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*Research article*

# On function spaces related to some kinds of weakly sober spaces

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**Abstract:** In this paper, we mainly study function spaces related to some kinds of weakly sober spaces, such as bounded sober spaces, $k$-bounded sober spaces and weakly sober spaces. For $T_0$ spaces $X$ and $Y$, it is proved that $Y$ is bounded sober iff the function space $\mathbf{Top}(X, Y)$ of all continuous functions $f : X \longrightarrow Y$ equipped with the pointwise convergence topology is bounded sober iff $\mathbf{Top}(X, Y)$ equipped with the Isbell topology is bounded sober. But for a $k$-bounded sober space $X$, the function space $\mathbf{Top}(X, Y)$ equipped with the pointwise convergence topology or the Isbell topology may not be $k$-bounded sober. It is shown that if the function space $\mathbf{Top}(X, Y)$ equipped with the pointwise convergence topology or the Isbell topology is weakly sober (resp., a cut space), then $Y$ is weakly sober (resp., a cut space). Relationships among some kinds of (weakly) sober spaces are also investigated.

**Keywords:** bounded sober space; $k$-bounded sober space; weakly sober space; cut space; function space
**Mathematics Subject Classification:** 54A05, 54B20, 54C35, 54D35, 06B30

## 1. Introduction

In domain theory and non-Hausdorff topology, sobriety is probably the most important and useful property of $T_0$ spaces (see [5, 6]). Sober spaces possess many nice properties (cf. [5, Exercise O-5.16]). They are closed under retracts and products, and are closed-hereditary and saturated-hereditary. More importantly, the function spaces of sober spaces equipped with the pointwise convergence topology are sober, and the category **Sob** of all sober spaces with continuous mappings is reflective in $\mathbf{Top}_0$ (see [5, 6]).

Recently, several weakly sober spaces have been introduced and extensively studied from various different perspectives (see [1, 2, 4, 13, 14, 17, 19, 20]). In [19], Zhao and Fan introduced bounded sobriety, a weak notion of sobriety, and proved that the category **BSob** of all bounded sober spaces with continuous mappings is also reflective in $\mathbf{Top}_0$. In [20], Zhao and Ho defined a new topology constructed from any given topology by irreducible sets and the derived topology leads to another weaker notion of sobriety of $T_0$ spaces, called $k$-bounded sobriety. In [1], Bezhanishvili and Harding gave an characterization of sobriety via Raney algebras. Such algebraic description of sober spaces induces a new type of weakly sober spaces — almost sober spaces. It was proved in [13] that the category **ASob** of all almost sober spaces is not a reflective subcategory of $\mathbf{Top}_0$.

In order to extend the theory of sober spaces and locally hypercompact spaces to situations where directed joins were missing, Erné [4] has weakened the concept of sobriety and introduced three kinds of weakly sober spaces—quasisober spaces, weakly sober spaces and cut spaces. These spaces are connected by the non-invertible implications: Sober ⇒ quasisober ⇒ weakly sober ⇒ cut space.

In this paper, we investigate some further properties of kinds of weakly sober spaces mentioned above, especially some properties of their function spaces endowed with the pointwise convergence topology or the Isbell topology. For $T_0$ spaces $X$ and $Y$, the following three main results are proved:

(1) $Y$ is bounded sober iff the function space $\mathbf{Top}(X, Y)$ of all continuous functions $f : X \longrightarrow Y$ equipped with the pointwise convergence topology is bounded sober iff $\mathbf{Top}(X, Y)$ equipped with the Isbell topology is bounded sober.

(2) For a $k$-bounded sober space $X$, the function space $\mathbf{Top}(X, Y)$ equipped with the pointwise convergence topology or the Isbell topology may not be $k$-bounded sober.

(3) If the function space $\mathbf{Top}(X, Y)$ equipped with the pointwise convergence topology or the the Isbell topology is weakly sober (resp., a cut space), then $Y$ is weakly sober (resp., a cut space).

Relationships among some kinds of (weakly) sober spaces are also investigated. It is also shown that a $T_0$ space $X$ is sober iff $X$ is $k$-bounded sober and almost sober iff its Smyth power space $P_S(X)$ is $k$-bounded sober and almost sober. By this, another example is given to show that the Smyth power space $P_S(X)$ of a weakly sober space $X$ may not be weakly sober.

## 2. Preliminaries

In this section, we briefly recall some standard definitions and notations that will be used in this paper, more details can be founded in [3, 5, 6].

In [5], for a poset $P$ and $A \subseteq P$, let $\uparrow A = \{x \in P : a \leq x$ for some $a \in A\}$ (dually $\downarrow A = \{x \in P : x \leq a$ for some $a \in A\}$). For $A = \{x\}$, $\uparrow A$ and $\downarrow A$ are shortly denoted by $\uparrow x$ and $\downarrow x$ respectively. A subset $A$ is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$).

As in [11], for a poset $P$ and $A \subseteq B \subseteq P$, $A^{\uparrow}$ and $A^{\downarrow}$ denote the sets of all upper and lower bounds of $A$ respectively, that is, $A^{\uparrow} = \{u \in P : A \subseteq \downarrow u\}$ and $A^{\downarrow} = \{v \in P : A \subseteq \uparrow v\}$. $A^{\uparrow_B}$ and $A^{\downarrow_B}$ denote the sets of all upper and lower bounds of $A$ in $B$, respectively. Let $A^{\delta} = (A^{\uparrow})^{\downarrow}$, $A^{\delta_B} = (A^{\uparrow_B})^{\downarrow_B}$ and $\delta(P) = \{A^{\delta} : A \subseteq P\}$. $\delta(P)$ is called the *Dedekind-Macneille* completion of $P$. $A^{\delta} = (A^{\uparrow})^{\downarrow}$ is called the *cut closure* of $A$ in $P$. If $A^{\delta} = A$, we say that $A$ is a *cut* in $P$.

