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Research article

A generalization of the q-Lidstone series

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Abstract: In this paper, we study the existence of solutions for the general q-Lidstone problem:

$$(D_{q^{-1}}^{r_n} f)(1) = a_n, \quad (D_{q^{-1}}^{s_n} f)(0) = b_n, \quad (n \in \mathbb{N})$$

where $(r_n)_n$ and $(s_n)_n$ are two sequences of non-negative integers and $(a_n)_n$ and $(b_n)_n$ are two sequences of complex numbers. We define a q^{-1}-standard set of polynomials and then we introduce a generalization of the q-Lidstone expansion theorem.

Keywords: q-difference equations; standard sets of polynomials; q-Lidstone series
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1. Introduction

The Lidstone series [12] expresses an analytic function in terms of the values of all its even order derivatives at two distinct points. In [17] Whittaker introduced a set of polynomials $\pi_n(z)$ and $\zeta_n(z)$ $(n \in \mathbb{N})$ and provided a generalization of the Lidstone expansion theorem that approximate an entire function f in a neighborhood of the points 0 and 1:

$$f(z) = \sum_{n=0}^{\infty} \left[f^{(p_n)}(1)\, \pi_n(z) + f^{(q_n)} f(0)\, \zeta_n(z) \right], \tag{1.1}$$

where $(p_n)_n$ and $(q_n)_n$ are two sequences of non-negative integers. Furthermore, Whittaker determined the class of functions for which (1.1) is valid. Recently, many new developments and applications of the Lidstone expansion have been realized; see for example [2, 4–7] and references therein.

We will interested to find a function $f(z)$ which satisfies the general q-Lidstone conditions:

$$(D_{q^{-1}}^{r_n} f)(1) = a_n, \quad (D_{q^{-1}}^{s_n} f)(0) = b_n, \quad (n \in \mathbb{N}) \tag{1.2}$$

where $(r_n; s_n)_n$ is a pair of strictly increasing sequences of integers numbers and, corresponding to them, $(a_n)_n$ and $(b_n)_n$ are two sequences of complex numbers. Throughout this paper q is a positive number less than one and $\mathbb{N}$ is the set of positive integers. The q-derivative D_q of the function f is defined by

$$D_q f(z) := \frac{f(z) - f(qz)}{z - qz} \text{ for } z \neq 0, \tag{1.3}$$

and the q-derivative at zero is defined to be $f'(0)$ if it exists, see [8, 11]. The q-shifted fractional $(a; q)_n$ of $a \in \mathbb{C}$ is defined by

$$(a;q)_0 := 1 \text{ and } (a;q)_n := \prod_{j=0}^{n}(1 - aq^j) \text{ for } n \in \mathbb{N},$$

and the q-number factorial $[n]_q!$ is defined for $q \neq 1$ by

$$[n]_q! = \prod_{j=0}^{n}[j]_q, \quad [j]_q = \frac{1 - q^j}{1 - q}.$$

Ismail and Mansour [9] provide the solution of Problem (1.2) when $(r_n)_n$ and $(s_n)_n$ are even positive integer sequences. More precisely, they expand a class of entire functions of q-exponential growth in terms of q^{-1}-derivatives of even orders at the points 0, 1:

$$f(z) = \sum_{n=0}^{\infty}\Big[(D^{2n}_{q^{-1}} f)(1)A_n(z) - (D^{2n}_{q^{-1}} f)(0)B_n(z)\Big], \tag{1.4}$$

where $(A_n)_n$ and $(B_n)_n$ are the q-Lidstone polynomials defined by the generating functions

$$\frac{E_q(zw) - E_q(-zw)}{E_q(w) - E_q(-w)} = \sum_{n=0}^{\infty} A_n(z)w^{2n}, \tag{1.5}$$

$$\frac{E_q(zw)E_q(-w) - E_q(-zw)E_q(w)}{E_q(w) - E_q(-w)} = \sum_{n=0}^{\infty} B_n(z)\frac{w^n}{[n]_q!}, \tag{1.6}$$

respectively. Here, $E_q(\cdot)$ and $e_q(\cdot)$ are the following q-exponential functions:

$$e_q(z) = \sum_{k=0}^{\infty}\frac{z^k}{[k]_q!}; \quad |z| < 1, \quad \text{and} \quad E_q(z) = \sum_{k=0}^{\infty}\frac{q^{\frac{k(k-1)}{2}}}{[k]_q!}z^k; \quad z \in \mathbb{C}. \tag{1.7}$$

Moreover, they proved that

$$A_0(z) = z, \quad B_0(z) = z - 1, \tag{1.8}$$

and for $n \in \mathbb{N}$, $A_n(z)$ and $B_n(z)$ satisfy the q-difference equation

$$(D^2_{q^{-1}} y_n)(z) = y_{n-1}(z) \quad \text{with} \quad y_n(0) = y_n(1) = 0. \tag{1.9}$$

The publications [13, 14] are the most affiliated with this work.

