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Research article

On a proximal point algorithm for solving common fixed point problems and convex minimization problems in Geodesic spaces with positive curvature

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Abstract: In this article, we present a new modified proximal point algorithm in the framework of CAT(1) spaces which is utilized for solving common fixed point problem and minimization problems. Also, we prove convergence results of the obtained process under some mild conditions. Our results extend and improve several corresponding results of the existing literature.

Keywords: minimization problem; resolvent operator; CAT(1) space; proximal point algorithm; nonexpansive mapping
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1. Introduction

Suppose $(\tilde{X}, \tilde{d})$ is a geodesic metric space and $\Omega \subset \tilde{X}$. Let $\tilde{F}(\theta)$ be the collection of fixed points of nonlinear mappings $\theta : \Omega \to \Omega$. It is a common knowledge that $\tilde{F}(\theta) := \{\alpha \in \Omega | \theta\alpha = \alpha\}$. The mapping θ is said to be nonexpansive if for all $\alpha, \overline{\alpha} \in \Omega$, $\|\theta\alpha - \theta\overline{\alpha}\| \leq \|\alpha - \overline{\alpha}\|$ holds. For a real number κ, a CAT(κ) space (named in honour of E. **C**artan, A. D. **A**lexanderov and V. A. **T**oponogov) is referred to a geodesic space such that its geodesic triangle is adequately thinner than the corresponding comparison triangle in a model space with curvature κ. The CAT(κ) has been of great interest to famous mathematicians and a number of contributions have been presented. For instance, by taking $\kappa \leq 0$, Kirk [1, 2] established the existence of fixed points for a nonexpansive mapping in CAT(κ). On the other hand, Espínola and Fernández-León [3] showed that if the space CAT(κ) has at least one fixed point, then it is possible to approximate it by using some suitable iterative techniques.

Optimization problem is considered a crucial problems due its appearance in many areas of applied sciences and engineering. As many optimization problems are nonlinear in nature, solving them analytically could be a cumbersome task or even impossible. Therefore, developing suitable iterative techniques for handling such problems becomes imperative. In this regards, many researchers have studied some famous iterative schemes in CAT(κ). For example, He et al. [4] extended the well-known Mann iterative scheme into CAT(κ) as follows: Given $\alpha_1 \in \tilde{X}$, the subsequent iterates can be calculated using

$$\alpha_{z+1} = \chi_z\alpha_z \oplus (1-\chi_z)\theta\alpha_z, \quad z \geq 1. \tag{1.1}$$

By taking $0 \leq \chi_z \leq 1$ for all z, they established $\triangle$-convergence for the iterative scheme (1.1) in CAT(κ) where θ is a nonexpasive mapping. To improve the work of He et al. [4], two $\triangle$-convergence results for the iterative scheme (1.1) in CAT(κ) was proved by Kimura et al. [5] for common fixed point of a countable family of nonexpansive mappings where the curvature is bounded above by a number greater than zero. Furthermore, Jun [6] extended the Ishikawa iterative scheme into CAT(κ) spaces as follows: Let $\{\chi_z\}$ and $\{\rho_z\}$ be some sequences in $[0,1]$ and $\alpha_1 \in \tilde{X}$ then, the sequence $\{\alpha_z\}$ can be calculated via

$$\begin{cases} \beta_z = \rho_z\alpha_z \oplus (1-\rho_z)\theta_1\alpha_z \\ \alpha_{z+1} = \chi_z\theta_2\beta_z \oplus (1-\chi_z)\theta_1\alpha_z, \end{cases} \quad z \geq 1. \tag{1.2}$$

He proved that the sequence of iterates generated by (1.2) $\triangle$-converges to some fixed point of some nonexpansive mapping in CAT(κ) spaces where $\kappa > 0$. For more details on some famous iterative schemes have been extended into the CAT(1) space, the reader may refer to [4,7–11] and the references therein.

Proximal point algorithms have gained tremendous attentions in recent time due to numerous applications. In 2013, Bačák [12], extended the proximal point algorithm in Hadamard space. The settings of his proximal algorithm is as follows: given $\alpha_1 \in \tilde{X}$ and $\tilde{\lambda}_z > 0$, for all $z > 0$, for which its series diverges, then the subsequent iterates are generated using

$$\alpha_{z+1} = \arg\min_{\beta\in\tilde{X}} \left[\tilde{g}(\beta) + \frac{1}{2\tilde{\lambda}_z}\tilde{d}^2(\beta,\alpha_z)\right]. \tag{1.3}$$

By assuming that the function $\tilde{g} : \tilde{X} \to (-\infty, \infty]$ is convex, lower semi-continuous and bounded below, the sequence $\{\alpha_z\}$ $\triangle$ −converges to its minimizer was proved. Following the successive of the work in [12], Cholamjiak [13] incorporated the proximal point algorithm into the famous Halpern iterative scheme in the settings of CAT(0) spaces and established strong convergence based on standard assumptions. Subsequently, Suparatulatorn and Cholamjiak [14] presented a modified proximal point algorithms in the framework of CAT(0) spaces involving nonexpansive mappings and obtained strong convergence results by imposing some standard conditions. The proximal point algorithm has been successfully incorporated into the S-iterative scheme [15] and SP-iteration [16] to approximate minimizers of a convex function and common fixed points of asymptotically nonexpansive as well as quasi-nonexpansive mappings in CAT(0) spaces [17, 18].

In another development, Chaipunya and Kumam [19] discussed general proximal point algorithm for obtaining the zero of maximal monotone set-valued vector field in complete CAT(0) spaces. With the aid of monotonicity and surjectivity assumptions, the weak convergence theorem of the proposed algorithm was proved under some conditions. Moreover, Kimura and Kohsaka [20] presented a proximal point algorithm that generates the sequence $\{\alpha_z\}$ in CAT(1) spaces in the following recursive formula

$$\alpha_{z+1} = \arg\min_{\beta\in\tilde{X}} \left[\tilde{g}(\beta) + \frac{1}{\tilde{\lambda}_z} \tan\tilde{d}(\beta,\alpha_z)\sin\tilde{d}(\beta,\alpha_z)\right], \quad z \geq 1, \tag{1.4}$$

where $\alpha_1 \in \tilde{X}$ is a given initial guess and the sequence $\{\tilde{\lambda}_z\}$ is positive for all z. By assuming that the sequence generated by (1.4) is bounded below, they showed there exists at least a minimizer of a convex function and subsequently proved that $\{\alpha_z\}$ converges to its minimizer under some appropriate conditions.

Recently, many convergence results by the proximal point algorithm for solving optimization problems have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces to the setting of manifolds [21–25]. The minimizers of the objective convex functionals in the space with nonlinearity play a crucial role in the branch of analysis and geometry. Numerous applications in computer vision, machine learning, electronic structure computation, system balancing and robot manipulation can be considered as solving optimization problems on manifolds see in [26–29].

