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Research article

Necessary optimality conditions for two-step descriptor systems

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Abstract: This paper is concerned with an optimal control problem for general two-step descriptor systems. Based on the nonsmooth analysis and variational techniques, we establish the first-order necessary optimality conditions. Then, due to the expression of the general solution for linear descriptor systems, which is established by virtue of the Drazin inverse, we derived the generalized second-order necessary conditions for the first time. Also, for unfixed switching point case, we give some discussions.

Keywords: steps descriptor systems; nonsmooth analysis; necessary optimality conditions; variational techniques
Mathematics Subject Classification: 34A38, 90C46, 35A15, 80M30

1. Introduction

Some applied problems in fields such as the economy, military defense and chemistry are inherently multistage optimization problems. In such problems, there are several stages that can be connected to each other by additional conditions, and they are characterized by their own equations, controls, constants, etc. And recently, descriptor systems which are generalizations of differential equations to singular leading-term case, have aroused a lot of attention due to their applications frequently in different research areas. In this paper, we consider a general two-step descriptor system with initial conditions containing control parameters of the form

$$\begin{cases} E_i\dot{x}_i(t) = f_i(t, x_i, u_i), \quad t \in [t_{i-1}, t_i], \quad i = 1, 2, \\ x_1(t_0) = g_1(v_1), \quad x_2(t_1) = g_2(x_1(t_1), v_2), \end{cases}$$

where $E_i \in R^{n\times n}, i = 1, 2$, are constant singular matrices, and $t_0 < t_1 < t_2$ are all given.

For steps nonlinear differential equations, that is $E_i = I_i$, there have been some researches. Regarding necessary conditions for steps systems in the smooth cost functional, it can be found in [9,10]. While for the nonsmooth discrete case, [14] has investigated its optimal control problem.

What's more, the problem of control with stepwise structure, which is described by a system of difference and integro-differential equations of the Volterra type has been considered in [17]. Under the assumption about openness of control domain and some modifications of the increment method, necessary conditions of optimality of the first- and second-order have been established simultaneously.

On the other hand, for one-step descriptor system case, the correlation theory is relatively mature [2–4]. Many researchers have investigated corresponding optimal control problems by means of dynamic programming or maximum principle (see, for example, [7,9,15,21–23] and references therein), and we have also done some works recently [19,20]. However, what we should point out is that these conditions are restricted to first-order necessary conditions, and there are no further discussions until now.

Being directly inspired by the works mentioned above, the purpose of this paper is to study the optimal control problem for two-step nonlinear descriptor system. With the methods of classical calculus of variations and nonsmooth analysis, we firstly establish the first-order necessary conditions in smooth and nonsmooth cases. Then we introduce the definitions of index and Drazin inverse for matrix, which are crucial to give the generalized second-order necessary conditions for steps descriptor systems. The main difficulty throughout our paper is that we should establish the proper formula of general solutions for descriptor system. It is worth mentioning that our second-order conditions consist of constraints on endpoints, which are resulted from the matching conditions. Moreover, we have pointed out that the unfixed switching point case can be transformed into a fixed one by some proper transformations.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries. First-order necessary optimality conditions are established in Section 3. In Section 4, we derive the generalized second-order necessary conditions for linear two-step descriptor systems. The main tools are the employment of Drazin inverse and index for matrices. Then in Section 5, we give some discussion for unfixed switching point case. Finally, the conclusion is made in Section 6.

2. Preliminaries

Firstly, some definitions for nonsmooth analysis are recalled, which are necessary for the discussion later.

Given a nonempty set $\Omega \subset R^n$, consider the associated distance function

$$dist(x;\Omega) = \inf_{w\in\Omega} \|x - w\|,$$

and define Euclidean projector of x onto Ω by $\Pi(x;\Omega) := \{w \in \Omega \,|\, \|x - w\| = dist(x;\Omega)\}$. If the set Ω is closed and bounded, then the set $\Pi(x;\Omega)$ is nonempty for every $x \in R^n$. The normal cone in finite dimensional space is defined using the Euclidean projector:

$$N(\bar{x};\Omega) := \limsup_{x\to\bar{x}} [cone(x - \Pi(x,\Omega))],$$

while the basic subdifferential $\partial\phi(\bar{x})$ is defined geometrically via the normal cone to the epigraph of ϕ. Here it is assumed that ϕ is a real finite function, $\partial\phi(\bar{x}) := \{x^* \in R^n | (x^*, -1) \in N((\bar{x}, \phi(\bar{x})); epi\phi)\}$ and $epi\phi := \{(x,\mu) \in R^{n+1} | \mu \geq \phi(x)\}$ is the epigraph of ϕ. This nonconvex cone to closed sets and corresponding subdifferential of lower semicontinuous extended real-valued functions satisfying these requirements were introduced by Mordukhovich at the beginning of 1975. The initial motivation came from the intention to derive necessary optimality conditions for optimal control problems with endpoint geometric constraints by passing to the limit from free endpoint control problems, which are much easier to handle.

To start our discussion, first we have describe certain points about functional analysis and nonsmooth analysis construction. For more infourmation we refer the readers to [17]. Note that this cone is nonconvex [10,16] and for the locally Lipschitz function, the convex hull of subdifferential is a Clarke generalized subdifferential, $\bar{\phi}_k(x^0) = co\partial\phi(x^0)$. If ϕ_k is lower semicontinuous around x, then its basic subdifferential can be shown by $\partial\phi(x^0) = \limsup_{x\to x^0} \hat{\partial}\phi(x)$. Here,

$$\hat{\partial}\phi(x^0) := \{x^* \in R^n | \liminf_{u\to x^0} \frac{\phi(u) - \phi(x^0) - \langle x^*, u - x^0\rangle}{|u - x^0|} \geq 0\}$$

is the Frechet subdifferential. By using plus-minus symmetric constructions, we can write

$$\partial^+\phi(x^0) := -\partial(-\phi)(x^0), \;\; \hat{\partial}^+\phi(x^0) := -\partial(-\hat{\phi})(x^0),$$

which are called basic superdifferential and Frechet superdifferential, respectively. Here,

$$\hat{\partial}\phi^+(x^0) := \{x^* \in R^n | \limsup_{u\to x^0} \frac{\phi(u) - \phi(x^0) - \langle x^*, u - x^0\rangle}{|u - x^0|} \leq 0\}.$$

What we should point out is that for a locally Lipschitz function, the basic subdifferential and the Frechet subdifferential may be different.

If ϕ is Lipschitz continuous around point x^0, then the strict differentiability of the function ϕ at x^0 is equivalent to $\partial\phi(x^0) = \partial^+\phi(x^0) = \{\nabla\phi(x^0)\}$. If $\partial\phi(x^0) = \hat{\partial}\phi(x^0)$, then this function is lower regular at x^0. Symmetrically, we can give upper regularity of the function at the point by using the definitions of superdifferential and Frechet superdifferential. Also, if ϕ is locally Lipschitz continuous around the given point and upper regular at this point, then the Frechet superdifferential is not empty at this point and coincides with the Clarke subdifferential at this point.

By using all these nonsmooth analysis tools, we will try to find the superdifferential form of the necessary optimality conditions for the steps descriptor systems in the following form.

$$\min J(u, v) = \sum_{i=1}^{2} \varphi_i(x_i(t_i)) + \sum_{i=1}^{2} \int_{t_{i-1}}^{t_i} f_i^0(t, x_i, u_i)dt, \tag{2.1}$$

subject to

$$\begin{cases} E_i\dot{x}_i(t) = f_i(t, x_i, u_i), \;\; t \in [t_{i-1}, t_i] = T_i, \;\; i = 1, 2, \\ x_1(t_0) = g_1(v_1), \\ x_2(t_1) = g_2(x_1(t_1), v_2), \\ u_i(t) \in U_i \subset R^r, \;\; t \in T_i, \;\; i = 1, 2, \\ v_i \in V_i \subset R^q, \;\; i = 1, 2, \end{cases} \tag{2.2}$$

where $E_i \in R^{n\times n}, i = 1, 2$, are constant matrices with $rank(E_i) = r_i \leq n$; $f_i(t, x_i, u_i), i = 1, 2$, are given n-dimensional vector-valued functions, which are at least twice continuously partially differentiable with respect to their variables; $g_1(v_1)$ and $g_2(x_1, v_2)$ are both given vector-valued functions that are at least twice continuously differentiable; $f_i^0(t, x_i, u_i), i = 1, 2$, are continuous, at least continuously partially differentiable vector-valued functions with respect to their variables; $\varphi_i, i = 1, 2$, are given twice continuously differentiable scalar functions. $u_i(t), i = 1, 2$, are r-dimensional measurable and bounded vector functions of controls, and $v_i, i = 1, 2$, are q-dimensional control parameters. U_i, V_i are assumed to be nonempty and bounded open sets for each $i, i = 1, 2$.

