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Research article

Unicity of transcendental meromorphic functions concerning differential-difference polynomials

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Abstract: Let f and g be two transcendental meromorphic functions of finite order with a Borel exceptional value ∞, let $\alpha\ (\not\equiv 0)$ be a small function of both f and g, let d, k, n, m and $v_j (j = 1, 2, \cdots, d)$ be positive integers, and let $c_j (j = 1, 2, \cdots, d)$ be distinct nonzero finite values. If $n \geq \max\{2k + m + \sigma + 5, \sigma + 2d + 3\}$, where $\sigma = v_1 + v_2 + \cdots + v_d$, and $(f^n(z)(f^m(z) - 1) \prod_{j=1}^{d} f^{v_j}(z + c_j))^{(k)}$ and $(g^n(z)(g^m(z) - 1) \prod_{j=1}^{d} g^{v_j}(z + c_j))^{(k)}$ share α CM then $f \equiv tg$, where $t^m = t^{n+\sigma} = 1$. This result extends and improves some restlts due to [1, 10, 14, 15, 19].

Keywords: meromorphic functions; small functions; unicity; differences; differential
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1. Introduction and main results

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory, see [9, 11, 16, 17]. In the following, a meromorphic function always means meromorphic in the whole complex plane. By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possible outside of an exceptional set E with finite logarithmic measure $\int_E dr/r < \infty$. A meromorphic function α is said to be a small function of f if it satisfies $T(r, \alpha) = S(r, f)$.

Let f be a nonconstant meromorphic function. The order of f is defined by

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ T(r, f)}{\log r}.$$

The exponent of convergence of poles of f is defined by

$$\lambda\left(\frac{1}{f}\right) = \overline{\lim_{r\to\infty}} \frac{\log^+ N(r, f)}{\log r}.$$

Let a be a complex number, and let f be a nonconstant meromorphic function. If

$$\overline{\lim_{r\to\infty}}\frac{\log^+ N\left(r,\frac{1}{f-a}\right)}{\log r}<\rho(f),$$

for $\rho(f)>0$; and $N\left(r,\frac{1}{f-a}\right)=O(\log r)$ for $\rho(f)=0$, then a is called a Borel exceptional value of f.

Let f and g be two meromorphic functions, and let α be a small functions of both f and g. We say that f and g share α CM(IM) if $f-\alpha$ and $g-\alpha$ have the same zeros counting multiplicities (ignoring multiplicities).

$N(r,\alpha)$ is the counting function of common zeros of $f-\alpha$ and $g-\alpha$ with the same multiplicities and the multiplicity is counted. If $N\left(r,\frac{1}{f-\alpha}\right)+N\left(r,\frac{1}{g-\alpha}\right)-2N(r,\alpha)\le S(r,f)+S(r,g)$, then we call that f and g share α CM almost.

Let $\overline{N}_{(k}(r,f)$ be the counting function for poles of f with multiplicity $\ge k$ where multiplicity is not counted. Set $N_k(r,f)=\overline{N}(r,f)+\overline{N}_{(2}(r,f)+\cdots+\overline{N}_{(k}(r,f)$.

Nevanlinna [17] proved the following famous five-value theorem.

Theorem A. *Let f and g be two nonconstant meromorphic functions, and let a_j $(j=1,2,3,4,5)$ be five distinct values in the extended complex plane. If f and g share a_j $(j=1,2,3,4,5)$ IM, then $f\equiv g$.*

In 2000, Li and Qiao [13] improved Theorem A as follows.

Theorem B. *Let f and g be two nonconstant meromorphic functions, and let α_j $(j=1,2,3,4,5)$ be five distinct small functions of both f and g. If f and g share α_j $(j=1,2,3,4,5)$ IM, then $f\equiv g$.*

Recently, value distribution in difference analogue of meromorphic functions has become a subject of some interests, see [1–4, 6–8, 10, 12, 15, 18, 19].

In 2010, Zhang [18] proved the following result.

Theorem C. *Let f and g be two transcendental entire functions of finite order, let α $(\not\equiv 0)$ be a small function of both f and g, and let c be a nonzero finite complex constant and $n\ge 7$ an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share α CM, then $f\equiv g$.*

In 2012, Chen and Chen [2] extended Theorem C as follows.

Theorem D. *Let f and g be two transcendental entire functions of finite order, let α $(\not\equiv 0)$ be a small function of both f and g, let d,n,m and $v_j(j=1,2,\cdots,d)$ be positive integers, and let $c_j(j=1,2,\cdots,d)$ be distinct nonzero finite values. If $n\ge m+8\sigma$, where $\sigma=v_1+v_2+\cdots+v_d$, and $f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f^{v_j}(z+c_j)$ and $g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g^{v_j}(z+c_j)$ share α CM, then $f\equiv tg$, where $t^m=t^{n+\sigma}=1$.*

Zhang and Yi [19], Banerjee and Majumder [1], Husna et al. [10], Sahoo and Biswas [15] continued to study this problem and proved:

Theorem E. [1] *Let f and g be two transcendental entire functions of finite order, let α $(\not\equiv 0)$ be a small function of both f and g with finitely many zeros, let d,k,n,m and $v_j(j=1,2,\cdots,d)$ be positive integers, and let $c_j(j=1,2,\cdots,d)$ be distinct nonzero finite values. If $n\ge\max\{2k+m+\sigma+5,\sigma+2d+3\}$, where $\sigma=v_1+v_2+\cdots+v_d$, and $(f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f^{v_j}(z+c_j))^{(k)}$ and $(g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g^{v_j}(z+c_j))^{(k)}$ share α CM, then $f\equiv tg$, where $t^m=t^{n+\sigma}=1$.*

In [1], Banerjee and Majumder posed a problem as follows.