Motivated by the ongoing above research, in this article, we introduce a new modified proximal point algorithm to solve common solution of the set of common fixed points of three nonexpansive mappings and the set of minimizer of a convex function in CAT(1) spaces. We also prove some Δ and strong convergence results of the presented algorithm under some mild conditions.

2. Preliminaries

In this section, we will present some basic notations, definitions, concepts and useful lemmas will be used in the next section.

Let $(\tilde{X},\tilde{d})$ be a metric space and $\alpha_1,\alpha_2 \in \tilde{X}$ such that $\tilde{d}(\alpha_1,\alpha_2) = \bar{r}$. A *geodesic path* from α_1 to α_2 is an isometry $\mu : [0,\bar{r}] \to \tilde{X}$ such that $\mu(0) = \alpha_1$ and $\mu(\bar{r}) = \alpha_2$. The image of a geodesic path is called the *geodesic segment*. The space $(\tilde{X},\tilde{d})$ is said to be a *geodesic space* if every two points of $\tilde{X}$ are joined by a geodesic. $(\tilde{X},\tilde{d})$ is called a *uniquely geodesic space* if every two points of $\tilde{X}$ are joined by exactly one geodesic segment and this unique geodesic segment is denoted by $[\alpha_1,\alpha_2]$. For all $\alpha_1,\alpha_2 \in \tilde{X}$ and $\bar{t} \in [0,1]$, there exists a unique $\alpha_3 \in [\alpha_1,\alpha_2]$ such that

$$\tilde{d}(\alpha_1,\alpha_3) = \bar{t}\tilde{d}(\alpha_1,\alpha_2) \text{ and } \tilde{d}(\alpha_2,\alpha_3) = (1-\bar{t})\tilde{d}(\alpha_1,\alpha_2).$$

We use the notation $(1-\bar{t})\alpha_1 \oplus \bar{t}\alpha_2$ for the above mentioned unique point α_3.

A subset Ω of $\tilde{X}$ is said to be *convex* if it contains every geodesic segment joining any two of its points. The set Ω is said to be *bounded* if

$$diam(\Omega) = \sup\{\tilde{d}(\alpha_1,\alpha_2) : \alpha_1,\alpha_2 \in \Omega\} < \infty.$$

Definition 2.1. For any $k \in \mathbb{R}$, we use M_k^n to denote the following metric spaces:

(i) If $k = 0$, then M_0^n is the Euclidean space E^n.

(ii) If $k > 0$, then M_k^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $\frac{1}{\sqrt{k}}$.

(iii) If $k < 0$, then M_k^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by the constant $\frac{1}{\sqrt{-k}}$.

A *geodesic triangle* $\Delta(\alpha_1, \alpha_2, \alpha_3)$ in a geodesic space $(\tilde{X}, \tilde{d})$ consists of three points $\alpha_1, \alpha_2, \alpha_3 \in \tilde{X}$ and three geodesic segments between each pair of vertices. A *comparison triangle* for a geodesic triangle $\Delta(\alpha_1, \alpha_2, \alpha_3)$ in $(\tilde{X}, \tilde{d})$ is a triangle $\Delta(\bar{\alpha_1}, \bar{\alpha_2}, \bar{\alpha_3})$ in M_k^2 such that

$$\tilde{d}(\alpha_i, \alpha_j) = \tilde{d}_{M_k^2}(\alpha_i, \alpha_j) \quad \text{for each } \ i, j = 1, 2, 3.$$

Also, if $k \leq 0$, then such a comparison triangle always exists in M_k^2 and, if $k < 0$, then such a triangle exists whenever $\tilde{d}(\alpha_1, \alpha_2) + \tilde{d}(\alpha_2, \alpha_3) + \tilde{d}(\alpha_3, \alpha_1) < 2D_k$, where $D_k = \frac{\pi}{\sqrt{k}}$.

A geodesic triangle $\Delta(\alpha_1, \alpha_2, \alpha_3)$ in $\tilde{X}$ is said to satisfy the *CAT(k) inequality* if, for any $p, q \in \Delta(\alpha_1, \alpha_2, \alpha_3)$ and for their comparison points $\bar{p}, \bar{q} \in \Delta(\bar{\alpha_1}, \bar{\alpha_2}, \bar{\alpha_3})$, we have

$$\tilde{d}(p, q) \leq \tilde{d}_{M_k^2}(\bar{p}, \bar{q}).$$

A metric space $(\tilde{X}, \tilde{d})$ is known as D-geodesic space if any two points of $\tilde{X}$ with distance less than D (where $D > 0$) are joined by a geodesic.

Definition 2.2. A metric space $(\tilde{X}, \tilde{d})$ is called a *CAT(k) space* if it is D_k-geodesic and any geodesic triangle $\Delta(\alpha_1, \alpha_2, \alpha_3)$ in $\tilde{X}$ with $\tilde{d}(\alpha_1, \alpha_2) + \tilde{d}(\alpha_2, \alpha_3) + \tilde{d}(\alpha_3, \alpha_1) < 2D_k$ satisfies the CAT(k) inequality.

Let $(\tilde{X}, \tilde{d})$ be a CAT(1) space such that $\tilde{d}(\alpha_1, \alpha_2) + \tilde{d}(\alpha_2, \alpha_3) + \tilde{d}(\alpha_3, \alpha_1) < 2D_1$ for all $\alpha_1, \alpha_2, \alpha_3 \in \tilde{X}$. Then the following holds for any $\chi \in [0, 1]$:

$$\cos \tilde{d}(\chi\alpha_1 \oplus (1 - \chi)\alpha_2, \alpha_3) \geq \chi \cos \tilde{d}(\alpha_1, \alpha_3) + (1 - \chi) \cos \tilde{d}(\alpha_2, \alpha_3). \tag{2.1}$$

Let $\{\alpha_z\}$ be a bounded sequence in a complete CAT(1) space $\tilde{X}$. For all $\alpha \in \tilde{X}$, we define:

$$r(\alpha, \{\alpha_z\}) = \limsup_{z\to\infty} \tilde{d}(\alpha, \alpha_z).$$

The *asymptotic radius* $r(\{\alpha_z\})$ is given by

$$r(\{\alpha_z\}) = \inf\{r(\alpha, \alpha_z) : \alpha \in \tilde{X}\}$$

and the *asymptotic center* $A(\{\alpha_z\})$ of $\{\alpha_z\}$ is defined as:

$$A(\{\alpha_z\}) = \{\alpha \in \tilde{X} : r(\alpha, \alpha_z) = r(\{\alpha_z\})\}.$$

Definition 2.3. Let $(\tilde{X}, \tilde{d})$ be a CAT(1) space. A sequence $\{\alpha_z\}$ in $\tilde{X}$ is said to be Δ-*convergent* to a point $\alpha \in \tilde{X}$ if α is the unique asymptotic center of every subsequence $\{\alpha_{z_k}\}$ of $\{\alpha_z\}$. In this case, we write $\Delta\text{-}\lim_{z\to\infty} \alpha_z = \alpha$.

