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Research article

Local Pre-Hausdorffness and D-connectedness in $\mathcal{L}$-valued closure spaces

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Abstract: Previously, several characterization of local Pre-Hausdorffness and D-connectedness have been examined in distinct topological categories. In this paper, we give the characterization of local T_0 (resp. local T_1) $\mathcal{L}$-valued closure spaces, examine how their mutual relationship. Furthermore, we give the characterization of a closed point and D-connectedness in $\mathcal{L}$-valued closure spaces and examine their relations with local T_0 and local T_1 objects. Finally, we examine the characterization of local Pre-Hausdorff and local Hausdorff $\mathcal{L}$-valued closure spaces and study their relationship with generic Hausdorff objects and D-connectedness.

Keywords: $\mathcal{L}$-valued closure space; $\mathcal{L}$- topological space; topological category; local separation; D-connected
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1. Introduction

Closure operators play a significant influence not just in mathematics, such as algebra [35], logic [25], calculus [32], and topology [19, 27], but also in physics, such as representation theory of physical systems and quantum logic [1, 2]. G.Birkhoff [5] discovered that a complete lattice is a class of all closed sets of closure space in the year 1940. Their relationships became key concerns for mathematicians [23] after that. Moreover, G. Aumann [3] also looked into the closure structures on contact relations which have applications in social science.

Due to the widely recognized usefulness of closure space in research, it has been generalized by introducing some suitable quantales on closure structure [29, 30, 33, 40].

Several generalizations of the classical separation axioms at some point p (locally) have been inspected in [6] by Baran where the primary purpose of this generalization was to interpret the notion of closed sets and strongly closed sets in the arbitrary set based topological categories. He also showed that these notions of closedness induce closure operators in the sense of Guili and

Dikranjan [21] in some well-known topological categories (see [13, 17, 22, 38]). In addition, Baran [6, 16] introduced local pre-Hausdorff objects in an arbitrary topological category which are reduced to local pre-Hausdorff topological space (Y, τ). The most important use of these local pre-Hausdorff objects is to define various forms of local Hausdorff objects [8], local T_3 and T_4 objects [11], regular, completely regular, normal objects [12] and the notion of compactness [10] and connectedness [15], Soberness [18] in Categorical Topology, and these notions have been studies in several topological categories (see [14, 28, 36]).

The main objectives of this paper are stated as under:

(i) to characterize local T_0 and local T_1 objects in $\mathcal{L}$-valued Closure Spaces and examine their mutual relationship;
(ii) to examine the characterization of the notion of closedness and D-connectedness in $\mathcal{L}$-valued Closure Spaces, and to show their relation with local T_0 and local T_1 objects;
(iii) to give the characterization of local Pre-Hausdorff (resp. Hausdorff) objects in $\mathcal{L}$-valued Closure Spaces, and to examine relationship among local Hausdorff (resp. Hausdorff) $\mathcal{L}$-valued Closure Spaces defined in [37] and D-connected $\mathcal{L}$-valued Closure Spaces.

2. Preliminaries

In this paper, let $\mathcal{L} = (L, \otimes, \lambda)$ be a quantale (unital, but not necessarily a commutative quantale), i.e., a complete lattice with a monoid structure and "$\otimes$" is binary operation satisfies the followings: for all $\psi_i, \eta \in L$, $\vee_{i\in I}(\psi_i \otimes \eta) = (\vee_{i\in I}\psi_i) \otimes \eta$ and $\vee_{i\in I}(\eta \otimes \psi_i) = \eta \otimes (\vee_{i\in I}\psi_i)$, where λ is an identity (neutral) element.

The quantale $\mathcal{L}$ is called an *integral* quantale if the identity element $\lambda = \top$, where $\top$ is the greatest element in L.

In a quantale $(L, \otimes, \lambda)$, if $s \in L$ and $s \neq \top$, then s is called the prime element if $y \wedge x \leq s$ implies $y \leq s$ or $x \leq s$ for all $y, x \in L$.

Let Y be a nonempty set, PY denotes the power set of Y and $\mathcal{L}^Y$ denotes the set of all mappings from Y to $\mathcal{L}$.

Definition 2.1. *(cf. [30]) An $\mathcal{L}$-valued closure structure on set Y is a mapping $C : PY \longrightarrow \mathcal{L}^Y$ satisfying*

(i) $\forall y \in A \subseteq Y : \lambda \leq (CA)(y)$ (Reflexivity),
(ii) $\forall A, B \subseteq Y, y \in Y$: $(\bigwedge_{x\in B}(CA)(x)) \otimes (CB)(y) \leq (CA)(y)$ (Transitivity).

The pair (Y, C) is called an $\mathcal{L}$-valued closure space.

Definition 2.2. *(cf. [30]) An $\mathcal{L}$-valued topological structure on set Y is a mapping $C : PY \longrightarrow \mathcal{L}^Y$ satisfying*

(i) C is an $\mathcal{L}$-valued closure structure on Y,
(ii) For all $y \in Y$ and $\emptyset$, the empty set: $(C\emptyset)(y) = \bot$,
(iii) For all $y \in Y$ and $\forall A, B \subseteq Y$: $C(A \cup B)(y) = (CA)(y) \vee (CB)(y)$.

The pair (Y, C) is called an $\mathcal{L}$-valued topological space.
A mapping $f : (Y, C) \longrightarrow (X, D)$ is called continuous if $(CA)(y) \leq D(fA)(fx)$ for all $A \subseteq Y$ and $y \in Y$.
*Let **$\mathcal{L}$-Cls** (resp.**$\mathcal{L}$-Top**) denotes the category with $\mathcal{L}$-valued closure spaces (resp. $\mathcal{L}$-valued topological*

*spaces) as objects and contractive mappings as morphisms. Note that **$\mathcal{L}$-Top** is the full subcategory of **$\mathcal{L}$-Cls** [30].*

Example 2.1. *(i) The quantale $\mathcal{L} = ([0,\infty], \geq, +, 0)$ is called Lawvere's quantale [24], then category of $\mathcal{L}$-valued topological spaces is equivalent to approach spaces (**App** denotes the category of approach spaces and morphisms are contraction mappings) [31] i.e., **$\mathcal{L}$-Top** $\cong$ **App** . Moreover, we have **$\mathcal{L}$-Cls** $\cong$ **Cls**$'$, where **Cls**$'$ is the category considered in [39].*