The purpose of this paper is to discuss the existence of solutions for Problem (1.2). We shall define a set of q-polynomials $\pi_n(z;q)$ and $\zeta_n(z;q)$, and show that there exists a generalization of q-Lidstone series

$$\sum_{n=1}^{\infty}\Big[(D^{r_n}_{q^{-1}} f)(1)\,\pi_n(z;q) + (D^{s_n}_{q^{-1}} f)(0)\,\zeta_n(z;q)\Big] \tag{1.10}$$

which, under certain conditions, converges to $f(z)$.

The structure of this paper is as follows. In Section 2, we determine the necessary conditions for a function f which satisfies q-Lidstone conditions:

$$(D^{2n}_{q^{-1}}f)(1) = a_n, \quad (D^{2n+1}_{q^{-1}}f)(0) = b_n \quad (n \in \mathbb{N}),$$

and introduce the q-Lidstone polynomials that appear as the q-Euler polynomials generated by the second Jackson q-Bessel function. In Section 3, we introduce a q^{-1}-standard set of polynomials as in the classical case [17]. In the last section, we shall study Problem (1.2) and conclude with a generalization of q-Lidstone series (1.10).

2. A q-Lidstone theorem

In this section, we solve a special case of Problem (1.2) for the sequences $\{2n\}_n$ and $\{2n+1\}_n$. That is

$$(D^{2n}_{q^{-1}}f)(1) = a_n, \quad (D^{2n+1}_{q^{-1}}f)(0) = b_n, \quad (n \in \mathbb{N}). \tag{2.1}$$

Recall the q-trigonometric functions

$$\mathrm{Sin}_q z := \frac{E_q(iz) - E_q(-iz)}{2i}, \ \sin_q z = \frac{e_q(iz) - e_q(-iz)}{2i},$$

$$\mathrm{Cos}_q z := \frac{E_q(iz) + E_q(-iz)}{2}, \ \cos_q z = \frac{e_q(iz) + e_q(-iz)}{2i},$$

where the functions $E_q(z)$ and $e_q(z)$ have the series representation in (1.7), and satisfy

$$e_q(w)E_q(zw) = \sum_{n=0}^{\infty} \frac{w^n(-z;q)_n}{[n]_q!}, \quad z, w \in \mathbb{C}, \tag{2.2}$$

(see [10]). Ismail and Mansour [9] defined a q-analog of Euler polynomials by the generating function

$$\frac{2\,E_q(xt)}{E_q(t/2)e_q(t/2) + 1} = \sum_{n=0}^{\infty} E_n(x;q)\frac{t^n}{[n]_q!}. \tag{2.3}$$

Clearly $E_0(z;q) = 1$ and for $n \in \mathbb{N}$, the polynomials $E_n(z;q)$ are given by

$$E_n(z;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \tilde{E}_{n-k}(q)\, z^k, \tag{2.4}$$

here, $\tilde{E}_n(q)$ denotes to $E_n(0;q)$ and the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} 1, & k = 0; \\ \frac{(1-q^n)(1-q^{n-1})...(1-q^{n-k+1})}{(q;q)_k}, & k \in \mathbb{N}. \end{cases}$$

We set

$$h(z,w) = Cosh_q(zw)\, Sech_q(w), \quad |w| < C_1,$$

where z and w are complex numbers and C_1 is the smallest positive zero of $Cos_q z$. Note that this function is an analytic function and it can be represented as

$$h(z,w) = \frac{E_q(zw) + E_q(-zw)}{E_q(w) + E_q(-w)}. \tag{2.5}$$

Lemma 2.1. *Let C_1 be the smallest positive zero of $Cos_q z$, z and w be complex numbers such that $|w| < C_1$. Then*

$$\frac{E_q(zw) + E_q(-zw)}{E_q(w) + E_q(-w)} = \sum_{n=0}^{\infty} M_n(z) w^{2n}, \tag{2.6}$$

where

$$M_n(z) := \frac{2^{2n}}{[2n]_q!} \sum_{j=0}^{2n} \begin{bmatrix} 2n \\ j \end{bmatrix}_q (-z;q)_j \left(\frac{1}{2}\right)^j \tilde{E}_{2n-j}(q)_j = \frac{2^{2n}}{[2n]_q!}{}_{q^{-1}}^{1} E_{2n}(z/2;q). \tag{2.7}$$

Here $\varepsilon^{y}_{q^{-1}}$ a q-translation operator defined by

$$\varepsilon^{y}_{q^{-1}} x^n = x^n(-y/x;q^{-1})_n = q^{-\frac{n(n-1)}{2}} y^n(-x/y;q)_n = q^{-\frac{n(n-1)}{2}} \varepsilon^{x}_{q} y^n.$$

