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Research article

Blow-up of solutions to fractional differential inequalities involving ψ-Caputo fractional derivatives of different orders

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Abstract: We consider a fractional differential inequality involving ψ-Caputo fractional derivatives of different orders, with a polynomial nonlinearity and a singular potential term. Using the test function method and some integral inequalities, we establish nonexistence criteria of global solutions in both cases: $\lim_{t\to\infty}\psi(t) = \infty$ and $\lim_{t\to\infty}\psi(t) < \infty$.

Keywords: fractional differential inequality; global solution; nonexistence; ψ-Caputo fractional derivative; singular potential term
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1. Introduction

The theory of fractional calculus has received a great attention from several researchers working in various disciplines. Namely, it was shown that many real-world phenomena can be better modeled by using fractional operators, such as viscoelastic phenomena [5], biological phenomena [8], diffusion-wave phenomena [18], Macroeconomic models [21], and bioelectrode behaviour [16]. Consequently, the study of fractional differential equations in both theoretical and numerical aspects has attracted the attention of several mathematicians, and many contributions have been published on this subject (see e.g. [2, 3, 17, 22] for theoretical aspects, and [15, 20, 23] for numerical aspects).

In this paper, we consider the nonlinear fractional differential inequality

$$\begin{cases} {}^{C}D_a^{\alpha,\psi} u(t) + {}^{C}D_a^{\beta,\psi} u(t) \geq V(t)|u(t)|^m, \quad t > a, \\ \left(\delta_\psi^k u\right)(a) = b_k, \quad k = 0, 1, \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, $0 < \beta < 1$, $\psi : [a,\infty) \to \mathbb{R}$ is a C^2-function, $\psi'(t) > 0$ for all $t \geq a$, ${}^{C}D_a^{\tau,\psi}$, $\tau \in \{\alpha,\beta\}$, is the ψ-Caputo fractional derivative of order τ, $V > 0$ is a measurable function, $m > 1$, and $\delta_\psi^k = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k$. Namely, we are interested in obtaining sufficient conditions for which (1.1) admits no global solution. We mention below some works from the literature, related to nonexistence results for fractional differential equations and inequalities.

The study of nonexistence of global solutions to fractional differential equations (or time-fractional evolution equations) was initiated by Kirane and his collaborators (see e.g. [14, 7, 13]). In particular, Furati and Kirane [7] (see also Kirane and Malik [13]) considered systems of fractional differential equations of the form

$$\begin{cases} au'(t) + b\,{}^{C}D_0^{\alpha}u(t) = f(t)|v(t)|^q + F(t), & t > 0, \\ cv'(t) + b\,{}^{C}D_0^{\beta}v(t) = g(t)|u(t)|^p + G(t), & t > 0, \end{cases} \tag{1.2}$$

where $0 < \alpha,\beta < 2$, ${}^{C}D_0^{\tau}$, $\tau \in \{\alpha,\beta\}$, is the Caputo fractional derivative of order τ, $p,q > 1$, a,b,c,d are constants, and f, g are positive functions, while F and G are given functions with nonnegative averages. Namely, they established necessary conditions for the existence of global solutions to (1.2) in both cases: $0 < \alpha,\beta < 1$ and $1 < \alpha,\beta < 2$.

In [10], Tatar et al. considered the special case of (1.1) when $\psi(t) = t$, $a = 0$, and $V(t) = t^\gamma$. Namely, they investigated the fractional differential inequality

$$\begin{cases} {}^{C}D_0^{\alpha}u(t) + {}^{C}D_0^{\beta}u(t) \geq t^\gamma |u(t)|^m, & t > 0, \\ u^{(k)}(0) = b_k, & k = 0, 1. \end{cases} \tag{1.3}$$

It was shown that the range of values of m ensuring nonexistence, depends only on the lower order derivative. More precisely, it was shown that, if

$$b_0, b_1 \geq 0, \quad m(1-\beta) - 1 < \gamma < m - 1,$$

then (1.3) does not admit nontrivial global solution in $AC^2([0,\infty))$.

Motivated by the above mentioned contributions, the nonexistence of global solutions to (1.1) is investigated in this paper. In the special case $\psi(t) = t$, our results extend and improve those obtained in [10].

Notice that in the special case $\psi(t) = \ln t$ and $a > 0$, (1.1) reduces to

$$\begin{cases} {}^{HC}D_a^{\alpha}u(t) + {}^{HC}D_a^{\beta}u(t) \geq V(t)|u(t)|^m, & t > a, \\ \left(\delta_{\ln}^k u\right)(a) = b_k, & k = 0, 1, \end{cases}$$

where ${}^{HC}D_a^{\tau}$, $\tau \in \{\alpha,\beta\}$, is the Hadamard-Caputo fractional derivative of order τ (see [1]). For nonexistence results for fractional differential inequalities involving Hadamard-type fractional derivatives, see Tatar and his collaborators [1, 9].

The rest of the paper is organized as follows: In Section 2, we recall some preliminaries on fractional calculus and provide some lemmas. In particular, we derive an integration by parts rule for fractional integrals of a function with respect to another function. In Section 3, we state our main results, discuss some special cases, and provide some examples. Section 4 is devoted to the proofs of our main results.

2. Some preliminaries on fractional calculus

Let $a, T \in \mathbb{R}$ be such that $a < T$. The left-sided and right-sided Riemann-Liouville fractional integrals of of order $\sigma > 0$ of a function $f \in L^1([a,T])$, are defined respectively by (see [12])

$$(I_a^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_a^t (t-s)^{\sigma-1} f(s)\, ds$$

and

$$(I_T^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_t^T (s-t)^{\sigma-1} f(s)\, ds,$$

for almost everywhere $t \in [a,T]$, where Γ is the Gamma function.

We have the following integration by parts rule.

Lemma 2.1 (see [12]). *Let $\sigma > 0$, $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \sigma$ ($p = 1$, $q = 1$, in the case $\frac{1}{p} + \frac{1}{q} = 1 + \sigma$). If $f \in L^p([a,T])$ and $g \in L^q([a,T])$, then*

$$\int_a^T (I_a^\sigma f)(t) g(t)\, dt = \int_a^T f(t)(I_T^\sigma g)(t)\, dt.$$

For a positive natural number n, let

$$AC^n([a,\infty)) = \left\{ f \in C^{n-1}([a,\infty)) :\ f^{(n-1)} \in AC([a,\infty)) \right\},$$

where $AC([a,\infty))$ is the space of real-valued and absolutely continuous functions in $[a,\infty)$. Let $n-1 < \kappa < n$. The Caputo fractional derivative of order κ of a function $f \in AC^n([a,\infty))$, is defined by (see [12])

$$ {}^{C}D_a^\kappa f(t) = \left(I_a^{n-\kappa} f^{(n)} \right)(t) = \frac{1}{\Gamma(n-\kappa)} \int_a^t (t-s)^{n-\kappa-1} f^{(n)}(s)\, ds, \tag{2.1}$$

for almost everywhere $t \geq a$.

