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Research article

Some linear differential equations generated by matrices

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Abstract: Given matrices $N \in C^{s\times s}$ and $S_0, \ldots, S_q \in C^{s\times s}$, we solve the linear differential equation

$$\sum_{n=0}^{q} T_n(t)\ (d/dt)^n f(t) = g(t),$$

where $t \in R$, $T_n(t) = e^{tN} S_n e^{-tN}$, and $f(t) : R \to C^s$, using the roots of $d(\nu) = \det\ D(\nu)$, where

$$D(\nu) = \sum_{n=0}^{q} S_n\ (\nu I_r + N)^n .$$

For example,

$$N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

implies

$$e^{tN} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

so that $T_n(t)$ are periodic, giving an explicit solution to a form of Floquet's theorem.

Keywords: characteristic matrix; Floquet's theorem; planetary perturbations
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1. Introduction

There is much research of linear differential equations involving or generated by matrices, see Derevenskii [2] and Elishevich [3]. But these and other papers are limited to the first few orders. We are not aware of papers giving solutions for linear differential equations of general order generated by matrices. In the paper, we provide solutions for such a class of linear differential equations.

Let R and C denote the real and complex numbers, respectively. For any function $f(t)$ of $t \in R$, set $f_{.n}(t) = (d/dt)^n f(t)$. Let $Y(t) = Y = (Y_1, \ldots, Y_r) \in C^{s\times r}$ be a function of $t \in R$, and a constant $N = \left(N_{j,k}\right) \in C^{r\times r}$, such that

$$Y_{.1}(t) = Y(t)N \in C^{s\times r}. \tag{1.1}$$

Choose $q \geq 1$. Suppose that for $0 \leq n \leq q$, and $T_n(t) \in C^{s\times s}$, there is a constant $S_n = \left(S_{n,j,k}\right) \in C^{r\times r}$ such that

$$T_n(t)\, Y(t) = Y(t)\, S_n. \tag{1.2}$$

If $r = s$, as in the abstract, we can take

$$Y(t) = e^{tN},\ T_n(t) = e^{tN}\, S_n\, e^{-tN}. \tag{1.3}$$

If $r > s$, we can take $Y(t)$ as the first s rows of e^{tN}. We wish to solve the differential equation $Lf = 0_s$ for $f = f(t) : R \to C^s$, where

$$L = L(t) = \sum_{n=0}^{q} T_n(t)\, (d/dt)^n,\ s \times s,$$

that is,

$$Lf = \sum_{n=0}^{q} T_n f_{.n}. \tag{1.4}$$

Theorem 2.1 gives solutions in terms of the roots of $d = d(\nu)$, where

$$d(\nu) = \det\ D(\nu) \text{ and } D(\nu) = D = \sum_{n=0}^{q} S_n\ (\nu I_r + N)^n \in C^{r\times r}. \tag{1.5}$$

We call $D(\nu)$ the *characteristic matrix* of the operator L. If ν is a multiple root, then other solutions are given by Theorem 2.2.

Section 3 chooses $r = s = 2$ and N of (3.1) below. $T_n(t)$ of (1.3) are then linear in $\cos 2t$ and $\sin 2t$ and so have period π. Example 3.2 gives for the first time the full solution of a well known example. Example 3.3 solves a differential equation that arises in the theory of planetary perturbations, and was the incentive for this paper. Section 4 shows how $D(\nu)$ and $T_n(t)$ are given by the Jordan form of N when $r = s$.

Section 5 solves $Lf = g$, any given function in C^s, when $r = s$ and $T_q = I_s$, by converting it to the standard form $X_{.1} = AX + F$. Its solution is then given by the variation of constants formula in terms of the solution of $Lf = 0_s$. When A is periodic, Floquet's theorem only gives the *form* of the solution of $X_{.1} = AX$. We give the *actual* solution. Examples 3.1-3.5 all have periodic A. Set $i = \sqrt{-1}$. Set $\delta_{j,k} = 1$ or 0 for $j = k$ or $j \neq k$.

2. Main results

Theorem 2.1. *Suppose that $Y(t)$, N satisfy (1.1), and that $T_n(t)$, S_n satisfy (1.2) for $0 \leq n \leq q$. Let ν be any of the qr roots of $d(\nu) = 0$ for $d(\nu)$of (1.5). Choose $a(\nu) = a = (a_1, \ldots, a_r)' \in C^r$ such that $E(\nu) = 0_r$, where*

$$E = E(\nu) = D(\nu)a(\nu).$$

Then for $f = f(t) \in C^s$ and L of (1.4), a solution of $Lf = 0_s$ is

$$f_0(t, \nu) = e^{\nu t} Y(t) a(\nu) = e^{\nu t} \sum_{k=1}^{r} a_k(\nu) Y_k(t). \tag{2.1}$$

Proof: Since $Y_{.n} = YN^n$, by Leibniz' rule, the nth derivative of $e^{\nu t} Y$ is

$$\left(e^{\nu t} Y\right)_{.n} = e^{\nu t} Y \ \left(\nu I_r + N\right)^n. \tag{2.2}$$

So,

$$L\, e^{\nu t} Y = \sum_{n=0}^{q} T_n(t) \ \left(e^{\nu t} Y\right)_{.n} = e^{\nu t} Y\, D(\nu),$$

and $L\, f_0 = e^{\nu t} Y\, D(\nu) a(\nu) = 0_r$. □

When the roots of $d(\nu)$ are distinct, this gives qr independent solutions of (2.1).

Theorem 2.2. *Take ν, $D = D(\nu)$, $a = a(\nu)$ and $E = E(\nu)$ of Theorem 2.1. Suppose that for some $k \geq 1$,*

$$E_{.m}(\nu) = 0_r \tag{2.3}$$

for $0 \leq m < k$, where $E_{.m}(\nu) = \partial_\nu^m E(\nu)$, and $\partial_\nu = d/d\nu$. Then for $0 \leq n < k$, $Lf = 0_s$ has solutions

$$f_n = f_n(t, \nu) = e^{\nu t} Y(t) z_n = e^{\nu t} \sum_{j=1}^{r} z_{n,j}(t, \nu) Y_j(t), \tag{2.4}$$

where $z_n = z_n(t, \nu) = (t + \partial_\nu)^n \ a(\nu)$.