Definition 2.4. A mapping $\theta : \tilde{X} \to \tilde{X}$ is said to be demi-compact if, for any sequence $\{\alpha_z\}$ in Ω such that $\lim_{z\to\infty} \tilde{d}(\alpha_z, \theta\alpha_z) = 0$, $\{\alpha_z\}$ has a convergent subsequence.

Definition 2.5. Let $(\tilde{X}, \tilde{d})$ be a geodesic metric space.

(i) Let $\alpha_1, \alpha_2, \alpha_3 \in \tilde{P}$, where $\tilde{P}$ is an open set in $\tilde{X}$. Then, for all $R \in [0,2]$, $\tilde{P}$ is said to be a C_R-*domain* if, for any minimal geodesic $\mu : [0,1] \to \tilde{X}$ between α_2 and α_3 with $\chi \in [0,1]$, we have the following:

$$\tilde{d}^2(\alpha_1, (1-\chi)\alpha_2 \oplus \chi\alpha_3) \leq (1-\chi)\tilde{d}^2(\alpha_1, \alpha_2) + \chi\tilde{d}^2(\alpha_1, \alpha_3) - \frac{R}{2}(1-\chi)\chi\tilde{d}^2(\alpha_2, \alpha_3). \tag{2.2}$$

(ii) A geodesic metric space $(\tilde{X}, \tilde{d})$ is known as *R-convex* if $\tilde{X}$ is itself a C_R-domain for any $R \in [0,2]$.

A CAT(1) space $\tilde{X}$ is said to be *admissible* if $\tilde{d}(\alpha_1, \alpha_2) < \frac{\pi}{2}$ for all $\alpha_1, \alpha_2 \in \tilde{X}$. Further, the sequence $\{\alpha_z\}$ is said to be *spherically bounded* in $\tilde{X}$ if

$$\inf_{\beta \in \tilde{X}} \limsup_{z \to \infty} \tilde{d}(\beta, \alpha_z) < \frac{\pi}{2}.$$

A function $\tilde{g} : \tilde{X} \to (-\infty, \infty]$ is said to be *proper* if

$$Dom(\tilde{g}) = \{\alpha \in \tilde{X} : \tilde{g}(\alpha) \in \mathbb{R}\} \neq \emptyset.$$

Also, $\tilde{g}$ is said to be *lower semi-continuous* if the set $K = \{\alpha \in \tilde{X} : \tilde{g}(\alpha) \leq \rho\}$ is closed in $\tilde{X}$ for all $\rho \in \mathbb{R}$.

For all $\tilde{\lambda} > 0$, define the *resolvent* of a proper lower semi-continuous function $\tilde{g}$ in admissible CAT(1) spaces as follows:

$$R_\lambda(\alpha) = arg \min_{\beta \in \tilde{X}} \Big[\tilde{g}(\beta) + \frac{1}{\tilde{\lambda}} \tan \tilde{d}(\alpha, \beta) \sin \tilde{d}(\alpha, \beta)\Big], \quad \text{for all } \alpha \in \tilde{X}.$$

The mapping $R_{\tilde{\lambda}}$ is well defined and the set of fixed points of the resolvent associated with $\tilde{g}$ coincides with the set of minimizers of $\tilde{g}$ [30].

Next, we have the following important lemmas:

Lemma 2.6. *[20] Let $(\tilde{X}, \tilde{d})$ be an admissible complete CAT(1) space and $\tilde{g} : \tilde{X} \to (-\infty, \infty]$ be a proper lower semi-continuous convex function. If $\tilde{\lambda} > 0$, $\alpha \in \tilde{X}$ and $u \in arg \min_{\tilde{X}} \tilde{g}$, then the following inequalities hold:*

$$\frac{\pi}{2}\Big(\frac{1}{\cos^2 \tilde{d}(R_{\tilde{\lambda}}\alpha, \alpha)} + 1\Big)(\cos \tilde{d}(R_{\tilde{\lambda}}\alpha, \alpha) \cos \tilde{d}(u, R_{\tilde{\lambda}}\alpha) - \cos \tilde{d}(u, \alpha)) \geq \lambda(\tilde{g}(R_{\tilde{\lambda}}\alpha) - \tilde{g}(u)) \tag{2.3}$$

and

$$\cos \tilde{d}(R_{\tilde{\lambda}}\alpha, \alpha) \cos \tilde{d}(u, R_{\tilde{\lambda}}\alpha) \geq \cos \tilde{d}(u, \alpha). \tag{2.4}$$

Lemma 2.7. *[30] Let $(\tilde{X}, \tilde{d})$ be a admissible complete CAT(1) space and $\tilde{g} : \tilde{X} \to (-\infty, \infty]$ be a proper lower semi-continuous convex function. Then, $\tilde{g}$ is Δ-lower semi-continuous.*

Lemma 2.8. *[5] Let $(\tilde{X}, \tilde{d})$ be a admissible complete CAT(1) space and $\{\alpha_z\}$ be a spherical bounded sequence in $\tilde{X}$. If $\{\tilde{d}(\alpha_z, \tilde{p})\}$ is convergent for all $\tilde{p} \in W_\Delta(\{\alpha_z\})$, then the sequence $\{\alpha_z\}$ is Δ-convergent.*

In 2014, Panyanak [7] obtained the demiclosedness principle for a total asymptotically mapping in CAT(k) spaces. Since every nonexpansive mapping is a total asymptotically mapping, we have the following result for a nonexpansive mappings:

Lemma 2.9. *Let $\theta : \Omega \to \Omega$ be a nonexpansive mapping defined on a nonempty closed convex subset of a complete CAT(1) space $(\tilde{X}, \tilde{d})$. If $\{\alpha_z\}$ is a bounded sequence with $\lim_{z\to\infty} \tilde{d}(\alpha_z, \theta\alpha_z) = 0$ and Δ-$\lim_{z\to\infty} \alpha_z = \beta$, then $\beta \in \Omega$ and $\theta\beta = \beta$.*