We call a pair $(u_1(t), u_2(t), v_1, v_2) \equiv (u(t), v)$ with above properties as admissible control, and the corresponding absolutely continuous solution $(x_1(t), x_2(t)) \equiv x(t)$ to system (2.2) is called an admissible trajectory.

An admissible control $(u(t), v)$ that solves the problem of minimizing functional (2.1) under constraints (2.2) is called an optimal control, and the corresponding solution $x(t)$ to systems (2.1) and (2.2) is called an optimal trajectory. For the fixed admissible process (x^0, u^0, v^0), we introduce the following notations:

$$
\begin{aligned}
H_i(t, x_i, u_i, \psi_i^0) &= \psi_i^{0*} f_i(t, x_i, u_i) - f_i^0(t, x_i, u_i),\\
\frac{\partial H_i[t]}{\partial u_i} &= \frac{\partial H_i(t, x_i^0, u_i^0, \psi_i^0)}{\partial u_i}, \quad \frac{\partial H_i[t]}{\partial x_i} = \frac{\partial H_i(t, x_i^0, u_i^0, \psi_i^0)}{\partial x_i},\\
\Delta_{v_1} g_1(v_1) &= g_1(v_1) - g_1(v_1^0),\\
\Delta_{v_2} g_2(x_1^0(t_1), v_2^0) &= g_2(x_1^0(t_1), v_2) - g_2(x_1^0(t_1), v_2^0),\\
L_1(v_1, \psi_1^0(t_0)) &= \psi_1^{0*}(t_0) E_1 g_1(v_1),\\
L_2(x_1^0(t_1), v_2, \psi_2^0(t_1)) &= \psi_2^{0*}(t_1) E_2 g_2(x_1^0(t_1), v_2).
\end{aligned}
$$

Remark 2.1. It is worth pointing out that more general problem with multistage processes could also be considered, but for simplicity of presentation, we analyze the problem stated above.

3. First-order necessary conditions

In the following, first-order necessary conditions for optimal control problems (2.1) and (2.2) will be established by using nonsmooth analysis and variational techniques.

Theorem 3.1. Let φ_i is Frechet superdifferentiable at the point $x_i^0(t_i)$ and $(u^0(t), v^0, x^0(t))$ be an optimal solution to the control problems (2.1) and (2.2). Then for every element from Frechet superdifferential $x_i^* \in \hat{\partial}^+ \varphi(x_i^0(t_i)), i = 1, 2$, there are vector functions $\psi_i(t), i = 1, 2$ such that the following conditions holds:

$$
\begin{aligned}
\frac{\partial H_i[\theta]}{\partial u_i} &= 0, \text{ for all } u_i(t) \in U_i, \quad i = 1, 2, \quad \theta \in T_i,\\
\frac{\partial L_1(v_1, \psi_1^0(t_0))}{\partial v_1} &= 0, \ \forall\, v_1 \in V_1,\\
\frac{\partial L_2(x_1^0(t_1), v_2, \psi_2^0(t_1))}{\partial v_2} &= 0, \ \forall\, v_2 \in V_2,
\end{aligned}
$$

where $\psi_i^0(t), i = 1,2$, are adjoint trajectories satisfying

$$\begin{cases} E_i^*\dot{\psi}_i^0(t) = -\frac{\partial H_i[t]}{\partial x_i}, \quad i = 1,2, \\ E_1^*\psi_1^0(t_1) = -x_1^* + \frac{\partial L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial x_1}, \\ E_2^*\psi_2^0(t_2) = -x_2^*, \end{cases}$$

and θ is an arbitrary regular point (see [18]) of $u^0(t)$.

Proof. Take arbitrary element from Frechet superdifferential $x_i^* \in \hat{\partial}^+\varphi_i(x_i^0(t_i)), i = 1,2$ and employ the smooth variational description of $-x_i^*$ from assertion (i) of Theorem 1.88 (see [17]) to the subgradients $-x_i^* \in \hat{\partial}^+(-\varphi_i(x_i^0(t_i)))$. As a result, we find functions s_i for $i = 1,2$ satisfying the relations

$$s_i(x_i^0(t_i)) = \varphi_i(x_i^0(t_i)), \quad s_i(x_i(t)) \geq \varphi_i(x_i(t))$$

in some neighbourhood of $x_i^0(t_i)$ and such that each of them is Frechet differentiable at $x_i^0(t_i)$ with $\nabla s_i(x_i^0(t_i)) = x_i^*, i = 1,2$. It is easy to check that $x_i^0(t_i)$ is a local solution to the following optimization problem of types (2.1) and (2.2) but with cost continuously differentiable around $x_i^0(t_i)$. This means that we deduce the optimal control problems (2.1) and (2.2) with the nonsmooth cost functional to the smooth cost functional data:

$$\min J'(u,v) = \sum_{i=1}^{2} s_i(x_i(t_i)) + \sum_{i=1}^{2}\int_{t_{i-1}}^{t_i} f_i^0(t, x_i, u_i)dt \tag{3.1}$$

subject to condition (2.2). Thus, by using the Taylor formula, for any admissible values of the control and the parameter (u, v) and optimal value of the control and the parameter (u^0, v^0), the increment of cost functional $\Delta J'$ can be written in the form:

$$\begin{aligned} &\Delta J'(u^0, v^0) \\ = \;& J'(u,v) - J'(u^0,v^0) \\ = \;& \sum_{i=1}^{2}[s_i(x_i(t_i)) - s_i(x_i^0(t_i))] + \sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}[f_i^0(t,x_i,u_i) - f_i^0(t,x_i^0,u_i^0)]dt \\ = \;& \sum_{i=1}^{2}[s_i(x_i(t_i)) - s_i(x_i^0(t_i))] + \sum_{i=1}^{2}[\psi_i^{0*}(t_i)E_i\Delta x_i(t_i) - \psi_i^{0*}(t_{i-1})E_i\Delta x_i(t_{i-1})] \\ & -\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}[H_i(t,x_i,u_i,\psi_i^0) - H_i(t,x_i^0,u_i^0,\psi_i^0)]dt - \sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}\dot{\psi}_i^{0*}(t)E_i\Delta x_i(t)dt, \end{aligned} \tag{3.2}$$

whence if

$$\Delta x_1(t_0) = g_1(v_1) - g_1(v_1^0), \quad \Delta x_2(t_1) = g_2(x_1(t_1), v_2) - g_2(x_1^0(t_1), v_2^0).$$