Proof: By (2.2) and Leibniz' rule,

$$\partial_t^o \left(e^{\nu t} Y z_n\right) = \sum_{m=0_p}^{o} \binom{o}{m} e^{\nu t} Y \ \left(\nu I_s + N\right)^{o-m} \partial_t^m z_n,$$
$$L f_n = e^{\nu t} Y G_n,$$

where

$$G_n = \sum_{o=0}^{q} S_o \sum_{m=0}^{o} \binom{o}{m} \left(\nu I_s + N\right)^{o-m} \partial_t^m \left(t + \partial_\nu\right)^n \ a$$

$$= \sum_{m,r} \binom{n}{m}\binom{n-m}{r} t^{n-m-r} D_{.m} a_{.r}.$$

Transform from m to $c = m + r$. Then $\binom{n}{m}\binom{n-m}{r} = \binom{n}{c}\binom{c}{r}$. So,

$$G_n = \sum_c \binom{n}{c} t^{n-c} H_c,$$

where

$$H_c = \sum_r \binom{c}{r} D_{.c-r} a_{.r} = \partial_\nu^c E(\nu) = 0_s$$

by (2.3). □

We call f_n a *characteristic solution* of $Lf = 0_s$. $D(\nu)$ and $d(\nu)$ expand as

$$D(\nu) = \sum_{k=0}^{q} \nu^k D_k, \; d(\nu) = \sum_{k=0}^{rq} \nu^k d_k,$$

where

$$D_k = \sum_{n=k}^{q} \binom{n}{k} S_n N^{n-k}, \; D_0 = D(0), \; D_q = S_q,$$
$$d_0 = \det\ D(0), \; d_q = \det\ S_q.$$

So,

$$D_{.m}(\nu) = \sum_{k=m}^{q} (k)_m \nu^{k-m} D_k, \; d_{.m}(\nu) = \sum_{k=m}^{rq} (k)_m \nu^{k-m} d_k. \tag{2.5}$$

Corollary 2.1. *Take Lf of (1.4). Consider the exceptional case when $D(\nu) \equiv 0_{s\times s}$. Suppose that*

$$D_{.m}(\nu) = 0_{s\times s}$$

for $0_q \leq m \leq n$. Then for any $a \in C^s$, a characteristic solution of $Lf = 0_s$ is

$$f_n(t, \nu) = t^n e^{\nu t} Y(t)\, a.$$

We now transfer the condition (2.3) from $E(\nu)$ to $d(\nu)$. Let M be the *adjoint* of D: $(-1)^{j+k} M_{k,j}$ is the determinant of D with its jth row and kth column deleted. If $d \neq 0$, then $M = dD^{-1}$.

Corollary 2.2. *Let $e_{1,s}, \ldots, e_{2,s}$ be any basis for R^s. For D and d of (1.5), set $a_{(j)} = Me_{j,s}$ and $E_{(j)} = Da_{(j)} = de_{j,s}$. So, for $m \in Z^q$, $E_{(j).m} = d_{.m} e_{j,s}$. Choose ν so that $d = 0$. Given $1 \leq j \leq s$ and $a = a_{(j)}$, $f_{n,j}(t) = f_n(t, \nu)$ of (2.4) is a characteristic solution of $Lf = 0_s$ if*

$$d_{.m}(\nu) = 0 \tag{2.6}$$

for $0 \leq m \leq n$.

By (2.5), (2.6) does not extend to $n = rq$ if $d_{r,q} \neq 0$. We now take $e_{j,s}$ as the jth unit vector in R^s. So, for $1 \leq j \leq s$, $a_{(j)}$ is the jth column of M. For $1 \leq k \leq s$, its kth element is $a_{(j)k} = M_{k,j} = a_{j,k}$ say. Corollary 2.2 breaks the solution $f_n(x, \nu)$ into s *basis solutions* $f_{n,j}(x)$, $1 \leq j \leq s$, of $Lf = 0_s$. For example, if $s = 2$ then

$$\begin{aligned} &d = D_{1,1}D_{2,2} - D_{1,2}D_{2,1},\ M = \begin{pmatrix} D_{2,2} & -D_{1,2} \\ -D_{2,1} & D_{1,1} \end{pmatrix}, \\ &e_{1,2} = (1,0)',\ e_{2,2} = (0,1)',\ a_{(1)} = (D_{2,2}, -D_{2,1})',\ a_{(2)} = (-D_{1,2}, D_{1,1})', \\ &d = (D_{2,2}, D_{1,2})\, a_{(1)} = (D_{2,1}, D_{2,2})\, a_{(2)}. \end{aligned} \tag{2.7}$$

$d = 0$ implies

$$f_{0,1} = e^{\nu t}\left(D_{2,2}Y_1 - D_{2,1}Y_2\right),\ f_{0,2} = e^{\nu t}\left(-D_{1,2}Y_1 + D_{1,1}Y_2\right). \tag{2.8}$$

$d = d_{.1} = 0$ implies

$$f_{1,1} = e^{\nu t}\left[\left(tD_{2,2} + D_{2,2.1}\right)Y_1 - \left(tD_{2,1} + D_{2,1.1}\right)Y_2\right], \tag{2.9}$$

$$f_{1,2} = e^{\nu t}\left[-\left(tD_{1,2} + D_{1,2.1}\right)Y_1 + \left(tD_{1,1} + D_{1,1.1}\right)Y_2\right]. \tag{2.10}$$

$d = d_{.1} = d_{.2} = 0$ implies

$$f_{2,1} = e^{\nu t}\left[\left(t^2D_{2,2} + 2tD_{2,2.1} + D_{2,2.2}\right)Y_1 - \left(t^2D_{2,1} + 2tD_{2,1.1} + D_{2,1.2}\right)Y_2\right], \tag{2.11}$$

$$f_{2,2} = e^{\nu t}\left[-\left(t^2D_{1,2} + 2tD_{1,2.1} + D_{1,2.2}\right)Y_1 + \left(t^2D_{1,1} + 2tD_{1,1.1} + D_{1,1.}\right)Y_2\right]. \tag{2.12}$$