Problem 1. *Whether Theorem E is valid or not for any small function?*

In 2021, Majumder and Saha [14] gave a positive answer to Problem 1 and proved:

Theorem F. *Let f and g be two transcendental entire functions of finite order, let α $(\not\equiv 0)$ be a small function of both f and g, let d,k,n,m and $v_j(j=1,2,\cdots,d)$ be positive integers, and let $c_j(j=$*

$1, 2, \cdots, d)$ *be distinct nonzero finite values. If* $n \geq \max\{2k + m + \sigma + 5, \sigma + 2d + 3\}$, *where* $\sigma = v_1 + v_2 + \cdots + v_d$, *and* $(f^n(z)(f^m(z) - 1) \prod_{j=1}^{d} f^{v_j}(z + c_j))^{(k)}$ *and* $(g^n(z)(g^m(z) - 1) \prod_{j=1}^{d} g^{v_j}(z + c_j))^{(k)}$ *share* α *CM, then* $f \equiv tg$, *where* $t^m = t^{n+\sigma} = 1$.

In this paper, we consider the case of meromophic functions and obtain:

Theorem 1. *Let* f *and* g *be two transcendental meromorphic functions of finite order with a Borel exceptional value* ∞, *let* α $(\not\equiv 0)$ *be a small function of both* f *and* g, *let* d, k, n, m *and* $v_j (j = 1, 2, \cdots, d)$ *be positive integers, and let* $c_j (j = 1, 2, \cdots, d)$ *be distinct nonzero finite values. If* $n \geq \max\{2k + m + \sigma + 5, \sigma + 2d + 3\}$, *where* $\sigma = v_1 + v_2 + \cdots + v_d$, *and* $(f^n(z)(f^m(z) - 1) \prod_{j=1}^{d} f^{v_j}(z + c_j))^{(k)}$ *and* $(g^n(z)(g^m(z) - 1) \prod_{j=1}^{d} g^{v_j}(z + c_j))^{(k)}$ *share* α *CM, then* $f \equiv tg$, *where* $t^m = t^{n+\sigma} = 1$.

Remark. *By Theorem 1, we get Theorem F.*

2. Some lemmas

Lemma 1. [9, 17] *Let* f *be a nonconstant meromorphic function, and let* k *be a positive integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f).$$

Lemma 2. [3, 6] *Let* f *be a nonconstant meromorphic function of finite order, and let* c *be a nonzero finite complex number. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f),$$

and for any $\varepsilon > 0$, *we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O\left(r^{\rho(f)-1+\varepsilon}\right).$$

Lemma 3. [16] *Let* k *be a positive integer, and let* f *be a nonconstant meromorphic function satisfying* $f^{(k)} \not\equiv 0$. *Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + T\left(r, f^{(k)}\right) - T(r, f) + S(r, f) \tag{2.1}$$

$$\leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f). \tag{2.2}$$

Lemma 4. [9, 17] *Let* f *be a nonconstant meromorphic function, and let* α, β *be two distinct small functions of* f. *Then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f - \alpha}\right) + \overline{N}\left(r, \frac{1}{f - \beta}\right) + S(r, f).$$

Lemma 5. [8] *Let* f *be a nonconstant meromorphic function of finite order, and let* c *be a nonzero finite complex number. Then*

$$N(r, f(z+c)) = N(r, f(z)) + S(r, f).$$

$$\overline{N}(r, f(z+c)) = \overline{N}(r, f(z)) + S(r, f).$$

$$T(r, f(z+c)) = T(r, f(z)) + S(r, f).$$

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r, f).$$
$$\overline{N}\left(r, \frac{1}{f(z+c)}\right) = \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f).$$
$$T\left(r, \frac{1}{f(z+c)}\right) = T\left(r, \frac{1}{f(z)}\right) + S(r, f).$$

Lemma 6. *Let f be a nonconstant meromorphic function of finite order, and let $F(z) = f^n(z)(f^m(z) - 1)\prod_{j=1}^{d} f^{v_j}(z + c_j)$, where d, n, m, c_j and $v_j (j = 1, 2, \cdots, d)$ are positive integers and $\sigma = v_1 + \cdots + v_d$. Then*

$$T(r, F) \leq (n + m + \sigma)T(r, f) + S(r, f),$$
$$(n + m + \sigma)T(r, f) \leq T(r, F) + (n + m + \sigma)N(r, f) + S(r, f).$$

Proof. By Lemma 2 and Lemma 5, we have

$$\begin{aligned}
T(r, F) = & N\left(r, f^n(z)(f^m(z) - 1)\prod\nolimits_{j=1}^{d} f^{v_j}(z + c_j)\right) \\
&+m\left(r, f^n(z)(f^m(z) - 1)f^{\sigma}(z)\prod\nolimits_{j=1}^{d}\left(\frac{f(z + c_j)}{f(z)}\right)^{v_j}\right) \\
\leq & N\left(r, f^n(z)(f^m(z) - 1)\prod\nolimits_{j=1}^{d} f^{v_j}(z + c_j)\right) \\
&+m(r, f^{n+\sigma}(f^m - 1)) + \sum_{j=1}^{d} v_j m\left(r, \frac{f(z + c_j)}{f(z)}\right) \\
\leq & (n + m + \sigma)N(r, f) + (n + m + \sigma)m(r, f) + S(r, f) \\
\leq & (n + m + \sigma)T(r, f) + S(r, f). \qquad (2.3)
\end{aligned}$$

