}{\left[\sum_{j=1}^m \lambda(x, d(x, y_j))\right]^{m-\alpha}} \\
&\leq C \prod_{i=1}^m \int_{\mathcal{X}^m} \frac{|f_i(y_i)|\, d\mu(y_i)}{[\lambda(x, d(x, y_i))]^{1-\alpha_i}} \\
&= C \prod_{i=1}^m I_{\alpha_i}(|f_i|)(x).
\end{aligned}
$$

Let $1 < q_i < \infty$ satisfy $\frac{1}{q_i} = \frac{1}{p_i} - \alpha_i$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, then I_{α_i} is bounded from $L^{p_i}(\mu)$ to $L^{q_i}(\mu), i = 1, \ldots, m$. Combining Lemma 1.1 and Hölder's inequality, we obtain

$$
\left\|\mathcal{I}_{\alpha,m}\vec{f}\right\|_{L^q(\mu)} \leq C \left\|\prod_{i=1}^m I_{\alpha_i}(|f_i|)\right\|_{L^q(\mu)} \leq C \prod_{i=1}^m \left\|I_{\alpha_i}(|f_i|)\right\|_{L^{q_i}(\mu)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mu)}.
$$

□

Proof of Theorem 1.3. Theorem 1.2, Lemmas 2.1, 2.2, 2.9 and Hölder's inequality give that

$$
\begin{aligned}
\left\|\mathcal{I}_{\alpha,m,\vec{b}_\sigma}\vec{f}\right\|_{L^q(\mu)} &\leq \left\|N\left(\mathcal{I}_{\alpha,m,\vec{b}_\sigma}\vec{f}\right)\right\|_{L^q(\mu)} \\
&\leq \left\|M^{\sharp,(\alpha/m)}\left(\mathcal{I}_{\alpha,m,\vec{b}_\sigma}\vec{f}\right)\right\|_{L^q(\mu)} \\
&\leq C \sum_{j=1}^r \sum_{\sigma(j)} \left(\prod_{\sigma_i \in \sigma(j)} \|b_{\sigma_i}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right) \left\|M_{t,(6)}\left(\mathcal{I}_{\alpha,m,\vec{b}_{\sigma(j)'}}\vec{f}\right)\right\|_{L^q(\mu)} \\
&\quad + C \left(\prod_{\sigma_i \in \sigma} \|b_{\sigma_i}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right) \left\|\prod_{l=1}^m M^{(\alpha/m)}_{p_l,(5)} f_l\right\|_{L^q(\mu)} \\
&\leq C \sum_{j=1}^r \sum_{\sigma(j)} \left(\prod_{\sigma_i \in \sigma(j)} \|b_{\sigma_i}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right) \left\|\mathcal{I}_{\alpha,m,\vec{b}_{\sigma(j)'}}\vec{f}\right\|_{L^q(\mu)} \\
&\quad + C \left(\prod_{\sigma_i \in \sigma} \|b_{\sigma_i}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right) \left(\prod_{l=1}^m \left\|M^{(\alpha/m)}_{p_l,(5)} f_l\right\|_{L^{q_l}(\mu)}\right) \\
&\leq C \left(\prod_{\sigma_i \in \sigma} \|b_{\sigma_i}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right) \left\|\mathcal{I}_{\alpha,m}\vec{f}\right\|_{L^q(\mu)} + C \left(\prod_{\sigma_i \in \sigma} \|b_{\sigma_i}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right) \left(\prod_{l=1}^m \|f_l\|_{L^{p_l}(\mu)}\right) \\
&\leq C \left(\prod_{\sigma_i \in \sigma} \|b_{\sigma_i}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right) \left(\prod_{l=1}^m \|f_l\|_{L^{p_l}(\mu)}\right),
\end{aligned}
$$

where, $\frac{1}{q} = \frac{1}{p} - \alpha$, and for $i = 1, \ldots, m$, q_i satisfy $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ and $\frac{1}{q_i} = \frac{1}{p_i} - \alpha_i$, $\alpha = \alpha_1 + \cdots + \alpha_m$.

The proof of Theorem 1.3 is finished. □

4. Conclusions

In this article, we discussed the boundedness for $\mathcal{I}_{\alpha,m}$ and its iterated multi-commutators $\mathcal{I}_{\alpha,m,\vec{b}_\sigma}$ with $\widetilde{\mathrm{RBMO}}(\mu)$-functions on Lebesgue spaces over non-homogeneous spaces. It is worth mentioning that the proof of the latter relies on an important lemma, that is, the estimation of the sharp maximal function. At the same time, the case of multilinearity is not replaced by bilinear case because the proof process of the bilinearity can not reflect the idea of the proof completely.

