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A regularity criterion via horizontal components of velocity and molecular orientations for the 3D nematic liquid crystal flows

Qiang Li[1,*] **and Mianlu Zou**[2]

[1] School of Mathematics and Statistics, Xinyang College, Henan 464000, China
[2] School of Big Data and Artificial Intelligence, Xinyang College, Henan 464000, China

* **Correspondence:** Email: liqiang5412@163.com.

Abstract: In this paper, we obtain a regularity criterion via horizontal components of velocity and molecular orientations for the 3D nematic liquid crystal flows. That is the smooth solution (u, d) can be extended beyond T, provided that $\int_0^T (\|u_h\|^2_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}})\mathrm{d}t < \infty$.

Keywords: nematic liquid crystal flows; Navier-Stokes equations; regularity criterion
Mathematics Subject Classification: 35B65, 35Q35, 76A15

1. Introduction

In this paper, we consider the following nematic liquid crystal flows in 3-dimensions:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = -\lambda \nabla \cdot (\nabla d \odot \nabla d), \\ \partial_t d + u \cdot \nabla d = \gamma(\Delta d + |\nabla d|^2 d), \\ \nabla \cdot u = 0, |d| = 1, \\ u(x, 0) = u_0(x), d(x, 0) = d_0(x), \end{cases} \tag{1.1}$$

here, $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ denotes the velocity field, $d = d(x, t) = (d_1(x, t), d_2(x, t), d_3(x, t))$ denotes the macroscopic average of molecular orientation field and $p = p(x, t)$ is the scalar pressure. And μ, λ, γ are positive constants, which will be assumed to be 1 because their specific values play no roles in our arguments. The notation $\nabla d \odot \nabla d$ represents the 3×3 matrix whose the $(i, j)th$ component is given by $\partial_i d_k \partial_j d_k (i, j \leq 3)$.

The above system (1.1) is a simplified version of the Ericksen-Leslie equations , which was first introduced by Lin [9] to describe the nematic liquid crystals flows. It is well-known that the system (1.1) has a unique local smooth solution (u, d) provided that initial data $u_0 \in H^s(\mathbb{R}^n, \mathbb{R}^n)$ with

$\nabla \cdot u_0 = 0$ and $d_0 \in H^{s+1}(\mathbb{R}^n, \mathbb{S}^2)$ for $s \geq n$. For the regularity criteria readers may refer to [7, 8, 10, 12, 14–16].

Obviously, the above system (1.1) reduce to the incompressible Navier-Stokes equations when the orientation field d equals a constant. For Navier-Stokes equations, in [2], Dong and Zhang established the following regularity criterion:

$$\int_0^T \|\nabla_h u_h\|_{\dot{B}^0_{\infty,\infty}} \mathrm{d}t < \infty, \tag{1.2}$$

where $\nabla_h = (\partial_1, \partial_2), u_h = (u_1, u_2)$. And they left a question that had been solved by Gala and Ragusa in [3], which is that whether (1.2) can be replaced by the following condition:

$$\int_0^T \|u_h\|^2_{\dot{B}^0_{\infty,\infty}} \mathrm{d}t < \infty. \tag{1.3}$$

In this paper, we are aimed to extend the criterion (1.3) to the system (1.1). Our main results are stated as follows:

Theorem 1.1. *Let initial data $u_0 \in H^3(\mathbb{R}^3)$, $d_0 \in H^4(\mathbb{R}^3, \mathbb{S}^2)$, $\nabla \cdot u_0 = 0$. Suppose (u, d) is a local smooth solution to the equations of (1.1) on $[0, T)$ for some $0 < T < \infty$. If (u, d) satisfies*

$$\int_0^T (\|u_h\|^2_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}})dt < \infty, \tag{1.4}$$

then (u, d) can be extended beyond T.

