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Research article

Some results on the convergence of Hessian operator and $m-$subharmonic functions

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Abstract: In this paper we treat the problem of connection between the convergence in $m-$capacity and the convergence of the Hessian measure for a sequence f_j of $m-$subharmonic functions. We prove first that, under some conditions, the convergence of f_j in capacity Cap_m implies the weak convergence of the Hessian measures $H_m(f_j)$. Then we show that the converse sense of convergence is also true in some particular cases.

Keywords: $m-$subharmonic function; $m-$Capacity; Hessian operator
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1. Introduction

The notion of capacity represents a very useful tool in the study of several problems in complex analysis regarding its effectiveness in the proof of the continuity for the Monge-Ampère operator and also in the resolution of the Dirichlet problem. In [1], Bedford and Taylor noticed that the weak convergence of a uniformly bounded sequence of plurisubharmonic (psh) functions f_j defined on a domain Ω of $\mathbb{C}^n$ does not necessarily imply the convergence of the associated Monge-Ampère measures $(dd^c f_j)^n$. This is why different works gave sufficient conditions to establish a suitable connection between the two notions of convergence. In [9], Xing proved that the convergence in capacity C_n introduced by Bedford and Taylor gives the continuity of the Monge-Ampère operator. This work was extended in [10] to the case of psh functions, that are only bounded near the boundary of Ω. In [3], Blocki introduced a more general notion called the $m-$subharmonic function ($m-$sh for short) for $1 \leq m \leq n$ which coincides with the psh functions in the limit case $m = n$. This has given rise to various works which aim to extend the results proved in the case of psh to the case of $m-$sh.

Some of those problems are linked to the complex Hessian operator which itself generalizes the famous Monge-Ampère operator. In this paper we deal with the problem of connection between the convergence in capacity Cap_m and the continuity of the associated Hessian operator in the general case of $m-$subharmonic functions that are bounded near $\partial\Omega$. To establish such relation we will prove firstly several results of convergence that represent itself a useful tool in the study of problems related to the Hessian operator and also a generalization of Xing's inequalities for the class of $m-$subharmonic function that are bounded only near the boundary of Ω. Based on the established inequalities and the works of Xing [10] and Lu [7] we will prove the following main result:

Theorem: Let $E \Subset \Omega$ and $\mathfrak{g} \in SH^m(\Omega)$ a bounded function on $\Omega \backslash E$. Assume that there is $f_k \in SH^m(\Omega)$ satisfying:

(1) $|f_k| \leq |\mathfrak{g}|$ in Ω for all k.
(2) There exists an $m-$subharmonic function f in Ω such that $f_k \to f$ in Cap_m on each $E \Subset F$,

then the sequence of measures $(dd^c f_k)^m \wedge \gamma^{n-m}$ converges weakly to $(dd^c f)^m \wedge \gamma^{n-m}$ in Ω.

We prove also that every sequence of $m-$subharmonic functions, that converges weakly (with respect to the Lebesgue measure $d\lambda$) converges with respect to any measure that has no mass on $m-$polar sets.

In the last part of this paper, we discuss the converse sense of the above theorem and we prove, under suitable conditions, that the weak convergence of the Hessian measures $(dd^c f_k)^m \wedge \gamma^{n-m}$ to $(dd^c f)^m \wedge \gamma^{n-m}$ implies the convergence of $(f_j)_j$ to f with respect to the capacity Cap_m.

2. Preliminaries

In this paper we denote by Ω a bounded domain of $\mathbb{C}^n$, $d := \partial + \overline{\partial}$, $d^c := i(\overline{\partial} - \partial)$ and $\Lambda_p(\Omega)$ the set of $(p, p)-$forms in Ω. The classic Kähler form γ defined on $\mathbb{C}^n$ will be denoted as $\gamma := dd^c|z|^2$.

Definition 1. [3] Let $\zeta \in \Lambda_1(\Omega)$ and $m \in \mathbb{N} \cap [1, n]$. The form ζ is called $m-$positive if it satisfies

$$\zeta^j \wedge \gamma^{n-j} \geq 0, \quad \forall j = 1, \cdots, m$$

at every point of Ω.

Definition 2. [3] Let $\zeta \in \Lambda_p(\Omega)$ and $m \in \mathbb{N} \cap [p, n]$. We say that ζ is $m-$positive on Ω if the following measure

$$\zeta \wedge \beta^{m-n} \wedge \psi_1 \wedge \cdots \wedge \psi_{m-p}$$

is positive at every point of Ω where $\psi_1, \cdots, \psi_{m-p} \in \Lambda_1(\Omega)$.

We will denote by $\Lambda_p^m(\Omega)$ the set of all $(p, p)-$forms on Ω that are $m-$positive. In 2005, Blocki [3] introduced the notion of $m-$subharmonic functions to generalize the plurisubharmonic functions and he developed an analogous pluripotential theory. This notion is given as follows.

Definition 3. Let $f : \Omega \to \mathbb{R} \cup \{-\infty\}$. The function f is called $m-$subharmonic if it satisfies the following:

(1) The function f is subharmonic.

(2) For all $\zeta_1, \cdots, \zeta_{m-1} \in \Lambda_1^m(\Omega)$ one has

$$dd^c f \wedge \gamma^{n-m} \wedge \zeta_1 \wedge \cdots \wedge \zeta_{m-1} \geq 0.$$

We denote by $SH^m(\Omega)$ the cone of $m-$subharmonic functions defined on Ω and $B_m(\Omega)$ the set of functions $u \in SH^m(\Omega)$ that are locally uniformly bounded.

Remark 1. In the case $m = n$ we have the following:

(1) The definition of $m-$positivity coincides with the classic definition of positivity given by Lelong [8] for forms.
(2) The set $SH^n(\Omega)$ coincides with the set of plurisubharmonic functions on Ω.

For more details about the properties of $m-$subharmonicity one can refer to [3, 5, 7].

Example 1. (1) If

$$\zeta := i(4.dz_1 \wedge d\overline{z}_1 + 4.dz_2 \wedge d\overline{z}_2 - dz_3 \wedge d\overline{z}_3)$$

then $\zeta \in \Lambda_1^2(\mathbb{C}^3) \setminus \Lambda_1^3(\mathbb{C}^3)$.
(2) If

$$f(z) := 2|z_1|^2 + 2|z_2|^2 - |z_3|$$

then $f \in SH^2(\mathbb{C}^3) \setminus SH^3(\mathbb{C}^3)$.

In the following we give the notion of $s-$capacity for every integer s. Such notion will be useful throughout this paper and was defined on every subset E as follows:

Definition 4. [5] The $s-$capacity of a compact subset K in Ω denoted by $Cap_s(K)$ is defined as

$$Cap_s(K, \Omega) = Cap_s(K) := \sup\{\int_K (dd^c f)^s \wedge \gamma^{n-s}, f \in SH^m(\Omega), 0 \leq f \leq 1\},$$

for $1 \leq s \leq m$. If $E \subset \Omega$, then $Cap_s(E, \Omega) = \sup\{Cap_s(K), K\ compact\ of\ E\}$.

One of the most known property for $m-$subharmonic functions is the continuity outside a subset of small capacity. Such property is known as the quasicontinuity and will represent an essential tool in the proof of several result in this paper.