3. Main results

Lemma 3.1. *Let $(\tilde{X}, \tilde{d})$ be an admissible complete CAT(1) space and $\tilde{g} : \tilde{X} \to (-\infty, \infty]$ be a proper lower semi-continuous convex function. Let θ_1, θ_2 and θ_3 be three nonexpansive mappings on $\tilde{X}$ such that $\omega = \tilde{F}(\theta_1) \cap \tilde{F}(\theta_2) \cap \tilde{F}(\theta_3) \cap arg \min_{\alpha \in \tilde{X}} \tilde{g}(\alpha) \neq \emptyset$. Assume that $\{\chi_z\}$, $\{\rho_z\}$ and $\{\mu_z\}$ are sequences in $[a, b]$ for some $a, b \in (0, 1)$ for all $z \geq 1$ and $\{\tilde{\lambda}_z\}$ is a sequence such that $\tilde{\lambda}_z \geq \lambda > 0$ for all $z \geq 1$ and for some λ. Suppose that the sequence $\{\alpha_z\}$ is generated in the following manner for $\alpha_1 \in \tilde{X}$:*

$$\begin{cases} \xi_z = arg \min_{\beta \in \tilde{X}} [\tilde{g}(\beta) + \frac{1}{\tilde{\lambda}_z} tan(\tilde{d}(\beta, \alpha_z)) sin(\tilde{d}(\beta, \alpha_z))], \\ \gamma_z = (1 - \chi_z)\alpha_z \oplus \chi_z \theta_1 \xi_z, \\ \beta_z = (1 - \rho_z)\theta_1 \alpha_z \oplus \rho_z \theta_2 \gamma_z, \\ \alpha_{z+1} = (1 - \mu_z)\theta_2 \beta_z \oplus \mu_z \theta_3 \beta_z \end{cases} \tag{3.1}$$

for each $z \geq 1$. Then, we have the following:

(i) $\lim_{z\to\infty} \tilde{d}(\alpha_z, \tilde{p})$ *exists for all* $\tilde{p} \in \omega$.

(ii) $\lim_{z\to\infty} \tilde{d}(\alpha_z, \xi_z) = 0$.

(iii) $\lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_1\alpha_z) = \lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_2\alpha_z) = \lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_3\alpha_z) = 0$.

Proof. First, we will show that $\{\alpha_z\}$ is spherically bounded. Note that $\xi_z = R_{\tilde{\lambda}_z}\alpha_z$ for all $z \geq 1$. Let $\tilde{p} \in \omega$. Then, from Lemma 2.6, we have

$$\min(cos\, \tilde{d}(\tilde{p}, \xi_z), cos\, \tilde{d}(\xi_z, \alpha_z)) \geq cos\, \tilde{d}(\tilde{p}, \xi_z) cos\, \tilde{d}(\xi_z, \alpha_z) \geq cos\, \tilde{d}(\tilde{p}, \alpha_z), \tag{3.2}$$

which implies that

$$\max\{\tilde{d}(\tilde{p}, \xi_z), \tilde{d}(\xi_z, \alpha_z)\} \leq \tilde{d}(\tilde{p}, \alpha_z). \tag{3.3}$$

Since, θ_1, θ_2 and θ_3 are nonexpansive mappings and $\tilde{X}$ is admissible, using (2.1), we get

$$\begin{aligned} cos\, \tilde{d}(\tilde{p}, \gamma_z) &= cos\, \tilde{d}(\tilde{p}, (1 - \chi_z)\alpha_z \oplus \chi_z \theta_1 \xi_z) \\ &\geq (1 - \chi_z) cos\, \tilde{d}(\tilde{p}, \alpha_z) + \chi_z cos\, \tilde{d}(\tilde{p}, \theta_1 \xi_z) \\ &\geq (1 - \chi_z) cos\, \tilde{d}(\tilde{p}, \alpha_z) + \chi_z cos\, \tilde{d}(\tilde{p}, \xi_z) \\ &\geq (1 - \chi_z) cos\, \tilde{d}(\tilde{p}, \alpha_z) + \chi_z cos\, \tilde{d}(\tilde{p}, \alpha_z) \\ &= cos\, \tilde{d}(\tilde{p}, \alpha_z), \end{aligned} \tag{3.4}$$

$$\begin{aligned} cos\, \tilde{d}(\tilde{p}, \beta_z) &= cos\, \tilde{d}(\tilde{p}, (1 - \rho_z)\theta_1 \alpha_z \oplus \rho_z \theta_2 \gamma_z) \\ &\geq (1 - \rho_z) cos\, \tilde{d}(\tilde{p}, \theta_1 \alpha_z) + \rho_z cos\, \tilde{d}(\tilde{p}, \theta_2 \gamma_z) \\ &\geq (1 - \rho_z) cos\, \tilde{d}(\tilde{p}, \alpha_z) + \rho_z cos\, \tilde{d}(\tilde{p}, \gamma_z) \\ &\geq (1 - \rho_z) cos\, \tilde{d}(\tilde{p}, \alpha_z) + \rho_z cos\, \tilde{d}(\tilde{p}, \alpha_z) \\ &= cos\, \tilde{d}(\tilde{p}, \alpha_z), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} cos\, \tilde{d}(\tilde{p}, \alpha_{z+1}) &= cos\, \tilde{d}(\tilde{p}, (1 - \mu_z)\theta_2 \beta_z \oplus \mu_z \theta_3 \beta_z) \\ &\geq (1 - \mu_z) cos\, \tilde{d}(\tilde{p}, \theta_2 \beta_z) + \mu_z cos\, \tilde{d}(\tilde{p}, \theta_3 \beta_z) \\ &\geq (1 - \mu_z) cos\, \tilde{d}(\tilde{p}, \alpha_z) + \mu_z cos\, \tilde{d}(\tilde{p}, \alpha_z) \\ &= cos\, \tilde{d}(\tilde{p}, \alpha_z) \end{aligned} \tag{3.6}$$

which yields

$$\tilde{d}(\tilde{p}, \alpha_{z+1}) \leq \tilde{d}(\tilde{p}, \alpha_z) \leq \tilde{d}(\tilde{p}, \alpha_1) < \frac{\pi}{2}. \tag{3.7}$$

It follows from (3.3) and (3.7) that

$$\limsup_{z\to\infty} \tilde{d}(\tilde{p}, \xi_z) \leq \limsup_{z\to\infty} \tilde{d}(\tilde{p}, \alpha_z) < \frac{\pi}{2}.$$

