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*Research article*

# An approximate approach for fractional singular delay integro-differential equations

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**Abstract:** In this article, we present Jacobi-Gauss collocation method to numerically solve the fractional singular delay integro-differential equations, because such methods have better superiority, capability and applicability than other methods. We first apply a technique to replace the delay function in the considered equation and suggest an equivalent system. We then propose a Jacobi-Gauss collocation approach to discretize the obtained system and to achieve an algebraic system. Having solved the algebraic system, an approximate solution is gained for the original equation. Three numerical examples are solved to show the applicability of presented approximate approach. Obtaining the approximations of the solution and its fractional derivative simultaneously and an acceptable approximation by selecting a small number of collocation points are advantages of the suggested method.

**Keywords:** Caputo and Riemann-Liouville fractional derivatives; Fractional singular delay integro-differential equations; Jacobi-Gauss points; Lagrange interpolation polynomial
**Mathematics Subject Classification:** 26A33, 34K37, 45G05

## 1. Introduction

Integro-differential equations (IDEs) are a type of differential equation that includes both integrals and derivatives. These equations are widely used in physical problems and have been studied by many scientists. Cerna and Finek used the Galerkin wavelet method to numerically solve the IDEs of second order and the linear Fredholm integral equations [8]. They first made a quadratic wavelet spline base on a unit interval and applied it to the homogeneous Dirichlet boundary conditions. Internal tensor product to construction a wavelet base on hyper-rectangle is then utilized. Also, they showed that the matrices obtained from discretization have uniformly bounded condition numbers and can be approximated by sparse matrices. In [4], Andreev and Peregudova Used Lyapunov functional method to investigate the partial stability of the Volterra IDEs. They proved the uniform asymptotic stability of these equations using this method and obtained new theorems that were applied in the study of the motion stability of a hereditary mechanical system. Sakran [17] used third type of Chebyshev polynomials for solving Volterra IDEs of first order and achieved an algorithm that led to the solution of equations.

In an integral equation, if one of the limits of integral is infinite, or the kernel of equation becomes infinite at one point or at points in the integration interval, then we call that the integral equation is singular. The singular IDEs (SIDEs) is often related to the second case and various numerical studies have been performed about it. In [21], a class of SIDEs is considered. Li used Riemann-Hilbert method for solving the non-normal kind of these equations. He turned these equations into a boundary value Riemann problem with nodal points using the Fourier transform. He also proved the existence of solutions and the Noether theorem for suggested problem and obtained its general solutions. Maurya et al. Investigated the SIDEs and the Abel integral equations using the stable Lagrangian matrix method [24]. They used Lagrangian interpolation polynomials and Gram-Schmidt algorithm to generate the basic interpolation functions and the orthonormal Lagrangian basis functions, respectively. Using these functions, they transformed their equations into a set of algebraic equations and obtained the error boundaries using several intermediate mild mathematical conditions. Yang and Chen [33] used the spectral collocation method to solve nonlinear Volterra IDEs with weakly singular kernel. By applying a suitable variable, they discretized these equations at Jacobi-Gauss (JG) points and attained a system of algebraic equations. Finally, they investigated the error analysis of these equations. In [32], the Galerkin pseudo-spectral and Jacobi spectral methods are used for solving nonlinear Volterra IDE with weakly singular kernel. By changing the appropriate variables and analyzing the error, smooth solutions to this equation is obtained. Zhang and Du used multi-wavelet Legendre to achieve a new collocation method for solving the weakly singular Fredholm IDEs. They increased the space of kernel to achieve this method [30]. Zemlyanova and Machina [6], introduced two schemes based on the B-spline collocation method for numerically solving systems of higher order SIDEs, which appear in fracture problems with Steigmann-Ogden surface energy. They obtained a system of linear algebraic problems that are well-conditioned, as well as demonstrated the efficiency of their method by solving wavelet specific problems. In [35], Chen and Cheng expressed a new method based on piecewise homotopy perturbation method to solve Fredholm IDEs with weakly singular kernels, numerically. Since the common homotopy perturbation method is divergent for such equations, they modified it. To implement their method, they used the interpolation of low order and Gaussian quadrature rule.

Fractional calculus is one of the branches of mathematical science that has many applications in engineering, physics, chemistry, economics, dynamic systems, etc. The fractional calculus is involved the integrals and derivatives of any arbitrary order of real or complex. With the expansion of these calculus and the expansion of IDEs, many researchers seek to solve a class of fractional IDEs. Rajagopal et al. [20] used the Bemoulli wavelet to make a new method for solving Volterra fractional IDEs (FIDEs). They transformed these equations into a system of algebraic equations using the

Caputo fractional derivative and the Gaussian quadrature rule, and finally examined their error analysis. In [5], Legendre Chebyshev spectral method has been used to solve FIDEs. Yousefi et al. Considered the above equations of the Fredholm type and obtained a system of linear equations using the concept of fractional derivative and appropriate changing the variables. Roohollahi et al. defined two numerical methods for approximately solving the Volterra-Fredholm FIDEs and of multi order with initial values. They used generalized operational matrices of block pulse functions. In the first method, they converted the original equations into algebraic equations and obtained a system of algebraic equations by placement the initial conditions into the above system and using the properties of block pulse functions. In the second method, they used the concept of Caputo fractional derivative and turned their initial value problem into a linear system by appropriate change the variable and using the operational matrix [3]. Babaei et al. [2], used the Chebyshev collocation method of sixth kind to solve a class of FIDEs. They considered the nonlinear quadratic FIDEs of variable order and obtained the corresponding operational matrix corresponding to the Chebyshev polynomials. They applied the collocation method to these matrices and finally obtained a system of nonlinear algebraic equations. Dai and Liu [22], considered a class of nonlinear FIDEs. They propounded the stabilities of Ulam-Hyers, Ulam-Rassias and semi-Ulam-Hyers for these equations in terms of weighted space and Banach fixed-point theorem. In [29], Wang et al. investigated the stability, existence and uniqueness of coupled impulsive FIDEs with Riemann-Liouville derivatives. They used techniques of Kransnoselskii's type fixed point theorem for proving the existence and uniqueness and discussed different kinds of Ulam stabilities. Haar wavelet method was considered for the solution of linear variable order FIDEs by Amin et al. [23]. They used Caputo fractional derivatives to transform their problem into a system of algebraic equations and solved the system by using the Gauss elimination algorithm. Also, they computed maximum absolute and mean square root errors for different collocation points. In [26], Panda et al. investigated the Atangana–Baleanu Willis aneurysm system and nonlinear singular perturbations of boundary value problem for second-order fuzzy differential equation by fixed point technique and studied the existence and convergence of the solutions of these problems.