The left-sided and right-sided Hadamard fractional integrals of order $\sigma > 0$ of a function $f \in L^1([a,T])$, are defined respectively by (see [12])

$$(J_a^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_a^t \left(\ln \frac{t}{s} \right)^{\sigma-1} f(s) \frac{1}{s}\, ds$$

and

$$(J_T^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_t^T \left(\ln \frac{s}{t} \right)^{\sigma-1} f(s) \frac{1}{s}\, ds,$$

for almost everywhere $t \in [a,T]$.

For a positive natural number n, let

$$AC_{\ln}^n([a,\infty)) = \left\{ f \in C^{n-1}([a,\infty)) : \left(t\frac{d}{dt} \right)^{n-1} f \in AC([a,\infty)) \right\}.$$

Let $n-1 < \kappa < n$. The Hadamard-Caputo fractional derivative of order κ of a function $f \in AC_{\ln}^n([a,\infty))$, is defined by (see [1])

$$ {}^{HC}D_a^\kappa f(t) = \left(J_a^{n-\kappa} \left(t\frac{d}{dt} \right)^n f \right)(t) = \frac{1}{\Gamma(n-\kappa)} \int_a^t \left(\ln \frac{t}{s} \right)^{n-\kappa-1} \left(s\frac{d}{ds} \right)^n f(s) \frac{1}{s}\, ds,$$

for almost everywhere $t \geq a$.

Let n be a positive natural number. Let $\psi : [a, \infty) \to \mathbb{R}$ be a C^n-function such that

$$\psi'(t) > 0, \quad t \in [a, T].$$

The left-sided and right-sided fractional integrals of of order $\sigma > 0$ of a function $f \in L^1([a, T])$ with respect to the function ψ, are defined respectively by (see [12])

$$(I_a^{\sigma,\psi} f)(t) = \frac{1}{\Gamma(\sigma)} \int_a^t (\psi(t) - \psi(s))^{\sigma-1} \psi'(s) f(s)\, ds \tag{2.2}$$

and

$$(I_T^{\sigma,\psi} f)(t) = \frac{1}{\Gamma(\sigma)} \int_t^T (\psi(s) - \psi(t))^{\sigma-1} \psi'(s) f(s)\, ds, \tag{2.3}$$

for almost everywhere $t \in [a, T]$. It can be easily seen that, if $f \in C([a, T])$, then

$$\lim_{t\to a^+} (I_a^{\sigma,\psi} f)(t) = \lim_{t\to T^-} (I_T^{\sigma,\psi} f)(t) = 0.$$

Remark that

$$I_a^{\sigma,\psi} = \begin{cases} I_a^\sigma & \text{if} \quad \psi(t) = t, \\ J_a^\sigma & \text{if} \quad \psi(t) = \ln t,\ a > 0. \end{cases}$$

Similarly,

$$I_T^{\sigma,\psi} = \begin{cases} I_T^\sigma & \text{if} \quad \psi(t) = t, \\ J_T^\sigma & \text{if} \quad \psi(t) = \ln t,\ a > 0. \end{cases}$$

The following integration by parts rule holds.

Lemma 2.2. *Let $\sigma > 0$, $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \sigma$ ($p = 1$, $q = 1$, in the case $\frac{1}{p} + \frac{1}{q} = 1 + \sigma$). If $f \circ \psi^{-1} \in L^p([\psi(a), \psi(T)])$ and $g \circ \psi^{-1} \in L^q([\psi(a), \psi(T)])$, then*

$$\int_a^T (I_a^{\sigma,\psi} f)(t) g(t) \psi'(t)\, dt = \int_a^T f(t) (I_T^{\sigma,\psi} g)(t) \psi'(t)\, dt,$$

where $\psi^{-1} : [\psi(a), \psi(T)] \to [a, T]$ is the inverse function of $\psi : [a, T] \to [\psi(a), \psi(T)]$.

Proof. Using the change of variable $x = \psi(s)$ in (2.2), we obtain

$$(I_a^{\sigma,\psi} f)(t) = \frac{1}{\Gamma(\sigma)} \int_{\psi(a)}^{\psi(t)} (\psi(t) - x)^{\sigma-1} (f \circ \psi^{-1})(x)\, dx,$$

that is,

$$(I_a^{\sigma,\psi} f)(t) = \left(I_{\psi(a)}^\sigma f \circ \psi^{-1} \right) (\psi(t)). \tag{2.4}$$

Using the same change of variable in (2.3), we obtain

$$(I_T^{\sigma,\psi} g)(t) = \left(I_{\psi(T)}^\sigma g \circ \psi^{-1} \right) (\psi(t)). \tag{2.5}$$

By (2.4), there holds

$$\int_a^T (I_a^{\sigma,\psi} f)(t)g(t)\psi'(t)\,dt = \int_a^T \left(I_{\psi(a)}^{\sigma} f \circ \psi^{-1}\right)(\psi(t))g(t)\psi'(t)\,dt.$$

Using the change of variable $x = \psi(t)$, we obtain

$$\int_a^T (I_a^{\sigma,\psi} f)(t)g(t)\psi'(t)\,dt = \int_{\psi(a)}^{\psi(T)} \left(I_{\psi(a)}^{\sigma} f \circ \psi^{-1}\right)(x)(g \circ \psi^{-1})(x)\,dx.$$

Since $f \circ \psi^{-1} \in L^p([\psi(a), \psi(T)])$ and $g \circ \psi^{-1} \in L^q([\psi(a), \psi(T)])$, by Lemma 2.1, we deduce that

$$\int_a^T (I_a^{\sigma,\psi} f)(t)g(t)\psi'(t)\,dt = \int_{\psi(a)}^{\psi(T)} (f \circ \psi^{-1})(x)\left(I_{\psi(T)}^{\sigma} g \circ \psi^{-1}\right)(x)\,dx.$$