A nonempty subset $D$ of a poset $P$ is called *directed* if every two elements in $D$ have an upper bound in $D$. The set of all directed sets of $P$ is denoted by $\mathcal{D}(P)$. $P$ is said to be a *directed complete poset*, a *dcpo* for short, if every directed subset of $P$ has the least upper bound in $P$. A subset $U$ of $P$

is *Scott open* if (i) $U = \uparrow U$, and (ii) for any directed subset $D$ for which $\vee D$ exists, $\vee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of $P$ form a topology. This topology is called the *Scott topology* on $P$ and denoted by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of $P$. The Alexandroff topology $\alpha(P)$ on a poset $P$ is the topology consisting of all its upper subsets. Let $|P|$ be the cardinality of $P$ and $\omega = |\mathbb{N}|$, where $\mathbb{N}$ is the set of all natural numbers.

Given a topological space $X$, we can define a preorder $\leq_X$, called the *specialization preorder*, which is defined by $x \leq y$ iff $x \in \overline{\{y\}}$. Clearly, each open set is an upper set and each closed set is a lower set with respect to the preorder $\leq_X$. It is easy to see that $\leq_X$ is a partial order if and only if $X$ is a $T_0$ space. Unless otherwise stated, throughout the paper, whenever an order-theoretic concept is mentioned in a topological space, it is to be interpreted with respect to the specialization preorder. Let $\mathcal{O}(X)$ (resp., $\mathcal{C}(X)$) be the set of all open subsets (resp., closed subsets) of $X$ and denote $\mathcal{S}(X) = \{\{x\} : x \in X\}$, $\mathcal{D}(X) = \{D \subseteq X : D \text{ is a directed set of } X\}$.

**Remark 2.1.** ( [15]) Let $X$ be a $T_0$ space, $C \subseteq X$ and $x \in X$. Then the followings are equivalent:

(1) $x \in C^{\uparrow}$;
(2) $C \subseteq \downarrow x$;
(3) $\overline{C} \subseteq \downarrow x$;
(4) $x \in \overline{C}^{\uparrow}$.

Therefore,

$$\bigcap_{c \in C} \uparrow c = C^{\uparrow} = \overline{C}^{\uparrow} = \bigcap_{b \in \overline{C}} \uparrow b,$$

$$C^{\delta} = \bigcap \{\downarrow x : C \subseteq \downarrow x\} = \bigcap \{\downarrow x : \overline{C} \subseteq \downarrow x\} = \overline{C}^{\delta}$$

and

$$\overline{C} \subseteq \overline{C}^{\delta} = C^{\delta}.$$

**Remark 2.2.** ( [5, 6]) Let $X$ be a $T_0$ space. For any subset $A \subseteq X$, $\vee A$ exists if and only if $\vee \overline{A}$ exists. Moreover, $\vee A = \vee \overline{A}$ if they exist.

A nonempty subset $A$ of a $T_0$ space $X$ is called *irreducible* if for any $\{F_1, F_2\} \subseteq \mathcal{C}(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. We denote by $\mathsf{Irr}(X)$ (resp., $\mathsf{Irr}_c(X)$) the set of all irreducible (resp., irreducible closed) subsets of $X$. Clearly, every directed subset of $X$ under $\leq_X$ is irreducible and the non-empty irreducible subsets of a poset with respect to the Alexandroff topology are exactly the directed sets. A topological space $X$ is called *sober*, if for any $F \in \mathsf{Irr}_c(X)$, there is a unique point $a \in X$ such that $F = \overline{\{a\}}$.

The following two lemmas on irreducible sets are well-known.

**Lemma 2.3.** ( [15]) *Let $X$ be a space and $Y$ a subspace of $X$. Then the following conditions are equivalent for a subset $A \subseteq Y$.*

(1) *$A$ is an irreducible subset of $Y$.*
(2) *$A$ is an irreducible subset of $X$.*
(3) $\mathrm{cl}_X A$ *is an irreducible closed subset of $X$.*

**Lemma 2.4.** ( [15]) (1) *If $f : X \longrightarrow Y$ is continuous and $A \in \mathsf{Irr}(X)$, then $f(A) \in \mathsf{Irr}(Y)$.*
(2) *If $\rho$ is a coarser topology than $\tau$ on $X$, then $\mathsf{Irr}_\tau(X) \subseteq \mathsf{Irr}_\rho(X)$.*

For a topological space $X$, $\mathcal{G} \subseteq 2^X$ and $A \subseteq X$, let $\Diamond_{\mathcal{G}} A = \{G \in \mathcal{G} : G \cap A \neq \emptyset\}$ and $\Box_{\mathcal{G}} A = \{G \in \mathcal{G} : G \subseteq A\}$. The sets $\Diamond_{\mathcal{G}} A$ and $\Box_{\mathcal{G}} A$ will be simply written as $\Diamond A$ and $\Box A$ respectively if there is no confusion. The *lower Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Diamond U : U \in \mathcal{O}(X)\}$ as a subbase, and the resulting space is denoted by $P_H(\mathcal{G})$. If $\mathcal{G} \subseteq \mathsf{Irr}(X)$, then $\{\Diamond_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ is a topology on $\mathcal{G}$.

**Remark 2.5.** Let $X$ be a $T_0$ space.

(1) If $\mathcal{S}_c(X) \subseteq \mathcal{G}$, then the specialization order on $P_H(\mathcal{G})$ is the order of set inclusion, and the *canonical mapping* $\eta_X : X \longrightarrow P_H(\mathcal{G})$, given by $\eta_X(x) = \overline{\{x\}}$, is an order and topological embedding (cf. [5,6]).
(2) The space $X^s = P_H(\mathsf{Irr}_c(X))$ with the canonical mapping $\eta_X : X \longrightarrow X^s$ is the *sobrification* of $X$. Obviously, for $\mathcal{A} \in \mathsf{Irr}(P_H(\mathsf{Irr}_c(X)))$, $\overline{\bigcup \mathcal{A}} \in \mathsf{Irr}_c(X)$ (cf. [5,6]).

For a space $X$, a subset $A$ of $X$ is called *saturated* if $A$ equals the intersection of all open sets containing it (equivalently, $A$ is an upper set in the specialization order). We denote by $\mathsf{K}(X)$ the poset of nonempty compact saturated subsets of $X$ with the order reverse to containment, i.e., for $K_1, K_2 \in \mathsf{K}(X)$, $K_1 \sqsubseteq K_2$ iff $K_2 \subseteq K_1$. We consider the upper Vietoris topology on $\mathsf{K}(X)$, generated by the sets $\Box U = \{K \in \mathsf{K}(X) : K \subseteq U\}$, where $U$ ranges over the open subsets of $X$. The resulting topological space is called the *Smyth power space* or *upper space* of $X$ and denoted by $P_S(X)$.