$d_{.m} = 0$ for $0 \leq m \leq 3$ implies $f_{3,j} = e^{\nu t}\left(z_{3,1}Y_1 + z_{3,2}Y_2\right)$, where, for $j = 1$,

$$z_{3,1} = t^3D_{2,2} + 3t^2D_{2,2.1} + 3tD_{2,2.2} + D_{2,2.3}, \tag{2.13}$$

$$z_{3,2} = -t^3D_{2,1} - 3t^2D_{2,1.1} - 3tD_{2,1.2} - D_{2,1.3}, \tag{2.14}$$

and, for $j = 2$,

$$\begin{aligned} z_{3,1} &= -t^3D_{1,2} - 3t^2D_{1,2.1} - 3tD_{1,2.2} - D_{1,2.3}, \\ z_{3,2} &= t^3D_{1,1} + 3t^2D_{1,1.1} + 3tD_{1,1.2} + D_{1,1.3}. \end{aligned}$$

Generally, each $f_{n,j}$ is a linear combination of $(f_{m,1},\ 0 \leq m \leq n)$. We shall give details in a later paper. For example, $M_{1,1} \neq 0$ implies, for $2 \leq j \leq s$, $f_{0,j}(t) = f_{0,1}(t)M_{j,1}/M_{1,1}$. $s = 2$ and $D_{2,2} \neq 0$ imply $f_{0,2}(t) = -f_{0,1}(t)D_{2,1}/D_{2,2}$.

For $L(t) = L$ of (1.4) and $\tau \neq 0$, set

$$L_\tau(t) = \tau^q L(\tau t) = \sum_{n=0}^{q} T_{\tau,n}(t)\ (d/dt)^n,$$

where $T_{\tau,n}(t) = \tau^{q-n}T_n(\tau t)$. So, $T_q = I_s$ implies $T_{\tau,q} = I_s$. For example, $q = s = 2$, $T_2 = I_2$ imply $L_\tau(t) = (d/dt)^2 + \tau T_1(\tau t) + \tau^2 T_0(\tau t)$.

Corollary 2.3. *Take ν, $D = D(\nu)$, $a = a(\nu)$ of Theorems 2.1-2.2, and $f(t) = f_0(t,\nu)$ of (2.1), or $f(t) = f_n(t,\nu)$ of Theorem 2.2. Then a solution of $L_\tau(t)X(t) = 0_s$ is $X(t) = f_n(\tau t, \nu)$.*

We have not assumed that $r = s$ or $T_q = I_s$. However, if $r = s$ and $T_q = I_s$, then $Lf = 0_s$ can be written in the standard form $X_{.1} = AX$, where

$$X = \begin{pmatrix} f \\ f_{.1} \\ \cdots \\ f_{.q-1} \end{pmatrix},\ A = \begin{pmatrix} 0 & I_s & 0 & \cdots & 0 \\ 0 & 0 & I_s & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & I_s \\ -T_0 & -T_1 & -T_2 & \cdots & -T_{q-1} \end{pmatrix} \in C^{qs\times qs}, \tag{2.15}$$

(to be read as $A = -T_0$ if $q = 1$), and each 0 is $s \times s$. So, (2.15) with f of Theorems 2.1, 2.2 give all qs linearly independent solutions of $X_{.1} = AX$.

3. An application to the unit circle

Set $c_t = \cos t$, $s_t = \sin t$. Here, we take $r = s = 2$ and

$$N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ Y = (Y_1, Y_2) = e^{tN} = \begin{pmatrix} c_t & -s_t \\ s_t & c_t \end{pmatrix} = c_t I_2 + s_t. \tag{3.1}$$

So,

$$Y_1 = \begin{pmatrix} c_t \\ s_t \end{pmatrix},\ Y_2 = Y_{1.1} = \begin{pmatrix} -s_t \\ c_t \end{pmatrix},\ NY_1 = Y_2,\ NY_2 = -Y_1.$$

Set

$$\Lambda = \text{diag}\,(1, -1),\ J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$R(t) = c_t\Lambda + s_t J = (Y_1, -Y_2) = \begin{pmatrix} c_t & s_t \\ s_t & -c_t \end{pmatrix},$$
$$Q(t) = \begin{pmatrix} -s_t & c_t \\ c_t & s_t \end{pmatrix} = (Y_2, Y_1) = YJ = R_{.1}(t).$$

N, $R(t)$, $Q(t)$, Λ and J all have determinant ± 1. Some properties are:

$$J^2 = \Lambda^2 = -N^2 = R(t)^2 = Q(t)^2 = I_2,$$
$$N\Lambda = -\Lambda N = J,\ JN = -NJ = \Lambda,\ J\Lambda = -\Lambda J = N,$$
$$R(t)Y(s) = R(t - s) = Y(t - s)\Lambda,\ R(2t)Y = Y\Lambda = R(t),$$
$$R(2t)Y_1 = Y_1,\ R(2t)Y_2 = -Y_2,\ \Lambda Y = R(-t),\ Y(s)R(t) = R(s + t),$$
$$JR(t) = \overline{Y}N = N\overline{Y},\ R(t)\Lambda = Y,\ \Lambda R(t) = \overline{Y},\ R(2t) = Y\Lambda Y',$$
$$I_2 + \Lambda = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ J - N = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ J + N = 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ I_2 - \Lambda = 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(1.3) holds for

$$(S_n, T_n) = (I_2, I_2)\,,\ (N, N),\ (\Lambda, R(2t))\,,\ (J, Q(2t))\,.$$

So, (1.3) also holds for any linear combination of these, say

$$S_n = b_{n,1}I_2 + b_{n,2}N + b_{n,3}\Lambda + b_{n,4}J = \begin{pmatrix} b_{n,1}+b_{n,3} & b_{n,2}-b_{n,4} \\ -b_{n,2}-b_{n,4} & b_{n,1}-b_{n,3} \end{pmatrix}, \tag{3.2}$$

$$T_n(t) = b_{n,1}I_2 + b_{n,2}N + b_{n,3}R(2t) + b_{n,4}Q(2t) \tag{3.3}$$

$$= \begin{pmatrix} b_{n,1}+b_{n,3}c_{2,t}-b_{n,4}s_{2,t} & -b_{n,2}+b_{n,3}s_{2,t}+b_{n,4}c_{2,t} \\ b_{n,2}+b_{n,3}s_{2,t}+b_{n,4}c_{2,t} & b_{n,1}-b_{n,3}c_{2,t}+b_{n,4}s_{2,t} \end{pmatrix}$$

for any constants $b_{n,j} \in C$. Any 2×2 matrix S_n can be put in this form: set

$$b_{n,1} = (S_{n,1,1}+S_{n,2,2})/2,\ b_{n,3} = (S_{n,1,1}-S_{n,1,2})/2,$$
$$b_{n,2} = (S_{n,2,1}-S_{n,1,2})/2,\ b_{n,4} = -(S_{n,2,1}+S_{n,1,2})/2.$$