Then, after some calculations, we can rewrite (3.2) as

$$\begin{aligned} &\Delta J'(u^0, v^0) \\ = \;& \sum_{i=1}^{2}[s_i(x_i(t_i)) - s_i(x_i^0(t_i))] + \sum_{i=1}^{2}\psi_i^{0*}(t_i)E_i\Delta x_i(t_i) \end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}[H_i(t,x_i,u_i,\psi_i^0)-H_i(t,x_i^0,u_i^0,\psi_i^0)]dt-\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}\dot{\psi}_i^{0*}(t)E_i\Delta x_i(t)dt\\
&-[L_1(v_1,\psi_1^0(t_0))-L_1(v_1^0,\psi_1^0(t_0))]-[L_2(x_1(t_1),v_2,\psi_2^0(t_1))-L_2(x_1^0(t_1),v_2^0,\psi_2^0(t_1))]\\
=\;&\sum_{i=1}^{2}[s_i(x_i(t_i))-s_i(x_i^0(t_i))]+\sum_{i=1}^{2}\psi_i^{0*}(t_i)E_i\Delta x_i(t_i)-\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}\dot{\psi}_i^{0*}(t)E_i\Delta x_i(t)dt\\
&-\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}[H_i(t,x_i,u_i,\psi_i^0)-H_i(t,x_i^0,u_i,\psi_i^0)]dt-\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}\frac{\partial H_i^*[t]}{\partial u_i}\Delta u_i(t)dt\\
&-\frac{\partial L_1^*(v_1,\psi_1^0(t_0))}{\partial v_1}\Delta v_1-\frac{\partial L_2^*(x_1^0(t_1),v_2^0,\psi_2^0(t_1))}{\partial v_2}\Delta v_2\\
&-[L_2(x_1(t_1),v_2,\psi_2^0(t_1))-L_2(x_1^0(t_1),v_2,\psi_2^0(t_1))]. \qquad (3.3)
\end{aligned}
$$

On the other hand, in consideration of $\nabla s_i(x_i^0(t_i))=x_i^*, i=1,2$ and taking $\psi_i^0(t), i=1,2$ as solutions of the following equations:

$$
\begin{cases}
E_i^*\dot{\psi}_i^0(t)=-\frac{\partial H_i[t]}{\partial x_i}, \quad i=1,2,\\
E_1^*\psi_1^0(t_1)=-x_1^*+\frac{\partial L_2(x_1^0(t_1),v_2^0,\psi_2^0(t_1))}{\partial x_1},\\
E_2^*\psi_2^0(t_2)=-x_2^*,
\end{cases} \qquad (3.4)
$$

then the increment formula (3.3) reduces to a simpler one:

$$
\begin{aligned}
\Delta J'(u^0,v^0)\;=\;&-\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}\frac{\partial H_i[t]}{\partial u_i}\Delta u_i(t)dt-\frac{\partial L_1(v_1^0,\psi_1^0(t_0))}{\partial v_1}\Delta v_1-\frac{\partial L_2(x_1^0(t_1),v_2^0,\psi_2^0(t_1))}{\partial v_2}\Delta v_2\\
&+\eta^1(u^0,v^0;\Delta u,\Delta v), \qquad (3.5)
\end{aligned}
$$

where by definition

$$
\begin{aligned}
\eta^1(u^0,v^0;\Delta u,\Delta v)\;=\;&\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}\frac{\partial H_i^*[t]}{\partial x_i}\Delta x_i(t)dt-o_3(\|\Delta x_1(t_1)\|)+\sum_{i=1}^{2}o_1^i(\|\Delta x_i(t_i)\|)\\
&-\frac{\partial L_2^*(x_1^0(t_1),v_2^0,\psi_2^0(t_1))}{\partial x_1}\Delta x_1(t_1)-\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}o_2^i(\|\Delta x_i(t)\|)dt.
\end{aligned}
$$

Here $o_i(\cdot), i=1,2,3$ are defined by the expressions

$$
\begin{cases}
s_i(x_i(t_i))-s_i(x_i^0(t_i))=x_i^*(t_i)\Delta x_i(t_i)+o_1^i(\|\Delta x_i(t_i)\|), \quad i=1,2,\\
H_i(t,x_i,u_i,\psi_i^0)-H_i(t,x_i^0,u_i,\psi_i^0)=\frac{\partial H_i^*(t,x_i^0,u_i,\psi_i^0)}{\partial x_i}\Delta x_i(t)+o_2^i(\|\Delta x_i(t)\|), \quad i=1,2,\\
L_2(x_1(t_1),v_2,\psi_2^0(t_1))-L_2(x_1^0(t_1),v_2,\psi_2^0(t_1))=\frac{\partial L_2^*(x_1^0(t_1),v_2,\psi_2^0(t_1))}{\partial x_1}\Delta x_1(t_1)+o_3(\|\Delta x_1(t_1)\|).
\end{cases}
$$

A special increment of the control $(u^0(t),v^0)$ is defined as

$$
\Delta u_i(t,\varepsilon)=\varepsilon\delta u_i(t), \quad \Delta v_i(\varepsilon)=\varepsilon\delta v_i, \quad i=1,2,
$$

where ε is a very small amount, $\delta u_i(t)\in R^r(i=1,2)$ are arbitrary measurable bounded vector functions and $\delta v_i\in R^q(i=1,2)$ are arbitrary constant vectors. Then the pair $(\delta u(t)=(\delta u_1(t),\delta u_2(t)),\delta v=(\delta v_1,\delta v_2))$ is called an admissible variation of the control $(u^0(t),v^0)$.

By taking into account (3.5) and following the schemes in [12], it can be shown that the first-order classical variations of function (3.1) have the form:

$$
\begin{aligned}
\delta^1 J'(u^0, v^0) &= J'(u^0(t) + \Delta u(t,\varepsilon)) - J'(v^0 + \Delta v(\varepsilon)) \\
&= -\sum_{i=1}^{2} \int_{t_{i-1}}^{t_i} \frac{\partial H_i[t]}{\partial u_i} \delta u_i(t) dt - \frac{\partial L_1(v_1^0, \psi_1^0(t_0))}{\partial v_1} \delta v_1 \\
&\quad - \frac{\partial L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2} \delta v_2. \qquad (3.6)
\end{aligned}
$$

Since control domains U_i and $V_i (i = 1, 2)$ are open along the optimal process $(u^0(t), v^0, x^0(t))$ for all admissible variations $(\delta u(t), \delta v)$, the first variations of function (3.1) is zero, i.e.,

$$
-\sum_{i=1}^{2} \int_{t_{i-1}}^{t_i} \frac{\partial H_i[t]}{\partial u_i} \delta u_i(t) dt - \frac{\partial L_1(v_1^0, \psi_1^0(t_0))}{\partial v_1} \delta v_1 - \frac{\partial L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2} \delta v_2 = 0. \qquad (3.7)
$$

Due to the arbitrary and independent of $\delta u_i(t)$ and δv_i, identity (3.7) yields the conclusion. This completes the proof.

Theorem 3.2. Assume that φ_i is locally Lipschitz continuous and upper regular around at $x_i^0(t_i)$. Let $(u^0(t), v^0, x^0(t))$ be an optimal solution to the control problems (2.1) and (2.2). Then, for any $\tilde{x}_i \in \bar{\partial}\varphi_i(x_i^0(t_i))$, the following conditions should be true:

$$
\begin{aligned}
\frac{\partial H_i[\theta]}{\partial u_i} &= 0, \text{ for all } u_i(t) \in U_i, \quad i = 1, 2, \quad \theta \in T_i, \\
\frac{\partial L_1(v_1, \psi_1^0(t_0))}{\partial v_1} &= 0, \ \forall\, v_1 \in V_1, \\
\frac{\partial L_2(x_1^0(t_1), v_2, \psi_2^0(t_1))}{\partial v_2} &= 0, \ \forall\, v_2 \in V_2,
\end{aligned}
$$

where $\psi_i^0(t), i = 1, 2$, are adjoint trajectories and satisfying (3.4).

Proof. From the nonsmooth analysis in Section 2, it is a known fact that if the function is upper regular, the Frechet superdifferential $\hat{\partial}^+\varphi$ coincides with the Clarke generalized gradient $\bar{\partial}\varphi$, so the conclusion is obvious. This completes the proof.

In particular, if we take smoothness on the cost functional φ_i, the following corollaries can be obtained obviously.