Fractional SIDEs (FSIDEs) are a novel class of FIDEs and SIDES that are a relatively new subject in mathematics and are used in various branches of physics, chemistry and engineering. Yi et al. [18] investigated nonlinear and linear FIDEs with weakly singular kernel. To solve these equations, they used the concept of Caputo fractional derivative and the Legendre-wavelets method. They applied block pulse functions to these polynomials, so obtained an operational matrix of fractional integration and then used this matrix to solve their main problem with initial conditions. In [28], a spectral method using operational matrices of the Chebyshev polynomials of type II for solving FIDEs with weakly singular kernels is obtained. Nemati et al. obtained these matrices based on the properties of the shifted Chebyshev polynomials and used them to arrive at a linear algebraic system. Du et al. proposed the least residual method to solve nonlinear SFDIEs with a weakly singular kernel [9]. They used multi-wavelet bases to reduce the space of kernel, and they applied it with cubic Legendre wavelets in space $L^2[0,1]$. They showed that their method is stable and the accuracy of solution is maintained with fast oscillations. The second kind of Chebyshev wavelets were used by Wang and Zhu to generate fractional operational matrices for solving FIDEs with a weakly singular kernel [31]. Wang and Zhu transformed these equations into an algebraic system of equations and obtained a suitable error boundary for their method. In [27], Nemati and Lima used modified hat functions to solve a class of nonlinear FIDEs with a weakly singular kernel. They obtained the fractional-order integration operational matrix for these functions and converted these equations into a second-order Volterra integral equation by using Caputo fractional derivative and finally, with this change that reduced the original equation, they achieved a solution for the original equation.

The fractional delay IDEs (FDIDEs) are one of other FIDEs which the delay time is appeared in such equations. Related to this equations, many works have been presented, for example, Ravichandran et al. investigated the exact controllability and continuous dependence of a class of neutral fractional integro-differential systems in Banach spaces by using Kranselskii's fixed point theorem and Leray-Schauder alternative theorem with a resolvent operator and some analytical methods and presented some applications of their work [7]. Also, in [14], Jothimani the existence of solution for a class of FDIDEs in Banach space based on the Banach contraction principle.

Fractional singular delay IDEs (FSDIDEs) are a special type of FDIDEs. So far, due to the complexity of FSDIDEs, no attempt has been made to solve them and no numerical or approximate method has been proposed related to them. In this paper, we intend to provide an accurate method with high convergence rate and low error for these equations. For this goal, we implement a Jacobi-Gauss collocation (JGC) method. Our process is to first convert the original equation into a time-dependent equivalent system by substituting the delay function, and then achieve an algebraic system by applying the JGC points. One of the advantages of this method is that we get the approximate solution and its fractional derivative at the collocation points, simultaneously. Here, we will demonstrate the high accuracy and efficiency of the method by providing some numerical examples. Since these types of complex equations including singularity, delay and fractional derivatives have not yet been solved, it is not possible to compare the presented work with any published work.

This paper is organized as follows. Section 2 contains some essential definitions and properties. In Section 3, we first introduce the original equation that we want to solve and implement our method. We give three numerical examples in Section 4 for displaying the preference and privilege of the method. Finally, we describe the conclusion of our work in section 5.

## 2. Preliminaries and notations

In this section, we describe some basic definitions used in this article and express some of their properties.

**Definition 2.1.** *The Caputo fractional derivative $\phi(\cdot)$ is defined*

$$ {}_a^C D_x^{\alpha}\phi(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{\phi^{(n)}(t)}{(x-t)^{(\alpha-n+1)}}dt, \quad x \in (a,b], \tag{2.1}$$

*where function $\phi(\cdot)$ is defined on interval $[a,b]$, $n-1<\alpha<n$ is the order of the Caputo fractional derivative, $n$ is a positive integer and $\Gamma(\cdot)$ is Gamma function.*

**Definition 2.2.** *Consider the above assumptions. By these, we can define the Riemann-Liouville fractional integral $\phi(\cdot)$ as follows*

$$ {}_a^C I_x^{\alpha}\phi(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\phi(t)dt, & \alpha>0, \\ \phi(x), & \alpha=0. \end{cases} \tag{2.2}$$

**Remark 2.1.** *The following are some properties of Caputo fractional derivative and Riemann-Liouville fractional integral and relations between them*

$$ {}_a^C I_x^\alpha \left(k \cdot \phi(x) + q \cdot \psi(x)\right) = k \cdot {}_a^C I_x^\alpha \phi(x) + q \cdot {}_a^C I_x^\alpha \psi(x), \tag{2.3} $$

$$ {}_a^C I_x^\alpha \left({}_a^C I_x^\beta \psi(x)\right) = {}_a^C I_x^\beta \left({}_a^C I_x^\alpha \psi(x)\right) = {}_a^C I_x^{\alpha+\beta} \psi(x), \tag{2.4} $$

$$ {}_a^C I_x^\alpha \left({}_a^C D_x^\alpha \psi(x)\right) = \psi(x) - \sum_{j=0}^{n-1} \psi^{(j)}(a) \frac{x^j}{j!} \tag{2.5} $$

$$ {}_a^C D_x^\beta \left({}_a^C I_x^\alpha \psi(x)\right) = {}_a^C I_x^{\alpha-\beta} \psi(x), \quad \beta \leq \alpha, \tag{2.6} $$

$$ {}_a^C I_x^\alpha (x^\ell) = \frac{\Gamma(\ell+1)}{\Gamma(\ell+\alpha+1)} x^{\ell+\alpha}, \quad \ell \in (-1, \infty), \tag{2.7} $$

$$ {}_a^C D_x^\alpha (x^\ell) = \begin{cases} 0, & q \in \mathbb{N}_0 \text{ and } \ell < \lceil \alpha \rceil, \\ \dfrac{\Gamma(\ell+1)}{\Gamma(\ell-\alpha+1)} t^{\ell-\alpha}, & \ell \in \mathbb{N}_0 \text{ and } \ell \geq \lceil \alpha \rceil \text{ or, } \ \ell \notin \mathbb{N}_0 \text{ and } \ell > \lceil \alpha \rceil - 1, \end{cases} \tag{2.8} $$

*where $k$ and $q$ are constants, and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.*

For further scrutiny and study of fractional derivatives and integrals, refer to the [1,10–13,15,25,34].