By Lemma 2 and Lemma 5, we have

$$\begin{aligned}
(n + m + \sigma)T(r, f) = & T(r, f^{n+\sigma}(f^m - 1)) + S(r, f) \\
= & m(r, f^{n+\sigma}(f^m - 1)) + N(r, f^{n+\sigma}(f^m - 1)) + S(r, f) \\
\leq & m\left(r, \frac{f^{n+\sigma}(z)(f^m(z) - 1)}{f^n(z)(f^m(z) - 1)\prod_{j=1}^{d} f^{v_j}(z + c_j)}\right) \\
&+m\left(r, f^n(z)(f^m(z) - 1)\prod\nolimits_{j=1}^{d} f^{v_j}(z + c_j)\right) \\
&+N(r, f^{n+\sigma}(f^m - 1)) + S(r, f) \\
\leq & m\left(r, f^n(z)(f^m(z) - 1)\prod\nolimits_{j=1}^{d} f^{v_j}(z + c_j)\right) \\
&+N(r, f^{n+\sigma}(f^m - 1)) + S(r, f) \\
\leq & T(r, F) + (n + m + \sigma)N(r, f) + S(r, f). \qquad (2.4)
\end{aligned}$$

Lemma 7. [5] *Let f and g be two nonconstant meromorphic functions. If f and g share 1 CM almost, then one of the following cases must occur*

(1) $T(r, f) + T(r, g) \leq 2\left\{N_2(r, f) + N_2(r, g) + N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right)\right\} + S(r, f) + S(r, g)$;

(2) $f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, *where $a(\neq 0)$ and b are two constants.*

3. Proof of Theorem 1

Since ∞ is a Borel exceptional value of both f and g, we have

$$\lambda\left(\frac{1}{f}\right)=\overline{\lim_{r\to\infty}}\frac{\log^+ N(r,f)}{\log r}<\rho(f),\ \lambda\left(\frac{1}{g}\right)=\overline{\lim_{r\to\infty}}\frac{\log^+ N(r,g)}{\log r}<\rho(g).$$

Thus for $\frac{\rho(f)-\lambda\left(\frac{1}{f}\right)}{2}$, then exist a positive number R such that $r\geq R$, we have

$$N(r,f)<r^{\frac{\rho(f)+\lambda\left(\frac{1}{f}\right)}{2}},r\geq R. \tag{3.1}$$

Similarly, we get

$$N(r,g)<r^{\frac{\rho(g)+\lambda\left(\frac{1}{g}\right)}{2}},r\geq R. \tag{3.2}$$

Set

$$F(z)=\frac{(f^n(z)(f^m(z)-1)\prod_{j=1}^d f^{v_j}(z+c_j))^{(k)}}{\alpha(z)},$$

$$G(z)=\frac{(g^n(z)(g^m(z)-1)\prod_{j=1}^d g^{v_j}(z+c_j))^{(k)}}{\alpha(z)}.$$

Since $(f^n(z)(f^m(z)-1)\prod_{j=1}^d f^{v_j}(z+c_j))^{(k)}$ and $(g^n(z)(g^m(z)-1)\prod_{j=1}^d g^{v_j}(z+c_j))^{(k)}$ share $\alpha(z)$ CM, then $F(z)$ and $G(z)$ share 1 CM almost. By Lemma 7, we consider two cases.

Case 1.

$$\begin{aligned}&T(r,F)+T(r,G)\leq 2N_2\left(r,\frac{1}{F}\right)+2N_2\left(r,\frac{1}{G}\right)\\&+2N_2(r,F)+2N_2(r,G)+S(r,f)+S(r,g).\end{aligned} \tag{3.3}$$

Set

$$F_1(z)=f^n(z)(f^m(z)-1)\prod_{j=1}^d f^{v_j}(z+c_j),$$

$$G_1(z)=g^n(z)(g^m(z)-1)\prod_{j=1}^d g^{v_j}(z+c_j).$$

By Lemma 1 and Lemma 3, we obtain

$$\begin{aligned}N_2\left(r,\frac{1}{F_1^{(k)}}\right)=&N\left(r,\frac{1}{F_1^{(k)}}\right)-\left[N\left(r,\frac{1}{F_1^{(k)}}\right)-N_2\left(r,\frac{1}{F_1^{(k)}}\right)\right]\\ \leq&N\left(r,\frac{1}{F_1}\right)+T\left(r,F_1^{(k)}\right)-T(r,F_1)\\ &-\left[N\left(r,\frac{1}{F_1^{(k)}}\right)-N_2\left(r,\frac{1}{F_1^{(k)}}\right)\right]+S(r,f)\\ \leq&N_{k+2}\left(r,\frac{1}{F_1}\right)+T\left(r,F_1^{(k)}\right)-T(r,F_1)+S(r,f)\end{aligned} \tag{3.4}$$

$$\leq N_{k+2}\left(r,\frac{1}{F_1}\right)+k\overline{N}(r,F_1)+S(r,f). \tag{3.5}$$

Similarly,

$$N_2\left(r,\frac{1}{G_1^{(k)}}\right)\leq N_{k+2}\left(r,\frac{1}{G_1}\right)+T\left(r,G_1^{(k)}\right)-T(r,G_1)+S(r,g) \tag{3.6}$$

$$\leq N_{k+2}\left(r,\frac{1}{G_1}\right)+k\overline{N}(r,G_1)+S(r,g). \tag{3.7}$$