*(ii) For terminal quantale **1**, **Set** $\cong$ **1-Cls** $\cong$ **1-Top** [30].*

*(iii) Consider $\mathcal{L} = (2, \wedge, \top)$, where $2 = \{\bot < \top\}$, then **2-Cls** $\cong$ **Cls** and **2-Top** $\cong$ **Top** [30], where **Top** is the category of topological spaces and continuous mappings, and **Cls** is the category of closure spaces and continuous mappings [20].*

*(iv) Consider the quantale $p_{\&} = (p, \otimes, \lambda)$ of all distance distribution functions $\psi : [0,\infty] \to [0,1]$ that satisfy $\psi(\pi_2) = \sup_{\pi_1 < \pi_2} \psi(\pi_1)$ for all $\pi_2 \in [0,\infty]$ with $(\psi \otimes \xi)(\gamma) = \sup_{\pi_1+\pi_2<\gamma} \psi(\pi_1)\&\xi(\pi_2)$, where & is Lukasiewicz t-norm on $[0,1]$ defined by $\pi_1\&\pi_2 = \min\{\pi_1, \pi_2\}$. The $\otimes$-neutral function λ satisfies $\lambda(0) = 0$ and $\lambda(\pi_1) = 1$ for all $\pi_1 > 0$. Then, $p_{\&}$-**Top** $\cong$ **ProbApp**$_{\&}$ [29, 30], where **ProbApp**$_{\&}$ is the category of probabilistic approach spaces and contraction mappings defined in [26].*

Recall, [4, 34], a functor $\mathcal{F} : C \to$ **Set** (the category of sets and functions) is called topological if (i) $\mathcal{F}$ is concrete (i.e., faithful and amnestic) (ii) $\mathcal{F}$ consists of small fibers and (iii) every $\mathcal{F}$-source has a unique initial lift, i.e., if for every source $(f_i : X \to (X_i, \zeta_i))_{i\in I}$ there exists a unique structure ζ on X such that $g : (Y, \eta) \to (X, \zeta)$ is a morphism iff for each $i \in I$, $f_i \circ g : (Y, \eta) \to (X_i, \zeta_i)$ is a morphism. Moreover, a topological functor is called a discrete (resp. indiscrete) if it has a left (resp. right) adjoint.

Lemma 2.1. *(cf. [30]) Let $\mathcal{L}$ be a quantale, (Y_i, C_i) be a collection of $\mathcal{L}$-valued closure spaces and $f_i : Y \longrightarrow (Y_i, C_i)$ be a source. Then, for all $y \in Y$ and $A \subseteq Y$,*

$$(CA)(y) \quad = \quad \bigwedge_{i\in I} C_i(f_iA)(f_iy)$$

is an initial structure on Y.

Lemma 2.2. *(cf. [30]) Let Y be a non-empty set and (Y, C) be an $\mathcal{L}$-valued closure space. For all $y \in Y$, $A \subseteq Y$,*

(i) the discrete $\mathcal{L}$-valued closure structure on Y is given by

$$(C_{dis}A)(y) = \begin{cases} \lambda, & y \in A, \\ \bot, & y \notin A. \end{cases}$$

(ii) the indiscrete $\mathcal{L}$-valued closure structure on Y is given by $(C_{ind}A)(y) = \top$.

Note that for a quantale $\mathcal{L}$, the category **$\mathcal{L}$-Cls** is a topological category over **Set** [30].

3. Local T_0 and local T_1 $\mathcal{L}$-valued closure spaces

Let Y be a non-empty set and the wedge product $Y \vee_p Y$ be two copies of Y which are identified at the point p. That is to say, the pushout of $p : Y \to Y^2$ along itself. More precisely, if i_1 and

$i_2 : Y \to Y \vee_p Y$ denote the inclusion of Y as the first and second factor, respectively, then $i_1 p = i_2 p$ is the pushout diagram [6].

A point y in $Y \vee_p Y$ is denoted by y_1 (resp. y_2) if it is in the first (resp. second) component.

Definition 3.1. *(cf. [6]) A mapping $A_p : Y \vee_p Y \longrightarrow Y^2$ is called principal at p-axis mapping satisfying*

$$A_p(y_i) = \begin{cases} (y, p), & i = 1, \\ (p, y), & i = 2. \end{cases}$$

Definition 3.2. *(cf. [6]) A mapping $S_p : Y \vee_p Y \longrightarrow Y^2$ is called skewed p-axis mapping satisfying*

$$S_p(y_i) = \begin{cases} (y, y), & i = 1, \\ (p, y), & i = 2. \end{cases}$$

Definition 3.3. *(cf. [6]) A mapping $\nabla_p : Y \vee_p Y \longrightarrow Y$ is called folding mapping at p satisfying $\nabla_p(y_i) = y$ for $i = 1, 2$.*

Definition 3.4. *Let $\mathcal{F} : \mathcal{C} \longrightarrow$* **Set** *be topological, and $Y \in Obj(\mathcal{C})$ with $\mathcal{F}(Y) = X$ and $p \in X$.*

(i) Y is T_0 at p or local T_0 iff the initial lift of the $\mathcal{F}$-source $\{A_p : X \vee_p X \longrightarrow \mathcal{F}(Y^2) = X^2$ and $\nabla_p : X \vee_p X \longrightarrow \mathcal{F}D(X) = X\}$ is discrete, where D is the discrete functor [6].

(ii) Y is T_1 at p or local T_1 iff the initial lift of the $\mathcal{F}$-source $\{S_p : X \vee_p X \longrightarrow \mathcal{F}(Y^2) = X^2$ and $\nabla_p : X \vee_p X \longrightarrow \mathcal{F}D(X) = X\}$ is discrete [6].