**Definition 2.3.** *Jacobi polynomials $\mathcal{J}_N^{\alpha,\beta}$ defined as follows*

$$ \begin{aligned} \mathcal{J}_{n+1}^{\alpha,\beta} &= \left(C_{1,n}^{\alpha,\beta} x - C_{2,n}^{\alpha,\beta}\right) \mathcal{J}_n^{\alpha,\beta}(x) - C_{3,n}^{\alpha,\beta} \mathcal{J}_{n-1}^{\alpha,\beta}, \quad n \geq 1, \\ \mathcal{J}_0^{\alpha,\beta}(x) &= 1, \quad \mathcal{J}_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta), \end{aligned} \tag{2.9} $$

*belong to category of classical orthogonal polynomials that are orthogonal with respect to weight function $\omega^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$. $I$ is the interval defined as $[-1, 1]$ for $\alpha, \beta$ and*

$$ \begin{aligned} C_{1,n}^{\alpha,\beta} &= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}, \\ C_{2,n}^{\alpha,\beta} &= \frac{(\beta^2-\alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \\ C_{3,n}^{\alpha,\beta} &= \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \end{aligned} \tag{2.10} $$

**Definition 2.4.** *For $1 \leq q \leq \infty$, The space $L_{\omega^{\alpha,\beta}}^q(I)$ is defined as follows*

$$ L_{\omega^{\alpha,\beta}}^q(I) = \{f : f \text{ is measurable and } \|f\|_{L_{\omega^{\alpha,\beta}}^q} < \infty\}, $$

*where*

$$ \begin{cases} \|f\|_{L_{\omega^{\alpha,\beta}}^q(I)} = \left(\int_{-1}^1 |f(x)|^q \omega^{\alpha,\beta}(x) dx\right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \|f\|_{L_{\omega^{\alpha,\beta}}^\infty(I)} = ess \ \sup_{x \in I} |f(x)|, & q = \infty. \end{cases} $$

*Inner product in the $L_{\omega^{\alpha,\beta}}^2(I)$ space is expressed as*

$$ (\phi, \psi)_{L_{\omega^{\alpha,\beta}}^2(I)} = \int_{-1}^1 \phi(x)\psi(x)\omega^{\alpha,\beta}(x) dx, \quad \forall \phi, \psi \in L_{\omega^{\alpha,\beta}}^2(I), $$

*and the set $\{\mathcal{J}_n^{\alpha,\beta}(x)\}_{n=0}^\infty$ constitutes a complete orthogonal $L_{\omega^{\alpha,\beta}}^2(I)$ space.*

**Definition 2.5.** *We have the JG integration formula as follows*

$$\int_{-1}^{1}\phi(x)\omega^{\alpha,\beta}(x)dx \simeq \sum_{j=0}^{N}\phi(x_j^{\alpha,\beta})\omega_j^{\alpha,\beta}, \tag{2.11}$$

*where* $\{x_j^{\alpha,\beta}\}_{j=0}^{N}$ *and* $\{\omega_j^{\alpha,\beta}\}_{j=0}^{N}$ *are the JG points and corresponding weights, respectively and N is a given positive integer.*

**Definition 2.6.** *The Lagrange interpolating polynomial* $\phi \in C(I)$ *for space* $\mathcal{P}_N$ *is expressed as follows*

$$I_N^{\alpha,\beta}\phi(x) = \sum_{j=0}^{N}\phi(t_j^{\alpha,\beta})L_j(x), \tag{2.12}$$

*where* $I_N^{\alpha,\beta}\phi(t_j^{\alpha,\beta})$ *satisfying in* $I_N^{\alpha,\beta}\phi(t_j^{\alpha,\beta}) = \phi(t_j^{\alpha,\beta}), 0 \le j \le N$, $L_j(x), j = 0, 1, \ldots, N$ *are the Lagrange interpolating basis functions associated with* $\{t_j^{\alpha,\beta}\}_{j=0}^{N}$ *and space* $\mathcal{P}_N$ *is the space of all polynomials that are almost N.*

## 3. Implementing the JGC method for FSDIDEs

In this paper, we consider the following nonlinear FSDIDE:

$$\begin{cases} {}_0^C D_z^{\alpha} f(z) = Y(z, f(z), f(z-\mu)) + \int_0^z (z-t)^{-\beta} X(z,t,f(t))dt, & 0 \le z \le T, \\ f(z) = g(z), \quad -\mu \le z \le 0, \end{cases} \tag{3.1}$$

where, $Y, X : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : [0,T] \to \mathbb{R}^n$ are the given continuously differentiable functions, $f : \mathbb{R}^n \to \mathbb{R}$ is an unknown function, $0 < \mu < T$ is a given delay parameter and $0 < \alpha \le 1$.

According to parameter $\mu$, we have the equivalent system

$$\begin{cases} {}_0^C D_z^{\alpha} f(z) = \begin{cases} Y(z, f(z), g(z-\mu)) + \int_0^z (z-t)^{-\beta} X(z,t,f(t))\,dt, & 0 \le z \le \mu, \\ Y(z, f(z), f(z-\mu)) + \int_0^z (z-t)^{-\beta} X(z,t,Y(t))\,dt, & \mu \le z \le T, \end{cases} \\ f(0) = g(0). \end{cases} \tag{3.2}$$

We want to apply the JG collocation approach to solve (3.2). For this goal, we need to convert the time interval of equation into the interval $[-1, 1]$. So, we first assume the following change of variables

$$z = \frac{T}{2}(1+\theta), \quad \theta = \frac{2z}{T} - 1, \quad t = \frac{T}{2}(1+\vartheta), \quad s = \frac{2t}{T} - 1, \quad \mu = \frac{T}{2}(1+\bar{\mu}). \tag{3.3}$$

We use the above relations for applying the Jacobi polynomials. Let $u(\theta) = f(\frac{T}{2}(1+\theta))$. By using it, we adopt $u(\theta - \bar{\mu} - 1) = f(z - \mu)$ and therefore we have

$$\begin{cases} \bar{g}(\theta - \bar{\mu}) = g(\frac{T}{2}(\theta - \bar{\mu})), \\ \bar{Y}(\theta, u(\theta), u(\theta - \bar{\mu} - 1)) = Y(\frac{T}{2}(1+\theta), f(\frac{T}{2}(1+\theta)), f(\frac{T}{2}(1+\theta) - \frac{T}{2}(1+\bar{\mu}))), \\ \bar{X}(\theta, \vartheta, u(\vartheta)) = X(\frac{T}{2}(1+\theta), \frac{T}{2}(1+\vartheta), f(\frac{T}{2}(1+\vartheta))). \end{cases} \tag{3.4}$$

With the help of the following Lemma, we will continue the process of implementing our method

**Lemma 3.1.** *Assume that* $f : [0,T] \to \mathbb{R}$ *is a given function that is differentiable and* $u(\theta) = f\left(\frac{T}{2}(1+\theta)\right)$, $\theta \in (-1,1]$ *for this function. We have the following for* $z = \frac{T}{2}(1+\theta)$ *and any* $\theta \in (-1,1]$

$$\begin{cases} {}_0^C I_z^\alpha f(z) = \left(\frac{T}{2}\right)^\alpha {}_{-1}^C I_\theta^\alpha u(\theta), \\ {}_0^C D_z^\alpha f(z) = \left(\frac{2}{T}\right)^\alpha {}_{-1}^C D_\theta^\alpha u(\theta), \quad \theta \in (-1,1]. \end{cases} \tag{3.5}$$