Proof. We have

$$\begin{aligned} h(z,w) &:= \frac{E_q(zw) + E_q(-zw)}{E_q(w) + E_q(-w)} \\ &= \frac{1}{2}\left[\frac{2}{e_q(w)E_q(w)+1} e_q(w)E_q(zw)\right] + \frac{1}{2}\left[\frac{2}{e_q(-w)E_q(-w)+1} e_q(-w)E_q(-zw)\right]. \end{aligned} \tag{2.8}$$

Thus, By Eq (2.3) and from (2.2) we get

$$\begin{aligned} h(z,w) &= \frac{E_q(zw) + E_q(-zw)}{E_q(w) + E_q(-w)} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \Big[\frac{w^k(-z;q)_k}{[k]_q!} \sum_{j=0}^{\infty} \tilde{E}_j(q) \frac{(2w)^j}{[j]_q!} + \frac{(-w)^k(-z;q)_k}{[k]_q!} \sum_{j=0}^{\infty} \tilde{E}_j(q) \frac{(-2w)^j}{[j]_q!} \Big] \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{w^n + (-w)^n}{[n]_q!} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (-z;q)_j 2^{n-j} \tilde{E}_{n-j}(q) \\ &= \sum_{n=0}^{\infty} \frac{w^{2n}}{[2n]_q!} \sum_{j=0}^{2n} \begin{bmatrix} 2n \\ j \end{bmatrix}_q (-z;q)_j 2^{2n-j} \tilde{E}_{2n-j}(q) \\ &= \sum_{n=0}^{\infty} M_n(z)\, w^{2n}. \end{aligned}$$

□

Remark 2.2. By using (2.6), we can verify that the polynomials $M_n(z)$ $(n \in \mathbb{N})$ satisfy the q^{-1}-difference equation

$$D^2_{q^{-1}}\, y_n(z) = y_{n-1}(z),$$

with the boundary conditions $D_{q^{-1}}\, y_n(0) = 0 = y_n(1)$, and $y_0(z) = 1$.

Similarly, as in the proof of Lemma 2.1, we can obtain the following result.

Lemma 2.3. *If z and w are complex numbers such that $|w| < C_1$ then*

$$\frac{E_q(zw)E_q(-w) - E_q(-zw)E_q(w)}{E_q(w) + E_q(-w)} = \sum_{n=0}^{\infty} N_{n+1}(z)\, w^{2n+1}, \tag{2.9}$$

where

$$N_{n+1}(z) = \frac{2^{2n+1}}{[2n+1]_q!} E_{2n+1}(z/2; q). \tag{2.10}$$

Lemma 2.4. *If z and w are complex numbers such that $|w| < C_1$ then*

$$E_q(zw) = E_q(w) \sum_{n=0}^{\infty} M_n(z) w^{2n} - \sum_{n=0}^{\infty} N_{n+1}(z) w^{2n+1}, \tag{2.11}$$

where $M_n(z)$ and $N_{n+1}(z)$ are the q-polynomials defined in (2.7) *and* (2.10)*, respectively.*

Proof. It follows immediately from Lemmas 2.1, 2.3 and the fact that

$$E_q(zw) = E_q(w) \frac{E_q(zw) + E_q(-zw)}{E_q(w) + E_q(-w)} + \frac{E_q(zw)E_q(-w) - E_q(-zw)E_q(w)}{E_q(-w) + E_q(w)}.$$

□

Recall that the function Ψ is called comparison if it can be represented as a power series

$$\Psi(t) = \sum_{n=0}^{\infty} \Psi_n t^n,$$

such that $\Psi_n > 0$ and $\left(\Psi_{n+1}/\Psi_n\right) \downarrow 0$. As an example, the q-exponential function $E_q(z)$ is a comparison. Indeed, we take

$$\Psi_n = \frac{q^{\frac{n(n-1)}{2}}}{[n]_q!},$$

and then the sequence

$$\frac{\Psi_{n+1}}{\Psi_n} = \frac{q^n(1-q)}{1-q^{n+1}} = \frac{q^{n/2}}{[n+1]_q}$$

is decreasing and vanishes at ∞.

We denote by $\mathcal{R}_\Psi$ the class of all entire functions f such that, for some number τ (depending on f),

$$|f(re^{i\theta})| \leq M\Psi(\tau r), \quad r \uparrow \infty. \tag{2.12}$$

The infimum of numbers τ for which (2.12) holds is the Ψ-type of the function f. This type can be computed by

$$\tau = \limsup_{n\to\infty} \left|\frac{f_n}{\Psi_n}\right|^{\frac{1}{n}},$$

where $f(z) = \sum_{n=0}^{\infty} f_n z^n$ (see [15]).

We will use the following result from [3].