Remark 3.1. *In* **Top** *(the category of topological spaces and continuous mappings), an object Y, i.e., $Y \in Obj($***Top***$)$ is local T_0 (resp. local T_1) in (classical sense) iff Y is local T_0 (resp. local T_1) [9].*

Theorem 3.1. *Let (Y, C) be an $\mathcal{L}$-valued closure space and $p \in Y$. (Y, C) is local T_0 iff $\forall y \in Y$ with $y \neq p$, there exists $U \subseteq Y$ with $y \in U$, $p \notin U$ or there exists $V \subseteq Y$ with $p \in V$, $y \notin V$ such that $\perp = \bigwedge\{C(U)(p), C(V)(y), \lambda\}$, where λ is an identity element.*

Proof. Suppose (Y, C) is local T_0 and for all $y \in Y$ with $y \neq p$. Let $B \subseteq Y \vee_p Y$ and $y_1 \in Y \vee_p Y$ with $y_1 \notin B$, and $proj_i : Y^2 \longrightarrow Y$, $i = 1, 2$ are projection maps. Note that

$$\begin{aligned} C_{dis}(\nabla_p B)(\nabla_p y_1) =& C_{dis}(\nabla_p B)(y) = \lambda, \\ \lambda \leq C(proj_1 A_p B)(proj_1 A_p y_1) =& C(proj_1 A_p B)(y) = C(V)(y), \end{aligned}$$

since y $\in proj_1 A_p B$,

$$C(proj_2 A_p B)(proj_2 A_p y_1) = C(proj_2 A_p B)(p) = C(U)(p).$$

Since $y_1 \notin B$ and (Y, C) is local T_0, by Lemma 2.1,

$$\begin{aligned} C(B)(y_1) = & \bigwedge\{C(proj_1 A_p B)(proj_1 A_p y_1), C(proj_2 A_p B)(proj_2 A_p y_1), \\ & C_{dis}(\nabla_p B)(\nabla_p y_1)\}. \\ = & \bigwedge\{C(proj_1 A_p B)(y), C(proj_2 A_p B)(p), C_{dis}(\nabla_p B)(y)\}, \end{aligned}$$

$$= \bigwedge\{C(V)(y), C(U)(p), \lambda\}.$$

Since (Y, C) is local T_0, it follows that $\bigwedge\{C(V)(y), C(U)(p), \lambda\} = \bot$.

Conversely, let $\overline{C}$ be an initial structure induced by $A_p : Y \vee_p Y \longrightarrow (Y^2, C^2)$ and ∇_p: $Y \vee_p Y \longrightarrow (Y, C_{dis})$, where C^2 is a product structure on Y^2 and $proj_i : Y^2 \longrightarrow Y$, $i = 1, 2$ are projection maps, and C_{dis} is a discrete structure on Y.

Suppose $w \in Y \vee_p Y$ and B is a non empty subset of $Y \vee_p Y$. We have the following cases.
Case I: If $\nabla_p w = p \in \nabla_p B$ for some $p \in Y$, then $w = p_1 = p_2 \in B$, it follows from Lemma 2.1, $(\overline{C}B)(w) = \lambda$.
Case II: If $\nabla_p w = p \notin \nabla_p B$, by Lemma 2.2, $(C_{dis}\nabla_p B)(\nabla_p w) = \bot$ and consequently,

$$(\overline{C}B)(w) = \bigwedge\{(C(proj_1 A_p B))(proj_1 A_p w), (C(proj_2 A_p B))(proj_2 A_p w), (C_{dis}(\nabla_p B))(\nabla_p w)\} = \bot.$$

Case III: Suppose $\nabla_p w = y$ for some $y \in Y$ with $y \neq p$ and it follows that $w = y_i$ for $i = 1, 2$.

(i) If $w = y_1 = y_2 \in B$, then $\nabla_p w \in \nabla_p B$ and $proj_i A_p w \in proj_i A_p B$ for $i = 1, 2$, by Lemma 2.1, $(\overline{C}B)(w) = \bigwedge\{(C(proj_i A_p B))(proj_i A_p w), (C_{dis}(\nabla_p A))(\nabla_p w)\} = \lambda$.

(ii) If $w = y_1, y_2 \notin B$, then $\nabla_p w \notin \nabla_p B$ and it follows by Lemma 2.1, $(\overline{C}B)(w) = \bot$.

(iii) Suppose that $w = y_1 \notin B$ but $y_2 \in B$, by Lemma 2.2

$$(C_{dis}\nabla_p B)(\nabla_p w) = \lambda.$$

and

$$C(proj_1 A_p B))(proj_1 A_p w) = C(proj_1 A_p B)(p),$$

$$C(proj_2 A_p B))(proj_2 A_p w) = C(proj_2 A_p B)(y).$$

By Lemma 2.2, it follows that

$$(\overline{C}B)(w) = \bigwedge\{C(proj_1 A_p B)(p), C(proj_2 A_p B)(y), C_{dis}(\nabla_p B)(\nabla_p w)\},$$
$$= \bigwedge\{C(U)(p), C(V)(y), \lambda\} = \bot.$$

Hence, for all $w \in Y \vee_p Y$ and $B \subseteq Y \vee_p Y$, we have

$$(\overline{C}B)(w) = \begin{cases} \lambda, & w \in B, \\ \bot, & w \notin B. \end{cases}$$

By Lemma 2.2 (i), $\overline{C}$ is an $\mathcal{L}$-valued discrete structure on $Y \vee_p Y$. Thus, (Y, C) is local T_0. □

Corollary 3.1. *Let (Y, C) be an $\mathcal{L}$-valued closure space and $p \in Y$, where $\mathcal{L}$ is an integral quantale and $\mathcal{L}$ has a prime bottom element. (Y, C) is local T_o iff $\forall y \in Y$ with $y \neq p$, there exists $U \subseteq Y$ with $y \in U$, $p \notin U$ or there exists $V \subseteq Y$ with $p \in V$, $y \notin V$ such that $C(U)(p) = \bot$ or $C(V)(y) = \bot$.*

Proof. It follows from definitions of prime bottom element, integral quantales and Theorem 3.1. □

Theorem 3.2. *Let (Y, C) be an $\mathcal{L}$-valued closure space and $p \in Y$. (Y, C) is local T_1 iff $\forall y \in Y$ with $y \neq p$, there exists $U \subseteq Y$ with $y \in U$, $p \notin U$ and there exists $V \subseteq Y$ with $p \in V$, $y \notin V$ such that $C(U)(p) \wedge \lambda = \perp = C(V)(y) \wedge \lambda$, where λ is an identity element.*