Proposition 1. *Every $f \in SH^m(\Omega)$ is Cap_m-quasicontinuous. That means for all $\varepsilon > 0$ there exists an open subset O_ε such that $Cap_m(O_\varepsilon) < \varepsilon$ and f is continuous on $\Omega \setminus O_\varepsilon$. As a consequence f can be written as follows*

$$f = f_1 + f_2$$

where f_1 is continuous on Ω and $f_2 \equiv 0$ on $\Omega \setminus O_\varepsilon$.

Definition 5. (1) A positive measure μ defined on Ω is said to be absolutely continuous with respect to the capacity Cap_m ($\mu << Cap_m$ for short) on a Borel subset E in Ω if

$$\forall t > 0,\ \exists s > 0\ such\ that\ for\ any\ E_1 \subset E;\ Cap_m(E_1) < s \Rightarrow \mu(E_1) < t.$$

(2) Let $f_j, f \in SH^m(\Omega)$, we say that $\liminf_{z \to \partial\Omega}(f_j - f) \geq 0$ if and only if

$$\forall \varepsilon > 0, \exists \Omega_1 \Subset \Omega,\ such\ that\ f_j(z) - f(z) \geq -\varepsilon$$

for every $z \in \Omega \setminus \Omega_1$ and $j \in \mathbb{N}$.

Definition 6. (1) The set Ω is said to be $m-$hyperconvex if it is open, bounded, connected and there exists a negative $m-$subharmonic function g such that for all $c < 0$, one has $\{z \in \Omega,\ g(z) < c\} \Subset \Omega$.

(2) A set $M \subset \Omega$ is called $m-$polar if there exist $u \in SH^m(\Omega)$ such that

$$M \subset \{u = -\infty\}.$$

(3) A sequence of functions $(f_j)_j$ defined on Ω is said to be convergent with respect to Cap_m to f on E if for all $t > 0$, one has

$$\lim_{j\to+\infty} Cap_m(E \cap \left\{|f - f_j| > t\right\}) = 0.$$

3. Convergence in Capacity and Hessian measure

In this section we prove that the convergence in $m-$capacity of a sequence $(f_j)_j \subset SH^m(\Omega)$ implies the convergence of the associated Hessian measure $H_m(f_j) := (dd^c f_j)^m \wedge \gamma^{n-m}$ for functions f_j that are only bounded near the boundary. We will start by establishing the following theorem.

Theorem 1. *Let $f \in SH^m(\Omega)$ and assume that there is a sequence $f_j \in SH^m(\Omega) \cap \mathbb{L}^\infty(\Omega)$ satisfying the following assumptions:*

(1) For all $j \in \mathbb{N}$, f_j is uniformly bounded near $\partial\Omega$.
(2) $f_j \to f$ in Cap_m on each $E \Subset \Omega$.
(3) For every $E \Subset \Omega$, one has $(dd^c f_j)^m \wedge \gamma^{n-m} \ll Cap_m$ uniformly.

Then the sequence of measures $(dd^c f_j)^m \wedge \gamma^{n-m}$ converges weakly to $(dd^c f)^m \wedge \gamma^{n-m}$ in Ω and $(dd^c f)^m \wedge \gamma^{n-m} \ll Cap_m$ on each $E \Subset \Omega$.

Proof. Using the assumption (1), we get that f is bounded near $\partial\Omega$. So the Borel measure $(dd^c f)^m \wedge \gamma^{n-m}$ is well defined, see [4]. To prove the convergence of $(dd^c f_j)^m \wedge \gamma^{n-m}$ toward $(dd^c f)^m \wedge \gamma^{n-m}$, we take a smooth function φ with compact support in Ω. So we have for all constant $r > 0$

$$\begin{aligned}\int_\Omega \varphi((dd^c f_j)^m - (dd^c f)^m) \wedge \gamma^{n-m} &= \int_\Omega \varphi((dd^c f_j)^m - (dd^c \max(f_j, -r))^m) \wedge \gamma^{n-m}\\ &\quad + \int_\Omega \varphi((dd^c \max(f_j, -r))^m - (dd^c \max(f, -r))^m) \wedge \gamma^{n-m}\\ &\quad + \int_\Omega \varphi((dd^c \max(f, -r))^m - (dd^c f)^m) \wedge \gamma^{n-m}\\ &:= A + B + C.\end{aligned}$$

Using Theorem 2.12 in [6], we obtain that for each $r > 0$ sufficiently large

$$\begin{aligned}|\, A\, | &= \left| \int_{f_j\le -r} \varphi((dd^c f_j)^m - (dd^c \max(f_j, -r))^m) \wedge \gamma^{n-m} \right|\\ &\le \max_\Omega |\varphi| \left(\int_{f_j\le -r} (dd^c f_j)^m \wedge \gamma^{n-m} + \int_{f_j\le -r} (dd^c \max(f_j, -r))^m \wedge \gamma^{n-m} \right).\end{aligned}$$

Now by Lemma 3 in [4] we get

$$\begin{aligned}\int_{f_j\le -r}(dd^c\max(f_j,-r))^m\wedge\gamma^{n-m} &\le \int_{f_j\le -r}(-1-\frac{2f_j}{r})^m(dd^c\max(f_j,-r))^m\wedge\gamma^{n-m}\\ &\le 2^m\int_{f_j<\frac{-r}{2}}(\frac{-r}{2}-f_j)^m(dd^c\max(\frac{f_j}{r},-1))^m\wedge\gamma^{n-m}\\ &\le 2^m(m!)^2\int_{f_j<\frac{-r}{2}}(dd^cf_j)^m\wedge\gamma^{n-m}.\end{aligned}$$

It follows that for each r large enough and all j

$$|A|\le(1+2^m(m!)^2)\max_{\Omega}|\varphi|\int_{f_j<\frac{-r}{2}}(dd^cf_j)^m\wedge\gamma^{n-m}.$$

As $Cap_m\{f<\frac{-r}{2}\}\to 0$ as $r\to\infty$ and $f_j\to f$ in Cap_m, we obtain that $Cap_m\{f_j<\frac{-r}{2}\}$ is uniformly convergent to zero for all j when $r\to\infty$. Using the assumption of the uniformly absolute continuity of $(dd^cf_j)^m\wedge\gamma^{n-m}$ we get that the integral $\int_{f_j<\frac{-r}{2}}(dd^cf_j)^m\wedge\gamma^{n-m}$ tends uniformly to zero for all j when $r\to\infty$.

Hence, for every $\varepsilon>0$ there exits a constant $r\ge 0$ such that $|A|\le\varepsilon$ for all j, and by Theorem 2 in [4] we can also require that $|C|\le\varepsilon$. However, for a such fixed constant r the assumption (2) implies that functions $\max(f_j,-r)$ converge to $\max(f,-r)$ in Cap_m on each $E\Subset\Omega$ as $j\to\infty$ and hence we conclude by Theorem 1.3.7 in [7] that $B\to 0$ as $j\to\infty$. Therefore, we obtain that $(dd^cf_j)^m\wedge\gamma^{n-m}$ converges weakly to $(dd^cf)^m\wedge\gamma^{n-m}$.