Theorem 2.5. *Let Ψ be a comparison function. Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ belong to the class $\mathcal{R}_\Psi$, and let $D(f)$ be the closed set which consists of the union of the set of all singular points of F and the set of all exterior points to the domain of F. Then*

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \Psi(zw)F(w)\,dw$$

where Γ encloses $D(f)$ and

$$F(w) = \sum_{n=0}^{\infty} \frac{f_n}{\Psi_n w^{n+1}}.$$

According to the above arguments and results we can define the polynomials π_n and ζ_n, and determine the class of functions for which the boundary conditions (2.1) are satisfied.

Theorem 2.6. *Let C_1 be the smallest positive zero of $Cos_q z$. Assume that one of the following conditions hold:*

(i) The function $f(z)$ is an entire function of q^{-1}-exponential growth of order 1 and a finite type α, where

$$\alpha < \left(\frac{1}{2} - \frac{\log C_1}{\log q}\right). \tag{2.13}$$

(ii) The function $f(z)$ is an entire function of q^{-1}- exponential growth of order less than 1.

Then $f(z)$ has the convergent representation

$$f(z) = \sum_{n=0}^{\infty} \left[(D_{q^{-1}}^{2n} f)(1)\, M_n(z) - (D_{q^{-1}}^{2n+1} f)(0)\, N_{n+1}(z)\right],$$

where M_n and N_{n+1} are q-Euler polynomials generated by the second Jackson q-Bessel function defined in (2.7) *and* (2.10).

Proof. We apply Theorem 2.5 when $\Psi(z)$ is chosen as $E_q(z)$. By using (4.2), we have that for any entire function $f(z) = \sum_{n=0}^{\infty} f_n z^n$ of q^{-1}- exponential growth of order k and a finite type α, there exists a real number $K > 0$ such that

$$|f_n| \le K q^{\frac{(n-\alpha)^2}{2k}}.$$

By assumption, we have two cases:

Case 1. If $k = 1$, then $|f_n| \le K q^{\frac{(n-\alpha)^2}{2}}$. This implies both that (2.12) holds and that $f \in \mathcal{R}_\Psi$. Here, the Ψ-type of the function f given by

$$\tau := \limsup \sqrt[n]{\frac{f_n}{\psi_n}} \le q^{\frac{1}{2}-\alpha} < C_1.$$

Case 2. If $k < 1$, then $\tau = 0$.

Hence, we can take $D(f)$ lies in the closed disk $|w| \le \tau$. So, any closed circle of the form $|w| = r$, $q^{\frac{1}{2}-\alpha} < r < C_1$ is an admissible curve for Γ (for any $k \le 1$). So, by Theorem 2.5, we obtain

$$f(z) = \frac{1}{2\pi i}\int_\Gamma E_q(zw)F(w)\,dw,$$

for any $w \in \Gamma$, $|w| < C_1$, where Γ is a closed contour which encloses $D(f)$. Therefore,

$$\begin{aligned}(D_{q^{-1}}^{2n+1} f)(0) &= \frac{1}{2\pi i}\int_\Gamma w^{2n+1} F(w)\,dw,\\ (D_{q^{-1}}^{2n} f)(1) &= \frac{1}{2\pi i}\int_\Gamma w^{2n} E_q(w)F(w)\,dw.\end{aligned}$$

By using Lemma 2.4, we have

$$\begin{aligned}f(z) &= \frac{1}{2\pi\, i}\int_\Gamma E_q(zw)F(w)\,dw\\ &= \frac{1}{2\pi\, i}\int_\Gamma \left\{E_q(w)\sum_{n=0}^{\infty} w^{2n} M_n(z) - w^{2n+1} N_{n+1}(z)\right\} F(w)\,dw\\ &= \sum_{n=0}^{\infty}\left[(D_{q^{-1}}^{2n} f)(1) M_n(z) - (D_{q^{-1}}^{2n+1} f)(0) N_{n+1}(z)\right].\end{aligned}$$

□

3. A q^{-1}-standard set of polynomials

Suppose we are given two strictly increasing sequences of non-negative integers numbers $(r_n)_n$ and $(s_n)_n$. We shall use the notation $(r; s)$ to denote these two sequences. We define the q-analog of standard set of polynomials with respect to the q-difference operator $D_{q^{-1}}$.

Definition 3.1. A set of polynomials $\pi_n(z; q)$ and $\zeta_n(z; q)$ are called a q-analog of standard set with respect to the $D_{q^{-1}}$ derivative (or a q^{-1}-standard set) in relation to the pair of sequences $(r; s)$ if

$$(D_{q^{-1}}^{r_k} \pi_n)(1) = \delta_{n,k} \quad\text{and}\quad (D_{q^{-1}}^{s_k} \pi_n)(0) = 0; \tag{3.1}$$

$$(D_{q^{-1}}^{s_k} \zeta_n)(0) = \delta_{n,k} \quad\text{and}\quad (D_{q^{-1}}^{r_k} \zeta_n)(1) = 0, \tag{3.2}$$

where $\delta_{n,k}$ is the Kronecker delta ($k \in \mathbb{N}$).