Therefore, the sequences $\{\xi_z\}$ and $\{\alpha_z\}$ are spherically bounded. Also, $\sup_{z\geq 1} \tilde{d}(\alpha_z, \xi_z) < \frac{\pi}{2}$ and $\lim_{z\to\infty} \tilde{d}(\tilde{p}, \alpha_z) < \frac{\pi}{2}$ exists for all $\tilde{p} \in \omega$. Let

$$\lim_{n\to\infty} \tilde{d}(\tilde{p}, \alpha_z) = \zeta \geq 0. \tag{3.8}$$

Now, we show that $\lim_{z\to\infty} \tilde{d}(\alpha_z, \xi_z) = 0$. It follows from

$$\begin{aligned} cos\, \tilde{d}(\tilde{p}, \alpha_{z+1}) &= cos\, \tilde{d}(\tilde{p}, (1-\mu_z)\theta_2\beta_z \oplus \mu_z\theta_3\beta_z) \\ &\geq (1-\mu_z)cos\, \tilde{d}(\tilde{p}, \theta_2\beta_z) + \mu_z cos\, \tilde{d}(\tilde{p}, \theta_3\beta_z) \\ &\geq cos\, \tilde{d}(\tilde{p}, \alpha_z) - \mu_z cos\, \tilde{d}(\tilde{p}, \alpha_z) + \mu_z cos\, \tilde{d}(\tilde{p}, \beta_z) \end{aligned}$$

which implies that

$$\mu_z cos\, \tilde{d}(\tilde{p}, \alpha_z) \geq cos\, \tilde{d}(\tilde{p}, \alpha_z) - cos\, \tilde{d}(\tilde{p}, \alpha_{z+1}) + \mu_z cos\, \tilde{d}(\tilde{p}, \beta_z)$$

i.e.,

$$cos\, \tilde{d}(\tilde{p}, \alpha_z) \geq \frac{1}{\mu_z}(cos\, \tilde{d}(\tilde{p}, \alpha_z) - cos\, \tilde{d}(\tilde{p}, \alpha_{z+1})) + cos\, \tilde{d}(\tilde{p}, \beta_z).$$

Since $\mu_z \geq a > 0$ for each $z \geq 1$, we get

$$cos\, \tilde{d}(\tilde{p}, \alpha_z) \geq \frac{1}{a}(cos\, \tilde{d}(\tilde{p}, \alpha_z) - cos\, \tilde{d}(\tilde{p}, \alpha_{z+1})) + cos\, \tilde{d}(\tilde{p}, \beta_z), \tag{3.9}$$

by using (3.8), which yields

$$\zeta = \liminf_{z\to\infty} cos\, \tilde{d}(\tilde{p}, \alpha_z) \geq \liminf_{z\to\infty} cos\, \tilde{d}(\tilde{p}, \beta_z). \tag{3.10}$$

Also, from (3.5), we have

$$\limsup_{z\to\infty} cos\, \tilde{d}(\tilde{p}, \beta_z) \geq \limsup_{z\to\infty} cos\, \tilde{d}(\tilde{p}, \alpha_z) = \zeta. \tag{3.11}$$

Thus, it follows from (3.10) and (3.11), we obtain

$$\lim_{z\to\infty} cos\, \tilde{d}(\tilde{p}, \beta_z) = \zeta. \tag{3.12}$$

Next, we consider

$$\begin{aligned} cos\, \tilde{d}(\tilde{p}, \beta_z) &= cos\, \tilde{d}(\tilde{p}, (1-\rho_z)\theta_1\alpha_z \oplus \rho_z\theta_2\gamma_z) \\ &\geq (1-\rho_z)cos\, \tilde{d}(\tilde{p}, \theta_1\alpha_z) + \rho_z cos\, \tilde{d}(\tilde{p}, \gamma_z) \\ &\geq cos\, \tilde{d}(\tilde{p}, \alpha_z) - \rho_z cos\, \tilde{d}(\tilde{p}, \alpha_z) + \rho_z cos\, \tilde{d}(\tilde{p}, \gamma_z), \end{aligned}$$

by using the fact that $\rho_z \geq a > 0$ for all $z \geq 1$ gives

$$cos\ \tilde{d}(\tilde{p}, \alpha_z) \geq \frac{1}{a}(cos\ \tilde{d}(\tilde{p}, \alpha_z) - cos\ \tilde{d}(\tilde{p}, \beta_z)) + cos\ \tilde{d}(\tilde{p}, \gamma_z), \tag{3.13}$$

it follows from (3.8) and (3.12), which yields

$$\zeta = \liminf_{z\to\infty} cos\ \tilde{d}(\tilde{p}, \alpha_z) \geq \liminf_{z\to\infty} cos\ \tilde{d}(\tilde{p}, \gamma_z). \tag{3.14}$$

Also, from (3.4), we have

$$\limsup_{z\to\infty} cos\ \tilde{d}(\tilde{p}, \gamma_z) \geq \limsup_{z\to\infty} cos\ \tilde{d}(\tilde{p}, \alpha_z) = \zeta. \tag{3.15}$$

Thus, from (3.14) and (3.15), we obtain

$$\lim_{z\to\infty} cos\ \tilde{d}(\tilde{p}, \gamma_z) = \zeta. \tag{3.16}$$

From (3.3) and (3.4), we get

$$\begin{aligned} cos\ \tilde{d}(\tilde{p}, \gamma_z) &\geq (1-\chi_z)cos\ \tilde{d}(\tilde{p}, \alpha_z) + \chi_z cos\ \tilde{d}(\tilde{p}, \xi_z) \\ &\geq (1-\chi_z)cos\ \tilde{d}(\tilde{p}, \alpha_z) + \chi_z \tfrac{cos\ \tilde{d}(\tilde{p}, \alpha_z)}{cos\ \tilde{d}(\xi_z, \alpha_z)} \\ &= cos\ \tilde{d}(\tilde{p}, \alpha_z) + \chi_z cos\ \tilde{d}(\tilde{p}, \alpha_z)[\tfrac{1}{cos\ \tilde{d}(\xi_z, \alpha_z)} - 1] \end{aligned}$$

i.e.,

$$\frac{cos\ \tilde{d}(\tilde{p}, \gamma_z)}{cos\ \tilde{d}(\tilde{p}, \alpha_z)} - 1 \geq \chi_z[\frac{1}{cos\ \tilde{d}(\xi_z, \alpha_z)} - 1]$$

Since $\chi_z \geq a > 0$ for each $z \geq 1$, from (3.8) and (3.16), we get

$$\lim_{z\to\infty} \tilde{d}(\xi_z, \alpha_z) = 0, \tag{3.17}$$

which implies that

$$\lim_{z\to\infty} \tilde{d}(R_{\tilde{\lambda}_z}\alpha_z, \alpha_z) = 0.$$

Also, as $\tilde{\lambda}_z \geq \lambda > 0$ for each $z \geq 1$, we obtain

$$\lim_{z\to\infty} \tilde{d}(R_\lambda \alpha_z, \alpha_z) = 0.$$

Next, we prove that $\lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_1\alpha_z) = \lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_2\alpha_z) = \lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_3\alpha_z) = 0$. From (2.2), we have