Proof. Suppose (Y, C) is local T_1 and $\forall\, y \in Y$ with $y \neq p$. Let $B \subseteq Y \vee_p Y$ and $y_1 \in Y \vee_p Y$ with $y_1 \notin B$. Note that

$$C_{dis}(\nabla_p B)(\nabla_p y_1) = C_{dis}(\nabla_p B)(y) = \lambda,$$

$$\lambda \leq C(proj_1 S_p B)(proj_1 S_p y_1) = C(proj_1 S_p B)(y) = C(V)(y),$$

since y $\in proj_1 S_p B$,

$$C(proj_2 S_p B)(proj_2 S_p y_1) = C(proj_2 S_p B)(y).$$

Since $y_1 \notin B$ and (Y, C) is local T_1, by Lemma 2.1,

$$\begin{aligned} C(B)(y_1) = & \bigwedge\{C(proj_1 S_p B)(proj_1 S_p y_1), C(proj_2 S_p B)(proj_2 S_p y_1), \\ & C_{dis}(\nabla_p B)(\nabla_p y_1)\}, \\ = & \bigwedge\{C(proj_1 S_p B)(y), (C(proj_2 S_p B)(y), \lambda\}, \\ = & \bigwedge\{C(V)(y), \lambda\}, \end{aligned}$$

and by assumption $C(B)(y_1) = \perp$ and consequently, $C(V)(y) \wedge \lambda = \perp$.

Similarly, suppose $B \subseteq Y \vee_p Y$ and $y_2 \in Y \vee_p Y$ with $y_2 \notin B$, then we have

$$\perp = \bigwedge\{C(U)(p), \lambda\},$$

and consequently, $C(U)(p) \wedge \lambda = \perp$.

Conversely, let $\overline{C}$ be an initial structure induced by $S_p : Y \vee_p Y \longrightarrow (Y^2, C^2)$ and $\nabla_p : Y \vee_p Y \longrightarrow$ (Y,C_{dis}), where C^2 is a product structure on Y^2 and $proj_i : Y^2 \longrightarrow Y$, $i = 1, 2$ are projection maps and C_{dis} is a discrete structure on Y and $w \in Y \vee_p Y$. We have the following cases.

Case I: If $\nabla_p w = p \in \nabla_p B$, then $w = p_1 = p_2 \in B$, it follows from Lemma 2.1, $(\overline{C}B)(w) = \lambda$.

Case II: If $\nabla_p w = p \notin \nabla_p B$, by Lemma 2.2

$$(C_{dis} \nabla_p B)(\nabla_p w) = \perp,$$

and consequently,

$$(\overline{C}B)(w) = \bigwedge\{C(proj_1 S_p B)(proj_1 S_p w), C(proj_2 S_p B)(proj_2 S_p w), C_{dis}(\nabla_p B)(\nabla_p w)\} = \perp.$$

Case III: If $\nabla_p w = y$ for some $y \in Y$ with $y \neq p$, it follows that, $w = y_1$ or $w = y_2$.

(i) If $w = y_i \in B$ for $i = 1, 2$, then $\nabla_p w \in \nabla_p B$ and $proj_i S_p w \in proj_i S_p B$, by Lemma 2.1,

$$(\overline{C}B)(w) = \bigwedge\{C(proj_i S_p B)(proj_i S_p w), C_{dis}(\nabla_p B)(\nabla_p w)\} = \lambda.$$

(ii) If $w = y_i \notin B$ for $i = 1, 2$, then $\nabla_p w \notin \nabla_p B$, by Lemma 2.2,

$$C_{dis}(\nabla_p B)(\nabla_p w) = C_{dis}(\nabla_p B)(y) = \perp,$$

and consequently, $(\overline{C}B)(w) = \perp$.

(iii) Suppose $w = y_1 \notin B$ but $y_2 \in B$, by Lemma 2.2

$$C_{dis}(\nabla_p B)(\nabla_p w) = C_{dis}(\nabla_p B)(y) = \lambda$$

and

$$C(proj_1 S_p B)(proj_1 S_p w) = C(proj_1 S_p B)(y) = C(V)(y),$$
$$C(proj_2 S_p B)(proj_2 S_p w) = C(proj_2 S_p B)(p) = C(U)(p).$$

By Lemma 2.1,

$$(\overline{C}B)(w) = \bigwedge\{C(proj_i S_p B)(proj_i S_p w), C_{dis}(\nabla_p B)(\nabla_p w)\},$$
$$(\overline{C}B)(w) = \bigwedge\{C(V)(y), \lambda\}$$

and by our assumption, $\bigwedge\{C(V)(y), \lambda\} = \bot$ and consequently, $(\overline{C}B)(w) = \bot$.
Similar to above, if $w = y_2 \notin B$ but $y_1 \in B$, then we have

$$(\overline{C}B)(w) = \bot.$$

Therefore, for all $w \in Y \vee_p Y$ and $B \subseteq Y \vee_p Y$, we have

$$(\overline{C}B)(w) = \begin{cases} \lambda, & w \in B, \\ \bot, & w \notin B. \end{cases}$$

By Lemma 2.2, $\overline{C}$ is an $\mathcal{L}$-valued discrete structure on $Y \vee_p Y$ and by Definition 3.4 (ii), (Y, C) is local T_1. □

Corollary 3.2. *Let (Y, C) be an $\mathcal{L}$-valued closure space and $p \in Y$, where $\mathcal{L}$ is an integral quantale. (Y, C) is local T_1 iff $\forall y \in Y$ with $y \neq p$, there exists $U \subseteq Y$ with $y \in U$, $p \notin U$ and there exists $V \subseteq Y$ with $p \in V$, $y \notin V$ such that $C(U)(p) = \bot = C(V)(y)$.*

Proof. It follows from Theorem 3.2, and definitions of prime bottom element and integral quantale. □

Corollary 3.3. *Every local T_1 $\mathcal{L}$-valued closure space is local T_0 but converse is not true, in general.*