Example 3.2. Let $A_n(z)$ and $B_n(z)$ be the q-Lidstone polynomials which defined in (1.5) and (1.6). From (1.8) and (1.9), we can verify that $A_n(z)$ and $-B_n(z)$ form a q^{-1}-standard set of polynomials in relation to the pair of sequences $(r; s) = (2n; 2n)_{n\in\mathbb{N}_0}$.

Proposition 3.3. *The polynomials of a q^{-1}-standard set are linearly independent.*

Proof. Let π_n and ζ_n be a q^{-1}-standard set of polynomials in relation to the pair of sequences $(r; s)$. If there exist some constants $c_1, c_2, \ldots, c_k, \tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_k \in \mathbb{C}$ ($k \in \mathbb{N}$) such that

$$\sum_{i=1}^{k}\Big(c_i\, \pi_i(z; q) + \tilde{c}_i\, \zeta_i(z; q)\Big) = 0, \tag{3.3}$$

then we obtain k equations of the form

$$\sum_{i=1}^{k} c_i\, (D_{q^{-1}}^{t}\, \pi_i)(z; q) + \sum_{i=1}^{k} \tilde{c}_i\, (D_{q^{-1}}^{t}\, \zeta_i)(z; q) = 0 \quad (1 \le n \le k). \tag{3.4}$$

Setting $t = r_n$ and $z = 1$ in (3.4), and then using (3.1) and (3.2) we get $c_n = 0$ for all n. Similarly, if we choose $t = s_n$ and and $z = 0$, we get $\tilde{c}_n = 0$. □

Definition 3.4. A pair of sequences $(r; s)$ is called

1) complete if one and only one q^{-1}-standard set of polynomials exists;
2) indeterminate if more than one of q^{-1}-standard set of polynomials exists;
3) redundant if no q^{-1}-standard set exists.

We denote by $R(m)$ the number of elements of $(r_n)_n$ which are less than m and $S(m)$ the number of elements of $(s_n)_n$ which are less than m. We say that the r-sequence and s-sequence are complementary if $R(m) + S(m) = m$ $(m \in \mathbb{N})$.

We need the following lemma (see [18]):

Lemma 3.5. *Let $a, b, \ldots, a', b', \ldots$ be positive integers satisfying the inequalities*

$$a < b < \ldots < k, \quad a' < b' < \ldots < k',$$
$$a' \leq a, \quad b' \leq b, \quad \ldots, \quad k' \leq k.$$

Then, the determinant

$$\begin{vmatrix} \{a, a'\}_q & \{b, a'\}_q & \ldots & \{k, a'\}_q \\ \{a, b'\}_q & \{b, b'\}_q & \ldots & \{k, b'\}_q \\ \ldots & \ldots & \ldots & \ldots \\ \{a, k'\}_q & \{b, k'\}_q & \ldots & \{k, k'\}_q \end{vmatrix}$$

is always positive, where $\{a, a'\}_q = \frac{[a]_q!}{[a-a']_q!} = [a]_q[a-1]_q[a-2]_q \ldots [a-a'+1]_q$.

In the following result, we present a necessary and sufficient condition for the pair $(r; s)$ to be complete.

Theorem 3.6. *A pair of sequences $(r; s)$ is complete if and only if it satisfies the following conditions:*

$$R(m) + S(m) \geq m \quad (m \in \mathbb{N}); \tag{3.5}$$
$$R(m_k) + S(m_k) = m_k, \tag{3.6}$$

for an infinite sequence $(m_k)_k$, $k \in \mathbb{N}$.

Proof. First, assume that (3.5) and (3.6) are satisfied. We want to prove that the pair $(r; s)$ is complete. For this, let $(u_k)_k$ be the sequence complementary to $(s_k)_k$ with respect to $0, 1, 2, \ldots$. If $U(m)$ denotes the number of elements of $(u_n)_n$ which are less than m, then we have $U(m) + S(m) = m$. So, from (3.5) we get $R(m) \geq U(m)$ for $m \in \mathbb{N}$, or equivalently

$$r_k \leq u_k, \quad k \in \mathbb{N}. \tag{3.7}$$

For some fixed value of n, we will take $N \in \{m_1, m_2, \ldots\}$ such that $N \geq n$, and we write $R(N) = R$, $S(N) = S$. Assume that the polynomial $\pi_n(z; q)$ of degree $\leq N$. Then, $\pi_n(z; q)$ is represented in the form $\pi_n(z; q) = \sum_{i=1}^{R} \alpha_i z^{u_i}$, and the q^{-1}-derivatives of π given by

$$(D_{q^{-1}}^{r_t} \pi_n)(z; q) \quad = \quad \sum_{i=1}^{R} \alpha_i D_{q^{-1}}^{r_t} z^{u_i}$$

