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Research article

Eigenvalues of fourth-order differential operators with eigenparameter dependent boundary conditions

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Abstract: This paper is concerned with a fourth-order differential operator with eigenparameter dependent boundary conditions. We prove that each of the eigenvalues of the problem can be embedded in a continuous eigenvalue branch. Furthermore, the differential expressions of the eigenvalues with respect to each of parameters are given.

Keywords: fourth-order differential operator; eigenparameter-dependent boundary condition; dependence of eigenvalue; differential expression
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1. Introduction

Sturm-Liouville problem originates from various fields such as physics, engineering, finance and medicine, and it has been widely researched [1, 2]. Nowadays, differential equations with boundary conditions depending on eigenparameter are also widely used in acoustic scattering, quantum mechanics theory and so on. Particularly, more and more researchers have paid close attention to Sturm-Liouville problems with boundary conditions depending on eigenparameter, the distribution of eigenvalues, asymptotic of eigenvalues and eigenfunctions, oscillation theory and inverse spectral theory of such problrm are deeply researched, and many results are obtained. Up to now, it has become an important research topic and has made great progress [3–7]. In recent years, the fourth-order differential operators with eigenparameter dependent boundary conditions appear in elastic beam models, the heat conduct problem and so on are also gained great progress. For more details, we refer the readers to [8–11].

In the last two decades, the dependence of eigenvalues on coefficients and parameters of differential operator has attracted lots of the attention by many researchers. In [12], Kong and Zettl obtained that the eigenvalues of regular Sturm-Liouville problems are differentiable functions with respect to all the data and they gave expressions for their derivatives. Later, this problem was extended to Sturm-Liouville operators with discontiuity, third-order and fourth-order differential operators etc. [13–18]. Recently, Zhang and Li in [19] showed that the eigenvalues of Sturm-Liouville problems with eigenparameter dependent boundary conditions are differentiable functions of all the data. In [20], Zinsou considered the dependence of eigenvalues of a general fourth-order differential equation with transmission conditions and obtained similar results. These results provide a theoretical support for the numerical calculation of eigenvalues and eigenfunctions [21, 22].

Inspired by the above mentioned results, a natural question is that whether similar results still true for fourth-order boundary value problems when eigenparameter appear in the the boundary conditions? In this paper, we give a confirm answer. As we know, the problems with spectral parameter arise from several physical or other applied problems, for instance, the free bending vibrations of rod [23, 24]. Therefore, in this paper, we try to discuss the dependence of eigenvalues of fourth-order differential equations with eigenparameter dependent boundary conditions. It is worth mentioning that we consider such a problem with both endpoints depending on the spectral parameter μ. Compared with the problem with spectral parameter at one end, the inner product and space constructed are different, and it is more troublesome in the process of deriving the differential expression of the eigenvalues with respect to the coefficient matrix of the boundary conditions with spectral parameter. The main result is that each of the eigenvalues of the fourth-order boundary value problem can be embedded in a continuous eigenvalue branch. Furthermore, we obtain the differential expression of the eigenvalues with respect to all data in the sense of ordinary or Fréchet derivatives.

The rest of this paper is organized as follows. In Section 2, we introduce a fourth-order boundary value problems and define a new self-adjoint operator $\mathcal{F}$ such that the eigenvalues of such a problem coincide with those of $\mathcal{F}$. In Section 3, we discuss the continuity of the eigenvalues and eigenfunctions. In Section 4, we give the differential expressions of the eigenvalues with respect to each of parameters.

2. Fourth-order boundary value problem

We consider the fourth-order differential equation

$$lf := (p(x)f''(x))'' - (q(x)f'(x))' + q_0(x)f(x) = \mu w(x)f(x), \tag{2.1}$$

on $[a, b]$, with eigenparameter dependent boundary conditions at endpoints

$$l_1 f := \mu f(a) - f^{[3]}(a) = 0, \tag{2.2}$$

$$l_2 f := \mu f^{[1]}(a) + f^{[2]}(a) = 0, \tag{2.3}$$

$$l_3 f := \mu(\tau_1 f(b) - \gamma_1 f^{[3]}(b)) - (\tau_2 f(b) - \gamma_2 f^{[3]}(b)) = 0, \tag{2.4}$$

$$l_4 f := \mu(\beta_1 f^{[1]}(b) - \alpha_1 f^{[2]}(b)) + (\beta_2 f^{[1]}(b) - \alpha_2 f^{[2]}(b)) = 0, \tag{2.5}$$

where $-\infty < a < b < +\infty$, $\mu \in \mathbb{C}$ is the spectral parameter,

$$\frac{1}{p}, q, q_0, w \in L^1[a, b],\ \ p, w > 0 \text{ a.e. on } [a, b], \tag{2.6}$$

$$\alpha_i, \beta_i,\ \gamma_i, \tau_i \in \mathbb{R},\ \ i = 1, 2,\ \ \rho_1 = \begin{vmatrix} \tau_1 & \tau_2 \\ \gamma_1 & \gamma_2 \end{vmatrix} > 0,\ \ \rho_2 = \begin{vmatrix} \beta_1 & \beta_2 \\ \alpha_1 & \alpha_2 \end{vmatrix} > 0. \tag{2.7}$$

Note that the quasi-derivatives associated to (2.1) are

$$f^{[0]} = f,\ f^{[1]} = f',\ f^{[2]} = pf'',\ f^{[3]} = (pf'')' - qf'. \tag{2.8}$$