Corollary 3.1. If $U_i, V_i, i = 1, 2$, are still given nonempty and bounded open sets, then for the optimality of the pair $(u^0(t), v^0)$, it is necessary that following equations hold:

$$
\frac{\partial H_i^*[\theta]}{\partial u_i} = 0, \quad \theta \in T_i, \quad i = 1, 2,
$$

$$
\frac{\partial L_1^*(v_1^0, \psi_1^0(t_0))}{\partial v_1} = 0,
$$

$$
\frac{\partial L_2^*(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2} = 0,
$$

where $\psi_i^0(t), i=1,2$, are adjoint trajectories and satisfying

$$\begin{cases} E_i^*\dot{\psi}_i^0(t) = -\frac{\partial H_i[t]}{\partial x_i}, \quad i=1,2, \\ E_1^*\psi_1^0(t_1) = -\frac{\partial \varphi_1(x_1(t_1))}{\partial x_1(t_1)} + \frac{\partial L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial x_1}, \\ E_2^*\psi_2^0(t_2) = -\frac{\partial \varphi_2(x_2(t_2))}{\partial x_2(t_2)}, \end{cases} \tag{3.8}$$

Corollary 3.2. If $U_i, V_i, i=1,2$, are given nonempty, bounded and convex sets, then for the optimality of the pair $(u^0(t), v^0)$, it is necessary that the following inequalities hold:

$$\int_{t_{i-1}}^{t_i} \frac{\partial H_i^*[t]}{\partial u_i}(u_i(t) - u_i^0(t))dt \leq 0, \text{ for all } u_i(t) \in U_i, t \in T_i, i=1,2,$$

$$\frac{\partial L_1^*(v_1^0, \psi_1^0(t_0))}{\partial v_1}(v_1 - v_1^0) \leq 0, \text{ for all } v_1 \in V_1,$$

$$\frac{\partial L_2^*(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2}(v_2 - v_2^0) \leq 0, \text{ for all } v_2 \in V_2,$$

where $\psi_i^0(t), i=1,2$, are adjoint trajectories and satisfying (3.8).

4. Generalized second-order necessary conditions

In this section, we will give some second-order necessary optimality conditions. In particular, we assume that the functions in problems (2.1) and (2.2) are smooth enough and the sets $U_i, V_i, i=1,2$, are nonempty, bounded and convex.

By means of Taylor's formula which expands to the second-order derivatives and similar with the discussion above, we can obtain

$$\begin{aligned} & \Delta J(u^0, v^0) \\ = \; & \frac{1}{2}\sum_{i=1}^{2} \Delta x_i^*(t_i) \frac{\partial^2 \varphi_i(x_i^0(t_i))}{\partial x_i^2} \Delta x_i(t_i) - \sum_{i=1}^{2} \int_{t_{i-1}}^{t_i} \frac{\partial H_i^*[t]}{\partial u_i} \Delta u_i(t) dt \\ & -\frac{1}{2}\sum_{i=1}^{2} \int_{t_{i-1}}^{t_i} \Big[\Delta x_i^*(t) \frac{\partial^2 H_i[t]}{\partial x_i^2} \Delta x_i(t) + 2\Delta u_i^*(t) \frac{\partial^2 H_i[t]}{\partial u_i \partial x_i} \Delta x_i(t) + \Delta u_i^*(t) \frac{\partial^2 H_i[t]}{\partial u_i^2} \Delta u_i(t)\Big] dt \\ & -\frac{\partial L_1^*(v_1^0, \psi_1^0(t_0))}{\partial v_1} \Delta v_1 - \frac{1}{2}\Delta v_1^* \frac{\partial^2 L_1(v_1^0, \psi_1^0(t_0))}{\partial v_1^2} \Delta v_1 - \frac{\partial L_2^*(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2} \Delta v_2 \\ & -\frac{1}{2}\Big[\Delta x_1^*(t_1) \frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial x_1^2} \Delta x_1(t_1) + 2\Delta v_2^* \frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2 \partial x_1} \Delta x_1(t_1) \\ & +\Delta v_2^* \frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2^2} \Delta v_2\Big] + \eta^2(u^0, v^0; \Delta u, \Delta v), \end{aligned}$$

where

$$\begin{aligned} \eta^2(u^0, v^0; \Delta u, \Delta v) \;=\; & \sum_{i=1}^{2} \int_{t_{i-1}}^{t_i} o_i([\|\Delta x_i(t)\| + \|\Delta u_i(t)\|]^2) dt + o_3(\sum_{i=1}^{2} \|\Delta x_i(t_i)\|^2) \\ & -o_4(\|\Delta v_1\|^2) - o_5([\|\Delta x_1(t_1)\| + \|\Delta v_2\|]^2). \end{aligned}$$

Moreover, values $o_i, i = 1, \dots, 5$ are determined correspondingly from decompositions

$$\begin{aligned}
&\varphi_i(x_i(t_i)) - \varphi_i(x_i^0(t_i)) \\
= \;& \frac{\partial \varphi_i^*(x_i^0(t_i))}{\partial x_i}\Delta x_i(t_i) + \frac{1}{2}\Delta x_i^*(t_i)\frac{\partial^2 \varphi_i(x_i^0(t_i))}{\partial x_i^2}\Delta x_i(t_i) + o_3(\|\Delta x_i(t_i)\|^2), \quad i = 1, 2, \\
&H_i(t, x_i, u_i, \psi_i^0) - H_i(t, x_i^0, u_i^0, \psi_i^0) \\
= \;& \frac{\partial H_i^*[t]}{\partial x_i}\Delta x_i(t) + \frac{\partial H_i^*[t]}{\partial u_i}\Delta u_i(t) + \frac{1}{2}[\Delta x_i^*(t)\frac{\partial^2 H_i[t]}{\partial x_i^2}\Delta x_i(t) + 2\Delta u_i^*(t)\frac{\partial^2 H_i[t]}{\partial u_i \partial x_i}\Delta x_i(t) \\
&+\Delta u_i^*(t)\frac{\partial^2 H_i[t]}{\partial u_i^2}\Delta u_i(t)] + o_i([\|\Delta x_i(t)\| + \|\Delta u_i(t)\|]^2), \quad i = 1, 2, \\
&L_1(v_1, \psi_1^0(t_0)) - L_1(v_1^0, \psi_1^0(t_0)) \\
= \;& \frac{\partial L_1^*(v_1^0, \psi_1^0(t_0))}{\partial v_1}\Delta v_1 + \frac{1}{2}\Delta v_1^*\frac{\partial^2 L_1(v_1^0, \psi_1^0(t_0))}{\partial v_1^2}\Delta v_1 + o_4(\|\Delta v_1\|^2), \\
&L_2(x_1(t_1), v_2, \psi_2^0(t_1)) - L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1)) \\
= \;& \frac{\partial L_2^*(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial x_1}\Delta x_1(t_1) + \frac{\partial L_2^*(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2}\Delta v_2 \\
&+\frac{1}{2}[\Delta x_1^*(t_1)\frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial x_1^2}\Delta x_1(t_1) + 2\Delta v_2^*\frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2 \partial x_1}\Delta x_1(t_1) \\
&+\Delta v_2^*\frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2^2}\Delta v_2] + o_5([\|\Delta x_1(t_1)\| + \|\Delta v_2\|]^2).
\end{aligned}$$