Using the above change of variable, there holds

$$\int_a^T (I_a^{\sigma,\psi} f)(t)g(t)\psi'(t)\,dt = \int_a^T f(t)\left(I_{\psi(T)}^{\sigma} g \circ \psi^{-1}\right)(\psi(t))\psi'(t)\,dt.$$

Thus, by (2.5), the desired result follows. □

Let us introduce the functional space (see [6])

$$AC_\psi^n([a,\infty)) = \left\{ f \in C^{n-1}([a,\infty)) : \delta_\psi^{n-1} f := \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n-1} f \in AC([a,\infty)) \right\}.$$

Let $n-1 < \kappa < n$. The ψ-Caputo fractional derivative of order κ of a function $f \in AC_\psi^n([a,\infty))$, is defined by (see [4])

$${}^C D_a^{\kappa,\psi} f(t) = \left(I_a^{n-\kappa,\psi}\delta_\psi^n f\right)(t) = \frac{1}{\Gamma(n-\kappa)}\int_a^t (\psi(t)-\psi(s))^{n-\kappa-1}\psi'(s)\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n f(s)\,ds,$$

for almost everywhere $t \geq a$.

Remark that

$${}^C D_a^{\kappa,\psi} = \begin{cases} {}^C D_a^{\kappa} & \text{if} \quad \psi(t) = t, \\ \\ {}^{HC} D_a^{\kappa} & \text{if} \quad \psi(t) = \ln t,\ a > 0. \end{cases}$$

For sufficiently large λ, let

$$\varphi(t) = (\psi(T) - \psi(a))^{-\lambda}(\psi(T) - \psi(t))^{\lambda}, \quad a \leq t \leq T. \tag{2.6}$$

By elementary calculations, we can prove the following lemma.

Lemma 2.3. *Let* $\sigma > 0$. *Then*

$$(I_T^{\sigma,\psi}\varphi)(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\sigma+\lambda+1)}(\psi(T)-\psi(a))^{-\lambda}(\psi(T)-\psi(t))^{\sigma+\lambda}, \tag{2.7}$$

$$\delta_\psi^1(I_T^{\sigma,\psi}\varphi)(t) = -\frac{\Gamma(\lambda+1)}{\Gamma(\sigma+\lambda)}(\psi(T)-\psi(a))^{-\lambda}(\psi(T)-\psi(t))^{\sigma+\lambda-1}, \tag{2.8}$$

$$\delta_\psi^2(I_T^{\sigma,\psi}\varphi)(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\sigma+\lambda-1)}(\psi(T)-\psi(a))^{-\lambda}(\psi(T)-\psi(t))^{\sigma+\lambda-2}. \tag{2.9}$$

The following estimate follows from Young's inequality.

Lemma 2.4. *Let $A, B > 0$ and $p > 1$. For all $\varepsilon > 0$, there holds*

$$AB \le \varepsilon A^p + C_\varepsilon B^{\frac{p}{p-1}},$$

where

$$C_\varepsilon = \frac{p-1}{p}(\varepsilon p)^{\frac{-1}{p-1}}.$$

3. Main results

Let $\psi : [a,\infty) \to \mathbb{R}$ be a C^2-function such that

$$\psi'(t) > 0, \quad t \ge a.$$

By a global solution to (1.1), we mean a function $u \in AC^2_\psi([a,\infty))$ satisfying

$${}^C D_a^{\alpha,\psi} u(t) + {}^C D_a^{\beta,\psi} u(t) \ge V(t)|u(t)|^m,$$

for almost everywhere $t > a$, and the initial conditions

$$\left(\delta_\psi^k u\right)(a) = b_k, \quad k = 0, 1.$$

We shall discuss two cases: $\lim_{t\to\infty} \psi(t) = \infty$ and $\lim_{t\to\infty} \psi(t) = \ell_\psi$, $\psi(a) < \ell_\psi < \infty$.

3.1. The case $\lim_{t\to\infty} \psi(t) = \infty$

Theorem 3.1. *Let $\lim_{t\to\infty} \psi(t) = \infty$. Assume that $V^{\frac{-1}{m-1}} \in L^1_{loc}([a,\infty), \psi'(t)\,dt)$.*

(i) *If $b_1 > 0$, $\alpha < \beta + 1$, and*

$$\liminf_{T\to\infty} \psi(T)^{\alpha-2-\frac{\beta m}{m-1}} \int_a^T V^{\frac{-1}{m-1}}(t)\psi'(t)\,dt = 0, \tag{3.1}$$

then (1.1) *admits no global solution.*

(ii) *If $b_0 > 0$, $\alpha > \beta + 1$, and*

$$\liminf_{T\to\infty} \psi(T)^{-1-\frac{\beta}{m-1}} \int_a^T V^{\frac{-1}{m-1}}(t)\psi'(t)\,dt = 0, \tag{3.2}$$

then (1.1) *admits no global solution.*

(iii) *If $b_0 + b_1 > 0$, $\alpha = \beta + 1$, and* (3.2) *holds, then* (1.1) *admits no global solution.*

(iv) *If $b_0 = b_1 = 0$ and*

$$\liminf_{T\to\infty} \psi(T)^{-\frac{\beta m}{m-1}} \int_a^T V^{\frac{-1}{m-1}}(t)\psi'(t)\,dt = 0, \tag{3.3}$$

then (1.1) *admits no nontrivial global solution.*

Consider the special case, where the potential function V satisfies

$$V(t) \geq C_V(\psi(t) - \psi(a))^\gamma, \quad t > a, \tag{3.4}$$

for some positive constant C_V and $\gamma < m - 1$. In this case, an elementary calculation shows that

$$\int_a^T V^{\frac{-1}{m-1}}(t)\psi'(t)\,dt \leq \frac{C_V}{1 - \frac{\gamma}{m-1}}(\psi(T) - \psi(a))^{1-\frac{\gamma}{m-1}}. \tag{3.5}$$

Therefore, since $\lim_{t\to\infty} \psi(t) = \infty$, if

$$(\alpha - \beta - 1)m < \gamma + \alpha - 1, \tag{3.6}$$

then (3.1) holds.