It follows from Lemma 6, (3.3)–(3.7) that

$$\begin{aligned}
&(n+m+\sigma)[T(r,f)+T(r,g)]\\
\leq &T(r,F_1)+T(r,G_1)+(n+m+\sigma)[N(r,f)+N(r,g)]+S(r,f)+S(r,g)\\
\leq &N_{k+2}\left(r,\frac{1}{F_1}\right)+T\left(r,F_1^{(k)}\right)-N_2\left(r,\frac{1}{F_1^{(k)}}\right)+(n+m+\sigma)N(r,f)+S(r,f)\\
&+N_{k+2}\left(r,\frac{1}{G_1}\right)+T\left(r,G_1^{(k)}\right)-N_2\left(r,\frac{1}{G_1^{(k)}}\right)+(n+m+\sigma)N(r,g)+S(r,g)\\
\leq &N_{k+2}\left(r,\frac{1}{F_1}\right)+N_2\left(r,\frac{1}{F_1^{(k)}}\right)+2N_2\left(r,F_1^{(k)}\right)+(n+m+\sigma)N(r,f)+S(r,f)\\
&+N_{k+2}\left(r,\frac{1}{G_1}\right)+N_2\left(r,\frac{1}{G_1^{(k)}}\right)+2N_2\left(r,G_1^{(k)}\right)+(n+m+\sigma)N(r,g)+S(r,g)\\
\leq &2N_{k+2}\left(r,\frac{1}{F_1}\right)+k\overline{N}(r,F_1)+2N_2\left(r,F_1^{(k)}\right)+(n+m+\sigma)N(r,f)+S(r,f)\\
&+2N_{k+2}\left(r,\frac{1}{G_1}\right)+k\overline{N}(r,G_1)+2N_2\left(r,G_1^{(k)}\right)+(n+m+\sigma)N(r,g)+S(r,g)\\
\leq &[(4+k)(1+d)+(n+m+\sigma)][N(r,f)+N(r,g)]\\
&+2(k+2+m+\sigma)[T(r,f)+T(r,g)]+S(r,f)+S(r,g).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&[n-(2k+m+\sigma+4)][T(r,f)+T(r,g)]\\
\leq &[(4+k)(1+d)+(n+m+\sigma)][N(r,f)+N(r,g)]+S(r,f)+S(r,g).
\end{aligned} \tag{3.8}$$

Without loss of generality we assume that $\rho(f)\leq\rho(g)$. By (3.1), (3.2) and (3.8), we have

$$\begin{aligned}
&[n-(2k+m+\sigma+4)]T(r,g)\\
\leq &[(4+k)(1+d)+(n+m+\sigma)]\left(r^{\frac{\rho(f)+\lambda\left(\frac{1}{f}\right)}{2}}+r^{\frac{\rho(g)+\lambda\left(\frac{1}{g}\right)}{2}}\right)+o\left(r^{\frac{\rho(f)+\lambda\left(\frac{1}{f}\right)}{2}}\right)+o\left(r^{\frac{\rho(f)+\lambda\left(\frac{1}{f}\right)}{2}}\right)\\
\leq &2[(4+k)(1+d)+(n+m+\sigma)]r^{\max\left\{\frac{\rho(f)+\lambda\left(\frac{1}{f}\right)}{2},\frac{\rho(g)+\lambda\left(\frac{1}{g}\right)}{2}\right\}}+o\left(r^{\max\left\{\frac{\rho(f)+\lambda\left(\frac{1}{f}\right)}{2},\frac{\rho(g)+\lambda\left(\frac{1}{g}\right)}{2}\right\}}\right).
\end{aligned} \tag{3.9}$$

Since $n \geq 2k+m+\sigma+5$, then by (3.9), we get $\rho(g) \leq \max\left\{\frac{\rho(f)+\lambda\left(\frac{1}{f}\right)}{2}, \frac{\rho(g)+\lambda\left(\frac{1}{g}\right)}{2}\right\}$ that is $\rho(g) < \rho(g)$, a contradiction.

Case 2.

$$F = \frac{(b+1)G+(a-b-1)}{bG+(a-b)}, \tag{3.10}$$

where $a(\neq 0)$ and b are two constants.

Obviously,

$$T(r,F) = T(r,G) + O(1). \tag{3.11}$$

Next, we consider three subcases.

Case 2.1. $b \neq 0, -1$. In the following, we consider two subcases.

Case 2.1.1. $a-b-1 \neq 0$. From (3.10), we have $\overline{N}\left(r, \frac{1}{G+\frac{a-b-1}{b+1}}\right) = \overline{N}\left(r, \frac{1}{F}\right)$.

By Nevanlinna's second fundamental theorem, we get

$$\begin{aligned} T(r,G) \leq & \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G+\frac{a-b-1}{b+1}}\right) + S(r,G) \\ \leq & \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{F}\right) + S(r,g). \end{aligned} \tag{3.12}$$

By (3.11) and (3.12), we have

$$T(r,F) + T(r,G) \leq \overline{N}(r,F) + \overline{N}(r,G) + 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}\left(r, \frac{1}{G}\right) + S(r,f) + S(r,g). \tag{3.13}$$

It follows from Lemma 3 that

$$\overline{N}\left(r, \frac{1}{F_1^{(k)}}\right) \leq N_{k+1}\left(r, \frac{1}{F_1}\right) + T\left(r, F_1^{(k)}\right) - T(r,F_1) + S(r,f). \tag{3.14}$$

$$\leq N_{k+1}\left(r, \frac{1}{F_1}\right) + k\overline{N}(r,F_1) + S(r,f). \tag{3.15}$$

Similarly,

$$\overline{N}\left(r, \frac{1}{G_1^{(k)}}\right) \leq N_{k+1}\left(r, \frac{1}{G_1}\right) + T\left(r, G_1^{(k)}\right) - T(r,G_1) + S(r,g). \tag{3.16}$$

$$\leq N_{k+1}\left(r, \frac{1}{G_1}\right) + k\overline{N}(r,G_1) + S(r,g). \tag{3.17}$$