Let us determine a special increment of the optimal control $(u^0(t), v^0)$ as before,

$$\Delta u_i(t, \varepsilon) = \varepsilon(u_i(t) - u_i^0(t)) = \varepsilon\delta u_i(t), \quad \Delta v_i(\varepsilon) = \varepsilon(v_i - v_i^0) = \varepsilon\delta v_i,$$

where $u_i(t) \in U_i, t \in T_i, i = 1, 2$, are arbitrary measurable and bounded vector functions, and $v_i \in V_i, i = 1, 2$, are arbitrary constant vectors. Then following the schemes from [12], we can show that

$$\begin{aligned}
&\Delta J_\varepsilon(u^0, v^0) = J(u^0(t) + \Delta u(t, \varepsilon), v^0 + \Delta v(\varepsilon)) - J(u^0(t), v^0) \\
= \;& -\varepsilon\Big[\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}\frac{\partial H_i^*[t]}{\partial u_i}\delta u_i(t)dt + \frac{\partial L_1^*(v_1^0, \psi_1^0(t_0))}{\partial v_1}\delta v_1 + \frac{\partial L_2^*(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2}\delta v_2\Big] \\
&+\frac{\varepsilon^2}{2}\Big\{\sum_{i=1}^{2} l_i^*(t_i)\frac{\partial^2 \varphi_i(x_i^0(t_i))}{\partial x_i^2}l_i(t_i) - \delta v_1^*\frac{\partial^2 L_1(v_1^0, \psi_1^0(t_0))}{\partial v_1^2}\delta v_1 \\
&-\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}\Big[l_i^*(t)\frac{\partial^2 H_i[t]}{\partial x_i^2}l_i(t) + 2\delta u_i^*(t)\frac{\partial^2 H_i[t]}{\partial u_i \partial x_i}l_i(t) + \delta u_i^*(t)\frac{\partial^2 H_i[t]}{\partial u_i^2}\delta u_i(t)\Big]dt \\
&-l_1^*(t_1)\frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial x_1^2}l_1(t_1) - 2\delta v_2^*\frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2 \partial x_1}l_1(t_1) \\
&-\delta v_2^*\frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2^2}\delta v_2\Big\} + o(\varepsilon^2), \qquad (4.1)
\end{aligned}$$

where $l_i(t), i = 1, 2$, are the variations of the trajectories being solutions of the variational equations

$$\begin{cases} E_i\dot{l}_i(t) = \frac{\partial f_i[t]}{\partial x_i} l_i(t) + \frac{\partial f_i[t]}{\partial u_i}\delta u_i(t), \quad i = 1, 2, \\ l_1(t_0) = \frac{\partial g_1(v_1^0)}{\partial v_1}\delta v_1, \\ l_2(t_1) = \frac{\partial g_2(x_1^0(t_1),v_2^0)}{\partial x_1} l_1(t_1) + \frac{\partial g_2(x_1^0(t_1),v_2^0)}{\partial v_2}\delta v_2. \end{cases} \tag{4.2}$$

As in corollary 3.2, the first-order necessary conditions have been established. Following [11], we give the following definition for descriptor systems.

Definition 4.1. We call the admissible control $(u^0(t), v^0)$ a quasisingular control in the problems (2.1) and (2.2), if the following relations hold along the process $(u^0(t), v^0, x^0(t))$:

$$\frac{\partial H_i^*[\theta]}{\partial u_i}(u_i - u_i^0(\theta)) = 0 \text{ for all } u_i(t) \in U_i, \quad \theta \in [t_{i-1}, t_i), \quad i = 1, 2, \tag{4.3a}$$

$$\frac{\partial L_1^*(v_1^0, \psi_1^0(t_0))}{\partial v_1}(v_1 - v_1^0) = 0 \text{ for all } v_1 \in V_1, \tag{4.3b}$$

$$\frac{\partial L_2^*(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2}(v_2 - v_2^0) = 0 \text{ for all } v_2 \in V_2. \tag{4.3c}$$

As we see, when these relations hold, i.e., in a quasisingular case, the statement of Corollary 3.2 loses its sense. Expansion (4.1) with (4.3) yields the following inequality along the quasisingular optimal control $(u^0(t), v^0)$ for all $(u(t), v)$:

$$\begin{aligned} &\sum_{i=1}^{2} l_i^*(t_i)\frac{\partial^2\varphi_i(x_i^0(t_i))}{\partial x_i^2} l_i(t_i) - l_1^*(t_1)\frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial x_1^2} l_1(t_1) \\ &-\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}\left[l_i^*(t)\frac{\partial^2 H_i[t]}{\partial x_i^2} l_i(t) + 2\delta u_i^*(t)\frac{\partial^2 H_i[t]}{\partial u_i \partial x_i} l_i(t) + \delta u_i^*(t)\frac{\partial^2 H_i[t]}{\partial u_i^2}\delta u_i(t)\right]dt \\ &-2\delta v_2^*\frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2 \partial x_1} l_1(t_1) - \delta v_1^*\frac{\partial^2 L_1(v_1^0, \psi_1^0(t_0))}{\partial v_1^2}\delta v_1 \\ &-\delta v_2^*\frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2^2}\delta v_2 \geq 0. \end{aligned} \tag{4.4}$$

Obviously, inequality (4.4) is an implicit necessary optimality condition for quasisingular controls. However, this result yields various necessary optimality conditions for quasisingular controls. To this end, we will need representations of the solutions of problem (4.2). Due to the limit of our knowledge, we only consider the linear time-invariant case of function $f_i(t, x_i, u_i), i = 1, 2$, that is,

$$f_i(t, x_i, u_i) = A_i x_i(t) + B_i u_i(t) + g_i(t), \quad i = 1, 2. \tag{4.5}$$

That is $\frac{\partial f_i[t]}{\partial x_i} = A_i, \frac{\partial f_i[t]}{\partial u_i} = B_i, i = 1, 2,$ $g_i(t)$ is the inhomogeneous term, and we are going to continue with the notations previously.

Definition 4.2. [8] Let A be a linear transformation on C^n. The smallest non-negative integer k such that $rank(A^k) = rank(A^{k+1})$, is called the index of A and is denoted by $Ind(A)$.

Definition 4.3. [8] If $A \in C^{n\times n}$ with $IndA = k$, and if $A^D \in C^{n\times n}$ is such that

$$A^D A A^D = A^D,\ \ AA^D = A^D A,\ \ A^{k+1}A^D = A^k,$$

then A^D is called the Drazin inverse of A.

Now we give some assumptions, which are necessary for the discussion later.

($\mathcal{H}_1$) (E_i, A_i) is regular for each $i, i = 1, 2$, i.e., $det(\lambda_i E_i + A_i) \neq 0$ for some complex number λ_i.

($\mathcal{H}_2$) The commutativity conditions hold: $E_iA_i = A_iE_i,\ \ i = 1, 2$.

Remark 4.1. ($\mathcal{H}_1$) is a basic assumption, which can guarantee the consistency of system (2.2), that is, it has at most one solution $x(t)$ satisfying the required initial conditions. Under this assumption, it obvious that ($\mathcal{H}_2$) is not restrictive [24].

Under the above conditions, the solution of system (4.2) can be given by ([1])

$$\begin{cases} l_1(t) = e^{E_1^D A_1(t-t_0)} E_1 E_1^D \frac{\partial g_1(v_1^0)}{\partial v_1}\delta v_1 + \int_{t_0}^t e^{E_1^D A_1(t-s)} E_1^D \frac{\partial f_1[s]}{\partial u_1}\delta u_1(s)ds \\ \quad +(I - E_1E_1^D)\sum\limits_{j=0}^{k_1-1}(-1)^j (E_1A_1^D)^j A_1^D \frac{\partial f_1[t]}{\partial u_1}\delta u_1^j(t) \\ \quad \triangleq M_1(t)\delta v_1 + \int_{t_0}^t R_1(t,s)\frac{\partial f_1[s]}{\partial u_1}\delta u_1(s)ds + \sum\limits_{j=0}^{k_1-1} T_j^1 \frac{\partial f_1[t]}{\partial u_1}\delta u_1^j(t),\ \ t \in T_1, \\ l_2(t) = e^{E_2^D A_2(t-t_1)} E_2 E_2^D [\frac{\partial g_2(x_1^0(t_1), v_2^0)}{\partial x_1} l_1(t_1) + \frac{\partial g_2(x_1^0(t_1), v_2^0)}{\partial v_2}\delta v_2] \\ \quad +(I - E_2E_2^D)\sum\limits_{j=0}^{k_2-1}(-1)^j (E_2A_2^D)^j A_2^D \frac{\partial f_2[t]}{\partial u_2}\delta u_2^j(t) + \int_{t_1}^t e^{E_2^D A_2(t-s)} E_2^D \frac{\partial f_2[s]}{\partial u_2}\delta u_2(s)ds \\ \quad \triangleq M_2(t)l_1(t_1) + M_3(t)\delta v_2 + \sum\limits_{j=0}^{k_2-1} T_j^2 \frac{\partial f_2[t]}{\partial u_2}\delta u_2^j(t) + \int_{t_1}^t R_2(t,s)\frac{\partial f_2[s]}{\partial u_2}\delta u_2(s)ds,\ \ t \in T_2. \end{cases} \tag{4.6}$$

where $Ind(E_i) = k_i$.