Let the weighted Hilbert space be defined as

$$H_1 = L^2_w[a,b] = \{f|\ f(x) \text{ is absolutely continuous and } \int_a^b |f(x)|^2 w(x)\mathrm{d}x < +\infty\}$$

with inner product $\langle f, g\rangle_1 = \int_a^b f(x)\bar{g}(x)w(x)\mathrm{d}x$ for any $f, g \in H_1$. We define a new Hilbert space

$$H = H_1 \oplus \mathbb{C}^4$$

with the inner product

$$\langle F, G\rangle = \langle f, g\rangle_1 + f_1\bar{g}_1 + f_2\bar{g}_2 + \frac{1}{\rho_1} f_3\bar{g}_3 + \frac{1}{\rho_2} f_4\bar{g}_4,$$

for $F = (f, f_1, f_2, f_3, f_4)^T, G = (g, g_1, g_2, g_3, g_4)^T \in H$. Define an operator $\mathcal{F}$ as

$$\mathcal{F}\begin{pmatrix} f \\ f(a) \\ f^{[1]}(a) \\ \tau_1 f(b) - \gamma_1 f^{[3]}(b) \\ \beta_1 f^{[1]}(b) - \alpha_1 f^{[2]}(b) \end{pmatrix} = \begin{pmatrix} w^{-1}[(pf'')'' - (qf')' + q_0 f] \\ f^{[3]}(a) \\ -f^{[2]}(a) \\ \tau_2 f(b) - \gamma_2 f^{[3]}(b) \\ \alpha_2 f^{[2]}(b) - \beta_2 f^{[1]}(b) \end{pmatrix},$$

with the domain

$$\begin{aligned} D(\mathcal{F}) =& \{(f, f_1, f_2, f_3, f_4)^T \in H|\ w^{-1}[(pf'')'' - (qf')' + q_0 f] \in L^2_w[a,b], f, f^{[1]}, f^{[2]}, f^{[3]} \in AC[a,b], \\ & f_1 = f(a), f_2 = f^{[1]}(a), f_3 = \tau_1 f(b) - \gamma_1 f^{[3]}(b), f_4 = \beta_1 f^{[1]}(b) - \alpha_1 f^{[2]}(b)\}. \end{aligned}$$

Lemma 2.1. *The operator $\mathcal{F}$ is a self-adjoint operator in H.*

Proof. The proof is similar to that of [25], the equation we considered is more complicated and the derivatives in boundary conditions are quasi-derivatives, here we omit the details. □

Lemma 2.2. *[25] The spectrum of $\mathcal{F}$ consists of isolated eigenvalues, which coincide with those of the fourth-order boundary value problems (2.1)–(2.5). Furthermore, all the eigenvalues are real-valued.*

3. Continuity of eigenvalues and eigenfunctions

Let $\chi_1(x,\mu), \chi_2(x,\mu), \chi_3(x,\mu), \chi_4(x,\mu)$ be the linearly independent solutions of Eq (2.1) satisfying the initial conditions

$$\begin{pmatrix} \chi_1(a,\mu) & \chi_2(a,\mu) & \chi_3(a,\mu) & \chi_4(a,\mu) \\ \chi_1^{[1]}(a,\mu) & \chi_2^{[1]}(a,\mu) & \chi_3^{[1]}(a,\mu) & \chi_4^{[1]}(a,\mu) \\ \chi_1^{[2]}(a,\mu) & \chi_2^{[2]}(a,\mu) & \chi_3^{[2]}(a,\mu) & \chi_4^{[2]}(a,\mu) \\ \chi_1^{[3]}(a,\mu) & \chi_2^{[3]}(a,\mu) & \chi_3^{[3]}(a,\mu) & \chi_4^{[3]}(a,\mu) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.1}$$

We define their Wronskian as

$$\Phi(x,\mu) := \begin{pmatrix} \chi_1(x,\mu) & \chi_2(x,\mu) & \chi_3(x,\mu) & \chi_4(x,\mu) \\ \chi_1^{[1]}(x,\mu) & \chi_2^{[1]}(x,\mu) & \chi_3^{[1]}(x,\mu) & \chi_4^{[1]}(x,\mu) \\ \chi_1^{[2]}(x,\mu) & \chi_2^{[2]}(x,\mu) & \chi_3^{[2]}(x,\mu) & \chi_4^{[2]}(x,\mu) \\ \chi_1^{[3]}(x,\mu) & \chi_2^{[3]}(x,\mu) & \chi_3^{[3]}(x,\mu) & \chi_4^{[3]}(x,\mu) \end{pmatrix}.$$

Lemma 3.1. *The number μ is an eigenvalue of operator $\mathcal{F}$ if and only if*

$$\Delta(\mu) = \det(M + N\Phi(b,\mu)) = 0,$$

where

$$M = \begin{pmatrix} \mu & 0 & 0 & -1 \\ 0 & \mu & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tau_1\mu - \tau_2 & 0 & 0 & -(\gamma_1\mu - \gamma_2) \\ 0 & \beta_1\mu + \beta_2 & -(\alpha_1\mu + \alpha_2) & 0 \end{pmatrix}.$$

Proof. By [26, Theorem 1.8], we see that boundary value problems (2.1)–(2.5) is well-posed. Let μ be an eigenvalue of (2.1)–(2.5), then there exists a non-trivial solution

$$f(x,\mu) = c_1\chi_1(x,\mu) + c_2\chi_2(x,\mu) + c_3\chi_3(x,\mu) + c_4\chi_4(x,\mu),$$

of (2.1), where c_1, c_2, c_3, c_4 are not all zero. Since $f(x,\mu)$ satisfies the boundary conditions (2.2)–(2.5), we have

$$M\left(c_1\begin{pmatrix} \chi_1(a) \\ \chi_1^{[1]}(a) \\ \chi_1^{[2]}(a) \\ \chi_1^{[3]}(a) \end{pmatrix} + c_2\begin{pmatrix} \chi_2(a) \\ \chi_2^{[1]}(a) \\ \chi_2^{[2]}(a) \\ \chi_2^{[3]}(a) \end{pmatrix} + c_3\begin{pmatrix} \chi_3(a) \\ \chi_3^{[1]}(a) \\ \chi_3^{[2]}(a) \\ \chi_3^{[3]}(a) \end{pmatrix} + c_4\begin{pmatrix} \chi_4(a) \\ \chi_4^{[1]}(a) \\ \chi_4^{[2]}(a) \\ \chi_4^{[3]}(a) \end{pmatrix}\right) +$$
$$N\left(c_1\begin{pmatrix} \chi_1(b) \\ \chi_1^{[1]}(b) \\ \chi_1^{[2]}(b) \\ \chi_1^{[3]}(b) \end{pmatrix} + c_2\begin{pmatrix} \chi_2(b) \\ \chi_2^{[1]}(b) \\ \chi_2^{[2]}(b) \\ \chi_2^{[3]}(b) \end{pmatrix} + c_3\begin{pmatrix} \chi_3(b) \\ \chi_3^{[1]}(b) \\ \chi_3^{[2]}(b) \\ \chi_3^{[3]}(b) \end{pmatrix} + c_4\begin{pmatrix} \chi_4(b) \\ \chi_4^{[1]}(b) \\ \chi_4^{[2]}(b) \\ \chi_4^{[3]}(b) \end{pmatrix}\right) = 0.$$

By the initial condition (3.1), we have

$$(M + N\Phi(b,\mu))(c_1, c_2, c_3, c_4)^T = 0. \tag{3.2}$$

Since not all c_1, c_2, c_3, c_4 are zero, we get that $\det(M + N\Phi(b,\mu)) = 0$.