Example 3.1. *Let $Y = \{a, b, c\}$ and $P(Y) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, Y\}$. Consider a quantale $\mathcal{L} = (([0, 1], \leq), ., 1)$, where $[0, 1]$ is a real unit interval with $\leq$ as partial order, "." the product i.e., the quantale operation and 1 is an identity element. Let $C : P(Y) \longrightarrow \mathcal{L}^Y$ be a map defined by $\forall y \in Y$, and $\forall\ \phi \neq U \subset Y$. $C(U)(y) = 1$ if $y \in U$ and $C(\{b\})(c) = C(\{a, b\})(c) = C(\{c\})(b) = C(\{a, c\})(b) = \frac{1}{2}$, $C(\{b\})(a) = C(\{c\})(a) = C(\{b, c\})(a) = 0$. Clearly, (Y, C) be an $\mathcal{L}$-valued closure space. Note that, it is T_0 at a but not T_1 at a.*

4. Notion of closedness and D-connectedness in $\mathcal{L}$-valued closure spaces

Definition 4.1. *Let $Y^\infty = Y \times Y \times \ldots$ be the cartesian product of countable copies of Y.*

(1) A mapping $A_p^\infty : \vee_p^\infty Y \rightarrow Y^\infty$ is said to be infinite principle p-axis mapping satisfying $A_p^\infty(y_i) = (p, p, ..., p, y, p, \ldots)$, where y is at the i-th place [7].

(2) A mapping $\nabla_p^\infty : \vee_p^\infty Y \to Y$ is called the infinite fold mapping at p satisfying $\nabla_p^\infty(y_i) = y$ for all $i \in I$ [7].

The unique map arising from the multiple pushout of $p : 1 \to Y$ is A_p^∞ for which $A_p^\infty(i_j) = (p, p, ..., p, id, p, ...) : Y \to Y^\infty$, where the identity map, id, is in the j-th place [14].

Definition 4.2. *Let $\mathcal{F} : C \longrightarrow$ **Set** be a topological functor, $Y \in Ob(C)$ with $\mathcal{F}(Y) = X$ and $p \in X$,*

(i) $\{p\}$ is closed iff the initial lift of the $\mathcal{F}$-source $\{A_p^\infty : \vee_p^\infty X \to X^\infty$ and $\nabla_p^\infty : \vee_p^\infty X \to UD(X)\}$ is discrete, where D is the discrete functor [7].
(ii) Y is D-connected if and only if any morphism from Y to any discrete object is constant [15, 34].

Theorem 4.1. *Let (Y, C) be an $\mathcal{L}$-valued closure space, $\{p\}$ is closed iff for all $y \in Y$ with $y \neq p$, there exist $U \subseteq Y$ with $y \in U$, $p \notin U$ and $V \subseteq Y$ with $p \in V$, $y \notin V$ such that $\perp = \bigwedge\{C(U)(p), C(V)(y), \lambda\}$, where λ is the identity element.*

Proof. Let (Y, C) be an $\mathcal{L}$-valued closure space and $p \in Y$ with $\{p\}$ is closed, for all $y \in Y$ with $y \neq p$. Suppose $B \subseteq \vee_p^\infty Y$ and $w = (y, p, p, ..., p, ...) \in \vee_p^\infty Y$ with $w \notin B$. Note that

$$(C_{dis}\nabla_p^\infty B)(\nabla_p^\infty w) = (C_{dis}\nabla_p^\infty B)(y) = \lambda,$$

since $y \in \nabla_p^\infty B$,

$$\begin{aligned} C(proj_1 A_p^\infty B)(proj_1 A_p^\infty w) &= C(proj_1 A_p^\infty B)(y) = C(V)(y), \\ C(proj_2 A_p^\infty B)(proj_2 A_p^\infty w) &= C(proj_2 A_p^\infty B)(p) = C(U)(p) \end{aligned}$$

and for $k \geq 3$,

$$\begin{aligned} C(proj_k A_p^\infty B)(proj_k A_p^\infty w) &= C(proj_k A_p^\infty B)(p) = C(U)(p). \\ \lambda \leq C(proj_k A_p^\infty B)(proj_k A_p^\infty w) &= C(proj_k A_p^\infty B)(p), \end{aligned}$$

as $p \in proj_k A_p^\infty B$. Since $w = (y, p, p, ..., p, ...) \notin B$ and $\{p\}$ is closed. By Lemma 2.1 for all $k \in I$,

$$\begin{aligned} (CB)(w) &= \bigwedge\{C_{dis}(\nabla_p^\infty B)(\nabla_p^\infty w), C(proj_k A_p^\infty B)(proj_k A_p^\infty w)\}, \\ \perp &= \bigwedge\{\lambda, C(U)(p), C(V)(y)\}. \end{aligned}$$

Conversely, let $\overline{C}$ be an initial structure on wedge $\vee_p^\infty Y$ induced by $A_p^\infty : \vee_p^\infty Y \longrightarrow (Y^\infty, C_*)$ and ∇_p^∞: $\vee_p^\infty Y \longrightarrow (Y, C_{dis})$, where C_* is a product $\mathcal{L}$- closure structure induced by $proj_k : Y^\infty \longrightarrow Y$, $\forall k \in I$ projection map and C_{dis} is the discrete $\mathcal{L}$-closure structure.

Suppose, $w \in \vee_p^\infty Y$ and $B \subseteq \vee_p^\infty Y$. We have the following cases.
Case I: If $\nabla_p^\infty w = p \in \nabla_p^\infty B$ for some $p \in Y$, $w = (p, p, p, ...) \in \vee_p^\infty Y$. It follows that, $(\overline{C}B)(w) = \lambda$.
Case II: If $\nabla_p^\infty w = p \notin \nabla_p^\infty B$, then $C_{dis}(\nabla_p^\infty B)(\nabla_p^\infty w) = \perp$ and consequently, $(\overline{C}B)(w) = \perp$.
Case III: Suppose $\nabla_p^\infty w = y$ for some $y \in Y$ and it follows that $w = y_i$ for all $i \in I$.