To finish the proof it suffices to show that $(dd^cf)^m\wedge\gamma^{n-m}\ll Cap_m$ on any open set $E\Subset\Omega$. Let $\varepsilon>0$ and take $\delta>0$ such that inequalities $(dd^cf_j)^m\wedge\gamma^{n-m}(F)\le\varepsilon$ hold for all j and all Borel sets $F\subset E$ with $Cap_m(F)<\delta$. Let $(\chi_k)_k$ be a sequence of non-negative smooth functions that increases to the characteristic function of F in Ω.

Then

$$\begin{aligned}\int_F(dd^cf)^m\wedge\gamma^{n-m} &= \lim_{k\to\infty}\int_\Omega\chi_k(dd^cf)^m\wedge\gamma^{n-m}\\ &= \lim_{k\to\infty}\lim_{j\to\infty}\int_\Omega\chi_k(dd^cf_j)^m\wedge\gamma^{n-m}\\ &\le \limsup_{j\to\infty}\int_F(dd^cf_j)^m\wedge\gamma^{n-m}\le\varepsilon.\end{aligned}$$

Hence $(dd^cf)^m\wedge\gamma^{n-m}\ll Cap_m$ on E and we have completed the proof of the Theorem. □

In the next lemmas we will be interested to prove some estimations known as Xing inequalities. Some of those inequalities were proved by Bedford and Taylor in [1] for bounded psh function and have several applications on the Dirichlet problem. In [9], Xing obtained a stronger version of those inequalities. In the following we will generalize those results to the class of $m-$subharmonic functions that are only bounded near the boundary.

Lemma 1. *Let $f_j,f\in SH^m(\Omega)$ such that $(dd^cf_j)^m\wedge\gamma^{n-m}\to(dd^cf)^m\wedge\gamma^{n-m}$ on Ω. Then the following assertions are equivalent*

(1) $(dd^cf)^m\wedge\gamma^{n-m}$ has zero mass on any $m-$polar set and $h(dd^cf_j)^m\wedge\gamma^{n-m}\to h(dd^cf)^m\wedge\gamma^{n-m}$ for every locally bounded $m-$sh function h on Ω.

(2) The sequence $(dd^c f_j)^m \wedge \gamma^{n-m}$ puts uniformly small mass on sets of small $m-$capacity.

The proof of the above result will be omitted since it is inspired from to the proof Theorem 3.2 in [2] which was established in the case of plurisubharmonic functions.

Lemma 2. *Let f_j be a sequence of bounded $m-$sh functions in Ω that decreases to $f \in SH^m(\Omega)$. Assume that*

(1) The function f is bounded near $\partial\Omega$.
(2) $(dd^c f)^m \wedge \gamma^{n-m} \ll Cap_m$ on any relatively compact subset of Ω.

Then $(dd^c f_j)^m \wedge \gamma^{n-m} \ll Cap_m$ uniformly for all j on each $E \Subset \Omega$.

Proof. Using the proof of Theorem 2 in [4] we obtain that $g(dd^c f_j)^m \wedge \gamma^{n-m} \to g(dd^c f)^m \wedge \gamma^{n-m}$ weakly in Ω for any locally bounded $m-$sh function g on Ω. Thus, the Lemma follows directly from Lemma 1. □

Remark 2. As a consequence of the previous lemma, we can deduce that "a function f is bounded near $\partial\Omega$ and have absolute continuous Hessian measure with respect to Cap_m if and only if f is the limit of functions given in Theorem 1". Indeed if we assume that f is bounded near $\partial\Omega$ and $(dd^c f) \wedge \gamma^{n-m} \ll Cap_m$ then the sequence $f_j := \max(f, -j)$ is bounded and decreases to f. Using the quasicontiuity combined with the Dini's theorem we deduce that f_j converges to f with respect to Cap_m. Now the lemma 2 implies that $(dd^c f_j) \wedge \gamma^{n-m} \ll Cap_m$. Hence the sequence f_j satisfies Theorem 1.

Conversely it is easy to check that every limit of functions in Theorem 1 is bounded near the boundary of Ω and with Hessian measure absolutely continuous with respect to Cap_m.

Lemma 3. *Let $f, g \in SH^m(\Omega)$ such that*

(1) $\liminf_{z \to \partial\Omega}(f(z) - g(z)) \geq 0$.
(2) The functions f and g are bounded near $\partial\Omega$ and with Hessian measure absolutely continuous with respect to Cap_m on each $E \Subset \Omega$.

Then for any constant $c \geq 1$ and all $h_j \in SH^m(\Omega)$ with $0 \leq h_j \leq 1$, $j = 1, 2, ..., m$, one has

$$\frac{1}{m!^2}\int_{f<g}(g-f)^m dd^c h_1 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} + \int_{f<g}(c-h_1)(dd^c g)^m \wedge \gamma^{n-m}$$

$$\leq \int_{f<g}(c-h_1)(dd^c f)^m \wedge \gamma^{n-m}.$$

Moreover if we assume that

$$(dd^c f)^m \wedge \gamma^{n-m} \geq (dd^c g)^m \wedge \gamma^{n-m}$$

in Ω, then $\{f < g\} = \emptyset$.

Proof. Replacing f by $f + 2t$ and then taking $t \searrow 0$, we may assume that there exists a subset $E \Subset \Omega$ such that $\{f < g\} \subset E$. Take $f_k := \max(f, -k)$ and $g_j = \max(g, -j)$. Then $\{f_k < g_j\} \subset E$ for k and j large enough. Using Lemma 3 in [4] we obtain that for any constant $c \geq 1$ and all $h_j \in SH^m(\Omega)$ such that $0 \leq h_j \leq 1$, $j = 1, 2, ..., m$

$$\frac{1}{m!^2}\int_{f_k<g_j}(g_j-f_k)^m dd^c h_1\wedge ...\wedge dd^c h_m\wedge\gamma^{n-m}+\int_{f_k<g_j}(c-h_1)(dd^c g_j)^m\wedge\gamma^{n-m}$$
$$\le\int_{f_k<g_j}(c-h_1)(dd^c f_k)^m\wedge\gamma^{n-m}$$

where k and j are large enough. Since $f_k\searrow f$ then $(dd^c f_k)^m\wedge\gamma^{n-m}$ tends weakly to $(dd^c f)^m\wedge\gamma^{n-m}$ then by Lemma 2 we get that $(dd^c f_k)^m\wedge\gamma^{n-m}\ll Cap_m$ uniformly for all k in the set E. Similarly, $(dd^c g_j)^m\wedge\gamma^{n-m}\ll Cap_m$ uniformly for all j in E. Now take $\varepsilon>0$, and let U be an open subset of Ω with $Cap_m(U)<\varepsilon$ such that f, g are continuous on $F=\Omega\backslash U$. Thus, we can write $g=\varphi_1+\varphi_2$ where φ_1 is continuous on $\overline{F}$ and $\varphi_2=0$ outside of U. Then

$$(c-h_1)(dd^c g_j)^m\wedge\gamma^{n-m}\to(c-h_1)(dd^c g)^m\wedge\gamma^{n-m}$$

weakly on Ω and we have

$$\int_{f_k<\varphi_1}(c-h_1)(dd^c g)^m\wedge\gamma^{n-m}\le\lim_{j\to\infty}\int_{f_k<\varphi_1}(c-h_1)(dd^c g_j)^m\wedge\gamma^{n-m}.$$