By Lemma 6, (3.13)–(3.17), we get

$$\begin{aligned} &(n+m+\sigma)[T(r,f)+T(r,g)] \\ \leq & T(r,F_1) + T(r,G_1) + (n+m+\sigma)[N(r,f)+N(r,g)] + S(r,f) + S(r,g) \\ \leq & N_{k+1}\left(r, \frac{1}{F_1}\right) + T\left(r, F_1^{(k)}\right) - \overline{N}\left(r, \frac{1}{F_1^{(k)}}\right) + (n+m+\sigma)N(r,f) + S(r,f) \end{aligned}$$

$$
\begin{aligned}
&+N_{k+1}\left(r,\frac{1}{G_1}\right)+T\left(r,G_1^{(k)}\right)-\overline{N}\left(r,\frac{1}{G_1^{(k)}}\right)+(n+m+\sigma)N(r,g)+S(r,g)\\
\leq&N_{k+1}\left(r,\frac{1}{F_1}\right)+\overline{N}\left(r,\frac{1}{F_1^{(k)}}\right)+\overline{N}\left(r,F_1^{(k)}\right)+(n+m+\sigma)N(r,f)+S(r,f)\\
&+N_{k+1}\left(r,\frac{1}{G_1}\right)+\overline{N}\left(r,\frac{1}{G_1^{(k)}}\right)+\overline{N}\left(r,G_1^{(k)}\right)+(n+m+\sigma)N(r,g)+S(r,g)\\
\leq&2N_{k+1}\left(r,\frac{1}{F_1}\right)+k\overline{N}(r,F_1)+\overline{N}\left(r,F_1^{(k)}\right)+(n+m+\sigma)N(r,f)+S(r,f)\\
&+2N_{k+1}\left(r,\frac{1}{G_1}\right)+k\overline{N}(r,G_1)+\overline{N}\left(r,G_1^{(k)}\right)+(n+m+\sigma)N(r,g)+S(r,g)\\
\leq&[(k+1)(1+d)+(n+m+\sigma)][N(r,f)+N(r,g)]\\
&+2(k+1+m+\sigma)[T(r,f)+T(r,g)]+S(r,f)+S(r,g).
\end{aligned}
\tag{3.18}
$$

Since $n\geq 2k+m+\sigma+5$, then by (3.1), (3.2), (3.18) and using the same argument as used in Case 1, we get $\rho(g)<\rho(g)$, a contradiction.

Case 2.1.2. $a-b-1=0$. Then by (3.10), we have $\overline{N}\left(r,\frac{1}{G+\frac{1}{b}}\right)=\overline{N}(r,F)$.

By Nevanlinna's second fundamental theorem, we get

$$
\begin{aligned}
T(r,G)\leq&\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G+\frac{1}{b}}\right)+S(r,G)\\
\leq&\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+S(r,g).
\end{aligned}
\tag{3.19}
$$

Similarly,

$$
T(r,F)\leq\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+S(r,f).
\tag{3.20}
$$

It follows from (3.19) and (3.20) that

$$
T(r,F)+T(r,G)\leq 2\overline{N}(r,F)+2\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g).
\tag{3.21}
$$

By Lemma 6, (3.14)–(3.17), (3.21), we get

$$
\begin{aligned}
&(n+m+\sigma)[T(r,f)+T(r,g)]\\
\leq&T(r,F_1)+T(r,G_1)+(n+m+\sigma)[N(r,f)+N(r,g)]+S(r,f)+S(r,g)\\
\leq&N_{k+1}\left(r,\frac{1}{F_1}\right)+T\left(r,F_1^{(k)}\right)-\overline{N}\left(r,\frac{1}{F_1^{(k)}}\right)+(n+m+\sigma)N(r,f)+S(r,f)\\
&+N_{k+1}\left(r,\frac{1}{G_1}\right)+T\left(r,G_1^{(k)}\right)-\overline{N}\left(r,\frac{1}{G_1^{(k)}}\right)+(n+m+\sigma)N(r,g)+S(r,g)\\
\leq&N_{k+1}\left(r,\frac{1}{F_1}\right)+2\overline{N}\left(r,F_1^{(k)}\right)+(n+m+\sigma)N(r,f)+S(r,f)
\end{aligned}
$$

$$+N_{k+1}\left(r,\frac{1}{G_1}\right)+2\overline{N}\left(r,G_1^{(k)}\right)+(n+m+\sigma)N(r,g)+S(r,g)$$
$$\leq(n+m+\sigma+2d+2)[N(r,f)+N(r,g)]$$
$$+(k+1+m+\sigma)[T(r,f)+T(r,g)]+S(r,f)+S(r,g). \tag{3.22}$$

Since $n \geq 2k+m+\sigma+5$, then by (3.1), (3.2), (3.22), and using the same argument as used in Case 1, we get $\rho(g) < \rho(g)$, a contradiction.

Case 2.2. $b = -1$. From (3.10), we have

$$F = \frac{a}{(a+1)-G}. \tag{3.23}$$

Next, we consider two subcases.

Case 2.2.1. $a + 1 \neq 0$. Then by (3.23), we have $\overline{N}\left(r, \frac{1}{G-(a+1)}\right) = \overline{N}(r, F)$. Next, by using the same argument as used in Case 2.1.2, we get $\rho(g) < \rho(g)$, a contradiction.

Case 2.2.2. $a + 1 = 0$. Then by (3.23), we get $FG \equiv 1$. That is

$$F_1^{(k)}G_1^{(k)} \equiv \alpha^2. \tag{3.24}$$

Since ∞ is a Borel exceptional value of both f and g, then by Hadamard's factorization theorem and (3.24), we have

$$f(z) = \beta(z)e^{p_1(z)},\ g(z) = \gamma(z)e^{p_2(z)}. \tag{3.25}$$

where $\beta(\not\equiv 0,\infty)$, $\gamma(\not\equiv 0,\infty)$ are two meromophic functions and p_1, p_2 are two nonconstant polynomials with $\deg p_1 = \deg p_2$.