On the other hand, if $\Delta(\mu) = 0$, then Eq (3.2) has non-zero solution c_1, c_2, c_3, c_4. Let

$$f(x,\mu) = c_1\chi_1(x,\mu) + c_2\chi_2(x,\mu) + c_3\chi_3(x,\mu) + c_4\chi_4(x,\mu),$$

then $f(x,\mu)$ satisfies (2.1)–(2.5) and thus μ is an eigenvalue. This completes the proof. □

Now, we consider the Banach space

$$B := L^1[a,b] \oplus L^1[a,b] \oplus L^1[a,b] \oplus L^1[a,b] \oplus \mathbb{R}^8$$

with norm

$$\|\xi\| := \int_a^b \frac{1}{|p|}\mathrm{d}x + \int_a^b |q|\mathrm{d}x + \int_a^b |q_0|\mathrm{d}x + \int_a^b |w|\mathrm{d}x$$
$$+ |\gamma_1| + |\gamma_2| + |\tau_1| + |\tau_2| + |\alpha_1| + |\alpha_2| + |\beta_1| + |\beta_2|,$$

for any $\xi = (\frac{1}{p}, q, q_0, w, \gamma_1, \gamma_2, \tau_1, \tau_2, \alpha_1, \alpha_2, \beta_1, \beta_2) \in B$. Let

$$\Omega = \{\tau \in B|\ (2.6), (2.7) \text{ hold}\}.$$

Theorem 3.1. *Let $\tilde{\xi} = (\frac{1}{\tilde{p}}, \tilde{q}, \tilde{q}_0, \tilde{w}, \tilde{\gamma_1}, \tilde{\gamma_2}, \tilde{\tau_1}, \tilde{\tau_2}, \tilde{\alpha_1}, \tilde{\alpha_2}, \tilde{\beta_1}, \tilde{\beta_2}) \in \Omega$ and $\mu(\tilde{\xi})$ be an isolated eigenvalue of (2.1)–(2.5) with $\tilde{\xi}$. Then μ is continuous on $\tilde{\xi}$. That is, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that the problems (2.1)–(2.5) has exactly an isolated eigenvalue $\mu(\xi)$ satisfying*

$$|\mu(\xi) - \mu(\tilde{\xi})| < \varepsilon,$$

if $\xi = (\frac{1}{p}, q, q_0, w, \gamma_1, \gamma_2, \tau_1, \tau_2, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies

$$\begin{aligned}|\xi - \tilde{\xi}| = &\int_a^b |\frac{1}{p} - \frac{1}{\tilde{p}}|\mathrm{d}x + \int_a^b |q - \tilde{q}|\mathrm{d}x + \int_a^b |q_0 - \tilde{q_0}|\mathrm{d}x + \int_a^b |w - \tilde{w}|\mathrm{d}x\\ &+ |\gamma_1 - \tilde{\gamma_1}| + |\gamma_2 - \tilde{\gamma_2}| + |\tau_1 - \tilde{\tau_1}| + |\tau_2 - \tilde{\tau_2}| + |\alpha_1 - \tilde{\alpha_1}|\\ &+ |\alpha_2 - \tilde{\alpha_2}| + |\beta_1 - \tilde{\beta_1}| + |\beta_2 - \tilde{\beta_2}|\\ &< \delta.\end{aligned}$$

Proof. By Lemma 3.1, $\mu(\tilde{\xi})$ is an eigenvalue of (2.1)–(2.5) if and only if $\Delta(\tilde{\xi}, \mu(\tilde{\xi})) = 0$. For any $\xi \in \Omega$, $\Delta(\xi, \mu)$ is an entire function of μ and is continuous on ξ (see [27, Theorems 2.7 and 2.8]). It is easy seen that $\Delta(\tilde{\xi}, \mu)$ is not a constant in μ because $\mu(\tilde{\xi})$ is an isolated eigenvalue. Therefore, there exists $\rho_0 > 0$ such that $\Delta(\tilde{\xi}, \mu) \neq 0$ for $\mu \in S_{\rho_0} := \{\mu \in \mathbb{C} : |\mu - \mu(\tilde{\xi})| = \rho_0\}$. By the continuity of the roots of an equation as a function of parameters (see [28, (9.17.4)]), the statement follows. □

By a normalized eigenvector $(m, m_1, m_2, m_3, m_4)^T \in H$, we mean m satisfies the problems (2.1)–(2.5), $m_1 = m(a), m_2 = m^{[1]}(a), m_3 = \tau_1 m(b) - \gamma_1 m^{[3]}(b),\ m_4 = \beta_1 m^{[1]}(b) - \alpha_1 m^{[2]}(b)$, and

$$\begin{aligned}\|(m, m_1, m_2, m_3, m_4)^T\|^2 &= \langle (m, m_1, m_2, m_3, m_4)^T, (m, m_1, m_2, m_3, m_4)^T \rangle\\ &= \int_a^b m\bar{m}w\mathrm{d}x + m_1\bar{m}_1 + m_2\bar{m}_2 + \frac{1}{\rho_1}m_3\bar{m}_3 + \frac{1}{\rho_2}m_4\bar{m}_4\\ &= 1.\end{aligned}$$

Now we give a result for normalized eigenfunctions.