Remark 1.1. *In [14], Yuan and Wei obtained a regularity condition that is*

$$\int_0^T (\|\omega\|^{\frac{2}{2-r}}_{\dot{B}^{-r}_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}})dt < \infty,\ 0 < r < 2.$$

Later, Li and Yuan [7] improved the above regularity condition by

$$\int_0^T (\|\omega_3\|^{\frac{2p}{2p-3}}_{L^p} + \|u_3\|^{\frac{2q}{q-3}}_{L^q} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}})dt < \infty,\ with\ \frac{3}{2} < p \leq \infty,\ 3 < q \leq \infty.$$

Compared to the two regularity conditons, (1.4) only contains two components of the velocity field u in Besov spaces, which is an improved result.

2. Preliminaries

In this section, we will give some useful inequalities which play an important role in our proof.

Lemma 2.1. *(Page 82 in [1]) Let $1 < q < p < \infty$ and α be a positive real number. Then there exists a constant C such that*

$$\|f\|_{L^p} \leq C\|f\|^{1-\theta}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|f\|^{\theta}_{\dot{B}^{\beta}_{q,q}}, with\ \beta = \alpha(\frac{p}{q} - 1), \theta = \frac{q}{p}.$$

In particular, when $\beta = 1$, $q = 2$ and $p = 4$, we have $\alpha = 1$ and

$$\|f\|_{L^4} \leq C\|f\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}} \|\nabla f\|^{\frac{1}{2}}_{L^2}. \tag{2.1}$$

Lemma 2.2. *(Product and commutator estimate [6,11]) Let $s > 0$, $1 < p < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$ with $p_2,\ p_3 \in (1, +\infty)$ and $p_1,\ p_4 \in [1, +\infty]$. Then,*

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|g\|_{L^{p_1}}\|\Lambda^s f\|_{L^{p_2}} + \|\Lambda^s g\|_{L^{p_3}}\|f\|_{L^{p_4}}), \tag{2.2}$$

$$\|[\Lambda^s, f\cdot\nabla]g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|\nabla g\|_{L^{p_4}}), \tag{2.3}$$

where $[\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g$.

Lemma 2.3. *([5], Theorem 2.1) Let $s > \frac{3}{2}$, then there exists a constant C such that*

$$\|f\|_{\dot{B}^0_{\infty,2}} \leq C(1 + \|f\|_{\dot{B}^0_{\infty,\infty}} \log^{\frac{1}{2}}(1 + \|f\|_{\dot{H}^s})), \tag{2.4}$$

for any $f \in \dot{H}^s$.

Lemma 2.4. *([13], Lemma 2.3 or page 21 in [17]) Let $f \in BMO$ and $g, h \in H^1(\mathbb{R}^3)$, we have*

$$\int_{R^3} f\nabla(gh)dx \leq C\|f\|_{BMO}(\|\nabla g\|_{L^2}\|h\|_{L^2} + \|g\|_{L^2}\|\nabla h\|_{L^2}). \tag{2.5}$$

3. Proof of Theorem 1.1

Let's recall a result that will be a bridge to prove our conclusion. In [4], Huang and Wang established a BKM type blow-up criterion for the system (1.1). If T is the maximal time, $0 < T < \infty$, one has

$$\int_0^T (\|\omega\|_{L^\infty} + \|\nabla d\|^2_{L^\infty})\mathrm{d}t = \infty, \tag{3.1}$$

where $\omega = \nabla \times u$. Under the conditions (1.4) and (3.1), if we can show

$$\int_0^T (\|\nabla\Delta u\|^2_{L^2} + \|\Delta^2 d\|^2_{L^2})\mathrm{d}t < C, \tag{3.2}$$

then Theorem 1.1 is valid by using the Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$.