The last inequality implies that

$$\begin{aligned}\int_{f_k<g}(c-h_1)(dd^c g)^m\wedge\gamma^{n-m} &\le \int_{f_k<\varphi_1}(c-h_1)(dd^c g)^m\wedge\gamma^{n-m}+\int_U(c-h_1)(dd^c g)^m\wedge\gamma^{n-m}\\ &\le \lim_{j\to\infty}\int_{f_k<\varphi_1}(c-h_1)(dd^c g_j)^m\wedge\gamma^{n-m}+Cap_m(U)\\ &\le \lim_{j\to\infty}\int_{f_k<g}(c-h_1)(dd^c g_j)^m\wedge\gamma^{n-m}+O(\varepsilon)\\ &\le \lim_{j\to\infty}\int_{f_k<g_j}(c-h_1)(dd^c g_j)^m\wedge\gamma^{n-m}+O(\varepsilon).\end{aligned}$$

Hence if we let $j\to\infty$, we get

$$\frac{1}{m!^2}\int_{f<g}(g-f)^m dd^c h_1\wedge ...\wedge dd^c h_m\wedge\gamma^{n-m}+\int_{f_k<g}(c-h_1)(dd^c g)^m\wedge\gamma^{n-m}$$
$$\le\int_{f_k\le g}(c-h_1)(dd^c f_k)^m\wedge\gamma^{n-m}+O(\varepsilon).$$

Since the functions f, g are continuous on the set Ω and $f_k\searrow f$, we get

$$\begin{aligned}\int_{f\le g}(c-h_1)(dd^c f)^m\wedge\gamma^{n-m} &\ge \int_{\{f\le g\}\cap F}(c-h_1)(dd^c f)^m\wedge\gamma^{n-m}\\ &\ge \lim_{k\to\infty}\int_{\{f\le g\}\cap F}(c-h_1)(dd^c f_k)^m\wedge\gamma^{n-m}\\ &\ge \lim_{k\to\infty}\int_{\{f_k\le g\}\cap F}(c-h_1)(dd^c f_k)^m\wedge\gamma^{n-m}\\ &\ge \lim_{k\to\infty}\int_{f_k\le g}(c-h_1)(dd^c f_k)^m\wedge\gamma^{n-m}-O(\varepsilon).\end{aligned}$$

Now let $k \to \infty$ and as $\varepsilon > 0$ is arbitrary, we obtain

$$\frac{1}{m!^2}\int_{f<g}(g-f)^m dd^c h_1 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} + \int_{f<g}(c-h_1)(dd^c g)^m \wedge \gamma^{n-m}$$

$$\leq \int_{f\leq g}(c-h_1)(dd^c f)^m \wedge \gamma^{n-m}.$$

If we apply the last inequality to $f+t$ instead of f and then letting $t \searrow 0$, we obtain the desired result. □

Our main result in this paper is the following theorem where we give sufficient conditions combined with the convergence in capacity Cap_m for a sequence of $m-$subharmonic functions f_k to guarantee the weak convergence of the Hessian measures $(dd^c f_k)^m \wedge \gamma^{n-m}$. Such result generalizes well known results in [7, 10]. It suffices to take $m = n$ in our result to recover it.

Theorem 2. *Let* $\mathfrak{g} \in SH^m(\Omega)$ *a bounded function on* $\Omega \setminus E$ *for some* $E \Subset \Omega$. *Assume that there is* $f_k \in SH^m(\Omega)$ *satisfying*

(1) $|f_k| \leq |\mathfrak{g}|$ *in* Ω *for all* k.
(2) There exists an $m-$*subharmonic function* f *in* Ω *such that* $f_k \to f$ *in* Cap_m *on each* $E \Subset \Omega$.

Then the sequence of measures $(dd^c f_k)^m \wedge \gamma^{n-m}$ *converges weakly to* $(dd^c f)^m \wedge \gamma^{n-m}$ *in* Ω.

Before giving the proof of the Theorem, we need to establish some intermediate lemmas.

Lemma 4. *Let* $f_1, f_2, ..., f_m \in SH^m(\Omega)$ *such that* f_1 *is bounded in* Ω *and the functions* $f_2, ..., f_m$ *are bounded near* $\partial\Omega$. *For every* $E \Subset \Omega$ *there exists* $C_E > 0$ *such that for all Borel subset* G *in* E *the following estimate holds*

$$\int_G dd^c f_1 \wedge dd^c f_2 \wedge ... \wedge dd^c f_m \wedge \gamma^{n-m} \leq C_E (Cap_m(G))^{\frac{1}{2^m}}.$$

Proof. Without loss of generality we can assume that for all i, the functions f_i can be written, near $\partial\Omega$, as follows

$$f_i = \alpha\varphi(z) + \beta$$

where $\alpha > 0, \beta > 0$ and φ is a defining function of Ω. Let $G \subset E$ be a Borel subset and

$$f_G(z) = \sup\{u(z) : u \in SH^m(\Omega), u \leq -1 \text{ on } G, u < 0 \text{ on } \Omega\}$$

and f_G^* the associated upper semicontinuous regularization of G defined by

$$f_G^*(z) = \limsup_{\zeta \to z} f_G(\zeta).$$

We have $Cap_m(G) = \int_\Omega (dd^c f_G^*)^m \wedge \gamma^{n-m}$, $\lim_{z\to\partial\Omega} f_G^*(z) = 0$ and $f_G^* = -1$ on $G \setminus M$ for some $m-$polar set M, (see [7]). By the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\int_G dd^c f_1 \wedge dd^c f_2 \wedge ... \wedge dd^c f_m \wedge \gamma^{n-m} &\leq \int_\Omega -f_G^* dd^c f_1 \wedge dd^c f_2 \wedge ... \wedge dd^c f_m \wedge \gamma^{n-m}\\ &= \int_\Omega df_G^* \wedge d^c f_2 \wedge dd^c f_1 \wedge dd^c f_3 \wedge ... \wedge dd^c f_m \wedge \gamma^{n-m}\\ &\leq A\left(\int_\Omega df_G^* \wedge d^c f_G^* \wedge dd^c f_1 \wedge dd^c f_3 \wedge ... \wedge dd^c f_m \wedge \gamma^{n-m}\right)^{\frac{1}{2}}\\ &= A\left(\int_\Omega -f_G^* dd^c f_G^* \wedge dd^c f_1 \wedge dd^c f_3 \wedge ... \wedge dd^c f_m \wedge \gamma^{n-m}\right)^{\frac{1}{2}}\end{aligned}$$

where

$$A = \left(\int_\Omega df_2 \wedge d^c f_2 \wedge dd^c f_1 \wedge dd^c f_3 \wedge ... \wedge dd^c f_m \wedge \gamma^{n-m}\right)^{\frac{1}{2}}$$

is a finite constant because of the bounded assumption of the function f_1. By repeating the same argument $m-1$ more times we get

$$\begin{aligned}\int_G dd^c f_1 \wedge dd^c f_2 \wedge ... \wedge dd^c f_m \wedge \gamma^{n-m} &\leq A_E\left(\int_\Omega -f_G^*(dd^c f_G^*)^m \wedge \gamma^{n-m}\right)^{\frac{1}{2^m}}\\ &\leq C_E\left(\int_\Omega (dd^c f_G^*)^m \wedge \gamma^{n-m}\right)^{\frac{1}{2^m}}\\ &= C_E(Cap_m(G))^{\frac{1}{2^m}}\end{aligned}$$

for some constant $C_E > 0$. This proves the lemma. □

Lemma 5. *Let $f, g, h_1, ..., h_{m-1} \in SH^m(\Omega)$ bounded functions near $\partial\Omega$ and $h_m \in SH^m(\Omega) \cap \mathbb{L}^\infty(\Omega)$. If $\liminf_{z\to\partial\Omega}(f(z) - g(z)) \geq 0$ and the set $\{f < g\}$ is open, then for any $r \geq \sup_\Omega h_1$ one has*

$$\int_{f<g}(g-f)dd^c h_1 \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} + \int_{f<g}(r-h_1)dd^c g \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}$$
$$\leq \int_{f<g}(r-h_1)dd^c f \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}.$$