Theorem 3.2. *Assume that $\mu(\xi)$ is an eigenvalue of (2.1)–(2.5) with $\xi \in \Omega$ and $(m, m_1, m_2, m_3, m_4)^T \in H$ is the corresponding normalized eigenvector for $\mu(\xi)$. Then there exists a normalized eigenvector $(n, n_1, n_2, n_3, n_4)^T \in H$ for $\mu(\tilde{\xi})$ with $\tilde{\xi} \in \Omega$, which is specified in Theorem 3.1, such that*

$$\begin{aligned}&n(x) \to m(x), n^{[1]}(x) \to m^{[1]}(x), n^{[2]}(x) \to m^{[2]}(x), n^{[3]}(x) \to m^{[3]}(x),\\ &n_1(x) \to m_1(x), n_2(x) \to m_2(x), n_3(x) \to m_3(x), n_4(x) \to m_4(x),\end{aligned} \tag{3.3}$$

as $\tilde{\xi} \to \xi$ both uniformly on $[a, b]$.

Proof. (i) We know that $\mu(\xi)$ is an isolated eigenvalue of multiplicity $j(j = 1,2,3,4)$ for all ξ in some neighborhood $\mathcal{N}$ of $\tilde{\xi}$ in Ω. Suppose $\mu(\xi)$ is simple. Let $(f(x,\xi), f_1(\xi), f_2(\xi), f_3(\xi), f_4(\xi))^T$ be an eigenvector for $\mu(\xi)$ with

$$\|f(x,\xi)\| = \int_a^b f(x,\xi)\bar{f}(x,\xi)w\mathrm{d}x = 1.$$

By Theorem 3.1, there exists $\mu(\tilde{\xi})$ such that

$$\mu(\tilde{\xi}) \to \mu(\xi) \text{ as } \tilde{\xi} \to \xi.$$

Define the boundary condition matrix as

$$(M,N)(\xi) = \begin{pmatrix} \mu(\xi) & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & \mu(\xi) & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_1\mu(\xi)-\tau_2 & 0 & 0 & -(\gamma_1\mu(\xi)-\gamma_2) \\ 0 & 0 & 0 & 0 & 0 & \beta_1\mu(\xi)+\beta_2 & -(\alpha_1\mu(\xi)+\alpha_2) & 0 \end{pmatrix},$$

then

$$(M,N)(\tilde{\xi}) \to (M,N)(\xi) \text{ as } \tilde{\xi} \to \xi.$$

By Theorem 3.2 of [12], we can obtain an eigenfunction $f(x,\tilde{\xi})$ for $\mu(\tilde{\xi})$ such that $\|f(x,\tilde{\xi})\| = 1$ and

$$f(x,\tilde{\xi}) \to f(x,\xi),\ f^{[1]}(x,\tilde{\xi}) \to f^{[1]}(x,\xi),\ f^{[2]}(x,\tilde{\xi}) \to f^{[2]}(x,\xi),\ f^{[3]}(x,\tilde{\xi}) \to f^{[3]}(x,\xi), \tag{3.4}$$

as $\tilde{\xi} \to \xi$ both uniformly on $[a,b]$. Then we obtain

$$f_1(\tilde{\xi}) \to f_1(\xi),\ f_2(\tilde{\xi}) \to f_2(\xi),\ f_3(\tilde{\xi}) \to f_3(\xi),\ f_4(\tilde{\xi}) \to f_4(\xi) \text{ as } \tilde{\xi} \to \xi. \tag{3.5}$$

Let

$$\begin{aligned}
(m,m_1,m_2,m_3,m_4)^T &= \frac{(f(x,\xi),f_1(\xi),f_2(\xi),f_3(\xi),f_4(\xi))^T}{\|(f(x,\xi),f_1(\xi),f_2(\xi),f_3(\xi),f_4(\xi))^T\|},\\
(n,n_1,n_2,n_3,n_4)^T &= \frac{(f(x,\tilde{\xi}),f_1(\tilde{\xi}),f_2(\tilde{\xi}),f_3(\tilde{\xi}),f_4(\tilde{\xi}))^T}{\|(f(x,\tilde{\xi}),f_1(\tilde{\xi}),f_2(\tilde{\xi}),f_3(\tilde{\xi}),f_4(\tilde{\xi}))^T\|},\\
m^{[1]} &= \frac{f^{[1]}(x,\xi)}{\|(f(x,\xi),f_1(\xi),f_2(\xi),f_3(\xi),f_4(\xi))^T\|}, n^{[1]} = \frac{f^{[1]}(x,\tilde{\xi})}{\|(f(x,\tilde{\xi}),f_1(\tilde{\xi}),f_2(\tilde{\xi}),f_3(\tilde{\xi}),f_4(\tilde{\xi}))^T\|},\\
m^{[2]} &= \frac{f^{[2]}(x,\xi)}{\|(f(x,\xi),f_1(\xi),f_2(\xi),f_3(\xi),f_4(\xi))^T\|}, n^{[2]} = \frac{f^{[2]}(x,\tilde{\xi})}{\|(f(x,\tilde{\xi}),f_1(\tilde{\xi}),f_2(\tilde{\xi}),f_3(\tilde{\xi}),f_4(\tilde{\xi}))^T\|},\\
m^{[3]} &= \frac{f^{[3]}(x,\xi)}{\|(f(x,\xi),f_1(\xi),f_2(\xi),f_3(\xi),f_4(\xi))^T\|}, n^{[3]} = \frac{f^{[3]}(x,\tilde{\xi})}{\|(f(x,\tilde{\xi}),f_1(\tilde{\xi}),f_2(\tilde{\xi}),f_3(\tilde{\xi}),f_4(\tilde{\xi}))^T\|}.
\end{aligned}$$

Then (3.3) holds by (3.4) and (3.5).

(ii) Assume that $\mu(\xi)$ is an eigenvalue of multiplicity $j(j = 2,3,4)$. Then we can choose eigenfunctions of $\mu(\xi)$ such that all of them satisfy the same initial conditions at c_0 for some $c_0 \in [a,b]$ since a linear combination of j linearly independent eigenfunctions can be chosen to satisfy arbitrary initial conditions.