Hence, by the simple analysis, we get

$$T(r,\beta) = S(r,e^{p_1}),\ T(r,f) = T(r,e^{p_1}) + S(r,f). \tag{3.26}$$
$$T(r,\gamma) = S(r,e^{p_2}),\ T(r,g) = T(r,e^{p_2}) + S(r,g). \tag{3.27}$$

By (3.25), we have

$$\begin{aligned}F_1(z) &= f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f^{v_j}(z+c_j)\\ &= \beta^n(z)e^{np_1(z)}(\beta^m(z)e^{mp_1(z)}-1)\prod_{j=1}^{d}\beta^{v_j}(z+c_j)e^{v_jp_1(z+c_j)}\\ &= \beta^{n+m}(z)e^{(n+m)p_1(z)}\prod_{j=1}^{d}\beta^{v_j}(z+c_j)e^{v_jp_1(z+c_j)}\\ &\quad -\beta^n(z)e^{np_1(z)}\prod_{j=1}^{d}\beta^{v_j}(z+c_j)e^{v_jp_1(z+c_j)}.\end{aligned} \tag{3.28}$$

Further, we have

$$\begin{aligned}&\beta^{n+m}(z)e^{(n+m)p_1(z)}\prod_{j=1}^{d}\beta^{v_j}(z+c_j)e^{v_jp_1(z+c_j)}\\ =&\beta^{n+m}(z)\prod_{j=1}^{d}\beta^{v_j}(z+c_j)e^{v_j[p_1(z+c_j)-p_1(z)]}e^{(n+m+\sigma)p_1(z)}.\end{aligned} \tag{3.29}$$

Obviously,

$$T\left(r, e^{v_j[p_1(z+c_j)-p_1(z)]}\right) = S(r, e^{p_1}), \tag{3.30}$$

where $j = 1, 2, \cdots, d$.

By (3.26), (3.30) and Lemma 5, we get

$$T\left(r, \beta^{n+m}(z)\prod_{j=1}^{d}\beta^{v_j}(z+c_j)e^{v_j[p_1(z+c_j)-p_1(z)]}\right) = S(r, e^{p_1}). \tag{3.31}$$

Set

$$\beta^{n+m}(z)\prod_{j=1}^{d}\beta^{v_j}(z+c_j)e^{v_j(p_1(z+c_j)-p_1(z))} = \beta_1(z). \tag{3.32}$$

Then it follows from (3.31) and (3.32) that $\beta_1(z)(\not\equiv 0)$ is a small function of $e^{p_1(z)}$.

By (3.29) and (3.32), we get

$$\beta^{n+m}(z)e^{(n+m)p_1(z)}\prod_{j=1}^{d}\beta^{v_j}(z+c_j)e^{v_j p_1(z+c_j)} = \beta_1(z)e^{(n+m+\sigma)p_1(z)}. \tag{3.33}$$

Similarly, we get

$$-\beta^{n}(z)e^{np_1(z)}\prod_{j=1}^{d}\beta^{v_j}(z+c_j)e^{v_j p_1(z+c_j)} = \beta_2(z)e^{(n+\sigma)p_1(z)}, \tag{3.34}$$

where $\beta_2(z) = -\beta^n(z)\prod_{j=1}^{d}\beta^{v_j}(z+c_j)e^{v_j[p_1(z+c_j)-p_1(z)]}$. By using the same argument as used in (3.29)–(3.32), we get $\beta_2(z)$ is a small function of $e^{p_1(z)}$.

By (3.28), (3.33) and (3.34), we have

$$F_1(z) = \beta_1(z)e^{(n+m+\sigma)p_1(z)} + \beta_2(z)e^{(n+\sigma)p_1(z)}. \tag{3.35}$$

It follows from (3.35) that

$$\begin{aligned} F_1'(z) =& [\beta_1(z)e^{(n+m+\sigma)p_1(z)} + \beta_2(z)e^{(n+\sigma)p_1(z)}]' \\ =& \beta_1'(z)e^{(n+m+\sigma)p_1(z)} + \beta_1(z)(n+m+\sigma)p_1'(z)e^{(n+m+\sigma)p_1(z)} \\ &+ \beta_2'(z)e^{(n+\sigma)p_1(z)} + \beta_2(z)(n+\sigma)p_1'(z)e^{(n+\sigma)p_1(z)} \\ =& [\beta_1'(z) + \beta_1(z)(n+m+\sigma)p_1'(z)]e^{(n+m+\sigma)p_1(z)} \\ &+ [\beta_2'(z) + \beta_2(z)(n+\sigma)p_1'(z)]e^{(n+\sigma)p_1(z)}. \end{aligned} \tag{3.36}$$

It is easy to show that

$$T(r, \beta_1' + \beta_1(n+m+\sigma)p_1') = S(r, e^{p_1}), \tag{3.37}$$

$$T(r, \beta_2' + \beta_2(n+\sigma)p_1') = S(r, e^{p_1}). \tag{3.38}$$

Set

$$\beta_1'(z) + \beta_1(z)(n+m+\sigma)p_1'(z) = \beta_3(z), \tag{3.39}$$

$$\beta_2'(z) + \beta_2(z)(n+\sigma)p_1'(z) = \beta_4(z). \tag{3.40}$$

By (3.36)–(3.40) we have

$$F_1'(z) = \beta_3(z)e^{(n+m+\sigma)p_1(z)} + \beta_4(z)e^{(n+\sigma)p_1(z)}, \tag{3.41}$$

where $\beta_3(z), \beta_4(z)$ are two nonzero small functions of $e^{p_1(z)}$.

By mathematical induction, we obtain

$$F_1^{(k)}(z) = \beta_{2k+1}(z)e^{(n+m+\sigma)p_1(z)} + \beta_{2k+2}(z)e^{(n+\sigma)p_1(z)}, \tag{3.42}$$

where $\beta_{2k+1}(z), \beta_{2k+2}(z)$ are two nonzero small functions of $e^{p_1(z)}$.