Suppose that $\alpha < \beta + 1$. Then, if $\gamma \geq 1 - \alpha$, then (3.6) is satisfied. Moreover, if $\gamma < 1 - \alpha$, then (3.6) is equivalent to $m > \frac{1-\alpha-\gamma}{1-\alpha+\beta}$. Hence, by the statement (i) of Theorem 3.1, we deduce that, if $b_1 > 0$, $\alpha < \beta + 1$, and

$$1 - \alpha \leq \gamma < m - 1 \quad \text{or} \quad \gamma < 1 - \alpha,\ m > \max\left\{1, \frac{1 - \alpha - \gamma}{1 - \alpha + \beta}\right\},$$

then (1.1) admits no global solution.

On the other hand, by (3.5), we deduce that, if $-\beta < \gamma$, then (3.2) is satisfied. Therefore, by the statement (ii) of Theorem 3.1, we deduce that, if $\alpha > \beta + 1$, $b_0 > 0$, and $-\beta < \gamma < m - 1$, then (1.1) admits no global solution. By the statement (iii) of Theorem 3.1, we deduce that, if $\alpha = \beta + 1$, $b_0 + b_1 > 0$, and $-\beta < \gamma < m - 1$, then (1.1) admits no global solution.

Next, by (3.6), if $m(1-\beta)-1 < \gamma < m-1$, then (3.3) holds. Hence, by the statement (iv) of Theorem 3.1, we deduce that, if $b_0 = b_1 = 0$ and $m(1-\beta)-1 < \gamma < m-1$, then (1.1) admits no nontrivial global solution

Summarizing the above results, we obtain the following corollary.

Corollary 3.1. *Let* $\lim_{t\to\infty} \psi(t) = \infty$ *and V be the potential function satisfying* (3.4).

(i) *If* $b_1 > 0$, $\alpha < \beta + 1$, *and*

$$1 - \alpha \leq \gamma < m - 1 \quad \text{or} \quad \gamma < 1 - \alpha,\ m > \max\left\{1, \frac{1 - \alpha - \gamma}{1 - \alpha + \beta}\right\},$$

then (1.1) *admits no global solution.*

(ii) *If* $b_0 > 0$, $\alpha > \beta + 1$, *or* $b_0 + b_1 > 0$, $\alpha = \beta + 1$, *and*

$$-\beta < \gamma < m - 1,$$

then (1.1) *admits no global solution.*

(iii) *If* $b_0 = b_1 = 0$ *and*

$$m(1 - \beta) - 1 < \gamma < m - 1,$$

then (1.1) *admits no nontrivial global solution.*

Remark 3.1. *Let* $\lim_{t\to\infty} \psi(t) = \infty$ *and V be the potential function satisfying* (3.4).

(a) *From the statement* (i) *of* Corollary 3.1*, if* $b_1 > 0$ *and*

$$\alpha < \beta + 1, \gamma = 0 \quad or \quad -\beta < 1 - \alpha \leq \gamma < 0 \quad or \quad -\beta \leq \gamma < 1 - \alpha,$$

then, for all $m > 1$*,* (1.1) *admits no global solution.*

(b) *From the statement* (ii) *of* Corollary 3.1*, if* $\alpha > \beta + 1$*,* $b_0 > 0$*, or* $\alpha = \beta + 1$*,* $b_0 + b_1 > 0$*, and*

$$-\beta < \gamma \leq 0,$$

then, for all $m > 1$*,* (1.1) *admits no global solution.*

(c) *From the statement* (iii) *of* Corollary 3.1*, if* $b_0 = b_1 = 0$ *and*

$$-\beta < \gamma \leq 0, \quad 1 < m < \frac{\gamma + 1}{1 - \beta},$$

then (1.1) *admits no nontrivial global solution.*

Remark 3.2. *When* $a = 0$*,* $\psi(t) = t$*, and the function* V *satisfies* (3.4)*,* (1.1) *reduces to* (1.3)*. Then* Corollary 3.1 *holds for* (1.3)*. As we mentioned in* Section 1*, in* [10]*, it was shown that, if*

$$b_0, b_1 \geq 0, \quad m(1 - \beta) - 1 < \gamma < m - 1,$$

then (1.3) *does not admit nontrivial global solution in* $AC^2([0, \infty))$*. Observe that* Corollary 3.1 *improves the obtained result in* [10] *(see for instance* Example 3.1*).*

We present below some examples to illustrate our results in the case $\lim_{t\to\infty} \psi(t) = \infty$.

Example 3.1. *Consider the fractional differential inequality*

$$\begin{cases} {}^{C}D_0^{\frac{5}{4}} u(t) + {}^{C}D_0^{\frac{1}{2}} u(t) \geq t^{\frac{-3}{8}} |u(t)|^m, & t > 0, \\ \\ (u(0), u'(0)) = (-1, 2) \end{cases} \tag{3.7}$$

where $m > 1$*. Problem* (3.7) *is a special case of problem* (1.1) *with*

$$\psi(t) = t, \quad a = 0, \quad \alpha = \frac{5}{4}, \quad \beta = \frac{1}{2}, \quad V(t) = t^\gamma, \quad \gamma = -\frac{3}{8}, \quad b_0 = -1, \quad b_1 = 2.$$

Problem (3.7) *is also a special case of problem* (1.3)*. Notice that, since* $b_0 < 0$*, the obtained result in* [10] *cannot be applied in this case (see* Remark 3.2*). On the other hand, since* $b_1 = 2 > 0$ *and* $-\beta = -\frac{1}{2} < -\frac{3}{8} = \gamma < -\frac{1}{4} = 1 - \alpha$*, by* Remark 3.1 (a)*, we deduce that for all* $m > 1$*,* (3.7) *admits no global solution.*

Example 3.2. *Consider the fractional differential inequality*

$$\begin{cases} {}^{HC}D_a^{\frac{3}{2}} u(t) + {}^{HC}D_a^{\frac{1}{2}} u(t) \geq \left(\ln \frac{t}{a}\right)^{\frac{-1}{4}} |u(t)|^m, & t > a, \\ \\ (u(a), au'(a)) = (-1, 2), \end{cases} \tag{3.8}$$

where $a > 0$ and $m > 1$. Problem (3.8) *is a special case of problem* (1.1) *with*

$$\psi(t) = \ln t, \quad \alpha = \frac{3}{2}, \quad \beta = \frac{1}{2}, \quad V(t) = \left(\ln \frac{t}{a}\right)^{\frac{-1}{4}}, \quad b_0 = -1, \quad b_1 = 2.$$

Notice that the potential function V satisfies (3.4) *with*

$$C_V = 1, \quad \gamma = -\frac{1}{4}.$$

Since $b_0 + b_1 = 1 > 0$, $\alpha = \beta + 1$, and $-\beta = -\frac{1}{2} < -\frac{1}{4} = \gamma < 0$, then, by Remark 3.1(b)*, we deduce that for all $m > 1$,* (3.8) *admits no global solution.*