Remark 4.2. For the singular matrices $E_i, i = 1, 2$, without loss of generality, we can only consider the following form:

$$E_i = \begin{pmatrix} S_i & 0 \\ 0 & 0 \end{pmatrix},\ \ i = 1, 2,$$

where $S_i \in R^{r_i\times r_i}, i = 1, 2$, are nonsingular matrices. Otherwise, we can firstly take some transformations. Thus, we have $Ind(E_i) = 1, i = 1, 2$.

Taking into account the independence of the variations $(\delta u(t), \delta v)$ of $(u^0(t), v^0)$, we consider four possible cases in the following.

Case I. Let $u_2(t) = u_2^0(t), t \in T_2; v_i = v_i^0, i = 1, 2$. Then inequality (4.4) becomes

$$\sum_{i=1}^{2} l_i^*(t_i)\frac{\partial^2\varphi_i(x_i^0(t_i))}{\partial x_i^2} l_i(t_i) - \sum_{i=1}^{2}\int_{t_{i-1}}^{t_i} l_i^*(t)\frac{\partial^2 H_i[t]}{\partial x_i^2} l_i(t)dt - 2\int_{t_0}^{t_1}\delta u_1^*(t)\frac{\partial^2 H_1[t]}{\partial u_1\partial x_1} l_1(t)dt$$
$$-\int_{t_0}^{t_1}\delta u_1^*(t)\frac{\partial^2 H_1[t]}{\partial u_1^2}\delta u_1(t)dt - l_1^*(t_1)\frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial x_1^2} l_1(t_1) \geq 0. \tag{4.7}$$

Representations (4.6) yields

$$\begin{cases} l_1(t) = \int_{t_0}^{t} R_1(t,s)\frac{\partial f_1[s]}{\partial u_1}\delta u_1(s)ds + T_0^1\frac{\partial f_1[t]}{\partial u_1}\delta u_1(t), & t \in T_1, \\ l_2(t) = M_2(t)l_1(t_1), & t \in T_2. \end{cases} \tag{4.8}$$

Using the scheme from [11,12] and the Dirichlet formula [11], the following identities can be proved:

$$\begin{aligned}
&\int_{t_0}^{t_1} l_1^*(t)\frac{\partial^2 H_1[t]}{\partial x_1^2}l_1(t)dt \\
=\ &\int_{t_0}^{t_1}\int_{t_0}^{t_1} \delta u_1^*(\tau)\frac{\partial f_1^*[\tau]}{\partial u_1}\Big[\int_{\max(\tau,s)}^{t_1} R_1^*(t,\tau)\frac{\partial^2 H_1[t]}{\partial x_1^2}R_1(t,s)dt\Big]\frac{\partial f_1[s]}{\partial u_1}\delta u_1(s)dsd\tau \\
&+2\int_{t_0}^{t_1}\Big[\int_{t_0}^{t_1} \delta u_1^*(\tau)\frac{\partial f_1^*[\tau]}{\partial u_1}T_0^{1*}\frac{\partial^2 H_1[\tau]}{\partial x_1^2}R_1(\tau,t)d\tau\Big]\frac{\partial f_1[t]}{\partial u_1}\delta u_1(t)dt \\
&+\int_{t_0}^{t_1} \delta u_1^*(t)\frac{\partial f_1^*[t]}{\partial u_1}T_0^{1*}\frac{\partial^2 H_1[t]}{\partial x_1^2}T_0^1\frac{\partial f_1[t]}{\partial u_1}\delta u_1(t)dt, \\
&\int_{t_1}^{t_2} l_2^*(t)\frac{\partial^2 H_2[t]}{\partial x_2^2}l_2(t)dt \\
=\ &\Big[\int_{t_0}^{t_1} R_1(t_1,s)\frac{\partial f_1[s]}{\partial u_1}\delta u_1^*(s)ds\Big]^*\int_{t_1}^{t_2} M_2^*(t)\frac{\partial^2 H_2[t]}{\partial x_2^2}M_2(t)dt\Big[\int_{t_0}^{t_1} R_1(t_1,s)\frac{\partial f_1[s]}{\partial u_1}\delta u_1^*(s)ds\Big] \\
&+2\delta u_1^*(t_1)\frac{\partial f_1^*[t_1]}{\partial u_1}T_0^{1*}\Big[\int_{t_1}^{t_2} M_2^*(t)\frac{\partial^2 H_2[t]}{\partial x_2^2}M_2(t)dt\Big]\Big[\int_{t_0}^{t_1} R_1(t_1,s)\frac{\partial f_1[s]}{\partial u_1}\delta u_1(s)ds\Big] \\
&+\delta u_1^*(t_1)\frac{\partial f_1^*[t_1]}{\partial u_1}T_0^{1*}\Big[\int_{t_1}^{t_2} M_2^*(t)\frac{\partial^2 H_2[t]}{\partial x_2^2}M_2(t)dt\Big]T_0^1\frac{\partial f_1[t_1]}{\partial u_1}\delta u_1(t_1), \\
&\int_{t_0}^{t_1} \delta u_1^*(t)\frac{\partial^2 H_1[t]}{\partial u_1\partial x_1}l_1(t)dt \\
=\ &\int_{t_0}^{t_1}\Big[\int_{t_0}^{t_1} \delta u_1^*(\tau)\frac{\partial^2 H_1[\tau]}{\partial u_1\partial x_1}R_1(\tau,t)d\tau\Big]\frac{\partial f_1[t]}{\partial u_1}\delta u_1(t)dt + \int_{t_0}^{t_1} \delta u_1^*(t)\frac{\partial^2 H_1[t]}{\partial u_1\partial x_1}T_0^1\frac{\partial f_1[t]}{\partial u_1}\delta u_1(t)dt.
\end{aligned}$$

Assuming

$$\begin{aligned}
K_{11}(\tau,s) &= -R_1^*(t_1,\tau)\Big[\frac{\partial^2\varphi_1(x_1^0(t_1))}{\partial x_1^2} + M_2^*(t_2)\frac{\partial^2\varphi_2(x_2^0(t_2))}{\partial x_2^2}M_2(t_2) \\
&\quad -\frac{\partial^2 L_2(x_1^0(t_1),v_2^0,\psi_2^0(t_1))}{\partial x_1^2}\Big]R_1(t_1,s) + \int_{\max(\tau,s)}^{t_1} R_1^*(t,\tau)\frac{\partial^2 H_1[t]}{\partial x_1^2}R_1(t,s)dt, \\
K_1(t) &= \frac{\partial^2 H_1[t]}{\partial u_1^2} + \frac{\partial^2 H_1[t]}{\partial u_1\partial x_1}T_0^1\frac{\partial f_1[t]}{\partial u_1} + \frac{\partial f_1^*[t]}{\partial u_1}T_0^{1*}\frac{\partial^2 H_1[t]}{\partial x_1^2}T_0^1\frac{\partial f_1[t]}{\partial u_1}, \\
K_{12}(\tau,t) &= \frac{\partial^2 H_1[\tau]}{\partial u_1\partial x_1}R_1(\tau,t) + \frac{\partial f_1^*[\tau]}{\partial u_1}T_0^{1*}\frac{\partial^2 H_1[\tau]}{\partial x_1^2}R_1(\tau,t), \\
K_1 &= \Big[\frac{\partial^2\varphi_1(x_1^0(t_1))}{\partial x_1^2} + M_2^*(t_2)\frac{\partial^2\varphi_2(x_2^0(t_2))}{\partial x_2^2}M_2(t_2) - \frac{\partial^2 L_2(x_1^0(t_1),v_2^0,\psi_2^0(t_1))}{\partial x_1^2}\Big], \\
K &= \int_{t_1}^{t_2} M_2^*(t)\frac{\partial^2 H_2[t]}{\partial x_2^2}M_2(t)dt - K_1.
\end{aligned}$$