(i) If $w = y_i \in B$, then $\nabla_p^\infty w \in \nabla_p^\infty B$ and $proj_i A_p^\infty w \in proj_i A_p^\infty B$, it follows that $(\overline{C}B)(w) = \lambda$.
(ii) If $w = y_i \notin B$, then $\nabla_p^\infty w \notin \nabla_p^\infty B$ and consequently, $C_{dis}(\nabla_p^\infty B)(\nabla_p^\infty w) = \perp$ and $(\overline{C}B)(w) = \perp$.

(iii) Suppose $w = y_i \notin B$ but $y_j \in B$ with $i \neq j$. For $i \neq k \neq j$, by Lemma 2.2.

$$C_{dis}(\nabla_p^\infty B)(\nabla_p^\infty w) = C_{dis}(\nabla_p^\infty B)(y) = \lambda,$$

since y $\in \nabla_p^\infty B$.

$$C(proj_i A_p^\infty B)(proj_i A_p^\infty w) = C(proj_i A_p^\infty B)(y) = C(V)(y),$$
$$C(proj_j A_p^\infty B)(proj_j A_p^\infty w) = C(proj_j A_p^\infty B)(p) = C(U)(p),$$

and for $k \geq 3$,

$$C(proj_k A_p^\infty B)(proj_k A_p^\infty w) = C(proj_k A_p^\infty B)(p).$$

Since $p \in proj_k A_p^\infty B$ and by Lemma 2.1, then we get

$$\lambda \leq C(proj_k A_p^\infty B)(p).$$

It follows from Lemma 2.1 and for $k \in I$,

$$\begin{aligned}(\overline{C}B)(w) &= \bigwedge\{C_{dis}(\nabla_p^\infty B)(\nabla_p^\infty w), C(proj_k A_p^\infty B)(proj_k A_p^\infty w)\},\\ &= \bigwedge\{\lambda, C(V)(y), C(U)(p)\}.\end{aligned}$$

By our assumption $\perp = \bigwedge\{\lambda, C(U)(p), C(V)(y)\}$ and consequently, $(\overline{C}B)(w) = \perp$. Similarly if $w = y_j \notin B$ but $y_i \in B$ with $i \neq j$. For $i \neq k \neq j$, it follows that

$$(\overline{C}B)(w) = \perp.$$

Then for all $w \in \vee_p^\infty Y$ and all non-empty subset B of $\vee_p^\infty Y$, we have

$$(\overline{C}B)(w) = \begin{cases}\lambda, & w \in B,\\ \perp, & w \notin B.\end{cases}$$

by Lemma 2.2, $\overline{C}$ is the discrete $\mathcal{L}$-closure structure and by Definition 4.2, $\{p\}$ is closed. □

Corollary 4.1. *Let (Y, C) be an $\mathcal{L}$-valued closure space, then following are equivalent.*

(i) (Y, C) *is* T_0 *at* p.
(ii) $\{p\}$ *is closed.*

Proof. It follows from Theorems 3.1 and 4.1. □

Theorem 4.2. *Let (Y, C) be an $\mathcal{L}$-valued closure space, Y is D-connected iff for any non-empty proper subset U of Y, $C(\{y\})(x) > \perp$ or $C(\{x\})(y) > \perp$ for some $y \in U$ and $x \in U^c$.*

Proof. Suppose (Y, C) is D-connected and there exists a proper subset U of Y, with $C(\{x\})(y)(y) = \perp = C(\{y\})(x)$ for all $y \in U$ and $x \in U^c$. Suppose (X, C_{dis}) is a discrete $\mathcal{L}$-valued closure space with cardinality greater than 1. Define $f : (Y, C) \to (X, C_{dis})$ by for all $y \in Y$,

$$f(y) = \begin{cases} w, & y \in U, \\ t, & y \notin U^c. \end{cases}$$

Case I: If $x, y \in U$, then

$$\begin{aligned} \perp &= C(\{x\})(y) \\ &\leq C_{dis}(f\{x\})(f(y)) \\ &= C_{dis}(\{w\})(w) \\ &= \lambda \end{aligned}$$

and it follows that

$$\perp = C(\{y\})(x) \leq C_{dis}(f\{y\})(f(x)) = C_{dis}(\{w\})(w) = \lambda.$$

where λ is an identity element. Similarly if $x, y \in U^c$,

$$\perp = C(\{x\})(y) \leq C_{dis}(f\{x\})(f(y)) = C_{dis}(\{t\})(t) = \lambda$$

and

$$\perp = C(\{y\})(x) \leq C_{dis}(f\{y\})(f(x)) = C_{dis}(\{t\})(t) = \lambda,$$

this implies f is continuous but not constant.
Case II: If $y \in U$ and $x \in U^c$, then

$$C(\{x\})(y) = \perp = C_{dis}(f\{x\})(f(y))$$

and

$$C(\{y\})(x) = \perp = C_{dis}(f\{y\})(f(x))$$

This implies f is continuous but not constant, a contradiction.

Conversely, suppose the condition holds. Let (X, C_{dis}) be an $\mathcal{L}$-valued closure space and $f : (Y, C) \longrightarrow (X, C_{dis})$ be a continuous map.
Case I: If Card $X = 1$, then f is constant.
Case II: Suppose if Card $X > 1$ and f is not constant then, there exist $t, w \in Y$ with $t \neq w$ such that $f(w) \neq f(t)$ and let $U = f^{-1}(\{f(w)\})$. Note that U is a proper subset of Y, with $w \in U$ and $t \notin U$. By our assumption, $\exists\, y \in U$ and $x \in U^c$ such that $C(\{x\})(y) > \perp$ or $C(\{y\})(x) > \perp$. By Lemma 2.2(i),

$$C_{dis}(f(\{y\}))(f(x)) = \perp = C_{dis}(f(\{x\}))(f(y)),$$

which implies that f is not a continuous map, a contradiction. Hence f must be constant. By Definition 4.2, (Y, C) is D-connected. □

5. Local Pre-Hausdorff and local Hausdorff $\mathcal{L}$-valued closure spaces

Definition 5.1. *Let $\mathcal{F} : C \to$ **Set** be a topological functor and $Y \in Obj(C)$, $\mathcal{F}(Y) = X \in Obj$ (**Set**) and $p \in X$.*

(1) Y is local Pre-Hausdorff iff initial lift of $\mathcal{F}$-source $\{A_p : X \vee_p X \to X^2$ and $S_p : X \vee_p X \to X^2\}$ coincide [6].
(2) Y is called local Hausdorff iff Y is local Pre-T_2 and local T_0 [6].