*Proof.* See Theorem 4.1 in [16] and Lemma 1 in [19]. □

We have the following equivalent system by (3.3)–(3.5)

$$\begin{cases} {}_{-1}^C D_\theta^\alpha u(\theta) = \begin{cases} \left(\frac{T}{2}\right)^\alpha \bar{Y}(\theta, u(\theta), \bar{g}(\theta-\bar{\mu})) + \left(\frac{T}{2}\right)^{\alpha-\beta+1} \int_{-1}^\theta (\theta-\vartheta)^{-\beta} \bar{X}(\theta,\vartheta,u(\vartheta))\,d\vartheta, & -1 \le \theta \le \bar{\mu}, \\ \left(\frac{T}{2}\right)^\alpha \bar{Y}(\theta, u(\theta), u(\theta-\bar{\mu}-1)) + \left(\frac{T}{2}\right)^{\alpha-\beta+1} \int_{-1}^\theta (\theta-\vartheta)^{-\beta} \bar{X}(\theta,\vartheta,u(\vartheta))\,d\vartheta, & \bar{\mu} \le \theta \le 1, \end{cases} \\ u(-1) = \bar{g}(-1). \end{cases} \tag{3.6}$$

Let $\Psi(\theta) = {}_{-1}^C D_\theta^\alpha u(\theta)$. We can convert system (3.6) into the following system by (2.2) and (2.5)

$$\begin{cases} \Psi(\theta) = \begin{cases} \left(\frac{T}{2}\right)^\alpha \bar{Y}(\theta, u(\theta), \bar{g}(\theta-\bar{\mu})) + \left(\frac{T}{2}\right)^{\alpha-\beta+1} \int_{-1}^\theta (\theta-\vartheta)^{-\beta} \bar{X}(\theta,\vartheta,u(\vartheta))\,d\vartheta, & -1 \le \theta \le \bar{\mu}, \\ \left(\frac{T}{2}\right)^\alpha \bar{Y}(\theta, u(\theta), u(\theta-\bar{\mu}-1)) + \left(\frac{T}{2}\right)^{\alpha-\beta+1} \int_{-1}^\theta (\theta-\vartheta)^{-\beta} \bar{X}(\theta,\vartheta,u(\vartheta))\,d\vartheta, & \bar{\mu} \le \theta \le 1, \end{cases} \\ u(\theta) = \frac{1}{\Gamma(\alpha)} \int_{-1}^\theta (\theta-\vartheta)^{\alpha-1}\Psi(\vartheta)d\vartheta + \bar{g}(-1). \end{cases} \tag{3.7}$$

In order to approximate the singular integrals with the JG formula, we use $\tau_1(\theta,\tau) = \frac{1+\theta}{2}\tau + \frac{\theta-1}{2}$, then we have

$$\begin{cases} \Psi(\theta) \begin{cases} \left(\frac{T}{2}\right)^\alpha \bar{Y}(\theta, u(\theta), \bar{g}(\theta-\bar{\mu})) + \left(\frac{T}{2}\right)^{\alpha-\beta+1} \left(\frac{1+\theta}{2}\right)^{1-\beta} \int_{-1}^1 (1-\tau)^{-\beta} \bar{X}(\theta,\tau_1(\theta,\tau),u(\tau_1(\theta,\tau)))\,d\tau, & -1 \le \theta \le \bar{\mu}, \\ \left(\frac{T}{2}\right)^\alpha \bar{Y}(\theta, u(\theta), u(\theta-\bar{\mu}-1)) + \left(\frac{T}{2}\right)^{\alpha-\beta+1} \left(\frac{1+\theta}{2}\right)^{1-\beta} \int_{-1}^1 (1-\tau)^{-\beta} \bar{X}(\theta,\tau_1(\theta,\tau),u(\tau_1(\theta,\tau)))\,d\tau, & \bar{\mu} \le \theta \le 1, \end{cases} \\ u(\theta) = \frac{1}{\Gamma(\alpha)} \left(\frac{1+\theta}{2}\right)^\alpha \int_{-1}^1 (1-\tau)^{\alpha-1}\Psi(\tau_1(\theta,\tau))\,d\tau + \bar{g}(-1). \end{cases} \tag{3.8}$$

We put $\zeta = 1-\alpha$ and select the JG points $\{\theta_i^{-\zeta,-\zeta}\}$ and let $\omega^{-\zeta,-\zeta}$ be their corresponding weight function. $u_N(\theta-\bar{\mu}-1) = \sum_{j=0}^N u_j \mathcal{L}_j(\theta-\bar{\mu}-1)$ is obtained by defining the following relations

$$u_i = u(\theta_i^{-\zeta,-\zeta}), \quad \Psi_i = \Psi(\theta_i^{-\zeta,-\zeta}), \quad u_N(\theta) = \sum_{j=0}^N u_j \mathcal{L}_j(\theta), \quad \Psi_N(\theta) = \sum_{j=0}^N \Psi_j \mathcal{L}_j(\theta). \tag{3.9}$$

By JG integration formula (2.11), we can approximate two integrals in (3.8) as follows

$$\int_{-1}^1 (1-\tau)^{-\beta} \bar{X}\left(\theta_i^{-\zeta,-\zeta}, \tau_1(\theta_i^{-\zeta,-\zeta},\tau), u\left(\tau_1(\theta_i^{-\zeta,-\zeta},\tau)\right)\right) d\tau \simeq \sum_{k=0}^N \bar{X}\left(\theta_i^{-\zeta,-\zeta}, \tau_1(\theta_i^{-\zeta,-\zeta},\tau_k), u_N\left(\tau_1(\theta_i^{-\zeta,-\zeta},\tau_k)\right)\right) \omega_k^{-\beta,0}$$

$$\simeq \sum_{k=0}^{N} \bar{X}\left(\theta_i^{-\zeta,-\zeta}, \tau_1(\theta_i^{-\zeta,-\zeta}, \tau_k), \sum_{j=0}^{N} u_j \mathcal{L}_j\left(\tau_1(\theta_i^{-\zeta,-\zeta}, \tau_k)\right)\right)\omega_k^{-\beta,0},$$

$$\int_{-1}^{1}(1-\tau)^{-\zeta}\Psi\left(\tau_1(\theta_i^{-\zeta,-\zeta},\tau)\right)d\tau \simeq \sum_{k=0}^{N}\Psi_N\left(\tau_1(\theta_i^{-\zeta,-\zeta},\hat{\tau}_k)\right)\omega_k^{-\zeta,0} \simeq \sum_{k=0}^{N}\sum_{j=0}^{N}\Psi_j\mathcal{L}_j\left(\tau_1(\theta_i^{-\zeta,-\zeta},\hat{\tau}_k)\right)\omega_k^{-\zeta,0}, \tag{3.10}$$

where $\{\tau_k\}_{k=0}^N$ and $\{\hat{\tau}_k\}_{k=0}^N$ are two sets of JG points and $\{\omega_k^{-\beta,0}\}_{k=0}^N$ and $\{\omega_k^{-\zeta,0}\}_{k=0}^N$ are their corresponding weights.