$$= \sum_{i=1}^{R} d_{i,r_t} \frac{[u_i]_q!}{[u_i - r_t]_q!} z^{u_i - r_t},$$

where $d_{i,r_t} = q^{\frac{r_t(r_t+1)}{2} - r_t u_i} \alpha_i$. Hence, the second condition of (3.1) is satisfied, and we can use the first condition to obtain $R = N - S$ equations which determine the coefficients d_{i,r_t} (and then α_i). That is,

$$\sum_{i=1}^{R} d_{i,r_t} [u_i]_q [u_i - 1]_q \ \ldots \ [u_i - r_t + 1]_q = \delta_{t,n}, \quad 1 \le t \le R, \tag{3.8}$$

with $\delta_{n,k}$ the Kronecker delta. By using (3.7) and according to Lemma 3.5, we conclude that Eq (3.8) have a non-zero determinant, and then a unique solution. This implies, there is one and only one polynomial $\pi_n(z)$ satisfying (3.1). Similarly, we can construct a unique set of polynomials $\zeta_n(z; q)$. Therefore, the pair $(r; s)$ is complete.

Next, we prove the sufficient condition. First, suppose that (3.5) is satisfied but (3.6) is not. Then, there is an integer K such that

$$R(m) + S(m) > m \quad \text{for } m \ge K. \tag{3.9}$$

We may take K to be the smallest integer for which (3.9) is true. Therefore,

$$R(K - 1) + S(K - 1) = K - 1$$
$$R(K) + S(K) > K,$$

i.e., $K - 1$ is the number of elements in both sequences $(r_n)_n$ and $(s_n)_n$ which are less than $K - 1$. If we omit one element from the pair of sequences $(r; s)$, say from $(s_n)_n$, then we get a new pair for which (3.5) is satisfied.

Again, if this pair does not satisfy (3.6), we obtain

$$R(m) + S(m) > m \quad (m \ge K'),$$

for some integer K'. By omitting another s and repeating this way, we can establish a pair of sequence $(r; v)$ where $(v_n)_n$ is a subsequence of $(s_k)_k$ for which (3.5) and (3.6) are satisfied. Then $(r; v)$ is a complete pair of sets.

Since the pair $(r; s)$ is complete, the standard set of q^{-1}-polynomials $\pi_n(z; q)$ and $\zeta_n(z; q)$ exist and satisfy (3.1) and (3.2). Assume that s_n is one of the omitted terms of the sequence $(s_k)_k$. Then, we obtain

$$(D^t_{q^{-1}} \zeta_n)(0) = 0, \quad \text{for } t \in (v_n)_n;$$
$$(D^t_{q^{-1}} \zeta_n)(1) = 0, \quad \text{for } t \in (r_n)_n.$$

Therefore, constant multiples of ζ_n can be added to the polynomials of q^{-1}-standard set attached to the pair $(r; v)$. Since $\zeta_n(z)$ is not identically zero, the pair $(r; v)$ can not be complete and this contradiction implies that $(r; s)$ is redundant.

On the other hand, if (3.5) is not satisfied, then there exists an integer M such that

$$R(M) + S(M) < M.$$

This implies that if $\rho(z)$ a nonzero polynomial of degree less than M, we get

$$(D^t_{q^{-1}}\rho)(1) = 0, \quad \text{for } t = r_1, r_2, ..., r_{R(M)};$$
$$(D^t_{q^{-1}}\rho)(0) = 0, \quad \text{for } t = s_1, s_2, ..., s_{S(M)}.$$

Thus, the pair $(r; s)$ can not be complete. □

Remark 3.7. Clearly, the conditions (3.5) and (3.6) are satisfied if

$$R(m) + S(m) = m \quad (m \geq 1).$$

Thus, by Theorem 3.6, the pair $(r; s)$ is complete if the r-sequence and the s-sequence are complementary.

4. Main results

We start this section by defining the two variables polynomials $\phi_n(z, a)$, $z, a \in \mathbb{C}$:

$$\phi_0(a,z) := 1, \quad \phi_n(a,z) := \begin{cases} a^n(\frac{z}{a}; q)_n, & a \neq 0; \\ (-1)^n q^{\frac{n(n-1)}{2}} z^n, & a = 0. \end{cases} \tag{4.1}$$

We can verify that

$$D^m_{q^{-1},z}\frac{\phi_n(a,z)}{[n]_q!} = \frac{(-1)^m}{[n-m]_q!}\, q^{\frac{m(1-m)}{2}}\, \phi_{n-m}(a,z). \tag{4.2}$$

We need the following result from [1]:

Theorem 4.1. *Let $f(z)$ be a function with q-exponential growth of order k, $k < \ln q^{-1}$ and a finite type α, $\alpha \in \mathbb{R}$. Then for $a \in \mathbb{C} - \{0\}$, $f(z)$ has expansion*

$$f(z) = \sum_{n=0}^{\infty}(-1)^n q^{-n(n-1)/2}\,\frac{D^n_q f(aq^{-n})}{[n]_q!}\,\phi_n(a,z),$$

absolutely and uniformly convergent on compact subsets of $\mathbb{C}$.