Theorem 5.1. *Let (Y, C) be an $\mathcal{L}$-valued closure space and $p \in Y$, where $\mathcal{L}$ is an integral quantale. (Y, C) is Local Pre-Hausdorff iff for all $y \in Y$ with $y \neq p$, there exists $U \subseteq Y$ with $y \in U$, $p \notin U$, and there exists $V \subseteq Y$ with $p \in V$ and $y \notin V$ such that $\bigwedge\{C(V)(y), C(U)(p)\} = C(V)(y) = C(U)(p)$.*

Proof. Suppose (Y, C) is local Pre-Hausdorff and let $proj_i : Y^2 \to Y$; $i = 1, 2$ be the projection map for all $y \in Y$ with $y \neq p$. Assume that $w = y_1 \in Y \vee_p Y, \{y_2\} \subseteq B \subseteq Y \vee_p Y$ such that

$$C(proj_1 A_p B)(proj_1 A_p(w)) = C(proj_1 A_p B)(y) = C(V)(y)$$

and

$$C(proj_2 A_p B)(proj_2 A_p(w)) = C(proj_2 A_p B)(p) = C(U)(p).$$

Where $U = proj_2 A_p B$ and $V = proj_1 A_p B$ with $y \notin V$ and $p \in U$ since Y is local Pre-Hausdorff. By Lemma 2.1, it follows that

$$\bigwedge\{C(proj_i A_p B)(proj_i A_p(w)); i = 1, 2\} = \bigwedge\{C(V)(p), C(U)(y)\}.$$

Similarly,

$$C(proj_1 S_p B)(proj_1 S_p(w)) = C(proj_1 S_p B)(y) = C(V)(y)$$

and

$$C(proj_2 S_p B)(proj_2 S_p(w)) = C(proj_2 S_p B)(y) = C(U)(y) = \top.$$

Since $y \in (proj_2 S_p B)$ and $y \notin (proj_1 S_p B)$, and $\mathcal{L}$ is an integral quantale. By Lemma 2.1, it follows that

$$\begin{aligned}&\bigwedge\{C(proj_i S_p B)(proj_i S_p(w)); i = 1, 2\}\\ &= \bigwedge\{C(U)(y) = \top, C(V)(y)\} = C(V)(y).\end{aligned}$$

and consequently, we get

$$\bigwedge\{C(U)(p), C(V)(y)\} = C(V)(y).$$

Similarly, for all $y \in Y$ with $y \neq p$, Let $w = y_2 \in Y \vee_p Y$ and $\{y_1\} \subseteq B \subseteq Y \vee_p Y$ such that,

$$C(proj_1 A_p B)(proj_1 A_p(w)) = C(proj_1 A_p B)(p) = C(U)(p),$$

and

$$C(proj_2A_pB)(proj_2A_p(w)) = C(proj_2A_pB)(y) = C(V)(y),$$

where $V = (proj_2A_pB)$ and $U = (proj_1A_pB)$ with $y \in U$, $y \notin V$ and $p \in V$, $p \notin U$. By Lemma 2.1, it follows that

$$\bigwedge\{C(proj_iA_pB)(proj_iA_p(w)); i = 1, 2\} = \bigwedge\{C(U)(p), C(V)(y)\}.$$

Similarly,

$$C(proj_1S_pB)(proj_1S_p(w)) = C(proj_1S_pB)(y) = C(U)(p)$$

and

$$C(proj_2S_pB)(proj_2S_p(w)) = C(proj_2S_pB)(y) = C(V)(y) = \lambda = \top.$$

Since $y \in (proj_2S_pB) = V$ and $y \notin (proj_1S_pB) = U$. By Lemma 2.1, we get

$$\bigwedge\{C(proj_iS_pB)(proj_iS_p(w)); i = 1, 2\}$$
$$= \bigwedge\{C(U)(p), C(V)(y) = \top\} = C(U)(p)$$

and consequently, we get

$$\bigwedge\{C(U)(p), C(V)(y)\} = C(U)(p).$$

Conversely, let $\overline{C}_{AP}$ and $\overline{C}_{SP}$ be initial $\mathcal{L}$-valued closure structures on $Y \vee_p Y$ induced by the projection map $A_p : Y \vee_p Y \longrightarrow (Y^2, C^2)$ and $S_p : Y \vee_p Y \longrightarrow (Y^2, C^2)$ respectively, where C^2 is the product quantale valued closure structure on Y^2 induced by the projection map $proj_i : Y^2 \longrightarrow Y$ for $i = 1, 2$. We need to show that $\forall\ w \in Y \vee_p Y$ and all non empty subset B of $Y \vee_p Y$.

$$\overline{C}_{AP}(B)(w) = \overline{C}_{SP}(B)(w)$$

Case I: If $w \in B$, then $\overline{C}_{AP}(B)(w) = \overline{C}_{SP}(B)(w) = \lambda = \top$.
Case II: Suppose $w \notin B$ and they both are in same component of $Y \vee_p Y$. It follows that $w = y_i$ and $\{z_i\} \subseteq B$ for $i = 1, 2$. If $i = 1$, we have

$$C(proj_1A_pB)(proj_1A_p(w)) = C(proj_1A_pB)(y)$$

and

$$C(proj_2A_pB)(proj_2A_p(w)) = C(proj_2A_pB)(p) = \top.$$

Since p $\in proj_2A_pB$. Similarly,

$$C(proj_1S_pB)(proj_1S_p(w)) = C(proj_1S_pB)(y)$$

and

$$C(proj_2S_pB)(proj_2S_p(w)) = C(proj_2S_pB)(y).$$

Note that

$$\overline{C}_{AP}(B)(w) = \bigwedge\{C(proj_iA_pB)(proj_iA_p(w); i = 1, 2\},$$
$$\overline{C}_{AP}(B)(w) = \bigwedge\{C(proj_1A_pB)(y), \top\}$$

and

$$\overline{C}_{SP}(B)(w) = \bigwedge\{C(proj_iS_pB)(proj_iS_p(w); i = 1, 2\}$$
$$= C(proj_1S_pB)(y).$$

By our assumption w and B are in same component of the wedge and by Definition 3.2, it follows that

$$\overline{C}_{AP}(B)(w) = \overline{C}_{SP}(B)(w).$$

Similarly, for $i = 2$, we have $\overline{C}_{AP}(B)(w) = \overline{C}_{SP}(B)(w)$.