Noticing that $\nabla \cdot u = 0$, we obtain

$$\int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot u\mathrm{d}x = 0, \int_{\mathbb{R}^3} \nabla p\cdot u\mathrm{d}x = 0.$$

Mutiplying u to Eq $(1.1)_1$ and integrating over $\mathbb{R}^3$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2} = -\int_{\mathbb{R}^3} \nabla d\cdot\Delta d\cdot u\mathrm{d}x.$$

Similarly, multiplying $(1.1)_2$ by $-\Delta d$ and integrating over $\mathbb{R}^3$ one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla d\|^2_{L^2} + \|\Delta d\|^2_{L^2} = \int_{\mathbb{R}^3} u\cdot\nabla d\cdot\Delta d - |\nabla d|^2 d\Delta d\mathrm{d}x.$$

By adding the above equalities and using the facts $|d| = 1,\ \Delta(|d|^2) = 0 \Rightarrow |\nabla d|^2 = -d \cdot \Delta d$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \le \|\Delta d\|_{L^2}^2. \tag{3.3}$$

Integrating (3.3) in time gives

$$\sup_{0<t<T}(\|u(t)\|_{L^2}^2 + \|\nabla d(t)\|_{L^2}^2) + \int_0^T \|\nabla u(t)\|_{L^2}^2 \mathrm{d}t \le C(\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2).$$

Similarly going on the above process, multiplying $(1.1)_1$ by $-\Delta u$ and integrating over $\mathbb{R}^3$, then applying Δ to Eq $(1.1)_2$, and taking the inner product with Δd, after that adding the resulting equalities, we achieve that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2 \\
=&\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u \mathrm{d}x + \int_{\mathbb{R}^3}\nabla\cdot(\nabla d\odot\nabla d)\cdot\Delta u \mathrm{d}x \\
&-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla d)\cdot\Delta d \mathrm{d}x + \int_{\mathbb{R}^3}\Delta(|\nabla d|^2 d)\cdot\Delta d \mathrm{d}x \\
:=&I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.4}$$

For I_1, by the incompressibility condition and integration by parts several times, we conclude that

$$\begin{aligned}
I_1 =& \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u \mathrm{d}x \\
=&\int_{\mathbb{R}^3}(\sum_{k,j=1}^{3}\sum_{i=1}^{2}u_j\partial_j u_i\partial_k\partial_k u_i + \sum_{k,j=1}^{3}u_j\partial_j u_3\partial_k\partial_k u_3)\mathrm{d}x \\
=&\int_{\mathbb{R}^3}(\sum_{k,j=1}^{3}\sum_{i=1}^{2}u_j\partial_j u_i\partial_k\partial_k u_i + \sum_{k=1}^{3}\sum_{j=1}^{2}u_j\partial_j u_3\partial_k\partial_k u_3 - \frac{1}{2}\sum_{k=1}^{3}\partial_3 u_3\partial_k u_3\partial_k u_3)\mathrm{d}x \\
=&\int_{\mathbb{R}^3}[(\sum_{k,j=1}^{3}\sum_{i=1}^{2}u_j\partial_j u_i\partial_k\partial_k u_i + \sum_{k=1}^{3}\sum_{j=1}^{2}u_j\partial_j u_3\partial_k\partial_k u_3 + \frac{1}{2}\sum_{k=1}^{3}\partial_k u_3(\partial_1 u_1 + \partial_2 u_2)\partial_k u_3]\mathrm{d}x \\
=&\int_{\mathbb{R}^3}\{\sum_{k,j=1}^{3}\sum_{i=1}^{2}u_i\partial_k(\partial_k u_j\partial_j u_i) + \sum_{k=1}^{3}\sum_{j=1}^{2}[-u_j\partial_j(\frac{\partial_k u_3\partial_k u_3}{2}) + u_j\partial_k(\partial_j u_3\partial_k u_3)] \\
&-\frac{1}{2}\sum_{k=1}^{3}[u_1\partial_1(\partial_k u_3\partial_3 u_3) + u_2\partial_2(\partial_k u_3\partial_3 u_3)]\}\mathrm{d}x \\
\le& C\int_{\mathbb{R}^3}|u_h||\nabla(\nabla u\nabla u)|\mathrm{d}x.
\end{aligned}$$