**Remark 2.6.** Let $X$ be a $T_0$ space. Then

(1) the specialization order on $P_S(X)$ is the Smyth order, that is, $\leq_{P_S(X)} = \sqsubseteq$ (cf. [5,6]);
(2) the *canonical mapping* $\xi_X : X \longrightarrow P_S(X)$, $x \mapsto \uparrow x$, is an order and topological embedding (cf. [7]).

## 3. Some kinds of weakly sober spaces

In this section, we begin by recalling the definitions and notations about some kinds of weakly sober spaces.

**Definition 3.1.** ( [4]) Let $X$ be a $T_0$ space.

(1) $X$ is called *quasisober*, if each irreducible closed set is the cut closure of a directed set.
(2) $X$ is called *weakly sober*, if for any irreducible closed set $A$, $A = A^\delta$ holds.
(3) $X$ is called a *cut space*, if for any directed subset $D$, $\overline{D} = D^\delta$ holds.

By Remark 2.1 and Lemma 2.3, we know that a $T_0$ space $X$ is weakly sober iff $\overline{A} = A^\delta$ for any irreducible subset $A$ of $X$; $X$ is quasisober iff for any $A \in \mathsf{Irr}(X)$, there exists a directed set $D \subseteq X$ such that $\overline{A} = D^\delta$.

A non-empty subset $A$ of a $T_0$ space $X$ is said to be *upper bounded* (*bounded* for short), if there is $x \in X$ with $A \subseteq \downarrow x = \mathrm{cl}\{x\}$.

**Definition 3.2.** ( [19]) A $T_0$ space $X$ is called *bounded sober*, if for each bounded irreducible closed set $A$ of $X$, there is a unique point $x \in X$ such that $A = \mathrm{cl}\{x\}$. Denote by $\mathsf{Irr}^b(X)$ (resp., $\mathsf{Irr}_c^b(X)$) the set of all bounded irreducible (resp., irreducible closed) subsets of $X$.

**Proposition 3.3.** *Let $X$ be a $T_0$ space. Then the following conditions are equivalent:*

(1) *$X$ is bounded sober.*
(2) *For any $A \in \mathsf{Irr}^b(X)$, $\overline{A} \cap \bigcap_{a\in A} \uparrow a \neq \emptyset$.*
(3) *For any $A \in \mathsf{Irr}_c^b(X)$, $A \cap \bigcap_{a\in A} \uparrow a \neq \emptyset$.*
(4) *For any $A \in \mathsf{Irr}^b(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{a\in A} \uparrow a \subseteq U$ implies $\uparrow a \subseteq U$ (i.e., $a \in U$) for some $a \in A$.*
(5) *For any $A \in \mathsf{Irr}_c^b(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{a\in A} \uparrow a \subseteq U$ implies $\uparrow a \subseteq U$ (i.e., $a \in U$) for some $a \in A$.*

*Proof.* (1) $\Rightarrow$ (2): If $X$ is bounded sober and $A \in \mathsf{Irr}^b(X)$, then there is an $x \in X$ such that $\overline{A} = \overline{\{x\}} = \downarrow x$, whence $x \in \overline{A} \cap \bigcap_{a\in A} \uparrow a$.

(2) $\Leftrightarrow$ (3): Clearly, we have (2) $\Rightarrow$ (3). Conversely, if condition (3) holds, then for $A \in \mathsf{Irr}^b(X)$, $\overline{A} \in \mathsf{Irr}_c^b(X)$, and

$$\emptyset \neq \overline{A} \cap \bigcap_{b\in \overline{A}} \uparrow b = \overline{A} \cap \bigcap_{a\in A} \uparrow a$$

by Remark 2.1.

(2) $\Rightarrow$ (4): If $\uparrow a \not\subseteq U$ for all $a \in A$, then $A \subseteq X \setminus U$, and hence $\overline{A} \subseteq X \setminus U$. By condition (2),

$$\emptyset \neq \overline{A} \cap \bigcap_{a\in A} \uparrow a \subseteq (X \setminus U) \cap U = \emptyset,$$

a contradiction.

(4) $\Leftrightarrow$ (5): Obviously, (4) $\Rightarrow$ (5). Conversely, if condition (5) holds, then for $A \in \mathsf{Irr}^b(X)$ and $U \in \mathcal{O}(X)$ with $\bigcap_{a\in A} \uparrow a \subseteq U$, we have $\overline{A} \in \mathsf{Irr}_c^b(X)$ and

$$\bigcap_{b\in \overline{A}} \uparrow b = \bigcap_{a\in A} \uparrow a \subseteq U$$

by Remark 2.1. By condition (5), $b \in U$ for some $b \in \overline{A}$, whence $A \cap U \neq \emptyset$. Condition (4) is thus satisfied.

(5) $\Rightarrow$ (1): Suppose $A \in \mathsf{Irr}_c^b(X)$. Then $A \cap \bigcap_{a\in A} \uparrow a \neq \emptyset$ (otherwise, by condition (5),

$$A \cap \bigcap_{a\in A} \uparrow a = \emptyset \Rightarrow \bigcap_{a\in A} \uparrow a \subseteq X \setminus A \Rightarrow \uparrow a \subseteq X \setminus A$$

for some $a \in A$, a contradiction). Select an $x \in A \cap \bigcap_{a\in A} \uparrow a$. Then $A \subseteq \downarrow x = \overline{\{x\}} \subseteq A$, and hence $A = \overline{\{x\}}$. Thus $X$ is bounded sober. □

**Definition 3.4.** ( [20]) A $T_0$ space $X$ is called *k-bounded sober*, if for any non-empty irreducible closed set $A$ whose supremum exists, there is a unique point $x \in X$ such that $A = \mathrm{cl}\{x\}$. Denote by $\mathsf{Irr}^{\vee}(X)$ (resp., $\mathsf{Irr}_c^{\vee}(X)$) the set of all irreducible (resp., irreducible closed) subsets which supremum exists in the specialization order of $X$.