Example 3.3. *Consider the fractional differential inequality*

$$\begin{cases} {}^{HC}D_a^{\frac{3}{2}} u(t) + {}^{HC}D_a^{\frac{1}{2}} u(t) \geq \left(\ln \frac{t}{a}\right)^{\frac{-1}{3}} |u(t)|^m, \quad t > a, \\ \\ (u(a), u'(a)) = (0, 0), \end{cases} \tag{3.9}$$

where $a > 0$ and $m > 1$. Then, problem (3.9) *is a special case of problem* (1.1) *with*

$$\psi(t) = \ln t, \quad \alpha = \frac{3}{2}, \quad \beta = \frac{1}{2}, \quad V(t) = \left(\ln \frac{t}{a}\right)^{\frac{-1}{3}}, \quad b_0 = b_1 = 0.$$

Observe that the potential function V satisfies (3.4) *with*

$$C_V = 1, \quad \gamma = -\frac{1}{3}.$$

Since $-\beta = -\frac{1}{2} < -\frac{1}{3} = \gamma < 0$, by Remark 3.1(c)*, we deduce that, if*

$$1 < m < \frac{4}{3},$$

then (3.9) *admits no nontrivial global solution.*

3.2. The case $\lim_{t\to\infty} \psi(t) = \ell_\psi < \infty$

Assume that

$$\lim_{t\to\infty} \psi(t) = \ell_\psi,$$

where $\psi(a) < \ell_\psi < \infty$. Let

$$L_\psi = \ell_\psi - \psi(a), \quad \lambda_{m,\alpha} = \frac{\alpha m}{m-1}, \quad C_m = 2^{\frac{1}{m-1}} (m-1) m^{\frac{-m}{m-1}}.$$

Theorem 3.2. *Let $V^{\frac{-1}{m-1}} \in L^1([a, \infty), \psi'(t)\, dt)$. If $b_0 \geq 0$ and*

$$\begin{aligned} b_1 \;\; > \;\; & C_m \Gamma(\lambda_{m,\alpha} + 3 - \alpha) \Gamma(\lambda_{m,\alpha} + 1)^{\frac{1}{m-1}} L_\psi^{\alpha-2} \\ & \times \left[\left(\frac{L_\psi^{-\beta}}{\Gamma(1 - \beta + \lambda_{m,\alpha})} \right)^{\frac{m}{m-1}} + \left(\frac{L_\psi^{-\alpha}}{\Gamma(1 - \alpha + \lambda_{m,\alpha})} \right)^{\frac{m}{m-1}} \right] \int_a^\infty V^{\frac{-1}{m-1}}(t) \psi'(t)\, dt, \end{aligned} \tag{3.10}$$

then (1.1) *admits no global solution.*

We illustrate the above result by the following example.

Example 3.4. *Consider the fractional differential inequality* (1.1) *with*

$$a=0,\quad \psi(t)=2\arctan t,\quad V\equiv 1.$$

In this case, we have $L_\psi=1$ *and*

$$\int_a^\infty V^{\frac{-1}{m-1}}(t)\psi'(t)\,dt=1.$$

Let $b_0\geq 0$ *and*

$$b_1>C_m\Gamma(\lambda_{m,\alpha}+3-\alpha)\Gamma(\lambda_{m,\alpha}+1)^{\frac{1}{m-1}}\left[\Gamma(1-\beta+\lambda_{m,\alpha})^{\frac{-m}{m-1}}+\Gamma(1-\alpha+\lambda_{m,\alpha})^{\frac{-m}{m-1}}\right].$$

Then (3.10) *is satisfied. Thus, by* Theorem 3.2*, we deduce that* (1.1) *admits no global solution.*

4. Proofs of the main results

In the proofs of our main results, we make use of the test function method introduced by Mitidieri and Pohozaev [19].

Proof of Theorem 3.1. Suppose that $u\in AC_\psi^2([a,\infty))$ is a global solution to (1.1). For sufficiently large T, we introduce the test function

$$\varphi(t)=(\psi(T)-\psi(a))^{-\lambda_{m,\alpha}}(\psi(T)-\psi(t))^{\lambda_{m,\alpha}},\quad a\leq t\leq T,$$

where $\lambda_{m,\alpha}=\frac{\alpha m}{m-1}$. Multiplying the inequality in (1.1) by $\varphi(t)\psi'(t)$, and integrating over (a,T), we obtain

$$\int_a^T V(t)|u(t)|^m\varphi(t)\psi'(t)\,dt\leq\int_a^T {}^C D_a^{\alpha,\psi}u(t)\varphi(t)\psi'(t)\,dt+\int_a^T {}^C D_a^{\beta,\psi}u(t)\varphi(t)\psi'(t)\,dt. \tag{4.1}$$

On the other hand, by Lemma 2.2, we have

$$\begin{aligned}\int_a^T {}^C D_a^{\alpha,\psi}u(t)\varphi(t)\psi'(t)\,dt &= \int_a^T\left(I_a^{2-\alpha,\psi}\delta_\psi^2u\right)\varphi(t)\psi'(t)\,dt\\ &= \int_a^T(\delta_\psi^2u)(t)\left(I_T^{2-\alpha,\psi}\varphi\right)(t)\psi'(t)\,dt\\ &= \int_a^T(\delta_\psi^1u)'(t)\left(I_T^{2-\alpha,\psi}\varphi\right)(t)\,dt.\end{aligned}$$

Integrating by parts and using the initial conditions, we obtain

$$
\begin{aligned}
&\int_a^T {}^C D_a^{\alpha,\psi} u(t)\varphi(t)\psi'(t)\,dt \\
&= \left[(\delta_\psi^1 u)(t)\left(I_T^{2-\alpha,\psi}\varphi\right)(t)\right]_{t=a}^T - \int_a^T (\delta_\psi^1 u)(t)\left(I_T^{2-\alpha,\psi}\varphi\right)'(t)\,dt \\
&= -b_1\left(I_T^{2-\alpha,\psi}\varphi\right)(a) - \int_a^T u'(t)\delta_\psi^1\left(I_T^{2-\alpha,\psi}\varphi\right)(t)\,dt \\
&= -b_1\left(I_T^{2-\alpha,\psi}\varphi\right)(a) - \left[u(t)\delta_\psi^1\left(I_T^{2-\alpha,\psi}\varphi\right)(t)\right]_{t=a}^T + \int_a^T u(t)\delta_\psi^2\left(I_T^{2-\alpha,\psi}\varphi\right)(t)\psi'(t)\,dt \\
&= -b_1\left(I_T^{2-\alpha,\psi}\varphi\right)(a) + b_0\delta_\psi^1\left(I_T^{2-\alpha,\psi}\varphi\right)(a) - u(T)\delta_\psi^1\left(I_T^{2-\alpha,\psi}\varphi\right)(T) \\
&\quad + \int_a^T u(t)\delta_\psi^2\left(I_T^{2-\alpha,\psi}\varphi\right)(t)\psi'(t)\,dt.
\end{aligned}
$$