Now, we can convert system (3.8) to the following linear algebraic system by using the aforementioned relations

$$\begin{cases} \Psi_i = \begin{cases} \left(\frac{T}{2}\right)^{\alpha}\bar{Y}\left(\theta_i^{-\zeta,-\zeta}, u_i, \bar{g}(\theta_i^{-\zeta,-\zeta}-\bar{\mu})\right) \\ \quad +\left(\frac{1+\theta_i^{-\zeta,-\zeta}}{2}\right)^{1-\beta}\left(\frac{T}{2}\right)^{\alpha-\beta+1}\sum_{k=0}^{N}\bar{X}\left(\theta_i^{-\zeta,-\zeta},\tau_1(\theta_i^{-\zeta,-\zeta},\tau_k),\sum_{j=0}^{N}u_j\mathcal{L}_j\left(\tau_1(\theta_i^{-\zeta,-\zeta},\tau_k)\right)\right)\omega_k^{-\beta,0}, \\ \qquad i=1,2,...,l_{\bar{\mu}}, \\ \left(\frac{T}{2}\right)^{\alpha}\bar{Y}\left(\theta_i^{-\zeta,-\zeta},u_i,\sum_{j=0}^{N}u_j\mathcal{L}_j(\theta_i^{-\zeta,-\zeta}-\bar{\mu}-1)\right)+\left(\frac{1+\theta_i^{-\zeta,-\zeta}}{2}\right)^{1-\beta}\left(\frac{T}{2}\right)^{\alpha-\beta+1} \\ \quad \cdot\sum_{k=0}^{N}\bar{X}\left(\theta_i^{-\zeta,-\zeta},\tau_1(\theta_i^{-\zeta,-\zeta},\tau_k),\sum_{j=0}^{N}u_j\mathcal{L}_j\left(\tau_1(\theta_i^{-\zeta,-\zeta},\tau_k)\right)\right)\omega_k^{-\beta,0}, \\ \qquad i=l_{\bar{\mu}}+1,\dots,N, \end{cases} \\ u_i = \frac{1}{\Gamma(\alpha)}\left(\frac{1+\theta_i^{-\zeta,-\zeta}}{2}\right)^{\alpha}\sum_{k=0}^{N}\sum_{j=0}^{N}\Psi_j\mathcal{L}_j\left(\tau_1(\theta_i^{-\zeta,-\zeta},\hat{\tau}_k)\right)\omega_k^{-\zeta,0}+\bar{g}(-1), \quad i=1,2,\dots,N, \end{cases} \tag{3.11}$$

where $(\Psi_0,\Psi_1,\dots,\Psi_N)$ and $(u_0,u_1,\dots,u_N)$ are the unknowns of problem. $l_{\bar{\mu}}$ applies in $\theta_{l_{\bar{\mu}}} \le \bar{\mu} < \theta_{l_{\bar{\mu}}+1} < T$. As we see, the approximate solution values and their fractional derivatives are obtained, synchronously.

## 4. Numerical examples

In this section, we demonstrate the efficiency and advantages of our method by providing three examples. Here, we consider the errors by introducing $f^*(\cdot)$ and ${}_0^C D_z^{\alpha} f^*(\cdot)$ as the exact solution and exact fractional derivative and $f(\cdot)$ and ${}_0^C D_z^{\alpha} f(\cdot)$ as the approximate solution and approximate fractional derivative

$$\begin{cases} E_{\alpha,N}(z) = |{}_0^C D_z^{\alpha} f(z) - {}_0^C D_z^{\alpha} f^*(z)|, \quad 0 < z \le T, \\ E_N(z) = |f(z) - f^*(z)|, \quad 0 \le z \le T, \\ E_{Res}(z) = |{}_0^C D_z^{\alpha} f(z) - Y(z, f(z), f(z-\mu)) - \int_0^z (z-t)^{-\beta} X(z,t,f(t))dt|. \end{cases}$$

**Example 4.1.** *Consider the following SFDDIE*

$$\begin{cases} {}_0^C D_z^{\alpha} f(z) = f(z)f(z-\mu) + \int_0^z (z-t)^{-\frac{1}{3}} f^3(t)dt + \frac{\Gamma(3)}{\Gamma(3-\alpha)} z^{2-\alpha} - z^2(z-\mu)^2 - \frac{19683 z^{20/3}}{52360}, \quad 0 \le z \le 1 \\ f(z) = g(z), \quad -\mu \le z \le 0, \end{cases} \tag{4.1}$$

*where $g(z) = z^2, z \le 0$ and the exact solution $f(z)$ is $z^2$. The equivalent form of the above system using* *system* (3.11) *is as follows*