In the following, we shall give an equivalent characterization of $k$-bounded sober spaces similar to that of bounded sober spaces (its proof is similar to that of Proposition 3.3 and hence is omitted).

**Proposition 3.5.** *Let $X$ be a $T_0$ space. Then the following conditions are equivalent:*

(1) *$X$ is $k$-bounded sober.*
(2) *For any $A \in \mathsf{Irr}^{\vee}(X)$, $\overline{A} \cap \bigcap_{a\in A} \uparrow a \neq \emptyset$.*
(3) *For any $A \in \mathsf{Irr}_c^{\vee}(X)$, $A \cap \bigcap_{a\in A} \uparrow a \neq \emptyset$.*
(4) *For any $A \in \mathsf{Irr}^{\vee}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{a\in A} \uparrow a \subseteq U$ implies $\uparrow a \subseteq U$ (i.e., $a \in U$) for some $a \in A$.*
(5) *For any $A \in \mathsf{Irr}_c^{\vee}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{a\in A} \uparrow a \subseteq U$ implies $\uparrow a \subseteq U$ (i.e., $a \in U$) for some $a \in A$.*

**Definition 3.6.** ( [1]) A $T_0$ space $X$ is called *almost sober*, if for each completely prime filter $\mathcal{F}$ of $\mathcal{O}(X)$, there is a unique point $x \in X$ such that $\bigcap \mathcal{F} = \uparrow x$.

The following important characterization of almost sober spaces was given in [1].

**Lemma 3.7.** ( [1]) *For a $T_0$ space $X$, the following conditions are equivalent:*

(1) *$X$ is almost sober.*
(2) *For any irreducible closed set $A$, $\vee A$ exists in the specialization order of $X$.*
(3) *For any irreducible set $A$, $\vee A$ exists in the specialization order of $X$.*

**Remark 3.8.** (1) Every sober space is almost sober. In fact, if $X$ is a sober space and $A \in \mathsf{Irr}_c(X)$, then there is an element $x \in X$ with $A = \overline{\{x\}}$, hence $\vee A = \vee\overline{\{x\}} = x$. But an almost sober space may not be sober. For example, if $L$ is the complete lattice constructed by Isbell [8], then $\Sigma L$ is almost sober but non-sober.

(2) Every weakly sober space is $k$-bounded sober. Indeed, suppose $X$ is weakly sober and $A \in \mathsf{Irr}_c(X)$ whose supremum exists, then $A = A^{\delta} = \downarrow \vee A = \mathrm{cl}\{\vee A\}$, whence $X$ is $k$-bounded sober. But a $k$-bounded sober space may not be weakly sober. The following is a counterexample.

Let

$$Q = \{a_1, a_2, \cdots, a_n, \cdots\} \cup \{b_1, b_2\} \cup \{c\}$$

with the order as follows (see Figure 1):

(i) $a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots$;
(ii) $a_n < b_1, a_n < b_2$ for all $n \in \mathbb{N}$ and $b_1, b_2$ are incomparable; and
(iii) $c < b_1$ and $c < b_2$.

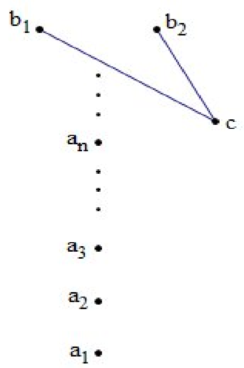


**Figure 1.** The poset $Q$ for which $(Q, \alpha(Q))$ is $k$-bounded sober but not weakly sober.

We consider the Alexandroff topology space $(Q, \alpha(Q))$. Then $(Q, \alpha(Q))$ is $k$-bounded sober, but is not weakly sober, because $A = \{a_n : n \in \mathbb{N}\} \in \mathsf{Irr}_c(Q, \alpha(Q))$, $A \neq A^\delta = A \cup \{c\}$.

(3) A weakly sober (resp., a $k$-bounded sober) space may not be an almost sober space and vice versa. Indeed, the Soctt space $\Sigma\mathbb{J}_{ab}$ in Example 3.15 is weakly sober and $k$-bounded sober, but is not almost sober. Conversely, let $P = \mathbb{N} \cup \{\top\}$ with the order generated by $1 < 2 < 3 < \cdots < n < n+1 < \cdots$ and $\forall n \in \mathbb{N}, n < \top$. Then the Alexandroff topology space $(P, \alpha(P))$ is almost sober. Clearly, $\mathbb{N} \in \mathsf{Irr}_c(P, \alpha(P))$ and $\vee\mathbb{N}$ exists, but $\mathbb{N} \neq \mathbb{N}^\delta = P$ and $\mathbb{N} \neq \mathrm{cl}_{\alpha(P)}\{x\}$ for each $x \in P$. So $(P, \alpha(P))$ is neither weakly sober nor $k$-bounded sober.

By Lemma 3.7 and Remark 3.8, we can easily obtain the following result.

**Corollary 3.9.** *For a $T_0$ space $X$, the following statements are equivalent.*

(1) *$X$ is sober.*
(2) *$X$ is weakly sober and almost sober.*
(3) *$X$ is $k$-bounded sober and almost sober.*

The non-empty irreducible subsets of a poset with respect to the Alexandroff topology are exactly the directed sets. Thus, we have the following:

**Lemma 3.10.** *For any poset $P$, $(P, \alpha(P))$ is quasisober iff $(P, \alpha(P))$ is weakly sober iff $(P, \alpha(P))$ is a cut space.*

**Lemma 3.11.** ( [20]) *For any poset $P$, $(P, \upsilon(P))$ is $k$-bounded sober.*

The following is a straightforward consequence of Lemma 3.7, Corollary 3.9 and Lemma 3.11.

**Proposition 3.12.** *For any poset $P$, the following statements are equivalent.*

(1) *$(P, \upsilon(P))$ is sober.*
(2) *$(P, \upsilon(P))$ is almost sober.*
(3) *Every irreducible subset of $(P, \upsilon(P))$ has the least upper bound in $P$.*

A summary on certain relations among some kinds of (weakly) sober spaces are shown in Figure 2.