Notice that by (2.7) and (2.8), we have

$$
\begin{aligned}
\left(I_T^{2-\alpha,\psi}\varphi\right)(a) &= \frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(3-\alpha+\lambda_{m,\alpha})}(\psi(T)-\psi(a))^{2-\alpha}, \\
\delta_\psi^1\left(I_T^{2-\alpha,\psi}\varphi\right)(a) &= -\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\alpha+\lambda_{m,\alpha})}(\psi(T)-\psi(a))^{1-\alpha}
\end{aligned}
$$

and

$$\delta_\psi^1\left(I_T^{2-\alpha,\psi}\varphi\right)(T) = 0.$$

Thus, there holds

$$
\begin{aligned}
&\int_a^T {}^C D_a^{\alpha,\psi} u(t)\varphi(t)\psi'(t)\,dt \\
&= -\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(3-\alpha+\lambda_{m,\alpha})}b_1(\psi(T)-\psi(a))^{2-\alpha} - \frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\alpha+\lambda_{m,\alpha})}b_0(\psi(T)-\psi(a))^{1-\alpha} \\
&\quad + \int_a^T u(t)\delta_\psi^2\left(I_T^{2-\alpha,\psi}\varphi\right)(t)\psi'(t)\,dt.
\end{aligned}
\tag{4.2}
$$

Similarly, by Lemma 2.2, we have

$$
\begin{aligned}
\int_a^T {}^C D_a^{\beta,\psi} u(t)\varphi(t)\psi'(t)\,dt &= \int_a^T \left(I_a^{1-\beta,\psi}\delta_\psi^1 u\right)\varphi(t)\psi'(t)\,dt \\
&= \int_a^T (\delta_\psi^1 u)(t)\left(I_T^{1-\beta,\psi}\varphi\right)(t)\psi'(t)\,dt \\
&= \int_a^T u'(t)\left(I_T^{1-\beta,\psi}\varphi\right)(t)\,dt.
\end{aligned}
$$

Integrating by parts and using the initial conditions, we obtain

$$\begin{aligned}&\int_a^T {}^{C}D_a^{\beta,\psi}u(t)\varphi(t)\psi'(t)\,dt\\&=\left[u(t)\left(I_T^{1-\beta,\psi}\varphi\right)(t)\right]_{t=a}^{T}-\int_a^T u(t)\left(I_T^{1-\beta,\psi}\varphi\right)'(t)\,dt\\&=-b_0\left(I_T^{1-\beta,\psi}\varphi\right)(a)-\int_a^T u(t)\delta_\psi^1\left(I_T^{1-\beta,\psi}\varphi\right)(t)\psi'(t)\,dt.\end{aligned}$$

On the other hand, by (2.7), we have

$$\left(I_T^{1-\beta,\psi}\varphi\right)(a)=\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\beta+\lambda_{m,\alpha})}(\psi(T)-\psi(a))^{1-\beta}.$$

Thus, we deduce that

$$\begin{aligned}&\int_a^T {}^{C}D_a^{\beta,\psi}u(t)\varphi(t)\psi'(t)\,dt\\&=-\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\beta+\lambda_{m,\alpha})}b_0(\psi(T)-\psi(a))^{1-\beta}-\int_a^T u(t)\delta_\psi^1\left(I_T^{1-\beta,\psi}\varphi\right)(t)\psi'(t)\,dt.\end{aligned}\tag{4.3}$$

Next, combining (4.1), (4.2), and (4.3), we obtain

$$\begin{aligned}&\int_a^T V(t)|u(t)|^m\varphi(t)\psi'(t)\,dt+\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(3-\alpha+\lambda_{m,\alpha})}b_1(\psi(T)-\psi(a))^{2-\alpha}\\&+\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\alpha+\lambda_{m,\alpha})}b_0(\psi(T)-\psi(a))^{1-\alpha}+\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\beta+\lambda_{m,\alpha})}b_0(\psi(T)-\psi(a))^{1-\beta}\\&\le\int_a^T|u(t)|\left|\delta_\psi^2\left(I_T^{2-\alpha,\psi}\varphi\right)(t)\right|\psi'(t)\,dt+\int_a^T|u(t)|\left|\delta_\psi^1\left(I_T^{1-\beta,\psi}\varphi\right)(t)\right|\psi'(t)\,dt.\end{aligned}\tag{4.4}$$

Now, using Lemma 2.4 with $p=m$ and $0<\varepsilon\le\frac{1}{2}$, we obtain

$$\begin{aligned}&\int_a^T|u(t)|\left|\delta_\psi^2\left(I_T^{2-\alpha,\psi}\varphi\right)(t)\right|\psi'(t)\,dt\\&\le\varepsilon\int_a^T V(t)|u(t)|^m\varphi(t)\psi'(t)\,dt+C_\varepsilon\int_a^T\left|\delta_\psi^2\left(I_T^{2-\alpha,\psi}\varphi\right)(t)\right|^{\frac{m}{m-1}}V^{\frac{-1}{m-1}}(t)\varphi^{\frac{-1}{m-1}}(t)\psi'(t)\,dt\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}&\int_a^T|u(t)|\left|\delta_\psi^1\left(I_T^{1-\beta,\psi}\varphi\right)(t)\right|\psi'(t)\,dt\\&\le\varepsilon\int_a^T V(t)|u(t)|^m\varphi(t)\psi'(t)\,dt+C_\varepsilon\int_a^T\left|\delta_\psi^1\left(I_T^{1-\beta,\psi}\varphi\right)(t)\right|^{\frac{m}{m-1}}V^{\frac{-1}{m-1}}(t)\varphi^{\frac{-1}{m-1}}(t)\psi'(t)\,dt.\end{aligned}\tag{4.6}$$