With the inequality (2.5), the Sobolev embedding $\dot{B}^0_{\infty,2} \hookrightarrow BMO$ and the inequality (2.4) , we have

$$\begin{aligned}
I_1 &\le C\|u_h\|_{BMO}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\\
&\le C\|u_h\|_{BMO}^2\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\Delta u\|_{L^2}^2\\
&\le C\|u_h\|_{\dot{B}^0_{\infty,2}}^2\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\Delta u\|_{L^2}^2\\
&\le C[1+\|u_h\|_{\dot{B}^0_{\infty,\infty}}^2\log(1+\|\Delta u\|_{L^2})]\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\Delta u\|_{L^2}^2.
\end{aligned}\tag{3.5}$$

Adding I_2 and I_3 together, it follows by the divergence free condition $\nabla\cdot u=0$

$$\begin{aligned}
I_2+I_3 &= \int_{\mathbb{R}^3}\sum_{i,j,k=1}^{3}[(\partial_i\partial_j d_k\partial_j d_k+\partial_i d_k\partial_j\partial_j d_k)\Delta u_i-(\Delta u_i\partial_i d_k\Delta d_k\\
&\quad+2\nabla u_i\partial_i\nabla d_k\Delta d_k+u_i\partial_i\Delta d_k\Delta d_k)]\mathrm{d}x\\
&= \int_{\mathbb{R}^3}\sum_{i,j,k=1}^{3}-2\nabla u_i\partial_i\nabla d_k\Delta d_k\mathrm{d}x\\
&\le C\int_{\mathbb{R}^3}|\nabla u||\nabla\nabla d||\Delta d|\mathrm{d}x.
\end{aligned}$$

Hence, it can be deduced from inequality (2.1) that

$$\begin{aligned}
I_2+I_3 &\le C\|\nabla u\|_{L^2}\|\Delta d\|_{L^4}^2\\
&\le C\|\nabla u\|_{L^2}\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\|\nabla\Delta d\|_{L^2}\\
&\le C\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\nabla\Delta d\|_{L^2}^2.
\end{aligned}\tag{3.6}$$

For I_4, by the product estimate (2.2) and inequality (2.1), we obtain

$$\begin{aligned}
I_4 &= \int_{\mathbb{R}^3}\Delta(|\nabla d|^2 d)\cdot\Delta d\mathrm{d}x\\
&\le \|\Delta(|\nabla d|^2 d)\|_{L^{\frac{4}{3}}}\|\Delta d\|_{L^4}\\
&\le C(\|\nabla\Delta d\|_{L^2}\|\nabla d\|_{L^4}\|d\|_{L^\infty}+\|\Delta d\|_{L^4}\|\nabla d\|_{L^4}^2)\|\Delta d\|_{L^4}\\
&\le C\|\Delta d\|_{L^4}^2\|\nabla d\|_{L^4}^2+\frac{1}{8}\|\nabla\Delta d\|_{L^2}^2\\
&\le C\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\|\nabla\Delta d\|_{L^2}\|d\|_{L^\infty}\|\Delta d\|_{L^2}+\frac{1}{8}\|\nabla\Delta d\|_{L^2}^2\\
&\le C\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2\|\Delta d\|_{L^2}^2+\frac{1}{4}\|\nabla\Delta d\|_{L^2}^2.
\end{aligned}\tag{3.7}$$

Combining (3.4)–(3.7), one has

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2)+\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2\\
\le\ & C(1+\|u_h\|_{\dot{B}^0_{\infty,\infty}}^2\log(1+\|\Delta u\|_{L^2})+\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2)(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2).
\end{aligned}\tag{3.8}$$

Noting (1.4), one concludes that for any small constant $\epsilon > 0$, there exists $T_0 < T$ such that

$$\int_{T_0}^{T} \|u_h\|^2_{\dot{B}^0_{\infty,\infty}} \mathrm{d}t < \epsilon. \tag{3.9}$$

For any $T_0 \leq t < T$, we set

$$M(t) = \sup_{T_0 \leq s \leq t} (\|\Delta u(s)\|^2_{L^2} + \|\nabla \Delta d(s)\|^2_{L^2}). \tag{3.10}$$