Similarly, we obtain (3.3) as (i), This completes the proof.

□

4. Differential expression of eigenvalues

In this section, we focus on giving the derivative formulas of the eigenvalues for all the parameters. First we give the definition of the Fréchet derivative.

Definition 4.1. *A map $\mathcal{F}$ from a Banach space X into Banach space Y is differentiable at a point $x \in X$, if there exists a bounded linear operator $\mathrm{d}\mathcal{F}_x : X \to Y$ such that for $k \in X$*

$$|\mathcal{F}(x+k) - \mathcal{F}(x) - \mathrm{d}\mathcal{F}_x(k)| = o(k) \ \text{as} \ k \to 0.$$

Theorem 4.1. *Assume that $\mu(\xi)$ is an eigenvalue of (2.1)–(2.5) with $\xi \in \Omega$ and $(m, m_1, m_2, m_3, m_4)^T \in H$ is the corresponding normalized eigenvector for $\mu(\xi)$. Suppose $\mu(\xi)$ is a simple eigenvalue or $\mu(\sigma)$ is an eigenvalue of multiplicity j ($j = 2, 3, 4$) for each σ in some neighborhood $\mathcal{N} \subset \Omega$ of ξ. Then μ is differentiable with respect to all the data in ξ.*

(1) Let all the data of ξ be fixed except the boundary condition parameter matrix

$$K_1 = \begin{pmatrix} \tau_1 & \tau_2 \\ \gamma_1 & \gamma_2 \end{pmatrix},$$

and $\mu(K_1) := \mu(\xi)$. Then

$$\mathrm{d}\mu_{K_1}(L) = (m(b), -m^{[3]}(b))[E - K_1(K_1 + L)^{-1}] \begin{pmatrix} \bar{m}^{[3]}(b) \\ \bar{m}(b) \end{pmatrix} \tag{4.1}$$

for all L satisfying $\det(K_1 + L) = \det K_1 = \rho_1$.

(2) Let all the data of ξ be fixed except the boundary condition parameter matrix

$$K_2 = \begin{pmatrix} \beta_1 & \beta_2 \\ \alpha_1 & \alpha_2 \end{pmatrix},$$

and $\mu(K_2) := \mu(\xi)$. Then

$$\mathrm{d}\mu_{K_2}(L) = (-m^{[1]}(b), m^{[2]}(b))[E - K_2(K_2 + L)^{-1}] \begin{pmatrix} \bar{m}^{[2]}(b) \\ \bar{m}^{[1]}(b) \end{pmatrix} \tag{4.2}$$

for all L satisfying $\det(K_2 + L) = \det K_2 = \rho_2$.

(3) Let all the data of ξ be fixed except p and $\mu(\frac{1}{p}) := \mu(\xi)$. Then

$$\mathrm{d}\mu_{\frac{1}{p}}(k) = -\int_a^b |pm''|^2 k \mathrm{d}x, \ k \in L^1[a, b]. \tag{4.3}$$

(4) Let all the data of ξ be fixed except q and $\mu(q) := \mu(\xi)$. Then

$$\mathrm{d}\mu_q(k) = \int_a^b |m^{[1]}|^2 k \mathrm{d}x, \ k \in L^1[a, b]. \tag{4.4}$$

(5) Let all the data of ξ be fixed except q_0 and $\mu(q_0) := \mu(\xi)$. Then

$$\mathrm{d}\mu_{q_0}(k) = \int_a^b |m|^2 k \mathrm{d}x, \ k \in L^1[a, b]. \tag{4.5}$$

(6) Let all the data of ξ be fixed except w and $\mu(w) := \mu(\xi)$. Then

$$\mathrm{d}\mu_w(k) = -\mu(w) \cdot \int_a^b |m|^2 k \mathrm{d}x, \ k \in L^1[a, b]. \tag{4.6}$$

Proof. Let all the data of ξ be fixed except one and $\mu(\tilde{\xi})$ be the eigenvalue satisfying Theorem 3.1 when $\|\tilde{\xi}-\xi\|<\varepsilon$ for sufficiently small $\varepsilon>0$. For the above six cases, we replace $\mu(\tilde{\xi})$ by $\mu(K_1+L)$, $\mu(K_2+L)$, $\mu(\frac{1}{p}+k)$, $\mu(q+k)$, $\mu(q_0+k)$, $\mu(w+k)$, respectively. Let $(n,n_1,n_2,n_3,n_4)^T$ be the corresponding normalized eigenvector.

(1) By (2.1) we have

$$(pm'')''-(qm')'+q_0m=\mu(K_1)wm, \tag{4.7}$$

$$(p\bar{n}'')''-(q\bar{n}')'+q_0\bar{n}=\mu(K_1+L)w\bar{n}. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$[\mu(K_1+L)-\mu(K_1)]m\bar{n}w=(p\bar{n}'')''m-(q\bar{n}')'m-(pm'')''\bar{n}+(qm')'\bar{n}.$$

Integrating from a to b implies that

$$\begin{aligned}[\mu(K_1+L)-\mu(K_1)]\int_a^b m\bar{n}w\mathrm{d}x =&m(b)[(p\bar{n}'')'(b)-(q\bar{n}')(b)]-m(a)[(p\bar{n}'')'(a)-(q\bar{n}')(a)]\\ &-[(pm'')'(b)-(qm')(b)]\bar{n}(b)+[(pm'')'(a)-(qm')(a)]\bar{n}(a)\\ &-p\bar{n}''(b)m'(b)+p\bar{n}''(a)m'(a)+(pm'')(b)\bar{n}'(b)-(pm'')(a)\bar{n}'(a)\\ =&m(b)\bar{n}^{[3]}(b)-m(a)\bar{n}^{[3]}(a)-m^{[3]}(b)\bar{n}(b)+m^{[3]}(a)\bar{n}(a)\\ &-m^{[1]}(b)\bar{n}^{[2]}(b)+m^{[1]}(a)\bar{n}^{[2]}(a)+m^{[2]}(b)\bar{n}^{[1]}(b)-m^{[2]}(a)\bar{n}^{[1]}(a).\end{aligned} \tag{4.9}$$