Acknowledgments

We would like to thank the editors and reviewers for their helpful suggestions. The first author would like to thank the National Natural Science Foundation of China under grant no. 11961056 and 11771399. And the second author would like to thank the New Talent Plan of Zhejiang Province under grant no. 2021R415027 and Zhejiang University of Science and Technology Graduate Research Innovation Fund under grant no. 2020yjskc06.

Conflict of interest

The authors declare no conflict of interest.

References

1. T. Hytönen, A framework for non-homogeneous analysis on metric spaces, and the RBMO space of Tolsa, *Publ. Mat.,* **54** (2010), 485–504. http://dx.doi.org/10.5565/PUBLMAT_54210_10
2. X. Fu, D. Yang, W. Yuan, Generalized fractional integrals and their commutators over non-homogeneous metric measure spaces, *Taiwanese J. Math.,* **18** (2014), 509–557. http://dx.doi.org/10.11650/tjm.18.2014.3651
3. T. Hytönen, D. Yang, D. Yang, The Hardy space H^1 on non-homogeneous metric spaces, *Math. Proc. Camb. Phil. Soc.,* **153** (2012), 9–31. https://doi.org/10.1017/S0305004111000776
4. G. Lu, S. Tao, Generalized Morrey spaces over non-homogeneous metric measure spaces, *J. Aust. Math. Soc.,* **103** (2017), 268–278. http://dx.doi.org/10.1017/S1446788716000483
5. R. Xie, L. Shu, A. Sun, Boundedness for commutators of bilinear θ-type Calderón-Zygmund operators on nonhomogeneous metric measure spaces, *J. Funct. Space.,* 2017, 1–10. http://dx.doi.org/10.1155/2017/3690452
6. T. Zheng, X. Tao, X. Wu, Bilinear Calderón-Zygmund operators of type $\omega(t)$ on non-homogeneous space, *J. Inequal. Appl.,* 2014, 1–18. http://dx.doi.org/10.1186/1029-242X-2014-113
7. C. Ri, Z. Zhang, Boundedness of θ-type Calderón-Zygmund operators on non-homogeneous metric measure space, *Front. Math. China,* **11** (2016), 141–153. http://dx.doi.org/10.1007/s11464-015-0464-0
8. S. Liu, D. Yang, D. Yang, Boundedness of Calderón-Zygmund operators on non-homogeneous metric measure spaces: Equivalent characterizations, *J. Math. Anal. Appl.,* **386** (2012), 258–272. http://dx.doi.org/10.1016/j.jmaa.2011.07.055
9. X. Tao, T. Zheng, Multilinear commutators of fractional integrals over Morrey spaces with non-doubling measures, *Nonlinear Differ. Equ. Appl.,* **18** (2011), 287–308. http://dx.doi.org/10.1007/s00030-010-0096-8

10. X. Fu, D. Yang, D. Yang, The molecular characterization of the Hardy space H^1 on non-homogeneous metric measure spaces and its application, *J. Math. Anal. Appl.,* **410** (2014), 1028–1042. http://dx.doi.org/10.1016/j.jmaa.2013.09.021

11. X. Tolsa, BMO, H^1, and Calderón-Zygmund operators for non doubling measures, *Math. Ann.,* **319** (2001), 89–149. http://dx.doi.org/10.1007/PL00004432

12. Y. Cao, J. Zhou, Morrey spaces for nonhomogeneous metric measure spaces, *Abstr. Appl. Anal.,* 2013, 1–8. http://dx.doi.org/10.1155/2013/196459

13. H. Lin, S. Wu, D. Yang, Boundedness of certain commutators over non-homogeneous metric measure spaces, *Anal. Math. Phys.,* **7** (2017), 187–218. http://dx.doi.org/10.1007/s13324-016-0136-6

14. X. Fu, D. Yang, W. Yuan, Boundedness of multilinear commutators of Calderón-Zygmund operators on Orlicz spaces over non-homogeneous spaces, *Taiwanese J. Math.,* **16** (2012), 2203–2238. http://dx.doi.org/10.11650/twjm/1500406848