$$\begin{aligned} \tilde{d}^2(\tilde{p}, \gamma_z) &= \tilde{d}^2(\tilde{p}, (1-\chi_z)\alpha_z \oplus \chi_z\theta_1\xi_z) \\ &\leq (1-\chi_z)\tilde{d}^2(\tilde{p}, \alpha_z) + \chi_z\tilde{d}^2(\tilde{p}, \theta_1\xi_z) - \tfrac{R}{2}(1-\chi_z)\chi_z\tilde{d}^2(\alpha_z, \theta_1\xi_z) \\ &\leq (1-\chi_z)\tilde{d}^2(\tilde{p}, \alpha_z) + \chi_z\tilde{d}^2(\tilde{p}, \alpha_z) - \tfrac{R}{2}ab\tilde{d}^2(\alpha_z, \theta_1\xi_z) \\ &= \tilde{d}^2(\tilde{p}, \alpha_z) - \tfrac{R}{2}ab\tilde{d}^2(\alpha_z, \theta_1\xi_z), \end{aligned}$$

which yields

$$\tilde{d}^2(\alpha_z, \theta_1\xi_z) \leq \frac{2}{Rab}[\tilde{d}^2(\tilde{p}, \alpha_z) - \tilde{d}^2(\tilde{p}, \gamma_z)].$$

Thus, we obtain

$$\lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_1\xi_z) = 0. \tag{3.18}$$

By using triangle inequality along with (3.17) and (3.18), we get

$$\begin{aligned}\tilde{d}(\alpha_z, \theta_1\alpha_z) &\leq \tilde{d}(\alpha_z, \theta_1\xi_z) + \tilde{d}(\theta_1\xi_z, \theta_1\alpha_z)\\ &\leq \tilde{d}(\alpha_z, \theta_1\xi_z) + \tilde{d}(\xi_z, \alpha_z)\\ &\to 0 \quad as \quad z \to \infty.\end{aligned} \tag{3.19}$$

Next, we consider

$$\begin{aligned}\tilde{d}^2(\tilde{p}, \beta_z) &= \tilde{d}^2(\tilde{p}, (1-\rho_z)\theta_1\alpha_z \oplus \rho_z\theta_2\gamma_z)\\ &\leq (1-\rho_z)\tilde{d}^2(\tilde{p}, \theta_1\alpha_z) + \rho_z\tilde{d}^2(\tilde{p}, \theta_2\gamma_z) - \tfrac{R}{2}(1-\rho_z)\rho_z\tilde{d}^2(\theta_1\alpha_z, \theta_2\gamma_z)\\ &\leq (1-\rho_z)\tilde{d}^2(\tilde{p}, \alpha_z) + \rho_z\tilde{d}^2(\tilde{p}, \alpha_z) - \tfrac{R}{2}ab\tilde{d}^2(\theta_1\alpha_z, \theta_2\gamma_z)\\ &= \tilde{d}^2(\tilde{p}, \alpha_z) - \tfrac{R}{2}ab\tilde{d}^2(\theta_1\alpha_z, \theta_2\gamma_z),\end{aligned}$$

which is equivalent to

$$\tilde{d}^2(\theta_1\alpha_z, \theta_2\gamma_z) \leq \frac{2}{Rab}[\tilde{d}^2(\tilde{p}, \alpha_z) - \tilde{d}^2(\tilde{p}, \beta_z)].$$

Implies that

$$\lim_{z\to\infty} \tilde{d}(\theta_1\alpha_z, \theta_2\gamma_z) = 0. \tag{3.20}$$

Also,

$$\tilde{d}(\gamma_z, \alpha_z) = \tilde{d}((1-\chi_z)\alpha_z \oplus \chi_z\theta_1\xi_z, \alpha_z) \leq \chi_z\tilde{d}(\theta_1\xi_z, \alpha_z),$$

by using (3.18), then we get

$$\lim_{z\to\infty} \tilde{d}(\gamma_z, \alpha_z) = 0. \tag{3.21}$$

By using triangle inequality along with (3.19)–(3.21), then we get

$$\begin{aligned}\tilde{d}(\alpha_z, \theta_2\alpha_z) &\leq \tilde{d}(\alpha_z, \theta_1\alpha_z) + \tilde{d}(\theta_1\alpha_z, \theta_2\gamma_z) + \tilde{d}(\theta_2\gamma_z, \theta_2\alpha_z)\\ &\leq \tilde{d}(\alpha_z, \theta_1\alpha_z) + \tilde{d}(\theta_1\alpha_z, \theta_2\gamma_z) + \tilde{d}(\gamma_z, \alpha_z)\\ &\to 0 \quad as \quad z \to \infty.\end{aligned} \tag{3.22}$$

Now, we have

$$\begin{aligned}\tilde{d}^2(\tilde{p}, \alpha_{z+1}) &= \tilde{d}^2(\tilde{p}, (1-\mu_z)\theta_2\beta_z \oplus \mu_z\theta_3\beta_z)\\ &\leq (1-\mu_z)\tilde{d}^2(\tilde{p}, \theta_2\beta_z) + \mu_z\tilde{d}^2(\tilde{p}, \theta_3\beta_z) - \tfrac{R}{2}(1-\mu_z)\mu_z\tilde{d}^2(\theta_2\beta_z, \theta_3\beta_z)\\ &\leq (1-\mu_z)\tilde{d}^2(\tilde{p}, \alpha_z) + \mu_z\tilde{d}^2(\tilde{p}, \alpha_z) - \tfrac{R}{2}ab\tilde{d}^2(\theta_2\beta_z, \theta_3\beta_z)\\ &= \tilde{d}^2(\tilde{p}, \alpha_z) - \tfrac{R}{2}ab\tilde{d}^2(\theta_2\beta_z, \theta_3\beta_z),\end{aligned}$$

which implies that

$$\tilde{d}^2(\theta_2\beta_z, \theta_3\beta_z) \leq \frac{2}{Rab}[\tilde{d}^2(\tilde{p}, \alpha_z) - \tilde{d}^2(\tilde{p}, \alpha_{z+1})].$$

Hence, we obtain

$$\lim_{z\to\infty} \tilde{d}(\theta_2\beta_z, \theta_3\beta_z) = 0. \tag{3.23}$$