According to the boundary condition (2.2), we have

$$\mu(K_1)m(a)\bar{n}(a)=m^{[3]}(a)\bar{n}(a),$$

$$\mu(K_1+L)m(a)\bar{n}(a)=m(a)\bar{n}^{[3]}(a).$$

Thus

$$[\mu(K_1+L)-\mu(K_1)]m_1\bar{n}_1=m(a)\bar{n}^{[3]}(a)-m^{[3]}(a)\bar{n}(a). \tag{4.10}$$

Analogously, the boundary condition (2.3) implies that

$$[\mu(K_1+L)-\mu(K_1)]m_2\bar{n}_2=m^{[2]}(a)\bar{n}^{[1]}(a)-m^{[1]}(a)\bar{n}^{[2]}(a). \tag{4.11}$$

Let $K_1+L=\begin{pmatrix}\tilde{\tau}_1 & \tilde{\tau}_2\\ \tilde{\gamma}_1 & \tilde{\gamma}_2\end{pmatrix}$. Then according to the boundary condition (2.4), we have

$$\mu(K_1)[\tau_1m(b)-\gamma_1m^{[3]}(b)]=\tau_2m(b)-\gamma_2m^{[3]}(b),$$

$$\mu(K_1+L)[\tilde{\tau}_1\bar{n}(b)-\tilde{\gamma}_1\bar{n}^{[3]}(b)]=\tilde{\tau}_2\bar{n}(b)-\tilde{\gamma}_2\bar{n}^{[3]}(b).$$

Thus

$$\begin{aligned}[\mu(K_1+L)-\mu(K_1)]\frac{1}{\rho_1}m_3\bar{n}_3 =&\frac{1}{\rho_1}[\tilde{\tau}_2\bar{n}(b)-\tilde{\gamma}_2\bar{n}^{[3]}(b)][\tau_1m(b)-\gamma_1m^{[3]}(b)]\\ &-\frac{1}{\rho_1}[\tau_2m(b)-\gamma_2m^{[3]}(b)][\tilde{\tau}_1\bar{n}(b)-\tilde{\gamma}_1\bar{n}^{[3]}(b)].\end{aligned} \tag{4.12}$$

Analogously, the boundary condition (2.5) implies that

$$[\mu(K_1+L)-\mu(K_1)]\frac{1}{\rho_2}m_4\bar{n}_4 = m^{[1]}(b)\bar{n}^{[2]}(b) - m^{[2]}(b)\bar{n}^{[1]}(b). \tag{4.13}$$

From (4.9)–(4.13), we get

$$\begin{aligned}
&[\mu(K_1+L)-\mu(K_1)]\Big[\int_a^b m\bar{n}w\mathrm{d}x + m_1\bar{n}_1 + m_2\bar{n}_2 + \frac{1}{\rho_1}m_3\bar{n}_3 + \frac{1}{\rho_2}m_4\bar{n}_4\Big]\\
=&m(b)\bar{n}^{[3]}(b) - m^{[3]}(b)\bar{n}(b)\\
&+\frac{1}{\rho_1}[\tilde{\tau_2}\bar{n}(b)-\tilde{\gamma_2}\bar{n}^{[3]}(b)][\tau_1 m(b)-\gamma_1 m^{[3]}(b)]\\
&-\frac{1}{\rho_1}[\tau_2 m(b)-\gamma_2 m^{[3]}(b)][\tilde{\tau_1}\bar{n}(b)-\tilde{\gamma_1}\bar{n}^{[3]}(b)]\\
=&(m(b),-m^{[3]}(b))E\begin{pmatrix}\bar{n}^{[3]}(b)\\ \bar{n}(b)\end{pmatrix}\\
&+\frac{1}{\rho_1}(m(b),-m^{[3]}(b))\begin{pmatrix}\tau_1\\ \gamma_1\end{pmatrix}(-\tilde{\gamma_2},\tilde{\tau_2})\begin{pmatrix}\bar{n}^{[3]}(b)\\ \bar{n}(b)\end{pmatrix}\\
&-\frac{1}{\rho_1}(m(b),-m^{[3]}(b))\begin{pmatrix}\tau_2\\ \gamma_2\end{pmatrix}(-\tilde{\gamma_1},\tilde{\tau_1})\begin{pmatrix}\bar{n}^{[3]}(b)\\ \bar{n}(b)\end{pmatrix}\\
=&(m(b),-m^{[3]}(b))\Big[E+\frac{1}{\rho_1}\begin{pmatrix}\tau_1\\ \gamma_1\end{pmatrix}(-\tilde{\gamma_2},\tilde{\tau_2})-\frac{1}{\rho_1}\begin{pmatrix}\tau_2\\ \gamma_2\end{pmatrix}(-\tilde{\gamma_1},\tilde{\tau_1})\Big]\begin{pmatrix}\bar{n}^{[3]}(b)\\ \bar{n}(b)\end{pmatrix}\\
=&(m(b),-m^{[3]}(b))\Big[E+\frac{1}{\rho_1}\begin{pmatrix}\tau_2\tilde{\gamma_1}-\tau_1\tilde{\gamma_2} & \tau_1\tilde{\tau_2}-\tau_2\tilde{\tau_1}\\ \gamma_2\tilde{\gamma_1}-\gamma_1\tilde{\gamma_2} & \gamma_1\tilde{\tau_2}-\gamma_2\tilde{\tau_1}\end{pmatrix}\Big]\begin{pmatrix}\bar{n}^{[3]}(b)\\ \bar{n}(b)\end{pmatrix}\\
=&(m(b),-m^{[3]}(b))[E-K_1(K_1+L)^{-1}]\begin{pmatrix}\bar{n}^{[3]}(b)\\ \bar{n}(b)\end{pmatrix}.
\end{aligned} \tag{4.14}$$

Dividing both sides of (4.14) by L and taking the limit as $L\to 0$, by Theorem 3.2, we get

$$\mathrm{d}\mu_{K_1}(L) = (m(b),-m^{[3]}(b))[E-K_1(K_1+L)^{-1}]\begin{pmatrix}\bar{m}^{[3]}(b)\\ \bar{m}(b)\end{pmatrix}.$$

Then (4.1) follows. In a similar discussion, we can obtain (4.2).