So,

$$T_n(t) = A_t(S_n),$$

where

$$2A_t(S) = B(S) + c_{2,t}\,C(S) + s_{2,t}\,G(S),$$
$$B(S) = \begin{pmatrix} B_1 & -B_2 \\ B_2 & B_1 \end{pmatrix},\ C(S) = \begin{pmatrix} C_1 & -C_2 \\ -C_2 & -C_1 \end{pmatrix},\ G(S) = \begin{pmatrix} C_2 & C_1 \\ C_1 & -C_2 \end{pmatrix},$$
$$B_1 = S_{1,1}+S_{2,2},\ B_2 = S_{1,2}-S_{2,1},\ C_1 = S_{1,1}-S_{2,2},\ C_2 = S_{1,2}+S_{2,1},$$
$$T_nY_1 = (b_{n,1}+b_{n,3})Y_1 + (b_{n,2}+b_{n,4})Y_2,\ T_nY_2 = (-b_{n,2}+b_{n,4})Y_1 + (b_{n,1}-b_{n,3})Y_2.$$

Corollary 3.1. *For $T_n(t)$ of (3.3), a solution of $Lf = 0_2$ is (2.1) with ν any of the $2q$ roots of $d = 0$,*

$$E = 0_2,\ D = c_1I_2 + c_2N + c_3\Lambda + c_4J = \begin{pmatrix} c_1+c_3 & -c_2+c_4 \\ c_2+c_4 & c_1-c_3 \end{pmatrix}, \tag{3.4}$$

where

$$c_j = \sum_{n=0}^{q} c_{n,j},\ d = c_1^2 + c_2^2 - c_3^2 - c_4^2, \tag{3.5}$$

where

$$c_{n,1} = g_nb_{n,1} - h_nb_{n,2},\ c_{n,2} = g_nb_{n,2} + h_nb_{n,1},\ c_{n,3} = g_nb_{n,3} + h_nb_{n,4},\ c_{n,4} = g_nb_{n,4} - h_nb_{n,3}, \tag{3.6}$$

$$g_n = g_n(\nu) = Real\left((\nu+i)^n\right) = \sum_j \binom{n}{2j}(-1)^j\nu^{n-2j}, \tag{3.7}$$

$$h_n = h_n(\nu) = Imag\left((\nu+i)^n\right) = \sum_j \binom{n}{2j+1}(-1)^j\nu^{n-2j-1}, \tag{3.8}$$

where the real and imaginary parts are taken as if ν were real.

Proof: Since $N^2 = -I_2$,

$$(\nu I_2 + N)^n = g_n I_2 + h_n N$$

for g_n and h_n of (3.7), (3.8). So, for S_n of (3.2),

$$S_n \ (\nu I_2 + N)^n = c_{n,1} I_2 + c_{n,2} N + c_{n,3} \Lambda + c_{n,4} J.$$

So, by (1.5), (3.4) holds. g_n, h_n, $c_{n,j}$, c_j, D, d are polynomials in ν of degree n, $n-1$, n, q, q, $2q$. So, $d = 0$ has $2q$ roots ν. □

By (3.7)-(3.8), when $q = 1$, the c_j needed for (3.5) are

$$c_1 = b_{0,1} + \nu b_{1,1} - b_{1,2}, \ c_2 = b_{0,2} + \nu b_{1,2} + b_{1,1},$$
$$c_3 = b_{0,3} + \nu b_{1,3} + b_{1,4}, \ c_4 = b_{0,4} + \nu b_{1,4} - b_{1,3}.$$

By (3.7)-(3.8), when $q = 2$, the c_j needed for (3.5) are

$$c_1 = b_{0,1} + \nu b_{1,1} - b_{1,2} + \left(\nu^2 - 1\right) b_{2,1} - 2\nu b_{2,2},$$
$$c_2 = b_{0,2} + \nu b_{1,2} + b_{1,1} + \left(\nu^2 - 1\right) b_{2,2} + 2\nu b_{2,1},$$
$$c_3 = b_{0,3} + \nu b_{1,3} + b_{1,4} + \left(\nu^2 - 1\right) b_{2,3} + 2\nu b_{2,4},$$
$$c_4 = b_{0,4} + \nu b_{1,4} - b_{1,3} + \left(\nu^2 - 1\right) b_{2,4} - 2\nu b_{2,3}.$$

When $q = 3$, we add $c_{3,j}$ of (3.6) to c_j for $j = 1, 2, 3, 4$.

$Lf = 0_2$ can only be reduced to the form $X_{.1} = AX$ if $e_q \neq 0$, where $e_n = \det\ T_n(t) = \det\ S_n = b_{n,1}^2 + b_{n,2}^2 - b_{n,3}^2 - b_{n,4}^2$, since then we can reduce $T_q(t)$ to I_2 by multiplying by $T_q(t)^{-1}$. Set

$$\overline{L} = T_q(t)^{-1} L = \sum_{n=0}^{q} \overline{T_n}(t) \ (d/dt)^n,$$

where $\overline{T_n}(t) = T_q(t)^{-1} T_n(t)$ is a linear combination of 1, $s_{2,t}$, $c_{2,t}$, $s_{4,t}$, $c_{4,t}$.

Example 3.1. *Take $q = 1$, $S_1 = I_2$, $S_0 = \lambda\,\Lambda$. So, (1.2) holds with $T_1 = I_2$ and $T_0(t) = \lambda R(2t)$. Further,*

$$D = \nu I_2 + \lambda\,\Lambda + N = \begin{pmatrix} \nu + \lambda & -1 \\ 1 & \nu - \lambda \end{pmatrix},$$

and $d = \nu^2 - \lambda^2 + 1$ with roots $\nu = \pm\left(\lambda^2 - 1\right)^{1/2} = \nu_1, \nu_2$, say. By (2.7), $a = a_{(1)} = \binom{\nu-\lambda}{-1}$ implies $a_{.1} = \binom{1}{0}$, $z_0 = \binom{\nu-\lambda}{-1}$ and $z_1 = \binom{t(\nu-\lambda)+1}{-t}$; $a = a_{(2)} = \binom{1}{\nu+\lambda}$ implies $a_{.1} = \binom{0}{1}$, $z_0 = \binom{1}{\nu+\lambda}$ and $z_1 = \binom{t}{t(\nu+\lambda)+1}$; $d = 0$ implies $\nu = \pm\left(1 - \lambda^2\right)^{1/2}$. So, by (2.8), solutions are

$$f_{0,1} = e^{\nu t}\left[(\nu - \lambda) Y_1 - Y_2\right], \ f_{0,2} = e^{\nu t}\left[Y_1 + (\nu + \lambda) Y_2\right].$$