then with (4.7) and (4.8) we have

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1}\delta u_1^*(\tau)\frac{\partial f_1^*[\tau]}{\partial u_1}K_{11}(\tau,s)\frac{\partial f_1[s]}{\partial u_1}\delta u_1(s)dsd\tau+\int_{t_0}^{t_1}\delta u_1^*(t)K_1(t)\delta u_1(t)dt$$
$$+2\int_{t_0}^{t_1}\left\{\int_{t_0}^{t_1}\delta u_1^*(\tau)K_{12}(\tau,t)d\tau\right\}\frac{\partial f_1[t]}{\partial u_1}\delta u_1(t)dt$$
$$+\Big[\int_{t_0}^{t_1}R_1(t_1,s)\frac{\partial f_1[s]}{\partial u_1}\delta u_1(s)ds\Big]^*\int_{t_1}^{t_2}M_2^*(t)\frac{\partial^2 H_2[t]}{\partial x_2^2}M_2(t)dt\Big[\int_{t_0}^{t_1}R_1(t_1,s)\frac{\partial f_1[s]}{\partial u_1}\delta u_1(s)ds\Big]$$
$$+\delta u_1^*(t_1)\frac{\partial f_1^*[t_1]}{\partial u_1}T_0^{1*}K\Big[2\int_{t_0}^{t_1}R_1(t_1,s)\frac{\partial f_1[s]}{\partial u_1}\delta u_1(s)ds+T_0^1\frac{\partial f_1[t_1]}{\partial u_1}\delta u_1(t_1)\Big]\le 0. \tag{4.9}$$

Case II. If we assume that $u_1(t)=u_1^0(t), t\in T_1; v_i=v_i^0, i=1,2$, inequality (4.4) becomes

$$l_2^*(t_2)\frac{\partial^2\varphi_2(x_2^0(t_2))}{\partial x_2^2}l_2(t_2)-2\int_{t_1}^{t_2}\delta u_2^*(t)\frac{\partial^2 H_2[t]}{\partial u_2\partial x_2}l_2(t)dt$$
$$-\int_{t_1}^{t_2}l_2^*(t)\frac{\partial^2 H_2[t]}{\partial x_2^2}l_2(t)dt-\int_{t_1}^{t_2}\delta u_2^*(t)\frac{\partial^2 H_2[t]}{\partial u_2^2}\delta u_2(t)dt\ge 0.$$

Representation (4.6) yields

$$\begin{cases} l_2(t)=\int_{t_1}^{t_2}R_2(t,s)\frac{\partial f_2[s]}{\partial u_2}\delta u_2(s)ds+T_0^2\frac{\partial f_2[t]}{\partial u_2}\delta u_2(t), & t\in T_2,\\ l_1(t)\equiv 0, & t\in T_1.\end{cases}$$

Then similar with Case I, assuming that

$$\begin{aligned} K_{21}(\tau,s) &= \int_{\max(\tau,s)}^{t_2}R_2^*(t,\tau)\frac{\partial^2 H_2[t]}{\partial x_2^2}R_2(t,s)dt,\\ K_2(t) &= \frac{\partial^2 H_2[t]}{\partial u_2^2}+\frac{\partial f_2^*[t]}{\partial u_2}T_0^{2*}\frac{\partial^2 H_2[t]}{\partial x_2^2}T_0^2\frac{\partial f_2[t]}{\partial u_2}+2\frac{\partial^2 H_2[t]}{\partial u_2\partial x_2}T_0^2\frac{\partial f_2[t]}{\partial u_2},\\ K_{22}(\tau,t) &= \Big[\frac{\partial^2 H_2[\tau]}{\partial u_2\partial x_2}+\frac{\partial f_2^*[\tau]}{\partial u_2}T_0^{2*}\frac{\partial^2 H_2[\tau]}{\partial x_2^2}\Big]R_2(\tau,t),\quad K_2=T_0^{2*}\frac{\partial^2\varphi_2(x_2^0(t_2))}{\partial x_2^2},\end{aligned}$$

we have

$$\int_{t_1}^{t_2}\int_{t_1}^{t_2}\delta u_2^*(\tau)\frac{\partial f_2^*[\tau]}{\partial u_2}K_{21}(\tau,s)\frac{\partial f_2[s]}{\partial u_2}\delta u_2(s)dsd\tau+\int_{t_1}^{t_2}\delta u_2^*(t)K_2(t)\delta u_2(t)dt$$
$$+2\int_{t_1}^{t_2}\left\{\int_{t_1}^{t_2}\delta u_2^*(\tau)K_{22}(\tau,t)d\tau\right\}\frac{\partial f_2[t]}{\partial u_2}\delta u_2(t)dt-\delta u_2^*(t_2)\frac{\partial f_2^*[t_2]}{\partial u_2}K_2T_0^2\frac{\partial f_2[t_2]}{\partial u_2}\delta u_2(t_2)$$
$$-\Big[\int_{t_1}^{t_2}R_2(t_2,s)\frac{\partial f_2[s]}{\partial u_2}\delta u_2(s)ds\Big]^*\frac{\partial^2\varphi_2(x_2^0(t_2))}{\partial x_2^2}\Big[\int_{t_1}^{t_2}R_2(t_2,s)\frac{\partial f_2[s]}{\partial u_2}\delta u_2(s)ds\Big]$$
$$-2\delta u_2^*(t_2)\frac{\partial f_2^*[t_2]}{\partial u_2}K_2\int_{t_1}^{t_2}R_2(t_2,s)\frac{\partial f_2[s]}{\partial u_2}\delta u_2(s)ds\le 0. \tag{4.10}$$

Case III. Let $u_i(t)=u_i^0(t), t\in T_i, i=1,2; v_2=v_2^0$. Then inequality (4.4) becomes

$$\sum_{i=1}^{2}l_i^*(t_i)\frac{\partial^2\varphi_i(x_i^0(t_i))}{\partial x_i^2}l_i(t_i)-\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i}l_i^*(t)\frac{\partial^2 H_i[t]}{\partial x_i^2}l_i(t)dt$$

$$-l_1^*(t_1)\frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial x_1^2}l_1(t_1) - \delta v_1^* \frac{\partial^2 L_1(v_1^0, \psi_1^0(t_0))}{\partial v_1^2}\delta v_1 \geq 0. \tag{4.11}$$

Representation (4.6) yields

$$l_1(t) = M_1(t)\delta v_1, \ \ t \in T_1; \ \ l_2(t) = M_2(t)l_1(t_1), \ \ t \in T_2. \tag{4.12}$$

Assuming

$$\begin{aligned} K_3 \ &= \ -M_1^*(t_1)\frac{\partial^2 \varphi_1(x_1^0(t_1))}{\partial x_1^2}M_1(t_1) - M_1^*(t_1)M_2^*(t_2)\frac{\partial^2 \varphi_2(x_2^0(t_2))}{\partial x_2^2}M_2(t_2)M_1(t_1) \\ &+\frac{\partial^2 L_1(v_1^0, \psi_1^0(t_0))}{\partial v_1^2} + \int_{t_0}^{t_1} M_1^*(t)\frac{\partial^2 H_1[t]}{\partial x_1^2}M_1(t)dt \\ &+ \int_{t_1}^{t_2} M_1^*(t_1)M_2^*(t)\frac{\partial^2 H_2[t]}{\partial x_2^2}M_2(t)M_1(t_1)dt + M_1^*(t_1)\frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial x_1^2}M_1(t_1), \end{aligned}$$

with (4.11) and (4.12) we get

$$\delta v_1^* K_3 \delta v_1 \leq 0 \text{ for all } v_1 \in V_1. \tag{4.13}$$

Case IV. If we assume that $u_i(t) = u_i^0(t), t \in T_i, i = 1, 2; v_1 = v_1^0$, inequality (4.4) becomes

$$l_2^*(t_2)\frac{\partial^2 \varphi_2(x_2^0(t_2))}{\partial x_2^2}l_2(t_2) - \int_{t_1}^{t_2} l_2^*(t)\frac{\partial^2 H_2[t]}{\partial x_2^2}l_2(t)dt - \delta v_2^* \frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2^2}\delta v_2 \geq 0.$$