Recall that if the power series expansion $\sum_{=0}^{\infty} a_n z^n$ is a function of q-exponential growth of order k and finite type α, then

$$|a_n| \leq K q^{-\frac{(n-\alpha)^2}{2k}}. \tag{4.3}$$

For more details see [16].

Lemma 4.2. *Let π_n and ζ_n ($n \in \mathbb{N}$) be a q^{-1}-standard set of polynomials. Assume that the associate pair of sequences $(r; s)$ is complete. Then, any polynomial $P(z)$ can be expressed in one and only one way as a linear combination of π_n and ζ_n, namely*

$$P(z) = \sum_{n=1}^{\infty}(D^{r_n}_{q^{-1}}P)(1)\,\pi_n(z;q) + \sum_{n=1}^{\infty}(D^{s_n}_{q^{-1}}P)(0)\,\zeta_n(z;q).$$

Proof. Let V be a space of all polynomials. Since the pair of sequences $(r; s)$ is complete, then by using Theorem 3.6, we can take $N = R + S$ $(n \leq N)$. Therefore, we have N polynomials $\pi_1(z;q), \pi_2(z;q), \ldots, \pi_R(z;q), \zeta_1(z;q), \zeta_2(z;q), \ldots, \zeta_S(z;q)$ of degree $N-1$. These polynomials can be regarded as a set of linear equations in the unknowns $z^0, z^1, \ldots, z^{N-1}$. By using Proposition 3.3, the determinant of these equations cannot be zero and then, the equations can be solved. Since $\{z^0, z^1, z^2, \ldots\}$ is a base of V, this implies an arbitrary polynomial $P(z) \in V$ can be expressed in the form

$$P(z) = A_1\,\pi_1(z;q) + A_2\,\pi_2(z;q) + \ldots + B_1\,\zeta_1(z;q) + B_2\,\zeta_2(z;q) + \ldots. \tag{4.4}$$

We obtain the values of the coefficients $A_1, A_2, \ldots, B_1, B_2, \ldots$, as in the required result, by acting the q^{-1}-difference operator on (4.4) r_i times and setting $z = 1$, then s_i times and setting $z = 0$. □

In the following result we prove that the entire function $f(z)$ may be expanded in terms of q^{-1}-standard polynomials and the coefficients are powers of a q^{-1}-derivative at 0 and 1.

Assume that π_n and ζ_n $(n \in \mathbb{N})$ are a q^{-1}-standard set of polynomials and the associate pair $(r; s)$. We take the compact set $|z-a| = M$ $(M > a)$ and put

$$\alpha_k(M) = \max_{|z-a|=M} |\pi_k(z;q)|, \quad \beta_k(M) = \max_{|z-a|=M} |\zeta_k(z;q)|.$$

Define the series

$$\begin{aligned}\Theta(k, M) := &\sum_{n=r_k}^{\infty} \frac{(-1)^{n+r_k}\,\alpha_k(M)}{[n-r_k]_q!}\, q^{\frac{-r_k(r_k-1)}{2}}\, a^{n-r_k}\Big(\frac{1}{a};q\Big)_{n-r_k}\\ &+\sum_{n=s_k}^{\infty} \frac{(-1)^{n+s_k}\,\beta_k(M)}{[n-s_k]_q!}\, q^{\frac{-s_k(s_k-1)}{2}}\, a^{n-s_k}.\end{aligned} \tag{4.5}$$

Theorem 4.3. *Let π_n and ζ_n $(n \in \mathbb{N})$ be a q^{-1}-standard set of polynomials. Assume that the associate pair of sequences $(r; s)$ is complete and the following conditions hold:*

(i) $f(z)$ is a function with q^{-1}-exponential growth of order k, $k < \ln q$ and a finite type α;
(ii) the series

$$\sum_{k=0}^{\infty} \Big|\Theta(k, M)\Big|\,\Big|(D^k_{q^{-1}} f)(a)\Big|, \quad a \in \mathbb{C} - \{0\} \tag{4.6}$$

converges, where $\Theta(k, M)$ is defined in (4.5).

Then $f(z)$ has a convergent representation

$$f(z) = \sum_{n=1}^{\infty} \Big[(D^{r_n}_{q^{-1}} f)(1)\,\pi_n(z;q) + (D^{s_n}_{q^{-1}} f)(0)\,\zeta_n(z;q)\Big].$$

Proof. By using Theorem 4.1, $f(z)$ has the expansion

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(D^n_{q^{-1}} f)(a)}{[n]_q!}\,\phi_n(a, z). \tag{4.7}$$

Consider the series

$$S = \sum_{k=0}^{\infty} \Big[(D^{r_k}_{q^{-1}} f)(1)\, \pi_k(z) + (D^{s_k}_{q^{-1}} f)(0)\, \zeta_k(z)\Big].$$