Employing Gronwall inequality for (3.8) in the interval $[T_0, t]$ and using (3.9), (3.10) yields

$$\begin{aligned}
&\|\nabla u(t)\|^2_{L^2} + \|\Delta d(t)\|^2_{L^2} + \int_{T_0}^{t} (\|\Delta u(s)\|^2_{L^2} + \|\nabla \Delta d(s)\|^2_{L^2}) \mathrm{d}s \\
\leq &(\|\nabla u(T_0)\|^2_{L^2} + \|\Delta d(T_0)\|^2_{L^2}) \exp\left\{ C \int_{T_0}^{t} (1 + \|u_h\|^2_{\dot{B}^0_{\infty,\infty}} \log(1 + \|\Delta u\|_{L^2}) + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}) \mathrm{d}s \right\} \\
\leq &C_0 C_1 \exp\left\{ C \int_{T_0}^{t} \|u_h\|^2_{\dot{B}^0_{\infty,\infty}} \log(1 + \|\Delta u\|^2_{L^2} + \|\nabla \Delta d\|^2_{L^2}) \mathrm{d}s \right\} \\
\leq &C_0 C_1 \exp\{ C\epsilon \log(1 + M(t)) \} \\
\leq &C_0 C_1 (1 + M(t))^{C\epsilon},
\end{aligned} \tag{3.11}$$

where the letter C_0 means a constant depending on $(\|\nabla u(T_0)\|^2_{L^2} + \|\Delta d(T_0)\|^2_{L^2})$, C_1 depends on $\exp\left\{ C \int_{T_0}^{t} (1 + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}) \mathrm{d}s \right\}$, and C is a generic constant which may be different from line to line.

Then we need to bound the norm $\|\Delta u\|_{L^2}$ and $\|\nabla \Delta d\|_{L^2}$ so as to confirm the validness of inequality (3.2). Applying Δ and $\nabla \Delta$ to Eqs $(1.1)_1$ and $(1.1)_2$ respectively, and taking the L^2 inner product with $(\Delta u, \nabla \Delta d)$, we obtain that

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\|\Delta u\|^2_{L^2} + \|\nabla \Delta d\|^2_{L^2}) + \|\nabla \Delta u\|^2_{L^2} + \|\Delta^2 d\|^2_{L^2} \\
= &- \int_{\mathbb{R}^3} \Delta(u \cdot \nabla u) \cdot \Delta u \mathrm{d}x - \int_{\mathbb{R}^3} \Delta(\nabla d_j \cdot \Delta d_j) \cdot \Delta u \mathrm{d}x \\
&- \int_{\mathbb{R}^3} \nabla \Delta(u \cdot \nabla d) \cdot \nabla \Delta d \mathrm{d}x - \int_{\mathbb{R}^3} \nabla \Delta(|\nabla d|^2 d) \cdot \nabla \Delta d \mathrm{d}x \\
:= &J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.12}$$

According to $\nabla \cdot u = 0$, Hölder inequality, the commutator estimate (2.3), interpolation inequality and Young inequality, J_1 can be estimated by

$$\begin{aligned}
J_1 &= -\int_{\mathbb{R}^3} [\Delta, u \cdot \nabla] u \cdot \Delta u \mathrm{d}x \\
&\leq \|[\Delta, u \cdot \nabla] u\|_{L^{\frac{4}{3}}} \|\Delta u\|_{L^4} \\
&\leq C(\|\nabla u\|_{L^2} \|\Delta u\|_{L^4} + \|\Delta u\|_{L^4} \|\nabla u\|_{L^2}) \|\Delta u\|_{L^4} \\
&\leq C \|\nabla u\|_{L^2} \|\Delta u\|^2_{L^4} \leq C \|\nabla u\|_{L^2} \|\nabla u\|^{\frac{1}{4}}_{L^2} \|\nabla \Delta u\|^{\frac{7}{4}}_{L^2} \\
&\leq C \|\nabla u\|^{10}_{L^2} + \frac{1}{6} \|\nabla \Delta u\|^2_{L^2}.
\end{aligned} \tag{3.13}$$