If $d = 0$ and $\lambda = \pm 1$, then $d_{.1} = \nu = 0$ so by (2.9) and (2.10) solutions are

$$f_{0,1} = -\lambda Y_1 - Y_2, \ f_{0,2} = Y_1 + \lambda Y_2 = -\lambda f_{0,1},$$
$$f_{1,1} = (1 - \lambda t) Y_1 - t Y_2, \ f_{1,2} = t Y_1 + (1 + \lambda t) Y_2.$$

An extension is

Example 3.2. *Given scalars b_0, b_1, b_2, we solve $f_{.1}(t) = A(t)f(t)$ for*

$$A(t) = \begin{pmatrix} b_0 + b_1 c_{2,t} & -b_2 + b_1 s_{2,t} \\ b_2 + b_1 s_{2,t} & b_0 - b_1 c_{2,t} \end{pmatrix} = B + b_1 R(2t), \tag{3.9}$$

$$B = b_0 I_2 + b_2 N = \begin{pmatrix} b_0 & -b_2 \\ b_2 & b_0 \end{pmatrix}.$$

So, $Lf = f_{.1} - A(t)f$. Take $q = 1$, $S_1 = I_2$ and

$$S_0 = \begin{pmatrix} -b_0 - b_1 & b_2 \\ -b_2 & -b_0 + b_1 \end{pmatrix} = -b_0 I_2 - b_1 \Lambda - b_2 N = -B - b_1 \Lambda.$$

Then (1.4) holds with $T_1(t) = I_2$, $T_0(t) = -A(t)$. So,

$$D(\nu) = S_0 + \nu I_2 + N = \begin{pmatrix} \nu - b_0 - b_1 & b_2 - 1 \\ -b_2 + 1 & \nu - b_0 + b_1 \end{pmatrix},$$

and $d(\nu) = (\nu - b_0)^2 - b_1^2 + (b_2 - 1)^2$ has roots

$$\nu_1 = b_0 + \delta^{1/2}, \ \nu_2 = b_0 - \delta^{1/2} \tag{3.10}$$

for $\delta = b_1^2 - (b_2 - 1)^2$. By (2.8), if $\nu = \nu_1$ or ν_2, then $d = 0$ and solutions are

$$f_{0,1}(t) = e^{\nu t}\left[(\nu - b_0 + b_1) Y_1 + (b_2 - 1) Y_2\right],$$
$$f_{0,2}(t) = e^{\nu t}\left[(1 - b_2) Y_1 + (\nu - b_0 - b_1) Y_2\right]. \tag{3.11}$$

Consider the case $\delta = 0$. So, $b_2 = 1 + \lambda b_1$, where $\lambda = \pm 1$,

$$f_{0,1}(t) = b_1 e^{b_0 t}(Y_1 + \lambda Y_2), \ f_{0,2}(t) = -b_1 e^{b_0 t}(\lambda Y_1 + Y_2),$$
$$f_{1,1}(t) = e^{b_0 t}\left[(b_1 t + 1) Y_1 + \lambda b_1 t Y_2\right], \ f_{1,2}(t) = e^{b_0 t}\left[-\lambda b_1 t Y_1 + (1 - b_1 t) Y_2\right].$$

Let us rescale this example by transforming to $T = t/\tau$, $x(T) = f(\tau T)$ for $\tau \neq 0$, then replacing T by t.

Example 3.3. *Set $A_\tau(t) = \tau A(\tau t)$. For $A(t)$ of (3.9), $x_{.1}(t) = A_\tau(t)x(t)$ has solutions $f_{0,j}(\tau t)$ for $f_{0,j}(t)$ of (3.11) with ν of (3.10). If $\delta = 0$, other solutions are $f_{1,j}(\tau t)$ for $f_{1,j}(t)$ of (2.9) and (2.10). We consider two cases. The first case is that $\tau = 1/2$. In this case,*

$$A_\tau(t) = \begin{pmatrix} b_0 + b_1 c_t & -b_2 + b_1 s_t \\ b_2 + b_1 s_t & b_0 - b_1 c_t \end{pmatrix}/2 = (b_0 I_2 + b_2 N + b_1 R(2t))/2.$$

The second case is that $\tau = -1$, $b_0 = 1/4$, $b_2 = 1$ and $b_1 = -3/4$. Then $\nu_1 = -1/2$, $\nu_2 = 1$ and independent solutions are

$$f_{0,1}(-t) = (-3/2)e^{t/2}\begin{pmatrix} c_t \\ -s_t \end{pmatrix}, \ f_{0,2}(-t) = (3/2)e^{-t}\begin{pmatrix} s_t \\ c_t \end{pmatrix}.$$

The other two $f_{0,j}$ *are* 0_2. *In this case,* $A_\tau(t) = -B - b_1R(-2t)$ *can be written*

$$A_\tau(t) = \begin{pmatrix} -1 + (3/2)c_t^2 & 1 - (3/2)c_t s_t, \\ -1 - (3/2)c_t s_t & -1 + (3/2)s_t^2 \end{pmatrix},$$

with period $T = \pi$. *Markus and Yamabe [5] used this form of* $A_\tau(t)$ *but only gave the first solution* $f_{0,1}(-t)$. *This example is quoted by Chicone [1], but again the second solution* $f_{0,2}(-t)$ *is not given.*

Example 3.4. *Take* $q = 2$, $S_2 = I_2$, $S_1 = 0_{2\times 2}$ *and* $S_0 = \text{diag } (b_0 + b_1, b_0 - b_1)$ *for scalars* b_0, b_1. *Then* $T_2 = I_2$, $T_1 = 0_{2\times 2}$ *and*

$$T_0 = b_0 I_2 + b_1 R(2t) = \begin{pmatrix} b_0 + b_1 c_{2,t} & b_1 s_{2,t} \\ b_1 s_{2,t} & b_0 - b_1 c_{2,t} \end{pmatrix}.$$