Thus, it follows from (4.4), (4.5), and (4.6) that

$$\begin{aligned}&(1-2\varepsilon)\int_a^T V(t)|u(t)|^m\varphi(t)\psi'(t)\,dt+\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(3-\alpha+\lambda_{m,\alpha})}b_1(\psi(T)-\psi(a))^{2-\alpha}\\&+\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\alpha+\lambda_{m,\alpha})}b_0(\psi(T)-\psi(a))^{1-\alpha}+\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\beta+\lambda_{m,\alpha})}b_0(\psi(T)-\psi(a))^{1-\beta}\\&\le C_\varepsilon(I_1+I_2),\end{aligned}\tag{4.7}$$

where

$$I_1 = \int_a^T \left|\delta_\psi^2 \left(I_T^{2-\alpha,\psi}\varphi\right)(t)\right|^{\frac{m}{m-1}} V^{\frac{-1}{m-1}}(t)\varphi^{\frac{-1}{m-1}}(t)\psi'(t)\,dt$$

and

$$I_2 = \int_a^T \left|\delta_\psi^1 \left(I_T^{1-\beta,\psi}\varphi\right)(t)\right|^{\frac{m}{m-1}} V^{\frac{-1}{m-1}}(t)\varphi^{\frac{-1}{m-1}}(t)\psi'(t)\,dt.$$

Taking $\varepsilon = \frac{1}{2}$ in the above inequality, there holds

$$\begin{aligned}
&\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(3-\alpha+\lambda_{m,\alpha})} b_1(\psi(T)-\psi(a))^{2-\alpha} \\
&+ \frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\alpha+\lambda_{m,\alpha})} b_0(\psi(T)-\psi(a))^{1-\alpha} + \frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\beta+\lambda_{m,\alpha})} b_0(\psi(T)-\psi(a))^{1-\beta} \\
&\le C_{\frac{1}{2}}(I_1+I_2).
\end{aligned} \tag{4.8}$$

Let us estimate the terms I_1 and I_2. By the definitions of the function φ and the constant $\lambda_{m,\alpha}$, and using (2.9), we get

$$\left|\delta_\psi^2 \left(I_T^{2-\alpha,\psi}\varphi\right)(t)\right|^{\frac{m}{m-1}} \varphi^{\frac{-1}{m-1}}(t)\psi'(t) = \left[\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(1-\alpha+\lambda_{m,\alpha})}\right]^{\frac{m}{m-1}} (\psi(T)-\psi(a))^{-\lambda_{m,\alpha}}\psi'(t),$$

which yields

$$I_1 = \left[\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(1-\alpha+\lambda_{m,\alpha})}\right]^{\frac{m}{m-1}} (\psi(T)-\psi(a))^{-\frac{\alpha m}{m-1}} \int_a^T V^{\frac{-1}{m-1}}\psi'(t)\,dt. \tag{4.9}$$

Similarly, using (2.8), we get

$$\begin{aligned}
&\left|\delta_\psi^1 \left(I_T^{1-\beta,\psi}\varphi\right)(t)\right|^{\frac{m}{m-1}} \varphi^{\frac{-1}{m-1}}(t)\psi'(t) \\
&= \left[\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(1-\beta+\lambda_{m,\alpha})}\right]^{\frac{m}{m-1}} (\psi(T)-\psi(a))^{-\lambda_{m,\alpha}}(\psi(T)-\psi(a))^{\lambda_{m,\alpha}-\frac{\beta m}{m-1}}\psi'(t),
\end{aligned}$$

which yields

$$I_2 \le \left[\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(1-\beta+\lambda_{m,\alpha})}\right]^{\frac{m}{m-1}} (\psi(T)-\psi(a))^{-\frac{\beta m}{m-1}} \int_a^T V^{\frac{-1}{m-1}}\psi'(t)\,dt. \tag{4.10}$$

Combining (4.9) with (4.10), there holds

$$\begin{aligned}
&I_1 + I_2 \\
&\le \left(\left[\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(1-\alpha+\lambda_{m,\alpha})}\right]^{\frac{m}{m-1}} (\psi(T)-\psi(a))^{-\frac{\alpha m}{m-1}} + \left[\frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(1-\beta+\lambda_{m,\alpha})}\right]^{\frac{m}{m-1}} (\psi(T)-\psi(a))^{-\frac{\beta m}{m-1}}\right) \\
&\quad \times \int_a^T V^{\frac{-1}{m-1}}\psi'(t)\,dt.
\end{aligned} \tag{4.11}$$

Since $\alpha > \beta$ and $\psi(T) \to \infty$ as $T \to \infty$, we deduce that there exists some constant $C > 0$ such that for sufficiently large T,

$$I_1 + I_2 \le C(\psi(T)-\psi(a))^{-\frac{\beta m}{m-1}} \int_a^T V^{\frac{-1}{m-1}}\psi'(t)\,dt. \tag{4.12}$$

Combining (4.8) with (4.12), we get

$$\begin{aligned} F(T) : = & \frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(3-\alpha+\lambda_{m,\alpha})} b_1(\psi(T)-\psi(a))^{2-\alpha} + \frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\alpha+\lambda_{m,\alpha})} b_0(\psi(T)-\psi(a))^{1-\alpha} \\ & + \frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\beta+\lambda_{m,\alpha})} b_0(\psi(T)-\psi(a))^{1-\beta} \\ \leq & C_{\frac{1}{2}} C(\psi(T)-\psi(a))^{-\frac{\beta m}{m-1}} \int_a^T V^{\frac{-1}{m-1}} \psi'(t)\, dt. \end{aligned} \tag{4.13}$$

Now, consider the case

$$b_1 > 0, \quad \alpha < \beta + 1.$$

In this case, there exists some constant $\overline{C} > 0$ such that for sufficiently large T,

$$F(T) \geq \overline{C}(\psi(T)-\psi(a))^{2-\alpha} b_1.$$

Thus, by (4.13), there holds

$$b_1 \leq \frac{C_{\frac{1}{2}} C}{\overline{C}} (\psi(T)-\psi(a))^{\alpha-2-\frac{\beta m}{m-1}} \int_a^T V^{\frac{-1}{m-1}} \psi'(t)\, dt.$$

Passing to the infimum limit as $T \to \infty$ in the above inequality, and using (3.1), we obtain a contradiction with $b_1 > 0$. This proves part (i) of Theorem 3.1.