Consider,

$$\begin{aligned}
\tilde{d}(\beta_z, \alpha_z) &= \tilde{d}((1-\rho_z)\theta_1\alpha_z \oplus \rho_z\theta_2\gamma_z, \alpha_z) \\
&\leq (1-\rho_z)\tilde{d}(\theta_1\alpha_z, \alpha_z) + \rho_z\tilde{d}(\theta_2\gamma_z, \alpha_z) \\
&\leq (1-\rho_z)\tilde{d}(\theta_1\alpha_z, \alpha_z) + \rho_z(\tilde{d}(\theta_2\gamma_z, \theta_1\alpha_z) + \tilde{d}(\theta_1\alpha_z, \alpha_z)),
\end{aligned}$$

it follows from (3.19) and (3.20). So, we get

$$\lim_{z\to\infty} \tilde{d}(\beta_z, \alpha_z) = 0. \tag{3.24}$$

By using triangle inequality along with (3.19)–(3.24), which yields

$$\begin{aligned}
\tilde{d}(\alpha_z, \theta_3\alpha_z) &\leq \tilde{d}(\alpha_z, \theta_1\alpha_z) + \tilde{d}(\theta_1\alpha_z, \theta_2\gamma_z) + \tilde{d}(\theta_2\gamma_z, \theta_2\beta_z) + \tilde{d}(\theta_2\beta_z, \theta_3\beta_z) + \tilde{d}(\theta_3\beta_z, \theta_3\alpha_z) \\
&\leq \tilde{d}(\alpha_z, \theta_1\alpha_z) + \tilde{d}(\theta_1\alpha_z, \theta_2\gamma_z) + \tilde{d}(\gamma_z, \alpha_z) + \tilde{d}(\alpha_z, \beta_z) + \tilde{d}(\theta_2\beta_z, \theta_3\beta_z) + \tilde{d}(\beta_z, \alpha_z) \qquad (3.25) \\
&\to 0 \quad as \quad z \to \infty.
\end{aligned}$$

Thus, the assertion (iii) is proven. This completes the proof. □

Theorem 3.2. *Let $(\tilde{X}, \tilde{d})$ be an admissible CAT(1) space and $\tilde{g} : \tilde{X} \to (-\infty, \infty]$ be a proper lower semi-continuous convex function. Then, the sequence $\{\alpha_z\}$ generated by (3.1) Δ-converges to an element of ω.*

Proof. Let $\tilde{p} \in \omega$, then $\tilde{g}(\tilde{p}) \leq \tilde{g}(\xi_z)$ for each $z \geq 1$. Now, from Lemma 2.6, we get

$$\tilde{\lambda}_z(\tilde{g}(\xi_z) - \tilde{g}(\tilde{p})) \leq \frac{\pi}{2}\Big(\frac{1}{cos^2\tilde{d}(\xi_z, \alpha_z)} + 1\Big)(cos\ \tilde{d}(\xi_z, \alpha_z) cos\ \tilde{d}(\tilde{p}, \xi_z) - cos\ \tilde{d}(\tilde{p}, \alpha_z)), \tag{3.26}$$

which yields

$$0 \leq \tilde{\lambda}_z(\tilde{g}(\xi_z) - \tilde{g}(\tilde{p})) \leq \frac{\pi}{2}\Big(\frac{1}{cos^2\tilde{d}(\xi_z, \alpha_z)} + 1\Big)(cos\ \tilde{d}(\xi_z, \alpha_z) cos\ \tilde{d}(\tilde{p}, \xi_z) - cos\ \tilde{d}(\tilde{p}, \alpha_z)). \tag{3.27}$$

Since $\tilde{\lambda}_z > \lambda > 0$ for each $z \geq 1$, from Lemma 3.1, we obtain

$$\lim_{z\to\infty} \tilde{d}(\xi_z, \alpha_z) = 0, \quad \lim_{z\to\infty} \tilde{d}(\tilde{p}, \alpha_z) \ \ and \ \ \lim_{z\to\infty} \tilde{d}(\tilde{p}, \xi_z) \ \ exist. \tag{3.28}$$

From (3.27) and (3.28), we have

$$\lim_{z\to\infty} \tilde{g}(\xi_z) = inf\,\tilde{g}(\tilde{X}). \tag{3.29}$$

Now, we claim that $W_\Delta(\{\alpha_z\}) \subset \omega$. Let $\bar{w} \in W_\Delta(\{\alpha_z\})$ then, there exists a subsequence $\{\alpha_{z_i}\}$ of $\{\alpha_z\}$ which Δ-converges to the point $\bar{w}$. Using the fact that $\lim_{z\to\infty} \tilde{d}(\xi_z, \alpha_z) = 0$, we can say that the subsequence $\{\xi_{z_i}\}$ of $\{\xi_z\}$ also Δ-converges to the point $\bar{w}$. From Lemma 2.7 and (3.29), we have

$$\tilde{g}(\bar{w}) \leq \liminf_{i\to\infty} \tilde{g}(\xi_{z_i}) \leq \lim_{z\to\infty} \tilde{g}(\xi_z) = inf\,\tilde{g}(\tilde{X}).$$

Thus, $\bar{w} \in \arg\min_{\alpha\in\tilde{X}}$ which yields $W_\Delta(\{\alpha_z\}) \subset arg\min_{\alpha\in\tilde{X}} \tilde{g}(\alpha)$. Also,

$$\lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_1\alpha_z) = \lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_2\alpha_z) = \lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_3\alpha_z) = 0$$

and $\{\alpha_z\}$ Δ-converges to $\bar{w}$. So, it follows from Lemma 2.9 that $\bar{w} \in \tilde{F}(\theta_1) \cap \tilde{F}(\theta_2) \cap \tilde{F}(\theta_3)$ implies that $W_\Delta(\{\alpha_z\}) \subset \omega$. It follows from (3.28) and $W_\Delta(\{\alpha_z\}) \subset \omega$, we can observe that $\tilde{d}(\bar{w}, \alpha_z)$ is convergent for all $\bar{w} \in W_\Delta(\{\alpha_z\})$. By using Lemma 2.8, we obtain that $\{\alpha_z\}$ Δ-converges to an element of ω. □

Theorem 3.3. *Let $(\tilde{X}, \tilde{d})$ be an admissible complete CAT(1) space and $\tilde{g} : \tilde{X} \to (-\infty, \infty]$ be a proper lower semi-continuous convex function. Then, the sequence $\{\alpha_z\}$ generated by (3.1) converges strongly to an element of ω if and only if $\liminf_{z\to\infty} \tilde{d}(\alpha_z, \omega) = 0$, where $\tilde{d}(\alpha_z, \omega) = \inf\{\tilde{d}(\alpha, \tilde{p}) : \tilde{p} \in \omega\}$.*