Proof. Without loss of generality, one can suppose that Ω is a $m-$hyperconvex domain and that there exists a function φ defined on Ω such that

$$f = g = h_1 = \varphi \text{ near } \partial\Omega.$$

Let $\{f_k\}$, $\{g_j\}$ and $\{h_1^l\}$ be a sequence of continuous $m-$sh functions such that $f_k = g_j = h_1^l = \varphi$ near $\partial\Omega$ and $f_k \to f$ as $k \to \infty$, $g_j \to g$ as $j \to \infty$ and $h_1^l \to h_1$ as $l \to \infty$. By Lemma 3 in [4] we get

$$\int_{f_k<g_j}(g_j-f_k)dd^c h_1^l \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} + \int_{f_k<g_j}(r-h_1^l)dd^c g_j \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}$$
$$\leq \int_{f_k<g_j}(r-h_1^l)dd^c f_k \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}.$$

If we let $k \to \infty$ then by Fatou's lemma we obtain

$$\int_{f<g_j} (g_j - f)dd^c h_1^l \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} + \int_{f<g_j} (r - h_1^l)dd^c g_j \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}$$
$$\leq \liminf_{k\to\infty} \int_{f<g_j} (r - h_1^l)dd^c f_k \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}.$$

Since

$$\lim_{k\to\infty}(r - h_1^l)dd^c f_k \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} = (r - h_1^l)dd^c f \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}$$

weakly in Ω we obtain using Lemma 4 that

$$(r - h_1^l)dd^c f_k \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} << Cap_m$$

for all k on each $E \Subset \Omega$. Hence $\forall \varepsilon > 0, \exists k_0 > 0$ such that

$$\liminf_{k\to\infty} \int_{f<g_j} (r - h_1^l)dd^c f_k \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}$$
$$\leq \liminf_{k\to\infty} \int_{f_{k_0}\leq g_j+\varepsilon} (r - h_1^l)dd^c f_k \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} + \varepsilon$$
$$\leq \int_{f_{k_0}\leq g_j+\varepsilon} (r - h_1^l)dd^c f \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} + \varepsilon.$$

As $f \leq f_{k_0}$ and $g \leq g_j$ and ε is arbitrary chosen, we obtain

$$\int_{f<g_j} (g_j - f)dd^c h_1^l \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} + \int_{f<g} (r - h_1^l)dd^c g_j \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}$$
$$\leq \int_{f\leq g_j} (r - h_1^l)dd^c f \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}.$$

By letting $l \to \infty$ and using the fact that

$$(g_j - f)dd^c h_1^l \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} \to (g_j - f)dd^c h_1 \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}$$

weakly in Ω when $l \to \infty$ we get

$$\int_{f<g_j} (g_j - f)dd^c h_1 \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} + \int_{f<g} (r - h_1)dd^c g_j \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}$$
$$\leq \int_{f\leq g_j} (r - h_1)dd^c f \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}.$$

Now if we let $j \to \infty$ and using the weak convergence of $(r - h_1)dd^c g_j \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}$ combined with the Fatou lemma and the fact that the set $\{f < g\}$ is supposed open, we get

$$\int_{f<g} (g - f)dd^c h_1 \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} + \int_{f<g} (r - h_1)dd^c g \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}$$
$$\leq \int_{f\leq g} (r - h_1)dd^c f \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m}.$$

To complete the proof it suffices to apply the previous inequality to $f + t$ instead of f and then we take $t \searrow 0$. □

Now we give the proof of the Theorem 2.

Proof. By hypothesis (1), we may assume without loss of generality that there exists a compact subset K in Ω such that $f_k = \mathfrak{g}$ in $\Omega \setminus K$ for all k and $\mathfrak{g} = f_k = 0$ on $\partial\Omega$. We will assume by induction that the current $(dd^c f_k)^l \wedge \gamma^{n-m}$ converges weakly to $(dd^c f)^l \wedge \gamma^{n-m}$ in Ω for $1 \leq l \leq m-1$. Using Lemma 5, we obtain that for any $r > 0$ and all k

$$\begin{aligned}
&\int_{f_k<-r} (-f_k)(dd^c f_k)^{m-p} \wedge \gamma^{n+p-m} \\
&\leq \int_{\mathfrak{g}<-r} (-\mathfrak{g})(dd^c f_k)^{m-p} \wedge \gamma^{n+p-m} \\
&\leq 2\int_{\mathfrak{g}<\frac{-r}{2}} (-\mathfrak{g}-\frac{r}{2})(dd^c f_k)^{m-p} \wedge \gamma^{n+p-m} \\
&\leq 2\int_{\mathfrak{g}<\frac{-r}{2}} (-f_k)dd^c\mathfrak{g} \wedge (dd^c f_k)^{m-p-1} \wedge \gamma^{n+p-m} \\
&\leq 2\int_{\mathfrak{g}<\frac{-r}{2}} -\mathfrak{g}dd^c\mathfrak{g} \wedge (dd^c f_k)^{m-p-1} \wedge \gamma^{n+p-m} \\
&\leq 2^2\int_{\mathfrak{g}<\frac{-r}{2^2}} -\mathfrak{g}(dd^c\mathfrak{g})^2 \wedge (dd^c f_k)^{m-p-2} \wedge \gamma^{n+p-m} \\
&\leq \ldots \leq 2^{m-p}\int_{\mathfrak{g}<\frac{-r}{2^{m-p}}} -\mathfrak{g}(dd^c\mathfrak{g})^{m-p} \wedge \gamma^{n+p-m}.
\end{aligned}$$

Hence we get that $(-\mathfrak{g})(dd^c\mathfrak{g})^{m-p} \wedge \gamma^{n+p-m} \ll (dd^c\mathfrak{g})^{m-p} \wedge \gamma^{n+p-m} \ll Cap_m$ on each $E \Subset \Omega$. So we obtain that

$$(-f_k)(dd^c f_k)^{m-p} \wedge \gamma^{n+p-m} \ll Cap_m \qquad (*)$$

on each $E \Subset \Omega$ uniformly for all k.