$$\begin{cases} \Psi_i = \begin{cases} \left(\frac{1}{2}\right)^{\alpha+2}(\theta_i^{-\zeta,-\zeta}-\bar{\mu})^2u_i+\left(\frac{1+\theta_i^{-\zeta,-\zeta}}{2}\right)^{\frac{2}{3}}\left(\frac{1}{2}\right)^{\alpha+\frac{2}{3}}\sum_{k=0}^{N}\sum_{j=0}^{N}\frac{(1+\theta_i^{-\zeta,-\zeta})(1+\tau_k)}{4}u_j^3\mathcal{L}_j\left(\tau_1(\theta_i^{-\zeta,-\zeta},\tau_k)\right)\omega_k^{-\frac{1}{3},0} \\ \quad +\frac{\Gamma(3)}{\Gamma(3-\alpha)}(\frac{1}{2}(1+\theta_i^{-\zeta,-\zeta}))^{2-\alpha}-\left(\frac{(1+\theta_i^{-\zeta,-\zeta})^2}{4}\right)(\frac{1+\theta_i^{-\zeta,-\zeta}}{2}-\mu)^2-\frac{19683(\frac{1}{2}(1+\theta_i^{-\zeta,-\zeta}))^{20/3}}{52360},\ i=1,2,...,l_{\bar{\mu}}, \\ \left(\frac{1}{2}\right)^{\alpha}\sum_{j=0}^{N}u_iu_j\mathcal{L}_j\left(\theta_i^{-\zeta,-\zeta}-\bar{\mu}-1\right)+\left(\frac{1+\theta_i^{-\zeta,-\zeta}}{2}\right)^{\frac{2}{3}}\left(\frac{1}{2}\right)^{\alpha+\frac{2}{3}}\sum_{k=0}^{N}\sum_{j=0}^{N}\frac{(1+\theta_i^{-\zeta,-\zeta})(1+\tau_k)}{4}u_j^3 \\ \quad \cdot\mathcal{L}_j\left(\tau_1(\theta_i^{-\zeta,-\zeta},\tau_k)\right)\omega_k^{-\frac{1}{3},0}+\frac{\Gamma(3)}{\Gamma(3-\alpha)}(\frac{1}{2}(1+\theta_i^{-\zeta,-\zeta}))^{2-\alpha}-\left(\frac{(1+\theta_i^{-\zeta,-\zeta})^2}{4}\right)(\frac{1+\theta_i^{-\zeta,-\zeta}}{2}-\mu)^2-\frac{19683(\frac{1}{2}(1+\theta_i^{-\zeta,-\zeta}))^{20/3}}{52360}, \\ \quad i=l_{\bar{\mu}}+1,\dots,N, \end{cases} \\ u_i=\frac{1}{\Gamma(\alpha)}\left(\frac{1+\theta_i^{-\zeta,-\zeta}}{2}\right)^{\alpha}\sum_{k=0}^{N}\sum_{j=0}^{N}\Psi_j\mathcal{L}_j\left(\tau_1(\theta_i^{-\zeta,-\zeta},\hat{\tau}_k)\right)\omega_k^{-\zeta,0},\ \ i=1,2,\dots,N, \end{cases} \tag{4.2}$$

*where* $\tau_1(\theta_i^{-\zeta,-\zeta},\tau_k)=\frac{1+\theta_i^{-\zeta,-\zeta}}{2}\tau_k+\frac{\theta_i^{-\zeta,-\zeta}-1}{2}$. *The approximate and exact solutions for* $\mu=0.25$*,* $\alpha=0.5$ *and* $N=7$ *are demonstrated in Figure 1. The exact and approximate solutions for* $\mu=0.25$*,* $\alpha=0.25,0.5,0.75$ *and* $N=7$ *are represented in Figure 2. The error of approximate solution for* $\mu=0.25$*,* $\alpha=0.5$ *and* $N=5,7,9$ *is illustrated in Figure 3. We can see the error of approximate solution with* $\mu=0.25$*,* $\alpha=0.25,0.5,0.75$ *and* $N=7$ *in Figure 4 and finally, the error with* $\mu=0.25,0.5,0.75$*,* $\alpha=0.5$ *and* $N=7$ *is showed in Figure 5.*