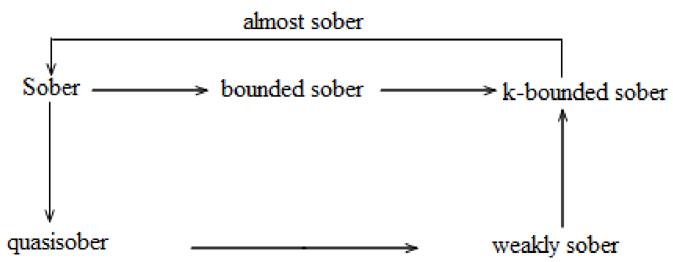


**Figure 2.** Certain relations among some kinds of (weakly) sober spaces.

For the sobriety of the Smyth power spaces, we have the following well-known result.

**Theorem 3.13.** ( [7]) *For a $T_0$ space $X$, the following conditions are equivalent:*

(1) *$X$ is sober.*
(2) *For any $\mathcal{A} \in \mathsf{Irr}(P_S(X))$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{A} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{A}$.*

(3) $P_S(X)$ *is sober.*

By Corollary 3.9, we get the following result.

**Corollary 3.14.** *For a $T_0$ space $X$, the following conditions are equivalent:*

(1) *$X$ is sober.*
(2) *$P_S(X)$ is weakly sober and almost sober.*
(3) *$P_S(X)$ is $k$-bounded sober and almost sober.*

It is known that if the Smyth power space $P_S(X)$ of a $T_0$ space $X$ is bounded sober (resp., $k$-bounded sober, weakly sober, a cut space), then so is $X$ (see [14, 18]). Ye and Xu [18] showed that there exists a bounded sober space $X$ whose Smyth power space $P_S(X)$ is not bounded sober. In [14], Wen and Xu constructed an example to show that the Smyth power space $P_S(X)$ of a quasisober space (resp., weakly sober space, cut space) $X$ may not be a quasisober space (resp., weakly sober space, cut space). In [13], it was shown that there exists a countable algebraic lattice $L$ for which the Alexandroff space $(L, \alpha(L))$ is almost sober but the Smyth power space $P_S(L, \alpha(L))$ is not almost sober, and on the other hand, there is a countable dcpo $P$ for which the Smyth power space $P_S(\Sigma P)$ is almost sober but the Scott space $\Sigma P$ itself is not almost sober. The following is another example that the Smyth power space $P_S(X)$ of a weakly sober space $X$ is not weakly sober.

**Example 3.15.** Let $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ with ordering defined by $(j, k) \leq (m, n)$ iff $j = m$ and $k \leq n$, or $n = \omega$ and $k \leq m$. $\mathbb{J}$ is a well-known dcpo constructed by Johnstone in [9]. And let $\mathbb{J}_{ab} = \mathbb{J} \cup \{a, b\}$. Define a partial order $\leq$ on $\mathbb{J}_{ab}$ as follows:

$$x \leq y \text{ if and only if } \begin{cases} x \leq y \text{ in } \mathbb{J}, & x, y \in \mathbb{J}, \\ x \in \mathbb{J}, & y \in \{a, b\}. \end{cases}$$

Consider the Scott topology on $\mathbb{J}_{ab}$. It was proved in [12, Lemma 3.1] that $\mathsf{Irr}_c(\Sigma\mathbb{J}_{ab}) = \{\downarrow x : x \in \mathbb{J}_{ab}\} \cup \{\mathbb{J}\}$. Then

(a) $\Sigma\mathbb{J}_{ab}$ is weakly sober (clearly, $\forall A \in \mathsf{Irr}_c(\Sigma\mathbb{J}_{ab})$, $A = A^\delta$ holds).
(b) $\Sigma\mathbb{J}_{ab}$ is $k$-bounded sober (see [12, Lemma 3.1]).
(c) $\Sigma\mathbb{J}_{ab}$ is not almost sober and hence non-sober.
Since $\mathbb{J} \in \mathsf{Irr}_c(\Sigma\mathbb{J}_{ab})$, but $\vee\mathbb{J}$ does not exists. By Lemma 3.7, $\Sigma\mathbb{J}_{ab}$ is not almost sober. And by Corollary 3.9, $\Sigma\mathbb{J}_{ab}$ is not sober.
(d) $P_S(\Sigma\mathbb{J}_{ab})$ is neither weakly sober nor $k$-bounded sober.
It was proved in [13, Example 5.8] that $P_S(\Sigma\mathbb{J}_{ab})$ is almost sober. Since $\Sigma\mathbb{J}_{ab}$ is not sober, $P_S(\Sigma\mathbb{J}_{ab})$ is neither weakly sober nor $k$-bounded sober by Proposition 3.14.

## 4. Function spaces of some kinds of weakly sober spaces

In this section, we study the function spaces endowed with the pointwise convergence topology or the Isbell topology of bounded sober spaces, $k$-bounded sober, weakly sober spaces and cut spaces.

**Definition 4.1.** ( [5]) Given topological spaces $X$ and $Y$, let $\mathbf{Top}(X, Y)$ be the set of all continuous functions from $X$ to $Y$.

(1) For a point $x \in X$ and an open set $U \in \mathcal{O}(Y)$, let $S(x, U) = \{f \in \mathbf{Top}(X, Y) : f(x) \in U\}$. The set $\{S(x, U) : x \in X, U \in \mathcal{O}(Y)\}$ is a subbasis for the *pointwise convergence topology* (i.e., the relative product topology) on $\mathbf{Top}(X, Y)$. Let $[X \to Y]_P$ denote the function space $\mathbf{Top}(X, Y)$ endowed with the topology of pointwise convergence.

(2) The *Isbell topology* on the set $\mathbf{Top}(X, Y)$ is generated by the subsets of the form $N(H \leftarrow V) = \{f \in \mathbf{Top}(X, Y) : f^{-1}(V) \in H\}$, where $H$ is a Scott open subset of the complete lattice $\mathcal{O}(X)$ and $V$ is open in $Y$. Let $[X \to Y]_I$ denote the function space $\mathbf{Top}(X, Y)$ endowed with the Isbell topology.

**Remark 4.2.** The specialization orders on $[X \to Y]_P$ and $[X \to Y]_I$ are both pointwise orders on $\mathbf{Top}(X, Y)$, i.e., for $f, g \in \mathbf{Top}(X, Y)$, $f \leq_{[X\to Y]_P} g$ iff $f \leq_{[X\to Y]_I} g$ iff $f(x) \leq g(x)$ for all $x \in X$.