(2) For $k\in L^1[a,b]$, let $\frac{1}{p}+k=\frac{1}{\tilde{p}}$. Using (2.1) and integration by parts, we have

$$\begin{aligned}
\Big[\mu\Big(\frac{1}{p}+k\Big)-\mu\Big(\frac{1}{p}\Big)\Big]\int_a^b m\bar{n}w\mathrm{d}x =& m(b)\bar{n}^{[3]}(b) - m(a)\bar{n}^{[3]}(a) - m^{[3]}(b)\bar{n}(b) + m^{[3]}(a)\bar{n}(a)\\
&- m^{[1]}(b)\bar{n}^{[2]}(b) + m^{[1]}(a)\bar{n}^{[2]}(a) + m^{[2]}(b)\bar{n}^{[1]}(b) - m^{[2]}(a)\bar{n}^{[1]}(a)\\
&+\int_a^b \tilde{p}\bar{n}''m''\mathrm{d}x - \int_a^b pm''\bar{n}''\mathrm{d}x,
\end{aligned}$$

where $\bar{n}^{[2]} = \tilde{p}\bar{n}''$, $\bar{n}^{[3]} = (\tilde{p}\bar{n}'')' - q\bar{n}'$. Then by (2.2)–(2.5), we obtain

$$\begin{aligned}
&\Big[\mu\Big(\frac{1}{p}+k\Big)-\mu\Big(\frac{1}{p}\Big)\Big]\Big[\int_a^b m\bar{n}w\mathrm{d}x + m_1\bar{n}_1 + m_2\bar{n}_2 + \frac{1}{\rho_1}m_3\bar{n}_3 + \frac{1}{\rho_2}m_4\bar{n}_4\Big]\\
&= \int_a^b \tilde{p}\bar{n}''m''\mathrm{d}x - \int_a^b pm''\bar{n}''\mathrm{d}x\\
&= \int_a^b (\tilde{p}-p)m''\bar{n}''\mathrm{d}x\\
&= \int_a^b [-p\tilde{p}km''\bar{n}'']\mathrm{d}x.
\end{aligned} \tag{4.15}$$

Dividing both sides of (4.15) by k and taking the limit as $k \to 0$, by Theorem 3.2 we get

$$\mathrm{d}\mu_{\frac{1}{p}}(k) = -\int_a^b |pm''|^2 k\mathrm{d}x.$$

Then (4.3) follows. Similar to the proof of (4.3), we can obtain (4.4).

(3) For $k \in L^1[a,b]$. By (2.1), we have

$$\begin{aligned}
[\mu(q_0+k)-\mu(q_0)]\int_a^b m\bar{n}w\mathrm{d}x =& m(b)\bar{n}^{[3]}(b) - m(a)\bar{n}^{[3]}(a) - m^{[3]}(b)\bar{n}(b) + m^{[3]}(a)\bar{n}(a)\\
&- m^{[1]}(b)\bar{n}^{[2]}(b) + m^{[1]}(a)\bar{n}^{[2]}(a) + m^{[2]}(b)\bar{n}^{[1]}(b) - m^{[2]}(a)\bar{n}^{[1]}(a)\\
&+ \int_a^b km\bar{n}\mathrm{d}x.
\end{aligned}$$

Using the boundary conditions (2.2)–(2.5), we have

$$[\mu(q_0+k)-\mu(q_0)]\Big[\int_a^b m\bar{n}w\mathrm{d}x + m_1\bar{n}_1 + m_2\bar{n}_2 + \frac{1}{\rho_1}m_3\bar{n}_3 + \frac{1}{\rho_2}m_4\bar{n}_4\Big] = \int_a^b km\bar{n}\mathrm{d}x. \tag{4.16}$$

Then (4.5) follows. The proof of (4.6) is similar as that of (4.5), hence we omit the details. □

Theorem 4.2. *Let $\mu(\xi)$ be an eigenvalue of (2.1)–(2.5) with $\xi \in \Omega$ and $(m, m_1, m_2, m_3, m_4)^T \in H$ be a normalized eigenvector for $\mu(\xi)$. Assume that $\mu(\xi)$ is a simple eigenvalue or $\mu(\sigma)$ is an eigenvalue of multiplicity j ($j = 2, 3, 4$) for each σ in some neighborhood $\mathcal{N} \subset \Omega$ of ξ. Then μ is differentiable with respect to the data in ξ.*

(1) Let all the data of ξ be fixed except τ_1 and $\mu(\tau_1) := \mu(\xi)$. Then

$$\mu'(\tau_1) = \frac{\mu}{\mu\gamma_1 - \gamma_2}|m(b)|^2, \tag{4.17}$$

where $\mu\gamma_1 - \gamma_2 \neq 0$.

(2) Let all the data of ξ be fixed except τ_2 and $\mu(\tau_2) := \mu(\xi)$. Then

$$\mu'(\tau_2) = -\frac{1}{\mu\gamma_1 - \gamma_2}|m(b)|^2, \tag{4.18}$$

where $\mu\gamma_1 - \gamma_2 \neq 0$.

(3) Let all the data of ξ be fixed except γ_1 and $\mu(\gamma_1) := \mu(\xi)$. Then

$$\mu'(\gamma_1) = -\frac{\mu}{\mu\tau_1 - \tau_2}|m^{[3]}(b)|^2, \tag{4.19}$$

where $\mu\tau_1 - \tau_2 \neq 0$.

(4) Let all the data of ξ be fixed except γ_2 and $\mu(\gamma_2) := \mu(\xi)$. Then

$$\mu'(\gamma_2) = \frac{1}{\mu\tau_1 - \tau_2}|m^{[3]}(b)|^2, \tag{4.20}$$

where $\mu\tau_1 - \tau_2 \neq 0$.

(5) Let all the data of ξ be fixed except α_1 and $\mu(\alpha_1) := \mu(\xi)$. Then

$$\mu'(\alpha_1) = \frac{\mu}{\mu\beta_1 + \beta_2}|m^{[2]}(b)|^2, \tag{4.21}$$

where $\mu\beta_1 + \beta_2 \neq 0$.