Similarly, we get

$$G_1^{(k)}(z) = \gamma_{2k+1}(z)e^{(n+m+\sigma)p_2(z)} + \gamma_{2k+2}(z)e^{(n+\sigma)p_2(z)}, \tag{3.43}$$

where $\gamma_{2k+1}(z), \gamma_{2k+2}(z)$ are two nonzero small functions of $e^{p_2(z)}$.

By (3.24), (3.42) and (3.43), we have

$$e^{(n+\sigma)p_1}[\beta_{2k+1}e^{mp_1} + \beta_{2k+2}]e^{(n+\sigma)p_2}[\gamma_{2k+1}e^{mp_2} + \gamma_{2k+2}] = \alpha^2. \tag{3.44}$$

Thus, we deduce that the zeros of $\beta_{2k+1}e^{mp_1} + \beta_{2k+2}$ are either the zeros of α or the poles of $\gamma_{2k+1}e^{mp_2} + \gamma_{2k+2}$. Hence, by Lemma 4 we get

$$\begin{aligned} mT(r, e^{p_1}) &= T(r, e^{mp_1}) \le T(r, \beta_{2k+1}e^{mp_1}) + S(r, e^{p_1}) \\ &\le \overline{N}(r, \beta_{2k+1}e^{mp_1}) + \overline{N}\left(r, \frac{1}{\beta_{2k+1}e^{mp_1}}\right) + \overline{N}\left(r, \frac{1}{\beta_{2k+1}e^{mp_1} + \beta_{2k+2}}\right) + S(r, e^{p_1}) \\ &\le \overline{N}\left(r, \frac{1}{\alpha}\right) + \overline{N}(r, \gamma_{2k+1}e^{mp_2} + \gamma_{2k+2}) + S(r, e^{p_1}) \le S(r, e^{p_1}). \end{aligned} \tag{3.45}$$

It follows $T(r, e^{p_1}) \le S(r, e^{p_1})$, a contradiction.

Case 2.3. $b = 0$. Then by (3.10), we obtain

$$F = \frac{G + (a-1)}{a}. \tag{3.46}$$

Next, we consider two subcases.

Case 2.3.1. $a - 1 \neq 0$. Then by (3.46), we have $\overline{N}\left(r, \frac{1}{G+(a-1)}\right) = \overline{N}(r, \frac{1}{F})$. Next, by using the same argument as used in Case 2.1.1, we get $\rho(g) < \rho(g)$, a contradiction.

Case 2.3.2. $a - 1 = 0$. Then by (3.46), we get $F \equiv G$.

It follows

$$F_1(z) = G_1(z) + p(z), \tag{3.47}$$

where p is a polynomial with $\deg p \le k - 1$.

Now, we prove $p \equiv 0$. Suppose on the contrary that $p \not\equiv 0$. Then by Lemma 4, Lemma 6 and (3.47), we have

$$\begin{aligned} (n+m+\sigma)T(r, f) &\le T(r, F_1) + (n+m+\sigma)N(r, f) + S(r, f) \\ &\le \overline{N}(r, F_1) + \overline{N}\left(r, \frac{1}{F_1}\right) + \overline{N}\left(r, \frac{1}{F_1 - p}\right) + (n+m+\sigma)N(r, f) + S(r, f) \\ &\le \overline{N}(r, F_1) + \overline{N}\left(r, \frac{1}{F_1}\right) + \overline{N}\left(r, \frac{1}{G_1}\right) + (n+m+\sigma)N(r, f) + S(r, f) \\ &\le (1+m+\sigma)[T(r, f) + T(r, g)] + (n+m+\sigma+1+d)N(r, f) + S(r, f). \end{aligned} \tag{3.48}$$

Likewise,

$$(n+m+\sigma)T(r,g) \leq (1+m+\sigma)[T(r,f)+T(r,g)] \\ +(1+n+m+\sigma+d)N(r,g)+S(r,g). \tag{3.49}$$

By (3.48) and (3.49), we get

$$(n+m+\sigma)[T(r,f)+T(r,g)] \leq 2(m+\sigma+1)[T(r,f)+T(r,g)] \\ +(n+m+\sigma+d+1)[N(r,f)+N(r,g)]+S(r,f)+S(r,g). \tag{3.50}$$

Since $n \geq 2k+m+\sigma+5$, then by (3.1), (3.2), (3.50), and using the same argument as used in Case 1, we get $\rho(g) < \rho(g)$, a contradiction.

Hence,

$$F_1 \equiv G_1. \tag{3.51}$$

Set $h = \frac{f}{g}$. From (3.51), we get

$$g^m(z)(h^{m+n}(z)\prod_{j=1}^{d} h^{v_j}(z+c_j)-1) = h^n(z)\prod_{j=1}^{d} h^{v_j}(z+c_j)-1. \tag{3.52}$$

We claim that h is a constant. Suppose on the contrary that h is a nonconstant.

We assert that both $h^{m+n}(z)\prod_{j=1}^{d} h^{v_j}(z+c_j)$ and $h^n(z)\prod_{j=1}^{d} h^{v_j}(z+c_j)$ are nonconstant. Without loss of generality, we suppose that $h^{m+n}(z)\prod_{j=1}^{d} h^{v_j}(z+c_j) \equiv c$, where c is a nonzero complex number. Then

$$h^{m+n}(z) \equiv \frac{c}{\prod_{j=1}^{d} h^{v_j}(z+c_j)}.$$

By Lemma 5 and Nevanlinna's first fundamental theorem, we get

$$(n+m)T(r,h) = T(r,h^{n+m}) = T\left(r, \frac{c}{\prod_{j=1}^{d} h^{v_j}(z+c_j)}\right) + S(r,h) \\ \leq \sum_{j=1}^{d} v_j T\left(r, \frac{1}{h(z+c_j)}\right) + S(r,h) \leq \sigma T(r,h) + S(r,h). \tag{3.53}$$

From (3.53), we have

$$(n+m-\sigma)T(r,h) \leq S(r,h).$$

It follows from $n \geq 2k+m+\sigma+5$ that $T(r,h) \leq S(r,h)$, a contradiction.