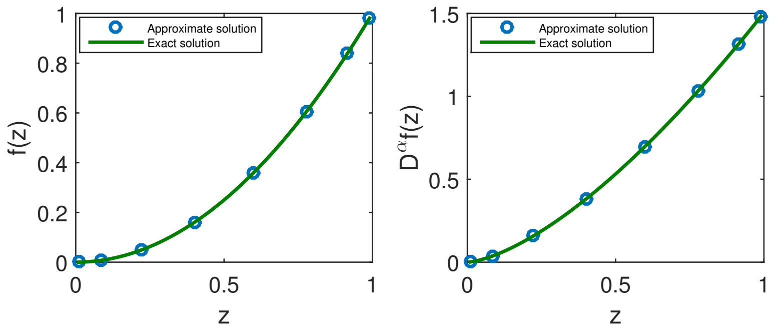


**Figure 1.** The exact and approximate solution with $\mu=0.25$, $\alpha=0.5$ and $N=7$ for Example 4.1.

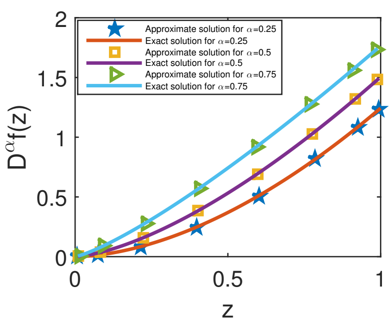


**Figure 2.** The exact and approximate solutions for $\mu=0.25$, $\alpha=0.25,0.5,0.75$ and $N=7$ for Example 4.1.

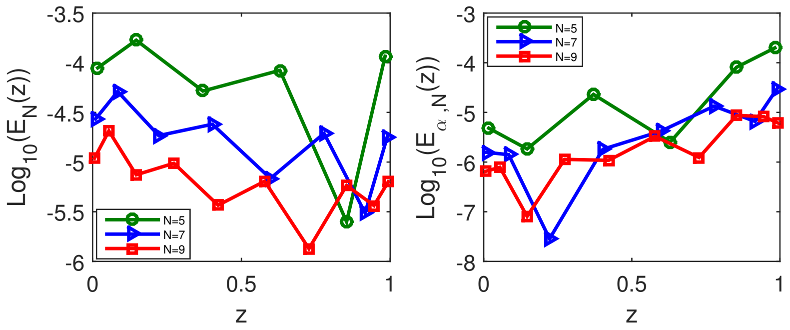


**Figure 3.** The absolute error with $\mu = 0.25$, $\alpha = 0.5$ and $N = 5, 7, 9$ for Example 4.1.

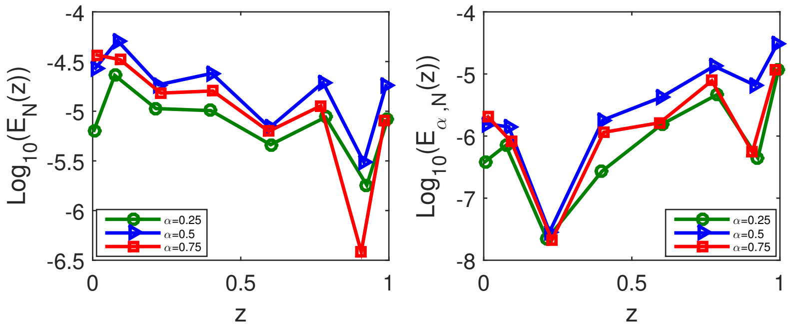


**Figure 4.** The absolute error for $\mu = 0.25$, $\alpha = 0.25, 0.5, 0.75$ and $N = 7$ for Example 4.1.

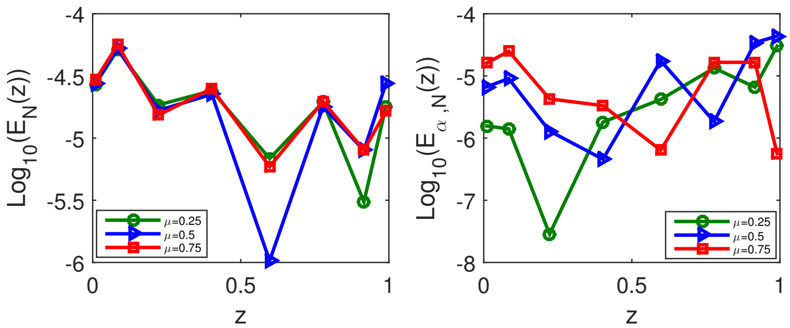


**Figure 5.** The absolute error for $\mu = 0.25, 0.5, 0.75$, $\alpha = 0.5$ and $N = 7$ for Example 4.1.

**Example 4.2.** *Consider the following SFDDIE*

$$\begin{cases} {}_{0}^{C}D_{z}^{\alpha} f(z) = f^{2}(z-\mu) + \int_{0}^{z} \dfrac{f(t)}{\sqrt{z-t}}dt + \dfrac{\Gamma(4)}{\Gamma(4-\alpha)} z^{3-\alpha} - (z-\mu)^{6} - \frac{32}{35} z^{7/2}, \quad 0 \le z \le 1, \\ f(z) = g(z), \quad -0.3 \le z \le 0. \end{cases} \tag{4.3}$$

*where* $g(z) = z^3, -0.3 \le t \le 0$. *For this example,* $f(z) = z^3$ *is the exact solution. The exact and approximate solutions for* $\alpha = 0.85$ *and* $N = 7$ *is showed in Figure 6. The exact and approximate solutions for* $\alpha = 0.75, 0.85, 0.95$ *and* $N = 7$ *is provided in Figure 7. The error of problem for* $\alpha = 0.85$ *and* $N = 5, 7, 9$ *are demonstrated in Figure 8. Moreover, in Figure 9, the error for* $\alpha = 0.75, 0.85, 0.95$ *and* $N = 7$ *are displayed.*

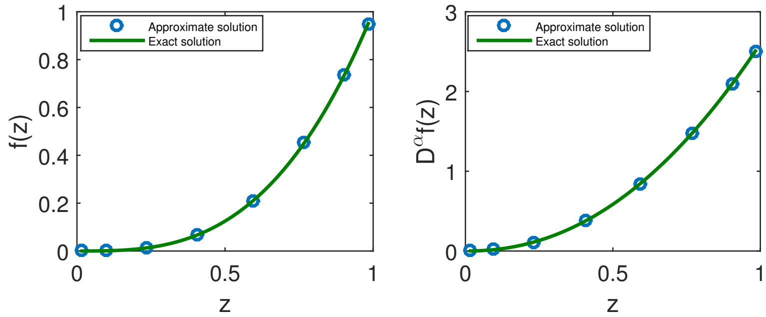


**Figure 6.** The approximate and exact solutions with $\alpha = 0.85$ and $N = 7$ for Example 4.2.

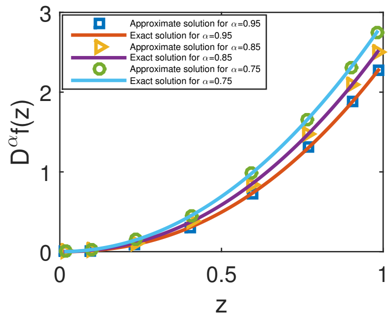


**Figure 7.** The exact and approximate solutions for $\alpha = 0.75, 0.85, 0.95$ and $N = 7$ for Example 4.2.

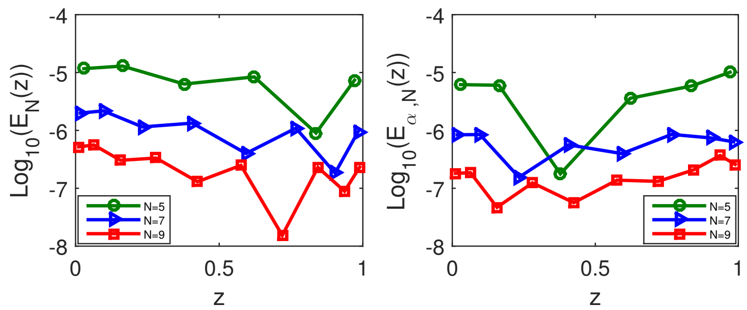


**Figure 8.** The absolute error for $\alpha = 0.85$ and $N = 5, 7, 9$ for Example 4.2.

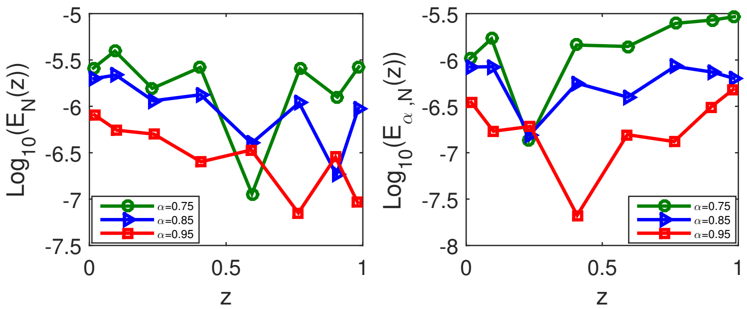


**Figure 9.** The absolute error for $\alpha = 0.75, 0.85, 0.95$ and $N = 7$ for Example 4.2.

**Example 4.3.** *Consider the following SFDDIE*

$$\begin{cases} {}_0^C D_z^\alpha f(z) = f(z-\mu) + \int_0^z \frac{t f(t)}{\sqrt[8]{z-t}} dt + \frac{\Gamma(2)}{\Gamma(2-\alpha)} z^{1-\alpha} + \frac{\Gamma(4)}{\Gamma(4-\alpha)} z^{3-\alpha} - z + \mu - (z-\mu)^3 \\ \qquad -\frac{262144 z^{39/8} + 412672 z^{32/8}}{973245}, \quad 0 \le z \le 1 \\ f(z) = g(z), \quad -\mu \le z \le 0, \end{cases} \tag{4.4}$$

*where* $g(z) = z + z^3, \mu \le z \le 0$. *The exact solution* $f(z)$ *is* $z + z^3$. *In figure 10, the exact and approximate solutions for* $\mu = 0.25$, $\alpha = 0.5$ *and* $N = 7$ *are presented. We can see the exact and approximate solutions for* $\mu = 0.25$, $\alpha = 0.25, 0.5, 0.75$ *and* $N = 7$ *in Figure 11. The error of* $\mu = 0.25$, $\alpha = 0.5$ *and* $N = 5, 7, 9$ *is provided in Figure 12. The error of* $\mu = 0.25$, $\alpha = 0.25, 0.5, 0.75$ *and* $N = 7$ *is illustrated in Figures 13 and 14, the error for* $\mu = 0.25, 0.5, 0.75$, $\alpha = 0.5$ *and* $N = 7$ *is demonstrated.*