**Lemma 4.3.** ( [5]) *For topological spaces X and Y, the Isbell topology on* $\mathbf{Top}(X, Y)$ *is finer than the pointwise convergence topology.*

**Lemma 4.4.** *Let $X, Y$ be topological spaces and $x \in X$. Consider the function*

$$\varphi_x^P : [X \to Y]_P \longrightarrow Y, \quad f \quad \mapsto f(x).$$

*Then $\varphi_x^P$ is continuous.*

*Proof.* Suppose $x \in X$, $f \in [X \to Y]_P$ and $\varphi_x^P(f) = f(x) \in V$, where $V$ is open in $Y$. Then $f \in S(x, V)$, $S(x, V)$ is open in $[X \to Y]_P$ and for any $g \in S(x, V)$, $\varphi_x^P(g) = g(x) \in V$, whence $\varphi_x^P(S(x, V)) \subseteq V$. It follows that $\varphi_x^P : [X \to Y]_P \longrightarrow Y$ is continuous. □

**Corollary 4.5.** *Let $X, Y$ be topological spaces and $x \in X$. Then the function $\varphi_x^I : [X \to Y]_I \longrightarrow Y$, defined by $\varphi_x^I(f) = f(x)$ for each $f \in [X \to Y]_I$, is continuous.*

*Proof.* By Lemmas 4.3 and 4.4. □

In the following, for topological spaces $X$, $Y$ and $y \in Y$, $c_y$ denotes the constant function from $X$ to $Y$ with value $y$, i.e., $c_y(x) = y$ for all $x \in X$.

**Lemma 4.6.** ( [10]) *Let X and Y be topological spaces. Consider the function*

$$\psi^I : Y \longrightarrow [X \to Y]_I, \quad y \mapsto c_y.$$

*Then $\psi^I$ is continuous.*

By Lemmas 4.3 and 4.6, we can directly obtain the following corollary.

**Corollary 4.7.** *Let X and Y be topological spaces. Then the function*

$$\psi^P : Y \longrightarrow [X \to Y]_P, \quad y \mapsto c_y$$

*is continuous.*

**Proposition 4.8.** ( [18]) *A retract of a bounded sober space is bounded sober.*

**Theorem 4.9.** *For $T_0$ spaces X and Y, if the function space* $\mathbf{Top}(X, Y)$ *of all continuous functions $f : X \to Y$ equipped with the pointwise convergence topology* (*resp., the Isbell topology*) *is bounded sober, then Y is bounded sober.*

*Proof.* Suppose $[X \to Y]_P$ is bounded sober. Select an $x \in X$. Then for any $y \in Y$, $y = \varphi_x^P \circ \psi^P(y)$ and $\varphi_x^P$, $\psi^P$ are continuous by Lemma 4.4 and Corollary 4.7. Thus $Y$ is a retract of $[X \to Y]_P$. So $Y$ is bounded sober by Proposition 4.8.

If the function space $\mathbf{Top}(X, Y)$ of all continuous functions $f : X \to Y$ equipped with the Isbell topology is bounded sober, then $Y$ is bounded sober by Corollary 4.5, Lemma 4.6 and Proposition 4.8. □

**Theorem 4.10.** ( [2]) *Let $X$ be a $T_0$ space and $Y$ a bounded sober space, then the function space $\mathbf{Top}(X, Y)$ of all continuous functions $f : X \longrightarrow Y$ equipped with the pointwise convergence topology is bounded sober.*

Next, we will discuss the question whether the function space $\mathbf{Top}(X, Y)$ equipped with the Isbell topology is bounded sober for a $T_0$ space $X$ and a bounded sober space $Y$.

Firstly, we need the following two lemmas.

**Lemma 4.11.** *Let $X$, $Y$ be topological spaces and $V \in \mathcal{O}(Y)$. Consider the function*

$$\zeta_V : [X \to Y]_I \longrightarrow \Sigma\mathcal{O}(X), \quad f \mapsto f^{-1}(V).$$

*Then $\zeta_V$ is continuous.*

*Proof.* Suppose $V \in \mathcal{O}(Y)$, $f \in [X \to Y]_I$ and $\zeta_V(f) = f^{-1}(V) \in H$, where $H$ is Scott open in $\mathcal{O}(X)$. Then $f \in N(H \leftarrow V)$, $N(H \leftarrow V)$ is open in $[X \to Y]_I$ and for any $g \in N(H \leftarrow V)$, $\zeta_V(g) = g^{-1}(V) \in H$, whence $\zeta_V(N(H \leftarrow V)) \subseteq H$. It follows that $\zeta_V : [X \to Y]_I \longrightarrow \Sigma\mathcal{O}(X)$ is continuous. □

**Lemma 4.12.** ( [19]) *For each $T_0$ space $X$, $P_H(\mathsf{Irr}_c^b(X))$ is a bounded sober space. In fact, for any $\{A_i : i \in I\} \in \mathsf{Irr}^b(P_H(\mathsf{Irr}_c^b(X)))$, $A = \overline{\bigcup_{i\in I} A_i} \in \mathsf{Irr}_c^b(X)$ and $\overline{\{A_i : i \in I\}} = \overline{\{A\}}$ in $P_H(\mathsf{Irr}_c^b(X))$. More precisely, $X^{bs} = P_H(\mathsf{Irr}_c^b(X))$ with the canonical topological embedding $\eta_X^{bs} : X \longrightarrow X^{bs}$ is the bounded sobrification of $X$, where $\eta_X^{bs}(x) = \overline{\{x\}}$ for all $x \in X$.*

Our main result is the following theorem.