(6) Let all the data of ξ be fixed except α_2 and $\mu(\alpha_2) := \mu(\xi)$. Then

$$\mu'(\alpha_2) = \frac{1}{\mu\beta_1 + \beta_2}|m^{[2]}(b)|^2, \tag{4.22}$$

where $\mu\beta_1 + \beta_2 \neq 0$.

(7) Let all the data of ξ be fixed except β_1 and $\mu(\beta_1) := \mu(\xi)$. Then

$$\mu'(\beta_1) = -\frac{\mu}{\mu\alpha_1 + \alpha_2}|m^{[1]}(b)|^2, \tag{4.23}$$

where $\mu\alpha_1 + \alpha_2 \neq 0$.

(8) Let all the data of ξ be fixed except β_2 and $\mu(\beta_2) := \mu(\xi)$. Then

$$\mu'(\beta_2) = -\frac{1}{\mu\alpha_1 + \alpha_2}|m^{[1]}(b)|^2, \tag{4.24}$$

where $\mu\alpha_1 + \alpha_2 \neq 0$.

Proof. (1) For $k \in L^1[a, b]$. Using (2.1) and integration by parts, we have

$$\begin{aligned}[\mu(\tau_1 + k) - \mu(\tau_1)] \int_a^b m\bar{n}w \mathrm{d}x =& m(b)\bar{n}^{[3]}(b) - m(a)\bar{n}^{[3]}(a) - m^{[3]}(b)\bar{n}(b) + m^{[3]}(a)\bar{n}(a) \\ &- m^{[1]}(b)\bar{n}^{[2]}(b) + m^{[1]}(a)\bar{n}^{[2]}(a) + m^{[2]}(b)\bar{n}^{[1]}(b) - m^{[2]}(a)\bar{n}^{[1]}(a).\end{aligned} \tag{4.25}$$

It follows from (2.2)–(2.5) that

$$[\mu(\tau_1 + k) - \mu(\tau_1)]m_1\bar{n}_1 = m(a)\bar{n}^{[3]}(a) - m^{[3]}(a)\bar{n}(a), \tag{4.26}$$

$$[\mu(\tau_1 + k) - \mu(\tau_1)]m_2\bar{n}_2 = m^{[2]}(a)\bar{n}^{[1]}(a) - m^{[1]}(a)\bar{n}^{[2]}(a), \tag{4.27}$$

$$[\mu(\tau_1 + k) - \mu(\tau_1)]\frac{1}{\rho_2}m_4\bar{n}_4 = m^{[1]}(b)\bar{n}^{[2]}(b) - m^{[2]}(b)\bar{n}^{[1]}(b), \tag{4.28}$$

and

$$
\begin{aligned}
&[\mu(\tau_1 + k) - \mu(\tau_1)]\frac{1}{\rho_1}m_3\bar{n}_4 \\
=&\frac{1}{\rho_1}[\tilde{\tau_2}\bar{n}(b) - \tilde{\gamma_2}\bar{n}^{[3]}(b)][\tau_1 m(b) - \gamma_1 m^{[3]}(b)] - \frac{1}{\rho_1}[\tau_2 m(b) - \gamma_2 m^{[3]}(b)][\tilde{\tau_1}\bar{n}(b) - \tilde{\gamma_1}\bar{n}^{[3]}(b)] \\
=&\frac{1}{\rho_1}(\tau_1\gamma_2 - \gamma_1\tau_2)[m^{[3]}(b)\bar{n}(b) - m(b)\bar{n}^{[3]}(b)] + \frac{1}{\rho_1}[-k\tau_2 m(b)\bar{n}(b) + k\gamma_2 m^{[3]}(b)\bar{n}(b)] \\
=&[m^{[3]}(b)\bar{n}(b) - m(b)\bar{n}^{[3]}(b)] + \frac{1}{\rho_1}\Big[- k\tau_2 m(b)\bar{n}(b) + k\gamma_2\frac{\mu\tau_1 - \tau_2}{\mu\gamma_1 - \gamma_2}m(b)\bar{n}(b)\Big] \\
=&[m^{[3]}(b)\bar{n}(b) - m(b)\bar{n}^{[3]}(b)] + \frac{1}{\rho_1}\Big[\frac{k\mu(\tau_1\gamma_2 - \gamma_1\tau_2)}{\mu\gamma_1 - \gamma_2}m(b)\bar{n}(b)\Big] \\
=&[m^{[3]}(b)\bar{n}(b) - m(b)\bar{n}^{[3]}(b)] + \frac{k\mu}{\mu\gamma_1 - \gamma_2}m(b)\bar{n}(b).
\end{aligned}
\tag{4.29}
$$

Combining (4.25)–(4.29), we obtain

$$
[\mu(\tau_1 + k) - \mu(\tau_1)]\Big[\int_a^b m\bar{n}w\,\mathrm{d}x + m_1\bar{n}_1 + m_2\bar{n}_2 + \frac{1}{\rho_1}m_3\bar{n}_3 + \frac{1}{\rho_2}m_4\bar{n}_4\Big] = \frac{k\mu}{\mu\gamma_1 - \gamma_2}m(b)\bar{n}(b). \tag{4.30}
$$

Dividing both sides of (4.30) by k and taking the limit as $k \to 0$, by Theorem 3.2, we get

$$
\mu'(\tau_1) = \frac{\mu}{\mu\gamma_1 - \gamma_2}|m(b)|^2, \tag{4.31}
$$

where $\mu\gamma_1 - \gamma_2 \neq 0$. Then (4.17) follows. The proofs of (4.18)–(4.24) are similar as that of (4.17), hence we omit the details. □

5. Conclusions

This paper gives the dependence of eigenvalues of a fourth-order differential operator with eigenparameter dependent boundary conditions. The novelty lies in the fact that the fourth-order differential operator we considered has eigenparameter dependent boundary conditions at two endpoints. By a newly defined operator $\mathcal{F}$ such that the eigenvalues of the fourth-order boundary problem being consistent with those of $\mathcal{F}$, we give the differential expressions of the eigenvalues with respect to all data.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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