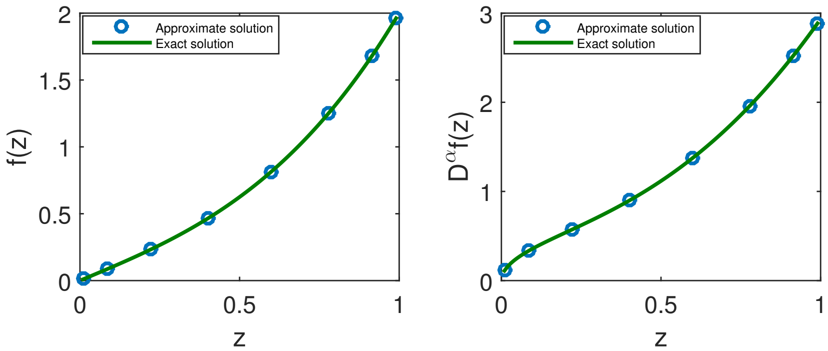


**Figure 10.** The approximate and exact solutions for $\mu = 0.25$, $\alpha = 0.5$ and $N = 7$ for Example 4.3.

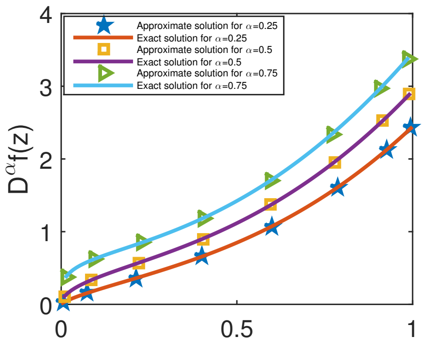


**Figure 11.** The exact and approximate solutions for $\mu = 0.25$, $\alpha = 0.25, 0.5, 0.75$ and $N = 7$ for Example 4.3.

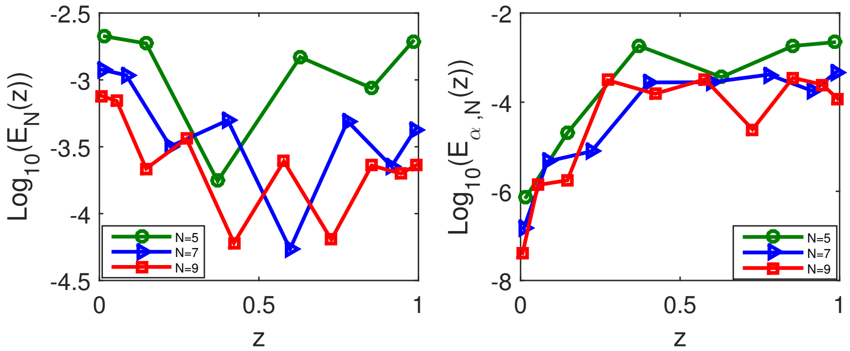


**Figure 12.** The absolute error for $\mu = 0.25$, $\alpha = 0.5$ and $N = 5, 7, 9$ for Example 4.3.

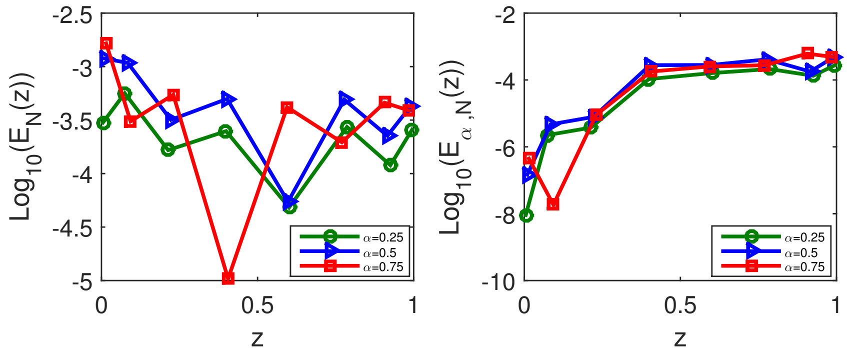


**Figure 13.** The absolute error with $\mu = 0.25$, $\alpha = 0.25, 0.5, 0.75$ and $N = 7$ for Example 4.3.

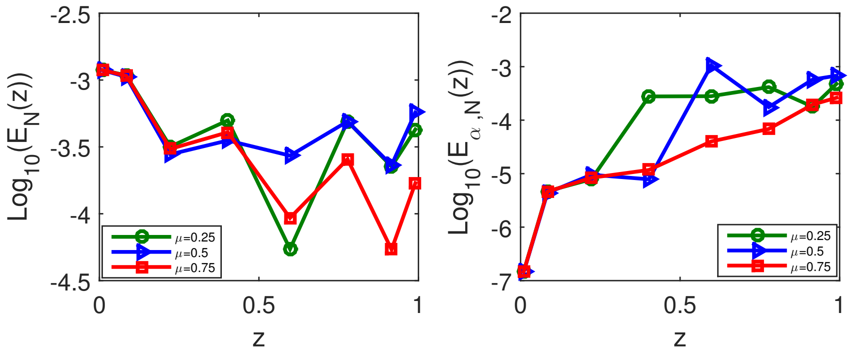


**Figure 14.** The absolute error for $\mu = 0.25, 0.5, 0.75$, $\alpha = 0.5$ and $N = 7$ for Example 4.3.

## 5. Conclusions

Jacobi-Gauss collocation methods are one of the accurate and efficient methods to solve continuous-time problems such as ordinary differential equations. Also, when delay parameters, singular integral or fractional derivatives appear in such problems, the superiority, capability and applicability of these methods show themselves better than other methods. Hence, in this paper, we showed Jacobi-Gauss collocation method can be utilized for a complex form of continuous-time problems including singular integral, fractional derivative and delay parameters, i.e., fractional singular delay integro-differential equations. Note that so far, no method has been presented to solve these problems numerically. Therefore, in this article, we proposed for the first time a high accurate numerical method to solve them. The implementation of the suggested method displayed that solving this class of problems can tend to obtain the solution of a system of algebraic equations. We applied a technique to insert the delay function in the main equation. One of the main advantages of the proposed method is to obtain the approximations of the solution and its fractional derivative, simultaneously. Also, obtaining an acceptable approximation by selecting a small number of collocation points is another advantage of the suggested method in this article. For future works, we will apply our method for solving optimal control problems under the aforementioned equations. We also will generalize the proposed method for fractional delay equations with variable order, fractional delay PDE equations with variable order and fractional delay algebraic dynamic systems.

## Conflict of interest

The authors declare that they have no conflict of interest.

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