Representation (4.6) yields

$$l_2(t) = M_3(t)\delta v_2, \ \ t \in T_2; \ \ l_1(t) \equiv 0, \ \ t \in T_1.$$

Assuming

$$\begin{aligned} K_4 \ &= \ -M_3^*(t_2)\frac{\partial^2 \varphi_2(x_2^0(t_2))}{\partial x_2^2}M_3(t_2) + \frac{\partial^2 L_2(x_1^0(t_1), v_2^0, \psi_2^0(t_1))}{\partial v_2^2} \\ &+ \int_{t_1}^{t_2} M_3^*(t)\frac{\partial^2 H_2[t]}{\partial x_2^2}M_3(t)dt, \end{aligned}$$

we can obtain

$$\delta v_2^* K_4 \delta v_2 \leq 0 \text{ for all } v_2 \in V_2. \tag{4.14}$$

Thus, the following statement is proved.

Theorem 4.1. For quasisingular control $(u^0(t), v^0)$ in the problems (2.1) and (2.2) with the linear form (4.5) to be optimal, it is necessary that relations (4.9), (4.10), (4.13) and (4.14) hold.

Remark 4.3. These relations are established based on the second-order variations of the cost functional, and compared with ordinary differential equations case, there are second-order constraints for controls $u_i(t_i), i = 1, 2$, which come from the matching conditions. Thus, we call them generalized second-order necessary optimality conditions for descriptor systems.

Remark 4.4. In particular, if the matrix $E_i = I_i, i = 1, 2$, then we can easily known that Theorem 4.1 can be reduced to Theorem 2 in [6]. And for the case that the control domains $U_i, v_i, i = 1, 2$, are nonempty bounded open sets, similar discussion can be established. So our result generalized the conclusions previously.

Remark 4.5. In fact, for specific problems, Theorem 4.1 may not be so complicated. For example, we can consider the case:

$$E_1 = E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; A_1 = A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We choose $x_1(0) = v_1$, $x_2(1) = 0$, $\varphi_1(x_1(1)) = 2x_1^2(1)$, $f_1^0 = x_1^2$. Let us think about scenario III, and then we are going to have to compute $M_1(t)$. After some manipulations we can get that

$$M_1(1) = e^{E_1 A_1} E_1.$$

Obviously, the expression is simple, and next we direct use the inequality (4.13) merely.

5. Unknown switching point case

All the discussion above is based on a fact that this is an initial problem where the switching point t_1 is fixed. In this section, we consider another different case:

$$\textit{Problem } (I) \begin{cases} \min J(u, x, t_1) = \sum\limits_{i=1}^{2} \varphi(x_i(t_i)) + \sum\limits_{i=1}^{2} \int_{t_{i-1}}^{t_i} f_i^0(t, x_i, u_i)dt, \\ s.t. \;\; E_i \dot{x}_i = f_i(t, x_i, u_i), \;\; t \in [t_{i-1}, t_i] = T_i, \;\; i = 1, 2, \end{cases} \tag{5.1}$$

where t_1 is a unfixed switching point, the conditions $x_1(t_0) = x^0, x_2(t_2) = x^T$ are given, and the others are defined as before.

Similar with [14], assume new variable x_{n+1} corresponding to the switching instant t_1. Let x_{n+1} satisfy

$$\frac{dx_{n+1}}{dt} = 0, \;\; x_{n+1}(t_0) = t_1.$$

It means x_{n+1} is constant in $[t_0, t_2]$. Next, a new independent time variable τ is introduced as

$$t = \begin{cases} t_0 + (x_{n+1} - t_0)\tau, \;\; 0 \le \tau < 1; \\ x_{n+1} + (t_2 - x_{n+1})(\tau - 1), \;\; 1 \le \tau \le 2. \end{cases}$$

Note $\tau = 0$ corresponds to $t = t_0$, $\tau = 1$ corresponds to $t = t_1$, and $\tau = 2$ to $t = t_2$. By introducing x_{n+1} and τ, and substitutions $y_i(\tau) = x_i(t(\tau)), v_i(\tau) = u_i(t(\tau)), i = 1, 2$, the problem (5.1) can be transformed into the following form

$$
Problem\ (II)\begin{cases}
E_1\frac{dy_1(\tau)}{d\tau} = (x_{n+1}-t_0)\tilde{f}_1(\tau, y_1, v_1),\\
\quad \frac{dx_{n+1}}{d\tau} = 0, \quad x_{n+1}(t_0) = t_1, \quad \tau \in [0,1),\\
\text{and}\\
E_2\frac{dy_2(\tau)}{d\tau} = (t_2 - x_{n+1})\tilde{f}_2(\tau, y_2, v_2),\\
\quad \frac{dx_{n+1}}{d\tau} = 0, \quad x_{n+1}(t_0) = t_1, \quad \tau \in [1,2],\\
\text{ and minimizing functional takes the form}\\
\tilde{J}(v, x_{n+1}) = \tilde{\varphi}(y_1(1), y_2(1)) + \int_0^1 (x_{n+1}-t_0)\tilde{f}_1^0(\tau, y_1, v_1)d\tau\\
\qquad\qquad + \int_1^2 (t_2 - x_{n+1})\tilde{f}_2^0(\tau, y_2, v_2)d\tau,
\end{cases} \tag{5.2}
$$

with the corresponding initial conditions $y_1(0) = x^0, y_2(2) = x^T$.

Since x_{n+1} is unknown constant in the interval [0, 2], the dimension of Problem (II) will be same as the dimension of the Problem (I). There is a one-to-one corresponding between admissible process $(t_1, x(t), u(t))$ and the admissible process $(y(\tau), v(\tau))$. That is, if the process $(t_1^0, x^0(t), u^0(t))$ gives the minimum for (5.1), then the process $(y^0(\tau), v^0(\tau))$, which is obtained after transformation, gives minimum value of (5.2), and vice versa.

For multiple unfixed switching points case, there is no difficulty in applying nonsmooth analysis and the Variational techniques to the problems with several subsystems similarly. To verify the efficiency of the procedure, readers can see [14] for details. Obviously, the equivalent problem is with fixed switching point, and we can consider the couple (v, t_1) as a new control. However, it is worth mentioning that the transformed system is time-variant, so our generalized second-order necessary optimality conditions can not be applied unfortunately.

6. Conclusions

We investigate a class of optimal control problems for multistage processes. By means of variational techniques and nonsmooth analysis, the first-order necessary optimality conditions for two-steps descriptor systems are established. Then, we also establish the generalized second-order necessary conditions for linear steps descriptor systems by using the definitions of Drazin inverse and the index for matrices. Finally, for the unfixed switching point case, some transformations can be implemented to transform it into a fixed one.

However, it is worth mentioning that we establish the generalized second-order necessary conditions for linear time-invariant steps descriptor systems only. The main difficulty for linear time-varying or nonlinear descriptor systems is that there is no formula for their general solutions. On the other hand, this paper discusses the continuous-time descriptor systems only. And we will leave these issues for future research.

Abbreviations

GDREs, generalized differential Riccati equations; GBDEs, generalized backward differential equations; LQ, linear quadratic. R^n, n-dimension Euclidean space; "$*$", transpose of a matrix or matric-valued function; "∇", gradient of a function; "o", infinitesimal of higher order; "∂", partial of a function; "∂^2", second-order partial derivative; "det", determinant of a matrix.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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