Then, $L = I_2(d/dt)^2 + T_0$. *(1.2) holds with*

$$D = (\nu I_2 + N)^2 + S_0 = \begin{pmatrix} \nu^2 - 1 + b_0 + b_1 & -2\nu \\ 2\nu & \nu^2 - 1 + b_0 - b_1 \end{pmatrix},$$

$$d = \left(\nu^2 - 1 + b_0\right)^2 - b_1^2 + 4\nu^2 = \nu^4 + 2b\nu^2 + c,$$

where $b = b_0 + 1$, $c = (b_0 - 1)^2 - b_1^2$. d *has four roots,* $\nu = \pm\nu_1$, $\pm\nu_2$, *where*

$$\nu_1 = \left(-b + \delta^{1/2}\right)^{1/2}, \ \nu_2 = \left(-b - \delta^{1/2}\right)^{1/2}, \ \delta = b^2 - c = 4b_0 + b_1^2. \tag{3.12}$$

So, $\nu^2 - 1 + b_0 = -2 \pm \delta^{1/2}$ *and solutions are given by* $f_{0,1}$, $f_{0,2}$ *of (2.8) with* $D_{2,2} = \nu^2 - 1 + b_0 - b_1$, $D_{2,1} = 2\nu$, $D_{1,2} = -2\nu$ *and* $D_{1,1} = \nu^2 - 1 + b_0 + b_1$. *If* δ *of (3.12) is 0 and* $b \neq 0$, *then there are two roots of multiplicity two,* $\nu = \lambda\nu_0$, *where* $\lambda = \pm 1$, $\nu_0 = (-b)^{1/2}$; *so other solutions are* $f_{1,1}$ *of (2.9) with* $\nu = \lambda\nu_0$,

$$D_{2,2} = -2 - b_1, \ D_{2,2.1} = D_{2,1} = 2\nu, \ D_{2,1.1} = 2, \tag{3.13}$$

and $f_{1,2}$ *of (2.10) with* $\nu = \lambda\nu_0$,

$$D_{1,2} = -2\nu, \ D_{1,2.1} = -2, \ D_{1,1} = b_1 - 2, \ D_{1,1.1} = 2\nu. \tag{3.14}$$

Now suppose that $b_0 = -1$, $b_1 = 2\lambda$, *where* $\lambda = \pm 1$. *Then* $\nu = 0$ *has multiplicity 4. So, other solutions are* $f_{2,1}$ *of (2.11), (3.13) with* $D_{2,2.2} = 2$, $D_{2,1.2} = 0$; $f_{2,2}$ *of (2.12), (3.14) with* $D_{1,2.2} = 0$, $D_{1,1.2} = 2$; $f_{3,1}$ *of (2.13) with*

$$z_{3,1} = -(2 + b_1)\, t^3 + 6\lambda\nu_0 t^2 + 6t, \ z_{3,2} = 2\lambda\nu_0 t^3 - 6t^2;$$

and $f_{3,2}$ *of (2.14) with*

$$z_{3,1} = 2\lambda\nu_0 t^3 + 6t^2, \ z_{3,2} = (b_1 - 2)\, t^3 + 6\lambda\nu_0 t^2 + 6t.$$

To solve $Lf = g$, *a given function in* C^2, *Section 5 will need its derivative,* $\partial_t\, f_0(t, \nu) = e^{\nu t} v(t, \nu)$, *where*

$$v(t, \nu) = \sum_{j=1}^{2} a_j \left(\nu Y_j + Y_{j.1}\right) = a_1 \ (\nu Y_1 + Y_2) + a_2 \ (\nu Y_2 - Y_1) = \sum_{j=1}^{2} e_j Y_j,$$

$$e_1 = a_1\nu - a_2 = -D_{1,2}\nu - D_{1,1} = \nu^2 + 1 - b_0 - b_1,$$
$$e_2 = a_1 + \nu a_2 = -D_{1,2} + \nu D_{1,1} = \nu\left(\nu^2 + 1 + b_0 + b_1\right).$$

By (2.15), $Lf = 0_2$ can be written $X_{.1} = AX$, where

$$X = \begin{pmatrix} f \\ f_{.1} \end{pmatrix},\ A = \begin{pmatrix} 0_{2\times2} & I_2 \\ -T_0 & 0_{2\times2} \end{pmatrix} \in C^{4\times4}.$$

Now suppose that b_0, b_1 are real. If $\delta < 0$, all four roots are complex. Suppose that $\delta \geq 0$. Then $\pm\nu_1$ are real if $b \leq \delta^{1/2}$, and $\pm\nu_2$ are real if $\delta^{1/2} \leq -b$. So, if $b \leq \delta^{1/2} \leq -b$ then all four roots are real.

The differential equation, $Lf = 0_2$, arises in the theory of planetary perturbations with $b_0 = -\gamma/2$, $b_1 = -3\gamma/2$, so that $b = 1 - \gamma/2$, $c = -2\gamma^2$, $\delta = 9\gamma(\gamma - 8/9)/4$, and $\gamma > 0$ real. So, $\delta \geq 0$ if and only if $\gamma \geq 8/9$. Also $\gamma \geq 1$ if and only if $\nu_1^2 \geq 0$. For $\gamma \geq 8/9$, $\nu_2^2 < 0$. So, $\gamma < 8/9$ implies four complex roots; $8/9 \leq \gamma < 1$ implies four imaginary roots; $1 \leq \gamma$ implies $\nu_1^2 \geq 0 > \nu_2^2$ so that two roots are real and two are imaginary. For large γ, ν_1 and ν_2 of (3.12) satisfy

$$\nu_1 = (2\gamma)^{1/2}\left[1 - 5\gamma^{-1}/12 + O\left(\gamma^{-2}\right)\right],\ \nu_2 = i\gamma^{1/2}\left[1 + \gamma^{-1}/6 + O\left(\gamma^{-2}\right)\right].$$

By Corollary 2.3, we have

Example 3.5. *For L of Example 3.4, $L_\tau(t) = (d/dt)^2 + \tau^2 T_0(\tau t)$, and $L_\tau(t)X(t) = 0_2$ has solution $X(t) = f_{n,j}(\tau t)$ when $f_{n,j}(t)$ is a solution to Example 3.4.*