Case III: Suppose $w \notin B$ and they both are in different component of wedge. We have following subcases.

(i) If $w = y_1$ and $\{y_2\} \subseteq B \subseteq Y \vee_p Y$. By Lemma 2.1,

$$\overline{C}_{AP}(B)(w) = \bigwedge\{C(proj_iA_pB)(proj_iA_p(w); i = 1, 2\}$$
$$= \bigwedge\{C(V)(y), C(U)(p)\},$$

where $proj_1A_pB = proj_1S_pB = V$ and $proj_2A_pB = U$ and

$$\overline{C}_{SP}(B)(w) = \bigwedge\{C(proj_iS_pB)(proj_iS_p(w); i = 1, 2\}$$
$$= \bigwedge\{\top, C(proj_1S_pB)(y)\} = C(proj_1S_pB)(y),$$

where $proj_1A_pB = proj_1S_pB = V$ since y $\notin$ V

$$\overline{C}_{SP}(B)(w) = C(V)(y).$$

By the assumption, we get

$$\overline{C}_{SP}(B)(w) = \overline{C}_{AP}(B)(w).$$

(ii) If $w = y_2$ and $\{y_1\} \subseteq B \subseteq Y \vee_p Y$, by Lemma 2.1, we have

$$\overline{C}_{AP}(B)(w) = \bigwedge\{C(proj_iA_pB)(proj_iA_p(w); i = 1, 2\}$$
$$= \bigwedge\{C(V)(y), C(U)(p)\},$$

where $proj_1A_pB = proj_1S_pB = U$ and $proj_2A_pB = V$ and

$$\overline{C}_{SP}(B)(w) = \bigwedge\{C(proj_iS_pB)(proj_iS_p(w); i = 1, 2\}$$
$$= \bigwedge\{\top, C(proj_1S_pB)(y)\} = C(proj_1S_pB)(y),$$

where $proj_1A_pB = proj_1S_pB = V$ since $y \notin V$,

$$\overline{C}_{SP}(B)(w) = C(U)(y).$$

By the assumption, we have

$$\overline{C}_{AP}(B)(w) = \overline{C}_{SP}(B)(w).$$

Therefore, for all $\emptyset \neq B \subset Y \vee_p Y$ and $\forall\ w \in Y \vee_p Y$,

$$\overline{C}_{AP}(B)(w) = \overline{C}_{SP}(B)(w).$$

Hence by Definition 5.1, (Y, C) is local Pre-Hausdorff.

□

Theorem 5.2. *Let (Y, C) be an $\mathcal{L}$-valued closure space, where $\mathcal{L}$ is an integral quantale and Y has a prime bottom element and $p \in Y$. (Y, C) is local Hausdorff iff (Y, C) is a discrete $\mathcal{L}$-closure structure at p, i.e.,*

$$C(U)(p) = \begin{cases} \bot, & p \notin U, \\ \top, & p \in U. \end{cases}$$

Proof. Combine Theorem 5.1 and Definition 5.1. □

Theorem 5.3. *Let (Y, C) be an $\mathcal{L}$-valued closure space and $p \in Y$. Then the followings are equivalent.*

(i) (Y, C) is local T_1, i.e., T_1 at p.
(ii) (Y, C) is local Hausdorff, i.e., Hausdorff at p.
(iii) (Y, C) is a discrete $\mathcal{L}$-closure structure at p.

Proof. It follows from Theorems 3.2 and 5.2. □

Theorem 5.4. *Let (Y, C) be an $\mathcal{L}$-valued closure space, where $\mathcal{L}$ is an integral quantale and Y has a prime bottom element and $p \in Y$. (Y, C) is Hausdorff iff (Y, C) is Hausdorff at p, for all $p \in Y$.*

Proof. It follows from Theorem 5.2 and Theorem 4.4 of [37]. □

Corollary 5.1. *(1) Every $\mathcal{L}$-valued closure space (Y, C) (except indiscrete $\mathcal{L}$-valued closure structure) is D-connected.*
(2) Every Hausdorff $\mathcal{L}$-valued closure space is D-connected but converse is not true in general.

Example 5.1. *Let $Y = \{l, m, n\}$, a quantale $\mathcal{L} = ([0, 1], \leq, \times, 1)$ where $[0, 1]$ is an integral quantale with $\leq$ as partial ordered, $\times$ as quantale operator and "1" is an identity element. Consider a map $C : P(Y) \longrightarrow \mathcal{L}^Y = ([0, 1], \leq, \times, 1)^Y$ defined by: for all $y \in Y$ and $\forall\ \emptyset \neq V \subseteq Y, C(V)(y) = 1$ if $y \in V$ and $C(\{l, m\})(n) = C(\{m\})(n) = C(\{m\})(m) = C(\{l, n\})(m) = \frac{1}{5}$ and $C(\{m\})(l) = C(\{n\})(l) = C(\{m, n\})(l) = 0$. It is obvious that (Y, C) is an $\mathcal{L}$-valued closure space. Note that (Y, C) is D-connected but not Hausdorff.*

6. Conclusions

First of all, we characterized local T_0 and local T_1 $\mathcal{L}$-valued closure spaces, and showed that every local T_1 $\mathcal{L}$-valued closure space is local T_0 but converse is not true in general and we provided a counter example. After that, we characterized closedness of a point and D-connectedness in $\mathcal{L}$-valued closure space, and show that a point p is closed iff (Y, C) is T_0 at p. Finally, we characterized local Pre-Hausdorff and Hausdorff objects in $\mathcal{L}$**-Cls** and showed that (Y, C) is local T_1 iff (Y, C) is local Hausdorff, and showed that every Hausdorff $\mathcal{L}$-valued closure space is D-connected but converse is not true in general and provided a counter example.

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Conflict of interest

We declare that we have no conflict of interest.

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