By using (4.2), we obtain

$$S = \sum_{k=1}^{\infty} \Big[\sum_{n=r_k}^{\infty} \frac{(-1)^{n+r_k}}{[n-r_k]_q!} q^{\frac{r_k(1-r_k)}{2}} a^{n-r_k} (\frac{1}{a}; q)_{n-r_k} (D^n_{q^{-1}} f)(a) \pi_k(z)$$
$$+ \sum_{n=s_k}^{\infty} \frac{(-1)^{n+s_k}}{[n-s_k]_q!} q^{\frac{s_k(1-s_k)}{2}} a^{n-s_k} (D^n_{q^{-1}} f)(a) \zeta_k(z)\Big].$$

Notice, the coefficient of $(D^n_{q^{-1}} f)(a)$ is

$$\sum_{r_k, s_k \le n} \Big[\frac{(-1)^{n+r_k}}{[n-r_k]_q!} q^{\frac{-r_k(r_k-1)}{2}} a^{n-r_k} (\frac{1}{a}; q)_{n-r_k} \pi_k(z) + \frac{(-1)^{n+s_k}}{[n-s_k]_q!} q^{\frac{-s_k(s_k-1)}{2}} a^{n-s_k} \zeta_k(z)\Big]. \tag{4.8}$$

By Lemma 4.2, we conclude that the polynomial $P(z) = \frac{(-1)^n}{[n]_q!} \phi_n(a, z)$ can be expressed by (4.8). Take the compact set $|z - a| = M$ $(M > a)$, and put

$$\alpha_k(M) = \max_{|z-a|=M} |\pi_k(z; q)|, \quad \zeta_k(M) = \max_{|z-a|=M} |\beta_k(z; q)|.$$

Therefore, if

$$\sum_{k=1}^{\infty} \Big\{ \sum_{n=r_k}^{\infty} \frac{(-1)^{n+r_k} \alpha_k(M)}{[n-r_k]_q!} q^{\frac{r_k(1-r_k)}{2}} a^{n-r_k} (\frac{1}{a}; q)_{n-r_k}$$
$$+ \sum_{n=s_k}^{\infty} \frac{(-1)^{n+s_k} \beta_k(M)}{[n-s_k]_q!} q^{\frac{-s_k(s_k-1)}{2}} a^{n-s_k} \Big\} (D^n_{q^{-1}} f)(a).$$

converges, the series S may be rearrangement as

$$S := \sum_{n=0}^{\infty} \frac{(-1)^n}{[n]_q!} \phi_n(a, z)\, D^n_{q^{-1}} f(a). \tag{4.9}$$

Comparing (4.7) with (4.9) we obtain the required result. □

Remark 4.4. If we return to the problem of finding a function $f(z)$ for which

$$\begin{cases} (D^{r_n}_{q^{-1}} f)(1) = a_n; \\ (D^{s_n}_{q^{-1}} f)(0) = b_n \quad (n \in \mathbb{N}), \end{cases} \tag{4.10}$$

then, by Theorem 4.3, we conclude that:

(1) If the pair $(r; s)$ is complete, there exist a unique function $f(z)$ with q^{-1}-exponential growth of order k, $k < \ln q$ and a finite type α satisfying (4.6) and (4.10):

$$f(z) = \sum_{n=1}^{\infty} a_n \pi_n(z; q) + \sum_{n=1}^{\infty} b_n \zeta_n(z; q);$$

(2) If the pair $(r; s)$ is indeterminate, more than one solution of (4.10) exists;

(3) If the pair $(r; s)$ is redundant, then no function satisfying (4.6) and (4.10) exists.

5. Conclusions

In [9], the authors determined the class of functions for which

$$f(z) = \sum_{n=0}^{\infty} \left[a_n\, A_n(z) - b_n\, B_n(z) \right]$$

is valid, where $A_n(z)$ and $B_n(z)$ are q-Lidstone polynomials generated by the second Jackson q-Bessel function, and

$$a_n = (D^{2n}_{q^{-1}} f)(1), \quad b_n = (D^{2n}_{q^{-1}} f)(0) \quad (n \in \mathbb{N}).$$

This paper generalizes this result. We introduce a q^{-1}-standard set of polynomials $\pi_n(z; q)$ and $\zeta_n(z; q)$, and then we proved there is a generalization of q-Lidstone series:

$$\sum_{n=1}^{\infty} a_n \pi_n(z; q) + \sum_{n=1}^{\infty} b_n \zeta_n(z; q);$$

which, under certain conditions, converges to the function $f(z)$ that satisfies general q-Lidstone conditions:

$$(D^{r_n}_{q^{-1}} f)(1) = a_n; \quad (D^{s_n}_{q^{-1}} f)(0) = b_n, \quad (n \in \mathbb{N})$$

where $(r_n; s_n)_n$ is the pair of strictly increasing sequences of non-negative integers.

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Conflicts of interest

The author declares that there is no conflict of interest.

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