4. Using the Jordan form when $r = s$

Suppose that $r = s$. Then D and $T_n(t)$ can be easily written using the Jordan form of N, in terms of block matrices, or scalars when the Jordan form is diagonal. Suppose that N has Jordan form $N = PJP^{-1}$, where $J = \text{diag}\left(J_1, \ldots, J_p\right)$, $J_j = J_{m_j}\left(\lambda_j\right)$, $J_m(\lambda) = \lambda I_m + U_m$, and U_m is the $m \times m$ matrix of zeros except for ones on the first super-diagonal, that is, $(U_m)_{j,k} = \delta_{k,j+1}$ for $1 \leq j < m$. So, $s = m_1 + \cdots + m_p$ and for $0 \leq n < m$, U_m^n is the $m \times m$ matrix of zeros except for ones on the nth super-diagonal, that is, $(U_m^n)_{j,k} = \delta_{k,j+n}$ for $1 \leq j < m - n$. Also for $n \geq m$, $U_m^n = 0_{m\times m}$, and

$$J_m(\lambda)^n = \sum_{j=0}^{\min(n,m-1)} \binom{n}{j} \lambda^{n-j} U_m^j, \tag{4.1}$$
$$\exp\{J_m(\lambda)t\} = e^{\lambda t} V_m(t), \tag{4.2}$$

where

$$U_m^0 = I_m,\ V_m(t) = \sum_{j=0}^{m-1} t^j U_m^j / j!.$$

So, $V_m(t)$ has zeros on its subdiagonals, and the elements of its jth superdiagonal are all t^j/j. That is, $V_m(t)_{j,j+n} = t^j/j!$ for $1 \leq j < m - n$. For example,

$$U_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ J_2(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},\ J_2(\lambda)^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix},$$

$$\exp\{J_2(\lambda)t\} = e^{\lambda t}V_2(t),\ V_2(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Partition P, P^{-1}, S_n and $T_n = T_n(t)$ of (1.3) as $m_j \times m_k$ blocks, $P = \left(P_{j,k}\right)$, $P^{-1} = \left(P^{j,k}\right)$, $S_n = \left(S_{n,j,k}\right)$, $T_n(t) = \left(T_n(t)_{j,k}\right)$ for $1 \leq j, k \leq p$, and do similarly for D, N^n and $Y = Y(t) = e^{tN}$. Then

$$\left(N^n\right)_{j,k} = \sum_{c=1}^{p} P_{j,c} J_c^n P^{c,k},\ Y_{j,k} = \sum_{c=1}^{p} P_{j,c} e^{tJ_c} P^{c,k},$$
$$T_n(t)_{j,k} = \sum_{a,b=1}^{p} Y_{j,a}(t) S_{n,a,b} Y_{b,k}(-t) = \sum_{c,d=1}^{p} P_{j,c} e^{tJ_c} Q_{d,n}^c e^{-tJ_d} P^{d,k},$$

where

$$Q_{d,n}^c = \sum_{a,b=1}^{p} P^{c,a} S_{n,a,b} P_{b,d},$$
$$D = \sum_{n=0}^{q} S_n P \text{ diag } \left(M_{n,1}, \ldots, M_{n,p}\right)\ P^{-1} = \left(D_{j,k}\right),\ M_{n,j} = J_{m_j}\left(\nu + \lambda_j\right)^n \text{ of (4.1)},$$
$$D_{j,k} = \sum_{n=0}^{q} \sum_{b,c=1}^{p} S_{n,j,b} P_{b,c} J_{m_c} \left(\nu + \lambda_c\right)^n\ P^{c,k},$$

and e^{tJ_c} is given by (4.2) with $m = m_c$, $\lambda = \lambda_c$. So, $T_n(t)$ is a mixture of polynomials in t and factors $e^{(\lambda_c - \lambda_d)t}$. For example, if $P = I_s$, then

$$\left(N^n\right)_{j,k} = J_j^n \delta_{j,k},\ Y_{j,k}(t) = e^{tJ_j}\delta_{j,k},\ T_n(t)_{j,k} = e^{tJ_j} S_{n,j,k} e^{-tJ_k},\ D_{j,k} = \sum_{n=0}^{q} S_{n,j,k} \left(\nu + \lambda_k\right)^n.$$

Consider the two extremes: first the one Jordan block case $J = J_s(\lambda)$. Then $p = 1$, $m_1 = s$, P is scalar, say 1,

$$N = \lambda I_s + U_s,\ T_n(t) = V_s(t) S_n V_s(-t),$$
$$T_n(t)_{j,k} = \sum_{a=j}^{s} \sum_{b=1}^{k} \left[t^{a-j}/(a-j)!\right]\ S_{n,a,b}\ (-t)^{k-b}/(k-b)!,$$
$$D = \sum_{n=0}^{q} S_n J_s(\nu + \lambda)^n \text{ of (4.1)}.$$

Second the diagonal Jordan form with $J = \text{diag }(\lambda_1, \ldots, \lambda_s)$. Then $p = s$, $m_j \equiv 1$, $J_c = \lambda_c$,

$$T_n(t)_{j,k} = \sum_{c,d=1}^{s} e^{(\lambda_c - \lambda_d)t} P_{j,c} Q_{d,n}^c P^{d,k}$$

with all components scalar, and

$$D_{j,k} = \sum_{n=0}^{q} \sum_{b,c=1}^{s} S_{n,j,b} P_{b,c} \left(\nu + \lambda_c\right)^n\ P^{c,k}.$$

For example, if $P = I_s$, then

$$T_n(t)_{j,k} = e^{\left(\lambda_j - \lambda_k\right)t} S_{n,j,k},\ D_{j,k} = \sum_{n=0}^{q} S_{n,j,k} \left(\nu + \lambda_k\right)^n .$$

Example 4.1. *Take* $J = \text{diag }(\lambda_1, \lambda_2)$. *So,* $s = 2$. *Set* $\delta = \lambda_1 - \lambda_2$. *Then*

$$T_n(t)_{j,k} = \sum_{c,d=1}^{2} e^{(\lambda_c - \lambda_d)t} P_{j,c} Q^c_{d,n} P^{d,k} = \sum_{c=1}^{2} P_{j,c} Q^c_{c,n} P^{c,k} + e^{\delta t} P_{j,1} Q^1_{2,n} P^{2,k} + e^{-\delta t} P_{j,2} Q^2_{1,n} P^{1,k}$$

and

$$T_n(t) = P H_n P^{-1},$$

where

$$H_n = \begin{pmatrix} q_{n,1,1} & e^{\delta t} q_{n,1,2} \\ e^{-\delta t} q_{n,2,1} & q_{n,2,2} \end{pmatrix}$$

and $q_n = P^{-1} S_n P$. *For N of Section 3, we can take*

$$J = \text{diag }(i, -i),\ P = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} / \sqrt{2},\ \det P = i,\ P^{-1} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} / \sqrt{2}.$$

In Section 3, we avoided having to use P and these eigenvalues by using $N^2 = -I_2$.