**Theorem 4.13.** *Let $X$ be a $T_0$ space and $Y$ a bounded sober space. Then the function space $\mathbf{Top}(X, Y)$ of all continuous functions $f : X \longrightarrow Y$ equipped with the Isbell topology is a bounded sober space.*

*Proof.* Let $F \in \mathsf{Irr}^b([X \to Y]_I)$. Since the specialization order on $[X \to Y]_I$ is the pointwise order on $\mathbf{Top}(X, Y)$, $[X \to Y]_I$ is $T_0$. For each $x \in X$, $\{f(x) : f \in F\}$ is a bounded irreducible subset of $Y$. As $Y$ is bounded sober, there is a unique element $a_x \in Y$ such that $\overline{\{f(x) : f \in F\}} = \overline{\{a_x\}}$. Now we can define a function

$$g : X \longrightarrow Y \ \text{ by } \ g(x) = a_x \ \text{ for each } \ x \in X.$$

For $x \in X$ and $V \in \mathcal{O}(Y)$ with $g(x) = a_x \in V$, we have $V \bigcap \{f(x) : f \in F\} \neq \emptyset$ since $\overline{\{f(x) : f \in F\}} = \overline{\{a_x\}}$, and hence there is an element $f \in F$ such that $f(x) \in V$. As $f : X \longrightarrow Y$ is continuous, there is a $U \in \mathcal{O}(X)$ with $x \in U$ such that $f(z) \in V$ for every $z \in U$. Since $f \in F$, we have

$$f(z) \in \overline{\{f(z) : f \in F\}} = \overline{\{a_z\}} = \overline{\{g(z)\}},$$

and hence $g(z) \in V$ for all $z \in U$. Thus $g$ is continuous.

**Claim:** For any subbasis open set $N(H \leftarrow W)$ in $[X \to Y]_I$ with $g \in N(H \leftarrow W)$, where $H$ is a Scott open subset of $\mathcal{O}(X)$ and $W$ is an open subset of $Y$, we have $F \bigcap N(H \leftarrow W) \neq \emptyset$.

Suppose $g \in N(H \leftarrow W)$ $(H \in \sigma(\mathcal{O}(X)), W \in \mathcal{O}(Y))$, then $g^{-1}(W) \in H$. For any $x \in g^{-1}(W)$, $g(x) \in W$, as $g(x) = a_x \in \overline{\{f(x) : f \in F\}}$, there exists an element $f_0 \in F$ such that $f_0(x) \in W$. So $x \in f_0^{-1}(W) \subseteq \bigcup_{f\in F} f^{-1}(W)$, whence $g^{-1}(W) \subseteq \bigcup_{f\in F} f^{-1}(W)$. Since $H$ is an upper set, we have $\bigcup_{f\in F} f^{-1}(W) \in H$.

Since $\Sigma\mathcal{O}(X)$ is $T_0$, one can directly deduce that $P_H(\mathsf{Irr}_c^b(\Sigma\mathcal{O}(X)))$ is $T_0$. By Lemmas 4.11 and 4.12, for each continuous function $f : X \longrightarrow Y$,

$$\zeta_W : [X \to Y]_I \longrightarrow \Sigma\mathcal{O}(X), \quad f \mapsto f^{-1}(W),$$

is continuous, and for any $U \in \mathcal{O}(X)$,

$$\eta^{bs}_{\Sigma\mathcal{O}(X)} : \Sigma\mathcal{O}(X) \longrightarrow P_H(\mathsf{Irr}_c^b(\Sigma\mathcal{O}(X))),$$
$$U \mapsto \mathrm{cl}_{\Sigma\mathcal{O}(X)}\{U\},$$

is continuous. Hence for each continuous function $f : X \longrightarrow Y$,

$$\eta^{bs}_{\Sigma\mathcal{O}(X)} \circ \zeta_W : [X \to Y]_I \longrightarrow P_H(\mathsf{Irr}_c^b(\Sigma\mathcal{O}(X))),$$
$$f \mapsto \mathrm{cl}_{\Sigma\mathcal{O}(X)}\{f^{-1}(W)\},$$

is continuous. Thus

$$\{\mathrm{cl}_{\Sigma\mathcal{O}(X)}\{f^{-1}(W)\} : f \in F\} \in \mathsf{Irr}^b(P_H(\mathsf{Irr}_c^b(\Sigma\mathcal{O}(X)))).$$

By Lemma 4.12 again,

$$\mathrm{cl}_{\Sigma\mathcal{O}(X)} \bigcup_{f\in F} \mathrm{cl}_{\Sigma\mathcal{O}(X)}\{f^{-1}(W)\} = \mathrm{cl}_{\Sigma\mathcal{O}(X)}\{\bigcup_{f\in F} f^{-1}(W)\} \in \mathsf{Irr}_c^b(\Sigma\mathcal{O}(X))$$

and in $P_H(\mathsf{Irr}_c^b(\Sigma\mathcal{O}(X)))$,

$$\overline{\{\mathrm{cl}_{\Sigma\mathcal{O}(X)}\{f^{-1}(W)\} : f \in F\}} = \overline{\{\mathrm{cl}_{\Sigma\mathcal{O}(X)}\{\bigcup_{f\in F} f^{-1}(W)\}\}}.$$

As $\bigcup_{f\in F} f^{-1}(W) \in H$, we get that $\mathrm{cl}_{\Sigma\mathcal{O}(X)}\{\cup_{f\in F} f^{-1}(W)\} \in \Diamond H$. And since $\Diamond H$ is open in $P_H(\mathsf{Irr}_c^b(\Sigma\mathcal{O}(X)))$, we have that $\{\mathrm{cl}_{\Sigma\mathcal{O}(X)}\{f^{-1}(W)\} : f \in F\} \cap \Diamond H \neq \emptyset$. It follows that there exists $f \in F$ such that $\mathrm{cl}_{\Sigma\mathcal{O}(X)}\{f^{-1}(W)\} \in \Diamond H$, and hence $\mathrm{cl}_{\Sigma\mathcal{O}(X)}\{f^{-1}(W)\} \cap H \neq \emptyset$, implying that $f^{-1}(W) \in H$. Therefore, $F \cap N(H \leftarrow W) \neq \emptyset$.

As the specialization order on $[X \to Y]_I$ is the pointwise order on $\mathbf{Top}(X, Y)$, $\mathrm{cl}_{[X\to Y]_I} F \subseteq \mathrm{cl}_{[X\to Y]_I}\{g\}$ by the definition of $g$. By the Claim and $F \in \mathsf{Irr}([X \to Y]_I)$, all basic open sets of $g$ in $[X \to Y]_I$ must meet $F$. It follows that $\mathrm{cl}_{[X\to Y]_I} F = \mathrm{cl}_{[X\to Y]_I}\{g\}$. Therefore, the function space $\mathbf{Top}(X, Y)$ of all continuous functions $f : X \longrightarrow Y$ equipped with the Isbell topology is a bounded sober space. □

The following corollary follows directly from Theorem 4.9, Theorem 4.10 and Theorem 4.13.