Replacing f and f_k by $\max(f, -c)$ and $\max(f_k, -c)$ respectively for a fixed constant c if necessary, we can assume that both f and f_k are locally uniformly bounded. So by assumption (1) and proposition 1 we get that for any $\varepsilon > 0$ the following writing hold

$$f_k = f_{k,1} + f_{k,2} \text{ and } f = f_1 + f_2$$

where f_1 is a continuous function in Ω and $f_{k,2} = f_2 = 0$ on $\Omega \setminus \mathcal{U}$ for some $\mathcal{U} \subset \Omega$ with $Cap_r(\mathcal{U}) < \varepsilon$. Furthermore, for each $E \Subset \Omega \setminus \mathcal{U}$, one has that $|f_{k,1} - f_1| < \varepsilon$ on E for large value of k and the functions $f_{k,1}$, $f_{k,2}$, f_1 and f_2 are bounded uniformly by a constant which does not depend on ε. If we consider the following decomposition

$$\begin{aligned}
f_k(dd^c f_k)^{m-p} \wedge \gamma^{n+p-m} - f(dd^c f)^{m-p} \wedge \gamma^{n+p-m} \quad = \quad & (f_{k,1} - f_1)(dd^c f_k)^{m-p} \wedge \gamma^{n+p-m} \\
+ \quad & f_1((dd^c f_k)^{m-p} - (dd^c f)^{m-p}) \wedge \gamma^{n+p-m} \\
+ \quad & (f_{k,2}(dd^c f_k)^{m-p} - f_2(dd^c f)^{m-p}) \wedge \gamma^{n+p-m}.
\end{aligned}$$

So the proof will be completed if we show that all three terms of the right hand side in the last equality tend weakly to 0. For the third term its suffices to use $(*)$ to get that it tends to zero weakly and uniformly for all k when ε goes to 0. Since we have

$$\int_E |f_{k,1} - f_1|(dd^c f_k)^{m-p} \wedge \gamma^{n+p-m} \leq \varepsilon \int_{E\setminus\mathcal{U}} (dd^c f_k)^{m-p} \wedge \gamma^{n+p-m} + \sup_k |f_{k,1} - f_1| \int_{\mathcal{U}} (dd^c f_k)^{m-p} \wedge \gamma^{n+p-m},$$

and $(dd^c f_k)^{m-p} \wedge \gamma^{n+p-m}$ converges weakly to $(dd^c f)^{m-p} \wedge \gamma^{n+p-m}$ by induction's assumption we deduce that the first and the second term in the last equality also converges weakly to zero uniformly for all k as $\varepsilon \to 0$. The result of the theorem follows. □

Remark 3. (1) Using the Theorem 2, we deduce that for all j

$$(r - h_1) dd^c g_j \wedge dd^c h_2 \wedge ... \wedge dd^c h_m \wedge \gamma^{n-m} << Cap_m$$

on every subset $E \Subset \Omega$. Then we can deduce that the assumption "the set $\{f < g\}$ is open" in Lemma 5 is superfluous. This implies that the Lemma 5 is an improved version of Lemma 3.

(2) The assumptions in the Theorem 2 can be replaced by the monotically convergence of f_k towards f for $f, f_k \in SH^m(\Omega) \cap \mathbb{L}^\infty(\Omega \setminus E)$.

Theorem 3. *Let $f_j \in B_m(\Omega)$ and $f \in SH^m(\Omega) \cap \mathbb{L}^\infty_{loc}(\Omega)$. The following assertions hold*

(1) If $f_j \to f$ in Cap_{m-1} in every $E \Subset \Omega$ then for all h in $B_m(\Omega)$ one has that $h(dd^c f_j)^m \wedge \gamma^{n-m}$ converges weakly to $h(dd^c f)^m \wedge \gamma^{n-m}$.

(2) If for every $E \Subset \Omega$ one has $f_j \to f$ in Cap_m then for every $\xi \in C_0^\infty(\Omega)$ we have that $\int_\Omega \xi h(dd^c f_j)^m \wedge \gamma^{n-m} \to \int_\Omega \xi h(dd^c f)^m \wedge \gamma^{n-m}$ uniformly for all h in $B_m(\Omega)$.

(3) If $f_j \to f$ in Cap_m on each $E \Subset \Omega$ and $h_j \in B_m(\Omega)$ converges weakly to $h \in B_m(\Omega)$, then $h_j(dd^c f_j)^m \wedge \gamma^{n-m}$ converges weakly to $h(dd^c f)^m \wedge \gamma^{n-m}$ in Ω.

Proof. To prove the assertion (1), it remains to show, by induction, that for each $k \leqslant m$, $(dd^c f_j)^k \wedge \gamma^{n-m}$ tends weakly to $(dd^c f)^k \wedge \gamma^{n-m}$. The case for $k = 1$ is obvious since the convergence assumption implies that $f_j \to f$ in $\mathbb{L}^1_{loc}(\Omega)$. Hence, it follows that $dd^c f_j \wedge \gamma^{n-m}$ converges weakly to $dd^c f \wedge \gamma^{n-m}$. Assume, by induction, that it is true for all $k = q < m$ and we have to show that $f_j(dd^c f_j)^q \wedge \gamma^{n-m}$ converges weakly to $f(dd^c f)^q \wedge \gamma^{n-m}$ and by taking the operator dd^c we will obtain the required statement for $k = q + 1$. Let $\varepsilon > 0$, the function f can be written as $f = h_1 + h_2$ on Ω, where h_1 is continuous, $h_2 = 0$ outside an open subset $U \subset \Omega$ with $cap_m(U) < \varepsilon$, and the supremum norm of h_2 depends only on the function h. We have

$$\begin{aligned} f_j(dd^c f_j)^q \wedge \gamma^{n-m} - f(dd^c f)^q \wedge \gamma^{n-m} &= (f_j - f)(dd^c f_j)^q \wedge \gamma^{n-m} \\ &+ h_2 \Big[(dd^c f_j)^q \wedge \gamma^{n-m} - (dd^c f)^q \wedge \gamma^{n-m} \Big] \\ &+ h_1 \Big[(dd^c f_j)^q \wedge \gamma^{n-m} - (dd^c f)^q \wedge \gamma^{n-m} \Big] \\ &= A_1 + A_2 + A_3. \end{aligned}$$

The inductive assumption gives that A_3 converges to 0 in the sense of currents. On the other hand, it is easy to check that

$$(dd^c f_j)^q \wedge \gamma^{m-q-1} \wedge \gamma^{n-m+1} \leqslant \left(dd^c (f_j + |z|^2) \right)^{m-1} \wedge \gamma^{n-m+1}.$$

The last term is dominated by a constant, independent on j, multiplied by Cap_m. Hence using the convergence assumption we obtain that A_1 converges in the sense of currents to 0. Now since $h_2 = 0$ outside U, then A_2 makes arbitrarily small mass for all j by choosing ε small enough. Hence we have obtained the weak convergence of $f_j(dd^c f_j)^q \wedge \gamma^{n-m}$ to $f(dd^c f)^q \wedge \gamma^{n-m}$. To finish the proof of the assertion (1) it suffices to use the quasicontinuity of the function h to get the desired result.