Proof. It is obvious that $\liminf_{z\to\infty} \tilde{d}(\alpha_z, \omega) = 0$ if the sequence $\{\alpha_z\}$ converges to a point $\tilde{p} \in \omega$.
For conversely part, let $\liminf_{z\to\infty} \tilde{d}(\alpha_z, \omega) = 0$. For all $\tilde{p} \in \omega$, we have

$$\tilde{d}(\alpha_{z+1}, \tilde{p}) \le \tilde{d}(\alpha_z, \tilde{p}),$$

implies that

$$\tilde{d}(\alpha_{z+1}, \omega) \le \tilde{d}(\alpha_z, \omega).$$

Thus, $\lim_{z\to\infty} \tilde{d}(\alpha_z, \omega) = 0$. Next, we show that $\{\alpha_z\}$ is a Cauchy sequence in $\tilde{X}$. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{z\to\infty} \tilde{d}(\alpha_z, \omega) = 0$, there exists z_0 such that for all $z \ge z_0$, we have

$$\tilde{d}(\alpha_z, \omega) < \frac{\epsilon}{4}.$$

In particular, we have

$$\inf\{\tilde{d}(\alpha_{z_0}, \tilde{p}) : \tilde{p} \in \omega\} < \frac{\epsilon}{4},$$

so there must exist a $\tilde{p}^* \in \omega$ such that

$$\tilde{d}(\alpha_{z_0}, \tilde{p}^*) < \frac{\epsilon}{2}.$$

Thus, for $m, z \ge z_0$, we have

$$\tilde{d}(\alpha_{z+m}, \alpha_z) \le \tilde{d}(\alpha_{z+m}, \tilde{p}^*) + \tilde{d}(\alpha_z, \tilde{p}^*) < 2\tilde{d}(\alpha_{z_0}, \tilde{p}^*) < 2(\frac{\epsilon}{2}) = \epsilon,$$

which implies that $\{\alpha_z\}$ is a Cauchy sequence in $\tilde{X}$. Thus, $\{\alpha_z\}$ converges to a point α^* in $\tilde{X}$ and so $\tilde{d}(\alpha^*, \omega) = 0$. Also, $\alpha^* \in \omega$ as ω is closed. This completes the proof. □

A family $\{P, Q, R, S\}$ of mappings is said to satisfy condition (τ) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(\zeta) > 0$ for all $\zeta \in (0, \infty)$ such that

$$\tilde{d}(\alpha, P\alpha) \ge f(\tilde{d}(\alpha, \tilde{F}))$$

or

$$\tilde{d}(\alpha, Q\alpha) \ge f(\tilde{d}(\alpha, \tilde{F}))$$

or

$$\tilde{d}(\alpha, R\alpha) \ge f(\tilde{d}(\alpha, \tilde{F}))$$

or

$$\tilde{d}(\alpha, S\alpha) \ge f(\tilde{d}(\alpha, \tilde{F}))$$

for all $\alpha \in \tilde{X}$, where $\tilde{F} = \tilde{F}(P) \cap \tilde{F}(Q) \cap \tilde{F}(R) \cap \tilde{F}(S)$.

Theorem 3.4. *Let $(\tilde{X}, \tilde{d})$ be an admissible complete CAT(1) space and $\tilde{g} : \tilde{X} \to (\infty, \infty]$ be a proper lower semi-continuous convex function. If the mappings R_λ, θ_1, θ_2 and θ_3 satisfy the condition (τ), then the sequence $\{\alpha_z\}$ generated by (3.1) converges strongly to an element of ω.*

Proof. From Lemma 3.1, $\lim_{z\to\infty} \tilde{d}(\alpha_z, \tilde{p})$ exists for all $\tilde{p} \in \omega$. So $\lim_{z\to\infty} \tilde{d}(\alpha_z, \omega)$ exists. Now, by using condition (τ), we get

$$\lim_{z\to\infty} f(\tilde{d}(\alpha_z, \omega)) \leq \lim_{z\to\infty} \tilde{d}(\alpha_z, R_\lambda \alpha_z) = 0$$

or

$$\lim_{z\to\infty} f(\tilde{d}(\alpha_z, \omega)) \leq \lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_1 \alpha_z) = 0$$

or

$$\lim_{z\to\infty} f(\tilde{d}(\alpha_z, \omega)) \leq \lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_2 \alpha_z) = 0$$

or

$$\lim_{z\to\infty} f(\tilde{d}(\alpha_z, \omega)) \leq \lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_3 \alpha_z) = 0.$$

Therefore, $\lim_{z\to\infty} f(\tilde{d}(\alpha_z, \omega)) = 0$ which by using property of f, we obtain $\lim_{z\to\infty} \tilde{d}(\alpha_z, \omega) = 0$. Thus, the proof follows from Theorem 3.3. □

Theorem 3.5. *Let $(\tilde{X}, \tilde{d})$ be an admissible complete CAT(1) space and $\tilde{g} : \tilde{X} \to (\infty, \infty]$ be a proper lower semi-continuous convex function. If the mappings R_λ or θ_1 or θ_2 or θ_3 is demi-compact, then the sequence $\{\alpha_z\}$ generated by (3.1) converges strongly to an element of ω.*

Proof. From Lemma 3.1, we have

$$\lim_{z\to\infty} \tilde{d}(\alpha_z, R_\lambda \alpha_z) = \lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_1 \alpha_z) = \lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_2 \alpha_z) = \lim_{z\to\infty} \tilde{d}(\alpha_z, \theta_3 \alpha_z) = 0. \tag{3.30}$$

Without loss of generality, we may assume that R_λ or θ_1 or θ_2 or θ_3 is demi-compact, then there exists a subsequence $\{\alpha_{z_i}\}$ of $\{\alpha_z\}$ such that $\{\alpha_{z_i}\}$ converges strongly to $\tilde{p}^* \in \tilde{X}$. By using (3.30) and the nonexpansiveness of the mappings $R_\lambda, \theta_1, \theta_2, \theta_3$, then we obtain

$$\tilde{d}(\tilde{p}^*, R_\lambda \tilde{p}^*) = \tilde{d}(\tilde{p}^*, \theta_1 \tilde{p}^*) = \tilde{d}(\tilde{p}^*, \theta_2 \tilde{p}^*) = \tilde{d}(\tilde{p}^*, \theta_3 \tilde{p}^*) = 0,$$

which yields $\tilde{p}^* \in \omega$. Further, we can prove the strong convergence of $\{\alpha_z\}$ to an element of ω. This completes the proof. □

4. Concluding remarks and open question

1) Our main results generalized and extended the results of Pakkaranang et al. [31, 32] from one onexpansive mapping and Wairojjana and Saipara [33] from two nonexpansive mappings to three nonexpansive mappings involving lower semi-continuous convex function in CAT(1) spaces.
2) Theorem 3.2 extends that of Bačák [12] in CAT(0) spaces and Kohsaka and Kimura [30] in CAT(1) spaces. In fact, we present a new modified proximal point algorithm for solving the convex minimization problem as sell as the common fixed point problem of nonexpansive mappings in CAT(1) spaces.

Question 1. Can we construct some examples or numerical results of the resolvent operator and a convex function in the setting of CAT(1) spaces?

5. Conclusions

In this paper, a new modified proximal point algorithm involving three nonexpansive mappings in the setting of CAT(1) spaces for solving convex minimization problem and common fixed point problem have been established. Strong and Δ-convergence theorems under mild conditions of the proposed algorithm converges to common solution between convex minimization problem and common fixed point problem have been proven.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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