By the product estimate (2.2), we have

$$
\begin{aligned}
J_2 &= \int_{\mathbb{R}^3} \partial_i(\nabla d_j \Delta d_j)\cdot \partial_i \Delta u \mathrm{d}x \\
&\leq \|\nabla(\nabla d\Delta d)\|_{L^2}\|\nabla\Delta u\|_{L^2} \\
&\leq (\|\nabla d\|_{L^4}\|\nabla\Delta d\|_{L^4} + \|\Delta d\|_{L^4}\|\Delta d\|_{L^4})\|\nabla\Delta u\|_{L^2} \\
&\leq C\|\nabla d\|_{L^4}^2\|\nabla\Delta d\|_{L^4}^2 + C\|\Delta d\|_{L^4}^4 + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2 \\
&\leq C\|d\|_{L^\infty}\|\Delta d\|_{L^2}\|\Delta d\|_{L^2}^{\frac{1}{4}}\|\Delta\Delta d\|_{L^2}^{\frac{7}{4}} + C\|\Delta d\|_{L^2}^{\frac{5}{2}}\|\Delta\Delta d\|_{L^2}^{\frac{3}{2}} + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2 \\
&\leq C\|\Delta d\|_{L^2}^{\frac{5}{4}}\|\Delta^2 d\|_{L^2}^{\frac{7}{4}} + C\|\Delta d\|_{L^2}^{\frac{5}{2}}\|\Delta^2 d\|_{L^2}^{\frac{3}{2}} + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2 \\
&\leq C\|\Delta d\|_{L^2}^{10} + \frac{1}{6}\|\Delta^2 d\|_{L^2}^2 + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2.
\end{aligned} \tag{3.14}
$$

Similar as (3.13), one may conclude

$$
\begin{aligned}
J_3 &= -\int_{\mathbb{R}^3} [\nabla\Delta, u\cdot\nabla]d\cdot\nabla\Delta d\mathrm{d}x \\
&\leq \|[\nabla\Delta, u\cdot\nabla]d\|_{L^{\frac{4}{3}}}\|\nabla\Delta d\|_{L^4} \\
&\leq (\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^4} + \|\nabla d\|_{L^4}\|\nabla\Delta u\|_{L^2})\|\nabla\Delta d\|_{L^4} \\
&\leq C\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^4}^2 + C\|\nabla d\|_{L^4}^2\|\nabla\Delta d\|_{L^4}^2 + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2 \\
&\leq C\|\nabla u\|_{L^2}\|\Delta d\|_{L^2}^{\frac{1}{4}}\|\Delta\Delta d\|_{L^2}^{\frac{7}{4}} + C\|d\|_{L^\infty}\|\Delta d\|_{L^2}\|\Delta d\|_{L^2}^{\frac{1}{4}}\|\Delta\Delta d\|_{L^2}^{\frac{7}{4}} + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2 \\
&\leq C\|\nabla u\|_{L^2}^8\|\Delta d\|_{L^2}^2 + C\|\Delta d\|_{L^2}^{10} + \frac{1}{6}\|\Delta^2 d\|_{L^2}^2 + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2.
\end{aligned} \tag{3.15}
$$

By the product estimate (2.2) and the fact $|\nabla d|^2 = -d\cdot\Delta d$, we infer that