To prove (2), thanks to the assertion (1) we have that $(dd^c f_j)^m \wedge \gamma^{n-m} \to (dd^c f)^m \wedge \gamma^{n-m}$ weakly in Ω and hence we may assume that $B_m(\Omega) = \{f \in SH^m(\Omega); 0 < f < 1\}$. Let $\xi \in C_0^\infty(\Omega)$ a test function. Changing the values of f_j and f near $\partial\Omega$, we can suppose that there exists a subset E such that $supp\, \xi \Subset E$ and $f_j = f$ in $\Omega \setminus E$. It follows that for every $\varepsilon > 0$ and all h in $B_m(\Omega)$, an integration by parts yields

$$\begin{aligned}
&\int_\Omega \xi h((dd^c f_j)^m - (dd^c f)^m) \wedge \gamma^{n-m}\\
&= \int_{E\cap\{|f_j - f|<\varepsilon\}} (f_j - f) dd^c(\xi h) \wedge (\Sigma_{k=0}^{m-1}(dd^c f_j)^k \wedge (dd^c f)^{m-1-k}) \wedge \gamma^{n-m}\\
&+ \int_{E\cap\{|f_j - f|\geq\varepsilon\}} (f_j - f) dd^c(\xi h) \wedge (\Sigma_{k=0}^{m-1}(dd^c f_j)^k \wedge (dd^c f)^{m-1-k}) \wedge \gamma^{n-m}\\
&:= A_{\varepsilon,j} + B_{\varepsilon,j}\ .
\end{aligned}$$

Let $\xi \in C_0^\infty(\Omega)$ and C_1 a constant sufficiently large satisfying $\xi = (\xi + C_1|z|^2) - C_1|z|^2 := \xi_1 - \xi_2$, where $0 \leq \xi_1, \xi_2 \in SH^m(\Omega) \cap \mathbb{L}^\infty(\Omega)$. For the cases $k = 1$ and $k = 2$ we get that $2dd^c(\xi_k h) = dd^c((\xi_k + h)^2) - dd^c(h^2) - dd^c(\xi_k^2)$. It follows that there exists a constant C_2 that does not depend on ε and $j \in \mathbb{N}$ such that

$$| A_{\varepsilon,j}(\xi) | \leq | A_{\varepsilon,j}(\xi_1) | + | A_{\varepsilon,j}(\xi_2) | \leq C_2 Cap_m(E)\varepsilon$$

and

$$| B_{\varepsilon,j}(\xi) | \leq | B_{\varepsilon,j}(\xi_1) | + | B_{\varepsilon,j}(\xi_2) | \leq C_2 Cap_m(E \cap \{|f_j - f| > \varepsilon\}) \to 0$$

as $j \to \infty$. This gives that

$$\int_\Omega \xi h(dd^c f_j)^m \wedge \gamma^{n-m} \to \int_\Omega \xi h(dd^c f)^m \wedge \gamma^{n-m}$$

as $j \to \infty$ uniformly in $B_m(\Omega)$.

For the assertion (3) we have

$$\begin{aligned}
&h_j(dd^c f_j)^m \wedge \gamma^{n-m} - h(dd^c f)^m \wedge \gamma^{n-m}\\
&= h_j((dd^c f_j)^m - (dd^c f)^m) \wedge \gamma^{n-m} + (h_j - h)((dd^c f)^m\\
&- (dd^c u_s)^m) \wedge \gamma^{n-m} + (h_j - h)(dd^c u_s)^m \wedge \gamma^{n-m} =: A + B + C
\end{aligned}$$

where u_s are smooth $m-$sh functions decreasing to f. Using the assertion (2) the term B goes weakly to zero as $s \to \infty$ uniformly for all j. Hence if s is a constant sufficiently large we get that both A and C converge weakly to zero as $j \to \infty$. So the assertion (3) follows. □

Using Theorem 3, one can get that the convergence with respect to the Lebegue measure of a sequence of $m-$sh functions f_j implies the weak convergence of f_j with respect to any measure that has nos mass on every $m-$polar sets.

Corollary 1. *If ν is a locally finite measure, f_k a sequence of $m-$sh functions in Ω and $f_0 \in SH^m(\Omega) \cap \mathbb{L}^1_{loc}(\Omega, \nu)$ satisfying the following assumptions.*

i) For every $m-$polar set $A \subset \Omega$ one has $\nu(A) = 0$.

ii) For all $k \in \mathbb{N}$, $|f_k| \leq |f_0|$.

iii) For every $E \Subset \Omega$, $\displaystyle\int_E | f_j - f | \, d\lambda \to 0$.

Then $\int_E | f_k - f | \, d\nu \to 0$ *as* $k \to \infty$ *on any* $E \Subset \Omega$.

Proof. Without loss of generality we may assume that for every $z \in \Omega$; $f_k(z) < 0$ and $f(z) < 0$. Using hypothesis *i*) it suffices to show that

$$\forall \zeta \in C_0^\infty(\Omega), \lim_{k\to+\infty} \int_\Omega \zeta f_k d\nu = \int_\Omega \zeta f d\nu.$$

For $\zeta \in C_0^\infty(\Omega)$, one has the following the following writing

$$\begin{aligned}
&\int_\Omega \zeta f_k d\nu - \int_\Omega \zeta f d\nu \\
&= \int_\Omega \zeta(f_k - \max(f_k, -s))d\nu + \int_\Omega \zeta(\max(f_k, -s) - \max(f, -s))d\nu + \int_\Omega \zeta(\max(f, -s) - f)d\nu \\
&\leq 2. \max | \zeta | \int_{supp\zeta \cap \{f_0 < -s\}} -f_0 d\nu + \int_\Omega \zeta(\max(f_k, -s) - \max(f, -s))d\nu.
\end{aligned}$$

As $\lim_{s\to+\infty} \max | \zeta | \int_{supp\zeta\cap\{f_0<-s\}} -f_0 d\nu = 0$ then

$$\int_\Omega \zeta f_k d\nu - \int_\Omega \zeta f d\nu \leq \int_\Omega \zeta(\max(f_k, -s) - \max(f, -s))d\nu.$$

On the other hand, using Theorem 5.3 in [6] there exists $\alpha \in SH^m(\Omega) \cap \mathbb{L}^\infty(\Omega)$ and $h \in \mathbb{L}^1(\Omega, (dd^c\alpha)^m \wedge \gamma^{n-m})$ such that $h \geq 0$ and $1\!\!1_{supp\zeta} d\nu = \mathrm{h}(\mathrm{dd^c}\alpha)^{\mathrm{m}} \wedge \gamma^{\mathrm{n-m}}$. So for every $\varepsilon > 0$ there exists $s, j > 0$ such that

$$\begin{aligned}
&| \int_\Omega \zeta f_k d\nu - \int_\Omega \zeta f d\nu| \\
&\leq | \int_\Omega \zeta \min(h, j)(\max(f_k, -s) - \max(f, -s))(dd^c\alpha)^m \wedge \gamma^{n-m}| + \varepsilon.
\end{aligned}$$

So one can take $g \in C(\Omega)$ such that $\int_{supp\zeta} | \min(h, k) - g|(dd^c\alpha)^m \wedge \gamma^{n-m} < \frac{\varepsilon}{s}$. it follows that

$$\begin{aligned}
&| \int_\Omega \zeta f_k d\nu - \int_\Omega \zeta f d\nu| \\
&\leq | \int_\Omega \zeta g(\max(f_k, -s) - \max(f, -s))(dd^c\alpha)^m \wedge \gamma^{n-m}| + (2 \max | \zeta | +1)\varepsilon.
\end{aligned}$$

The last integral tends to 0 when $k \to \infty$ by Theorem 3. Therefore the proof of the desired Theorem is completed. □

In the following theorem we treat the converse sense. So we will prove that the convergence of the hessian measure associated to a sequence of $m-$sh functions implies, under some conditions, the convergence in capacity Cap_m for a such sequence.