Hence, from (3.52), we get

$$g^m(z) = \frac{h^n(z)\prod_{j=1}^{d} h^{v_j}(z+c_j)-1}{h^{m+n}(z)\prod_{j=1}^{d} h^{v_j}(z+c_j)-1}. \tag{3.54}$$

Thus, the zeros of $h^{m+n}(z)\prod_{j=1}^{d} h^{v_j}(z+c_j)-1$ are either the poles of $g(z)$ or the zeros of $h^n(z)\prod_{j=1}^{d} h^{v_j}(z+c_j)-1$, that is the zeros of $(h^m(z)-1)$.

By (3.54), we have

$$\begin{aligned}\frac{1}{g^m(z)} =&\frac{h^{m+n}(z)\prod_{j=1}^{d}h^{v_j}(z+c_j)-1}{h^n(z)\prod_{j=1}^{d}h^{v_j}(z+c_j)-1}\\ =&\frac{h^m(z)(h^n(z)\prod_{j=1}^{d}h^{v_j}(z+c_j)-1)+(h^m(z)-1)}{h^n(z)\prod_{j=1}^{d}h^{v_j}(z+c_j)-1}\\ =&(h^m(z)-1)\left(1+\frac{1}{h^n(z)\prod_{j=1}^{d}h^{v_j}(z+c_j)-1}\right)+1.\end{aligned} \tag{3.55}$$

It follows from Lemma 6, (3.55) and Nevanlinna's first fundamental theorem that

$$\begin{aligned}&mT(r,g)=mT\left(r,\frac{1}{g}\right)+O(1)=T\left(r,\frac{1}{g^m}\right)+O(1)\\ =&T\left(r,(h^m(z)-1)\left(1+\frac{1}{h^n(z)\prod_{j=1}^{d}h^{v_j}(z+c_j)-1}\right)\right)+O(1)\\ \geq&T(r,h^n(z)\prod_{j=1}^{d}h^{v_j}(z+c_j))-T(r,h^m)+S(r,h)\\ \geq&T(r,h^n)-T(r,\prod_{j=1}^{d}h^{v_j}(z+c_j))-T(r,h^m)+S(r,h)\\ \geq&(n-\sigma-m)T(r,h)+S(r,h).\end{aligned} \tag{3.56}$$

By Lemma 6, we obtain

$$\begin{aligned}mT(r,g)=&T(r,g^m)=T\left(r,\frac{h^n(z)\prod_{j=1}^{d}h^{v_j}(z+c_j)-1}{h^{m+n}(z)\prod_{j=1}^{d}h^{v_j}(z+c_j)-1}\right)\\ \leq&T(r,h^n(z)\prod_{j=1}^{d}h^{v_j}(z+c_j)-1)\\ &+T(r,h^{m+n}(z)\prod_{j=1}^{d}h^{v_j}(z+c_j)-1)+S(r,h)\\ \leq&(2n+m+2\sigma)T(r,h)+S(r,h).\end{aligned} \tag{3.57}$$

Since $n\geq 2k+m+\sigma+5$, then by (3.56) and (3.57), we get

$$(n-\sigma-m)T(r,h)+S(r,h)\leq mT(r,g)\leq(2n+m+2\sigma)T(r,h)+S(r,h). \tag{3.58}$$

It follows from Lemma 5 and Nevanlinna's second fundamental theorem that

$$\begin{aligned}&(m+n-\sigma)T(r,h)\leq T(r,h^{m+n}(z)\prod_{j=1}^{d}h^{v_j}(z+c_j))+S(r,h)\\ \leq&\overline{N}(r,h^{m+n}(z)\prod_{j=1}^{d}h^{v_j}(z+c_j))+\overline{N}\left(r,\frac{1}{h^{m+n}(z)\prod_{j=1}^{d}h^{v_j}(z+c_j)}\right)\\ &+\overline{N}\left(r,\frac{1}{h^{m+n}(z)\prod_{j=1}^{d}h^{v_j}(z+c_j)-1}\right)+S(r,h)\end{aligned}$$

$$\leq 2(1+d)T(r,h)+\overline{N}\left(r,\frac{1}{h^m-1}\right)+\overline{N}(r,g)+S(r,h)$$
$$\leq (2+2d+m)T(r,h)+N(r,g)+S(r,h).$$

Since $n\geq \sigma+2d+3$, then by (3.58), we have

$$\frac{m}{2n+m+2\sigma}T(r,g)\leq [n-(2+2d+\sigma)]T(r,h)\leq N(r,g)+S(r,h). \tag{3.59}$$

It follows from (3.2), (3.58) and (3.59) that $\rho(g)<\lambda\left(\frac{1}{g}\right)$, a contradiction. Therefore, h is a nonzero constant.

From (3.51), we get

$$h^{n+\sigma}(z)g^n(z)(h^m(z)g^m(z)-1)\prod_{j=1}^{d}g^{v_j}(z+c_j)=g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g^{v_j}(z+c_j).$$

Then

$$h^m\equiv 1, h^{n+\sigma}\equiv 1.$$

Therefore, $f\equiv tg$, where t is a constant such that $t^m=t^{n+\sigma}=1$.

The proof of Theorem 1 is complete.

4. Conclusions

In this paper, by using Nevanlinna theory, we study the uniqueness problem of certain type of differential-difference polynomials sharing a small function and extend the existing results to the case of meromorphic functions with a Borel exceptional value ∞.

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Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

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