Next, suppose that

$$b_0 > 0, \quad \alpha > \beta + 1.$$

In this case, there exists some constant $\widetilde{C} > 0$ such that for sufficiently large T,

$$F(T) \geq \widetilde{C}(\psi(T)-\psi(a))^{1-\beta} b_0.$$

Thus, by (4.13), there holds

$$b_0 \leq \frac{C_{\frac{1}{2}} C}{\widetilde{C}} (\psi(T)-\psi(a))^{-1-\frac{\beta}{m-1}} \int_a^T V^{\frac{-1}{m-1}} \psi'(t)\, dt.$$

Passing to the infimum limit as $T \to \infty$ in the above inequality, and using (3.2), we obtain a contradiction with $b_0 > 0$. This proves part (ii) of Theorem 3.1.

Now, suppose that

$$b_0 + b_1 > 0, \quad \alpha = \beta + 1.$$

In this case,

$$F(T) = \frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\beta+\lambda_{m,\alpha})} (\psi(T)-\psi(a))^{1-\beta}(b_0+b_1) + \frac{\Gamma(\lambda_{m,\alpha}+1)}{\Gamma(2-\alpha+\lambda_{m,\alpha})} (\psi(T)-\psi(a))^{1-\alpha} b_0.$$

Hence, there exists some constant $C > 0$ such that for sufficiently large T,

$$F(T) \geq C(\psi(T)-\psi(a))^{1-\beta}(b_0+b_1).$$

Thus, by (4.13), there holds

$$b_0 + b_1 \leq \frac{C_{\frac{1}{2}} C}{C} (\psi(T) - \psi(a))^{-1-\frac{\beta}{m-1}} \int_a^T V^{\frac{-1}{m-1}} \psi'(t)\, dt.$$

Passing to the infimum limit as $T \to \infty$ in the above inequality, and using (3.2), we obtain a contradiction with $b_0 + b_1 > 0$. This proves part (iii) of Theorem 3.1.

Finally, consider the case

$$b_0 = b_1 = 0.$$

Taking $0 < \varepsilon < \frac{1}{2}$ in (4.7), and using (4.12), we get

$$\int_a^T V(t)|u(t)|^m \varphi(t)\psi'(t)\, dt \leq \frac{C_\varepsilon C}{1 - 2\varepsilon} (\psi(T) - \psi(a))^{\frac{-\beta m}{m-1}} \int_a^T V^{\frac{-1}{m-1}} \psi'(t)\, dt.$$

Passing to the infimum limit as $T \to \infty$ in the above inequality, and using (3.3), we obtain $u \equiv 0$. This proves part (iv) of Theorem 3.1. □

Proof of Theorem 3.2. Suppose that $u \in AC_\psi^2([a, \infty))$ is a global solution to (1.1). Since $b_0 \geq 0$, by (4.8) and (4.11), we deduce that

$$\begin{aligned}
&\frac{\Gamma(\lambda_{m,\alpha} + 1)}{C_{\frac{1}{2}} \Gamma(3 - \alpha + \lambda_{m,\alpha})} b_1 (\psi(T) - \psi(a))^{2-\alpha} \\
&\leq \left(\left[\frac{\Gamma(\lambda_{m,\alpha} + 1)}{\Gamma(1 - \alpha + \lambda_{m,\alpha})} \right]^{\frac{m}{m-1}} (\psi(T) - \psi(a))^{-\frac{\alpha m}{m-1}} + \left[\frac{\Gamma(\lambda_{m,\alpha} + 1)}{\Gamma(1 - \beta + \lambda_{m,\alpha})} \right]^{\frac{m}{m-1}} (\psi(T) - \psi(a))^{-\frac{\beta m}{m-1}} \right) \\
&\times \int_a^T V^{\frac{-1}{m-1}} \psi'(t)\, dt.
\end{aligned}$$

Passing to the limit as $T \to \infty$ in the above inequality, we obtain

$$\begin{aligned}
&\frac{\Gamma(\lambda_{m,\alpha} + 1)}{C_{\frac{1}{2}} \Gamma(3 - \alpha + \lambda_{m,\alpha})} b_1 L_\psi^{2-\alpha} \\
&\leq \left(\left[\frac{\Gamma(\lambda_{m,\alpha} + 1)}{\Gamma(1 - \alpha + \lambda_{m,\alpha})} \right]^{\frac{m}{m-1}} L_\psi^{-\frac{\alpha m}{m-1}} + \left[\frac{\Gamma(\lambda_{m,\alpha} + 1)}{\Gamma(1 - \beta + \lambda_{m,\alpha})} \right]^{\frac{m}{m-1}} L_\psi^{-\frac{\beta m}{m-1}} \right) \int_a^\infty V^{\frac{-1}{m-1}} \psi'(t)\, dt,
\end{aligned}$$

that is (notice that $C_{\frac{1}{2}} = C_m$)

$$\begin{aligned}
b_1 \quad \leq \quad & C_m \Gamma(\lambda_{m,\alpha} + 3 - \alpha) \Gamma(\lambda_{m,\alpha} + 1)^{\frac{1}{m-1}} L_\psi^{\alpha-2} \\
& \times \left[\left(\frac{L_\psi^{-\beta}}{\Gamma(1 - \beta + \lambda_{m,\alpha})} \right)^{\frac{m}{m-1}} + \left(\frac{L_\psi^{-\alpha}}{\Gamma(1 - \alpha + \lambda_{m,\alpha})} \right)^{\frac{m}{m-1}} \right] \int_a^\infty V^{\frac{-1}{m-1}}(t) \psi'(t)\, dt,
\end{aligned}$$

which contradicts with (3.10). The proof of Theorem 3.2 is completed. □

5. Conclusions

Using the test function method, we investigated the nonexistence of global solutions to the fractional differential inequality (1.1). We discussed two cases: $\lim_{t\to\infty} \psi(t) = \infty$ and $\lim_{t\to\infty} \psi(t) = \ell_\psi$, $\psi(a) < \ell_\psi < \infty$. In the first case, according to the signs of the initial values b_i, $i = 0, 1$, and the fractional orders α and β, sufficient conditions for the nonexistence of global solutions are obtained (see Theorem 3.1). In the second case, we proved that, if $b_0 \geq 0$ and b_1 is sufficiently large, then (1.1) admits no global solution (see Theorem 3.2).

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Conflict of interest

The authors declare that they have no competing interests.

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