Theorem 4. *Let $(f_j)_j \subset SH^m(\Omega) \cap \mathbb{L}^\infty(\Omega)$ be a sequence of locally uniformly bounded functions that converges weakly to $f \in SH^m(\Omega)$. Assume that*

(1) $\liminf_{z\to\partial\Omega}(f_j - f) \geq 0$ *uniformly for all j.*

(2) There exists a positive measure $d\mu$ in Ω such that $h(dd^c f_j)^m \wedge \gamma^{n-m}$ converges weakly to $hd\mu$ in Ω uniformly for all $h \in SH^m(\Omega)$ with $0 \leq h \leq 1$.

Then $(dd^c f)^m \wedge \gamma^{n-m} = d\mu$ and $f_j \to f$ in Cap_m on each $E \Subset \Omega$. Hence, if furthermore $\liminf_{z\to\partial\Omega}(f - f_j) \geq 0$ uniformly for all j then $f_j \to f$ in Cap_m on Ω.

Proof. Let $\varphi \in SH^m(\Omega)$ such that $0 < \varphi < 1$ and $E \Subset \Omega$. For every $t > 0$ one has

$$\int_{E\cap\{|f_j-f|>t\}} (dd^c\varphi)^m \wedge \gamma^{n-m} \leq Cap_m(E \cap \{f_j > f + t\}) + \int_{E\cap\{f_j<f-t\}} (dd^c\varphi)^m \wedge \gamma^{n-m}.$$

Using the quasicontinuity of $m-$sh functions and the Hartogs Lemma we get that $Cap_m(E \cap \{f_j > f + t\}) \to 0$ when $j \to \infty$. Hence, by the assumption (2) and Lemma 3 [4] we obtain

$$\int_{f_j<f-t} (dd^c\varphi)^m \wedge \gamma^{n-m} \leq \frac{1}{t^m} \int_{f_j<f-t} (f - f_j)^m (dd^c\varphi)^m \wedge \gamma^{n-m}$$

$$\leq \tfrac{m!^2}{t^m} \int_{f_j<f-t} (dd^c f_j)^m \wedge \gamma^{n-m} \leq \frac{m!^2}{t^{m+1}} \int_{f_j<f-t} (u - f_j)(dd^c\varphi)^m \wedge \gamma^{n-m}.$$

Take $\varepsilon > 0$, and $F_1 \Subset F_2 \Subset \Omega$ such that $f_j - f \geq -\varepsilon$ in $\Omega \setminus F_1$ and $\{f_j < f - t\} \Subset F_1$ for all j. Again by the Hartogs Lemma and the quasicontinuity of $m-$sh functions we get that there exist $j_0 > 0$ and $A \subset F_2$ with $Cap_m(A) < \varepsilon$ such that $\varepsilon + f(z) - f_j(z) \geq 0$ in $F_2 \setminus A$ for all $j \geq j_0$. Let $\chi \in C_0^\infty(F_2)$ such that $\chi \geq 0$ and $\chi = 1$ in F_1. Since all the functions f_j and f are uniformly bounded in F_2, then for $j \geq j_0$

$$\int_{f_j<f-t} (dd^c f_j)^m \wedge \gamma^{n-m} \leq \frac{m!^2}{t^{m+1}} \int_{f_j<f-t} (f - f_j)(dd^c\varphi)^m \wedge \gamma^{n-m}$$

$$\leq \int_{F_1\setminus A} \chi(\varepsilon + f - f_j)(dd^c f_j)^m \wedge \gamma^{n-m} + O(\varepsilon) \leq \int_{F_2} \chi(f - f_j)(dd^c f_j)^m \wedge \gamma^{n-m} + O(\varepsilon)$$

$$= \int_{F_2} \chi(f - f_j)((dd^c f_j)^m \wedge \gamma^{n-m} - d\mu) + \int_{F_2} \chi(f - f_j)d\mu + O(\varepsilon).$$

Now using assumption (3) and Corollary 1, we obtain that the last two integrals go to zero when $j \to \infty$. Hence $f_j \to f$ in Cap_m on each $E \Subset \Omega$. Then by [6], we get $(dd^c f)^m \wedge \gamma^{n-m} = d\mu$ and the proof of the Theorem is complete. □

4. Conclusions

In this paper we have dealt with a problem related to the convergence of a sequence of complex Hessian measures given by a sequence of $m-$subharmonic functions f_j. By introducing some conditions, we have shown that if f_j converges in Cap_m then the associated sequence of measures converges in the weak sense. In addition we have shown that the converse sense still true form some particular classes of $m-$subharmonic functions. The established results in this paper may be useful not only in the problem related to the convergence of the Hessian measure but also in the resolution of the famous complex Hessian equations.

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Conflict of interest

Authors declare that no conflicts of interest in this manuscript.

References

1. E. Bedford, B. A. Taylor, A new capacity for plurisubharmonic functions, *Acta. Math.*, **149** (1982), 1–40. https://doi.org/10.1007/BF02392348
2. E. Bedford, B. A. Taylor, Fine topology, *Šilov* boundary and $(dd^c)^n$, *J. Funct. Anal.*, **72** (1987), 225–251. https://doi.org/10.1016/0022-1236(87)90087-5
3. Z. Blocki, Weak solutions to the complex Hessian equation, *Ann. Inst. Fourier, Grenoble*, **55** (2005), 1735–1756.
4. A. Dhouib, F. Elkhadhra, $m-$Potential theory associated to a positive closed current in the class of $m-$sh functions, *Complex Var. Elliptic Eq.*, **61** (2016), 875–901. https://doi.org/10.1080/17476933.2015.1133615
5. S. Kolodziej, *The complex Monge-Ampére equation and theory*, American Mathematical Soc., 2005.
6. H. C. Lu, A variational approach to complex Hessian equations in $\mathbb{C}^n$, *J. Math. Anal. Appl.*, **431** (2015), 228–259. https://doi.org/10.1016/j.jmaa.2015.05.067
7. H. C. Lu, *Equations Hessiennes complexes,* Ph.D Thesis, Université de Toulouse, Université Toulouse III-Paul Sabatier, 2012.
8. P. Lelong, Discontinuité et annulation de l'opérateur de Monge-Ampére complexe, In: P. Lelong, P. Dolbeault, H. Skoda, *Séminaire d'analyse*, Springer, **1028** (1983), 219–224. https://doi.org/10.1007/BFb0071683
9. Y. Xing, Continuity of the complex Monge-Ampére operator, *Proc. Amer. Math. Soc.*, **124** (1996), 457–467. https://doi.org/10.1090/S0002-9939-96-03316-3
10. Y. Xing, Complex Monge-Ampére measures of plurisubharmonic functions with bounded values near the boundary, *Can. J. Math.*, **52** (2000), 1085–1100. https://doi.org/10.4153/CJM-2000-045-x