$$
J_4 = -\int_{\mathbb{R}^3} \nabla\Delta(|\nabla d|^2 d)\cdot\nabla\Delta d\mathrm{d}x \tag{3.16}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} \Delta(|\nabla d|^2 d)\cdot(\nabla\nabla\Delta d)\mathrm{d}x \\
&\leq \|\Delta(|\nabla d|^2 d)\|_{L^2}\|\nabla\nabla\Delta d\|_{L^2} \\
&\leq C(\|\Delta(|\nabla d|^2)d\|_{L^2} + \||\nabla d|^2\Delta d\|_{L^2})\|\Delta^2 d\|_{L^2} \\
&\leq C(\|\nabla d\|_{L^4}\|\nabla\Delta d\|_{L^4}\|d\|_{L^\infty} + \|d\cdot\Delta d\Delta d\|_{L^2})\|\Delta^2 d\|_{L^2} \\
&\leq C(\|\nabla d\|_{L^4}\|\nabla\Delta d\|_{L^4} + \|\Delta d\|_{L^4}^2)\|\Delta^2 d\|_{L^2} \\
&\leq C\|d\|_{L^\infty}^{\frac{1}{2}}\|\Delta d\|_{L^2}^{\frac{1}{2}}\|\Delta d\|_{L^2}^{\frac{1}{8}}\|\Delta^2 d\|_{L^2}^{\frac{7}{8}}\|\Delta^2 d\|_{L^2} + C\|\Delta d\|_{L^2}^{\frac{5}{4}}\|\Delta^2 d\|_{L^2}^{\frac{3}{4}}\|\Delta^2 d\|_{L^2} \\
&\leq C\|\Delta d\|_{L^2}^{10} + \frac{1}{6}\|\Delta^2 d\|_{L^2}^2.
\end{aligned} \tag{3.17}
$$

Inserting the above estimates (3.13)–(3.16) to (3.12), and combining (3.11), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(1+\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2)+\|\nabla\Delta u\|_{L^2}^2+\|\Delta^2 d\|_{L^2}^2\\ \leq\ & C(\|\nabla u\|_{L^2}^{10}+\|\Delta d\|_{L^2}^{10}+\|\nabla u\|_{L^2}^{8}\|\Delta d\|_{L^2}^{2})\\ \leq\ & CC_0C_1(1+M(t))^{5C\epsilon}.\end{aligned}$$

Integrating the above inequality with respect to time from T_0 to t, $T_0 \leq t < T$, it follows that

$$\begin{aligned}&(1+\|\Delta u(t)\|_{L^2}^2+\|\nabla\Delta d(t)\|_{L^2}^2)+\int_{T_0}^{t}(\|\nabla\Delta u\|_{L^2}^2+\|\Delta^2 d\|_{L^2}^2)\mathrm{d}\tau\\ \leq\ & 1+\|\Delta u(T_0)\|_{L^2}^2+\|\nabla\Delta d(T_0)\|_{L^2}^2+\int_{T_0}^{t}CC_0C_1(1+M(\tau))^{5C\epsilon}\mathrm{d}\tau\\ =\ & 1+\|\Delta u(T_0)\|_{L^2}^2+\|\nabla\Delta d(T_0)\|_{L^2}^2+\int_{T_0}^{t}CC_0C_1(1+M(\tau))\mathrm{d}\tau\end{aligned}$$

by choosing $\epsilon = \frac{1}{5C}$. And we can derive from the above inequality and (3.10) that

$$\begin{aligned}&(1+M(t))+\int_{T_0}^{t}(\|\nabla\Delta u\|_{L^2}^2+\|\Delta^2 d\|_{L^2}^2)\mathrm{d}\tau\\ \leq\ & 1+\|\Delta u(T_0)\|_{L^2}^2+\|\nabla\Delta d(T_0)\|_{L^2}^2+\int_{T_0}^{t}CC_0C_1(1+M(\tau))\mathrm{d}\tau.\end{aligned}$$

Gronwall's inequality implies

$$\begin{aligned}&(1+M(t))+\int_{T_0}^{t}(\|\nabla\Delta u\|_{L^2}^2+\|\Delta^2 d\|_{L^2}^2)\mathrm{d}\tau\\ \leq\ & (1+\|\Delta u(T_0)\|_{L^2}^2+\|\nabla\Delta d(T_0)\|_{L^2}^2)\exp\{CC_0C_1(T-T_0)\}.\end{aligned}$$

The proof of Theorem 1.1. is thus completed.

4. Conclusions

In this paper, we establish a regularity criterion for the 3D nematic liquid crystal flows via velocity component u_h and orientation field ∇d in Besov space. Furthermore, there is a lack of references about regularity criterion via component of orientation field d for the system (1.1) and we hope to weaken the condition (1.4) by the components of both u and d in more general spaces in future study.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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