5. Solving $Lf = g$ when $T_q = I_s$ and $r = s$

We now give the solution of $Lf = g$ when $T_q = I_s$ and $r = s$. This can be written $X_{.1} = AX + F$ for X, A of (2.15) and $F = (0'_s, 0'_s, \ldots, g')'$ as noted on page 90 of Hale [4] for the case $s = 1$.

For $j = 1, \ldots, qr$, set $f_j = f\left(t, \nu_j\right)$ of (2.1) if ν_j are distinct. Otherwise choose f_j using Theorem 2.2. Set

$$y(t,\nu) = \left((Ya)', \ldots, \left(Y\left(\nu I_r + N\right)^{q-1} a\right)'\right)',\ U_j = U\left(t, \nu_j\right),$$

where

$$U(t,\nu) = \left(\left(e^{\nu t} Y a\right)', \ldots, \left(e^{\nu t} Y a\right)'_{.q-1}\right)' = e^{\nu t} y(t,\nu).$$

So, $U_j \in C^{qs}$. For example, if $s = 2$ and $\left\{\nu_j\right\}$ are distinct, we can take $a = a_{(1)}$ or $a_{(2)}$ of (2.7). Now suppose that $r = s$. Then $U(t) = U = \left(U_1, \ldots, U_{qs}\right) \in C^{qs \times qs}$ is a *fundamental matrix solution* of $X_{.1} = AX$. That is, $\det\ U(t) \neq 0$. Its (j,k) element is

$$U_{j,k} = e^{\nu_j t} y_k\left(t, \nu_j\right). \tag{5.1}$$

So, $\widetilde{U}(t) = U(t)U(0)^{-1}$ is the *principal matrix solution at 0* as it satisfies $U(0) = I_{qs}$: see, for example, page 80 of Hale [4]. So, by the *variation of constants formula*, see, for example, his page 81, $X_{.1} = AX + F$ has solution

$$X(t) = U(t)\left[\widetilde{X}(0) + \int_0^t U^{-1}\, F\right], \tag{5.2}$$

where $\widetilde{X}(0) = U(0)^{-1}X(0)$. Write U^{-1} as a 2×2 block matrix with (j,k) element $U^{j,k} \in C^{s\times s}$. Then $U^{-1}F = \binom{U^{1,2}}{U^{2,2}} f$, so that $Lf = g$ has solution

$$f(t) = \sum_{j=1}^{2} U_{1,j}(t)\, \left[\widetilde{X}_j(0) + \int_0^t U^{j,2} g\right], \tag{5.3}$$

the first of the two block rows of

$$U(t)\, \left[\widetilde{X}(0) + \int_0^t \widetilde{U}_2 g\right],$$

where $\left(\widetilde{U}_1, \widetilde{U}_2\right) = U^{-1}$. So, the solutions (5.2) and (5.3) have qs unknowns $X(0)$ for X of (2.15), that is, the initial values of f and its first $q-1$ derivatives.

We now illustrate a use of Floquet's theorem. Set $r = qs$. Suppose that $r = s$, $T_q = I_s$, and $\{T_n\}$ have period T. (This holds for Example 1.1 and Section 3 with $T = \pi$ or π/τ for Example 3.4.) (2.15) puts $Lf = 0_s$ in the standard form $X_{.1} = AX$, where A is periodic. According to Floquet's theorem (see, for example, page 118 of Hale [4] or page 164 of Chicone [1]), since $U(t)$ is a fundamental matrix solution of $X_{.1} = AX \in C^{r\times r}$ for (X, A) of (2.15), and A has period T, there exists a constant $B \in C^{r\times r}$ and $P = P(t)$ with period T such that $U(t) = P(t)e^{Bt}$. However, Floquet's theorem does not give $P(t)$, B while our method does, as we now show. $\left\{\nu_j\right\}$ are the eigenvalues of B. Since these eigenvalues are distinct, B has diagonal Jordan form, say $Q\Lambda Q^{-1}$, where $\Lambda = \text{diag }(\nu_1, \ldots, \nu_r)$. So,

$$e^{Bt} = Qe^{\Lambda t}Q^{-1},\ e^{\Lambda t} = \text{diag }\left(e^{\nu_1 t}, \ldots, e^{\nu_r t}\right),\ U(t) = R(t)e^{\Lambda t}Q^{-1},$$

where $R(t) = PQ$. The (j,k) element of $U(t)Q$ is

$$\sum_{l=1}^{r} U_{j,l}(t)Q_{l,k} = R_{j,k}(t)e^{\nu_k t}.$$

So, $R_{j,k}(t)$ is the coefficient of $e^{\nu_k t}$ in $U(t)Q$. This gives $R(t)$. Set $\left(Q^{j,k}\right) = Q^{-1}$. By (5.1), the coefficient of $e^{\nu_b t}$ in

$$U_{j,k}(t) = \sum_{b=1}^{r} R_{j,b}(t)e^{\nu_b t}Q^{b,k}$$

is $\delta_{b,j}x_{j,k}(t) = R_{j,b}(t)Q^{b,k}$. So,

$$x_{j,k}(t) = y_k\left(t, \nu_j\right) = \sum_{b=1}^{r} R_{j,b}(t)Q^{b,k},\ x(t) = R(t)Q^{-1},\ P(t)Q = R(t) = x(t)Q.$$

$P(t) = x(t) = \left(y_k\left(t, \nu_j\right)\right)$ is of (5.1). Also, $U(t) = P(t)e^{Bt}$, so that $Qe^{\Lambda t}Q^{-1} = e^{Bt} = UP(t)^{-1} = Q(t)$ say. So, $Qe^{\Lambda t} = Q(t)Q$. Write Q as $(q_1, \ldots, q_r)$. So, for all t, q_j is an eigenvector of $Q(t)$ with eigenvalue $e^{\nu_j t}$. So, we can take q_j as the eigenvector of $Q(T)$ with eigenvalue $e^{\nu_j T}$. So, now we have Λ, Q and B.

Conflict of interest

The authors declare no conflicts of interest in this paper.

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