**Corollary 4.14.** *For $T_0$ spaces $X$ and $Y$, the following conditions are equivalent:*

(1) *Y is bounded sober.*
(2) *The function space* **Top**$(X, Y)$ *equipped with the pointwise convergence topology is bounded sober.*
(3) *The function space* **Top**$(X, Y)$ *equipped with the Isbell topology is bounded sober.*

The following example shows that for a $k$-bounded sober $X$, the function space **Top**$(X, Y)$ equipped with the pointwise convergence topology or the Isbell topology may not be $k$-bounded sober.

**Example 4.15.** *( [14])* Let $P = \mathbb{N} \cup \{a_1, a_2\}$ with the partial order defined by $n < a_1, n < a_2$ and $n < n+1$ for all $n \in \mathbb{N}$. It is clear that the Alexandroff topological space $X = (P, \alpha(P))$ is $k$-bounded sober. For any $n \in \mathbb{N}$, define the mapping $f_n : X \longrightarrow X$ by

$$f_n(x) = \begin{cases} a_2, & x \in P \backslash \{1\}, \\ n, & x = 1. \end{cases}$$

Then $f_n$ is continuous and $D = \{f_n : n \in \mathbb{N}\}$ is an irreducible subset of $[X \to X]_P$. One readily sees that the constant constant function $c_{a_2}$ with value $a_2$ (i.e., $c_{a_2}(x) = a_2$ for any $x \in P$) is the only upper bound of $D$, whence $c_{a_2} = \vee D$. If there exists a continuous function $h$ such that $\overline{D} = \overline{\{h\}}$, then $h = c_{a_2} = \vee D$. But $c_{a_2} \in S(1, \{a_2\})$ and $D \cap S(1, \{a_2\}) = \emptyset$, a contradiction. It follows that **Top**$(X, X)$ equipped with the pointwise convergence topology is not a $k$-bounded sober space.

As $D = \{f_n : n \in \mathbb{N}\}$ is also an irreducible subset whose supremum exists in $[X \to X]_I$, we have that **Top**$(X, X)$ endowed with the Isbell topology is not a $k$-bounded sober space, too.

In [14], Wen and Xu presented an example (see Example 3.12) to illustrate that for a $T_0$ space $X$ and a weakly sober space $Y$, the function space **Top**$(X, Y)$ equipped with the pointwise convergence topology is not a weakly sober space.

Now we consider the converse problem: For $T_0$ spaces $X$ and $Y$, if **Top**$(X, Y)$ equipped with the pointwise convergence topology or the Isbell topology is weakly sober, is $Y$ weakly sober? We will give a positive answer to this question.

**Theorem 4.16.** *For $T_0$ topological spaces $X$ and $Y$, if $[X \to Y]_P$ is weakly sober, then $Y$ is weakly sober.*

*Proof.* Let $[X \to Y]_P$ be weakly sober and $A \in \mathsf{Irr}(Y)$. According to Corollary 4.7, $\psi^P(A) = \{c_a : a \in A\} \in \mathsf{Irr}([X \to Y]_P)$, where $c_a$ is the constant function with value $a$. Whence $\mathrm{cl}_{[X \to Y]_P}\psi^P(A) = (\psi^P(A))^{\delta_{[X \to Y]_P}}$. Now we prove that $\overline{A} = A^{\delta}$. If not, then there exists $y \in A^{\delta} \backslash \overline{A}$ by Remark 2.1. Suppose $f$ is any upper bound of $\psi^P(A) = \{c_a : a \in A\}$ in **Top**$(X, Y)$. Then for any $a \in A$ and $x \in X$, $a = c_a(x) \leq f(x)$, hence $f(X) \subseteq A^{\uparrow}$. As $y \in A^{\delta}$, $c_y(x) = y \leq f(x)$ for each $x \in X$. It follows that $c_y \leq f$. Therefore, $c_y \in (\psi^P(A))^{\delta_{[X \to Y]_P}}$. Since $y \in Y \backslash \overline{A}$, then for each $x \in X$, we have $c_y(x) = y \in Y \backslash \overline{A}$, i.e., $c_y \in S(x, Y \backslash \overline{A})$. By $\mathrm{cl}_{[X \to Y]_P}\psi^P(A) = \psi^P(A)^{\delta_{[X \to Y]_P}}$, there exists $a \in A$ such that $\psi^P(a) = c_a \in S(x, Y \backslash \overline{A})$, and hence $a \in Y \backslash \overline{A}$, which is a contradiction. Therefore, $\overline{A} = A^{\delta}$. Thus $Y$ is weakly sober. □

By Lemma 4.3, Lemma 4.6 and Theorem 4.16, we get the following result.

**Corollary 4.17.** *Let $X$ and $Y$ be $T_0$ spaces. If $[X \to Y]_I$ is weakly sober space, then $Y$ is a weakly sober space.*

Replacing irreducible sets by directed sets, one can deduce the following theorem by Theorem 4.16 and Corollary 4.17.

**Theorem 4.18.** *Let $X$ and $Y$ be $T_0$ spaces. If $[X \to Y]_I$ or $[X \to Y]_P$ is a cut space, then $Y$ is a cut space.*

## 5. Conclusions

In this paper, we mainly prove that for $T_0$ spaces $X$ and $Y$, $Y$ is bounded sober iff the function space $\mathbf{Top}(X, Y)$ of all continuous functions $f : X \longrightarrow Y$ equipped with the pointwise convergence topology is bounded sober iff $\mathbf{Top}(X, Y)$ equipped with the Isbell topology is bounded sober. And if the function space $\mathbf{Top}(X, Y)$ equipped with the pointwise convergence topology or the the Isbell topology is weakly sober (resp., a cut space), then $Y$ is weakly sober (resp., a cut space).


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## Conflict of interest

All authors declare that there is no conflict of interest.


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