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Research article

Cauchy problem for non-autonomous fractional evolution equations with nonlocal conditions of order $(1, 2)$

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Abstract: This article contracts through Cauchy problems in infinite-dimensional Banach spaces towards a system of nonlinear non-autonomous mixed type integro-differential fractional evolution equation by nonlocal conditions through noncompactness measure (MNC). We demonstrate the existence of novel mild solutions in the condition that the nonlinear function mollifies generally adequate, an MNC form and local growth form, using evolution families and fractional calculus theory, as well as the fixed-point theorem w.r.t. K-set-contractive operator and another MNC assessment procedure. Our findings simplify and improve upon past findings in this area. Finally, towards the end of this article, as an example of submissions, we use a fractional non-autonomous partial differential equation (PDE) with nonlocal conditions and a homogeneous Dirichlet boundary condition.

Keywords: initial value problem; non-autonomous fractional evolution equations; measure of noncompactness; nonlocal conditions; mild solution; analytic semigroup
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1. Introduction

The qualitative actions of fractional evolution differential and difference equations, which is clearly time-dependent can be defined via non-autonomous dynamics. In past era, the concept of such structures has formulated into an extremely functioning field associated towards, still recognizably discrete from that of traditional autonomous dynamic frameworks. This development was driven by issues of applied math, in the existence sciences where straightforwardly non-autonomous structures feature. On another side, solution's existence of the fractional differential equations (FDE) via nonlocal conditions has been concentrated ordinarily by numerous writers in this way, nonlocal conditions are further practical than the classical initial conditions for example, in managing numerous physical issues [2, 19, 32, 33].

Fractional calculus (FC) is a simplification of calculus ordinary integer. In past years, impulsive differential equations have becoming functioning in research field because of incontestable submissions in extensive fields of engineering and science such as bio, chemistry, physics, economy, population dynamics, control theory, chemical technology, medical science and many more [3, 22, 40]. Various natural structures that are categorized by the incidence of sudden modification in the condition of the framework can be characterized by impulsive differential equations. The above mentioned variations happen at definite time moments for a time of minimal interval. Impulsive differential equations are too suitable form to genetic development for which a delay row originates in carving equations. Such equations depict the evolution procedures that are reliant on sudden variations and discontinuous hurdles in their circumstances. Many physical structures as the function of pendulum clock, the effect of mechanical systems, protection of species with the use of repeated stocking or collecting etc. naturally involve the impulsive marvels. Also, in several further states, the evolution procedures have impulsive actions. Such as, the disruptions in cellular nervous networks, electromechanical systems depending on easing oscillations, dynamical systems having automatic procedures etc., have the impulsive marvels. The uniqueness, existence and stability of mild solutions to functional differential equations with impulsive conditions have been concentrated by numerous authors in writing (refer [20, 27, 28, 41]).

Fractional calculus's benefit concluded that integer-order calculus offers an extraordinary agreement for the sort of hereditary and conviction properties of extended techniques and materials. Since previous twenty years, Fractional Calculus (FC) has pulled in investigation consideration by itself because of its significance in a few pieces of science technology, similar to Fluid Mechanics (FM), Physics, conduction of heat [14, 21, 31]. We can relate to the monographs [8] for the basics and to reference [5] for the recent progresses in fractional calculus field. In reference [25], existence and controllability of nonlocal mixed VolterraFredholm type fractional delay integrodifferential equations of order $1 < r < 2$. Nowadays non-autonomous differential equations with integer order have been deliberated by numerous scientists. Reference [35] discussed a new method for fractional differential evolution equations by approximating the there controllability having order $1 < r < 2$ in Hilbert spaces. Reference [36] investigated a novel way to estimate controllability of fractional evolution inclusions having order $1 < r < 2$with indefinite delay.

To determine the controllability of a nonautonomous nonlinear differential system involving non-instant impulses in the space R^n, Malik et al. [28] used the Rothe's fixed point theorem. Kucche [12] studied the uniqueness and existence of mild solutions for impulsive delay integro-differential equations with integral impulses in Banach spaces by using Krasnoselskii-Schaefer fixed point theorem. Let δ be a bounded and convex set in O, and O be a real Banach space with norm $\|.\|$. Operator $N : \delta \longrightarrow \delta$ is entirely continuous in Schauder's fixed point theorem, then N has fixed-point in δ in any case. It is widely identified as Schauders fixed point theorem. That is very well-known and essential fixed-point theorem, and this is highly extensive application. However, Schauder's fixed point theorem requires that the operator be totally continuous, which imposes very strong conditional constraints. Hence, the fixed point theorem of Sadovskii has been proven to be correct. In 1981, Lakshmikantham and Leela [26] designed the next mentioned IVP of ODE in the Banach space O.

$$\begin{cases} u'(s) = \ f(s,\ u(s)),\ s \in [0, b] \\ u(0) = \ u_0, \end{cases} \tag{1.1}$$

here $b > 0$ is constant. The writers showed that, f is said to be uniformly continuous on [0,b]$\times Y_E$ if for somewhat constant $E > 0$ then satisfies the MNC condition.

$$\tau(f(s, \nu)) \leq j\tau(\nu), \forall s \in [0, b], \nu \subset Y_E, \tag{1.2}$$

where Y_E = {u ∈ O: $\|u\|$ ≤ E}, $\tau(.)$ demonstrate Kuratowski MNC, j is a +ve constant, then initial value problem Eq (1.1) has a global solution if j mollifies the following condition

$$bj < 1. \tag{1.3}$$

Guo [16] explored the global solutions of the initial value problem (IVP) in the Banach space O for nonlinear integro-differential equations of first-order mixed type in 1989.

$$\begin{cases} u'(s) = f(s, u(s), (Su)(s), (Tu)(s)),\ s \in [0, b] \\ u(0) = u_0, \end{cases} \tag{1.4}$$

where

$$(Su)(s) = \int_0^s H(s, t)u(t)dt, \tag{1.5}$$

is a Volterra integral operator with H ∈ C ($\varDelta$ kernel, R),$\varDelta$=(s,t) — 0 ≤ t ≤ s ≤ b and

$$(Tu)(s) = \int_0^b K(s, t)u(t)dt, \tag{1.6}$$

is a Fredholm integral operator with K ∈ C ($\varDelta_0$,E) kernel , $\varDelta_0$={(s,t) — 0 ≤ s ≤ t ≤ b}. Represent $H_0 = \max_{(s,t)\in\varDelta} |H(s, t)|$ and $K_0 = \max_{(s,t)\in\varDelta_0} |K(s, t)|$ then Guo proved that initial value problem (1.4) if for $E > 0$ then there will exist a minimum one global solution, f is said to be uniformly continuous on [0,b]$\times Y_E \times Y_E \times Y_E$ then there exist positive constants j_i in such a way ($i = 0, 1, 2, 3$).

$$\tau(f(t, U_1, U_2, U_3)) \leq j_1\tau(U_1) + j_2\tau(U_2) + j_3\tau(U_3), \tag{1.7}$$

for ∀ s ∈ [0, b] and bounded sets U_1,U_2,U_3 ⊂ O and

$$2b(j_1 + bK_o j_2 + bH_0 j_3) < 1. \tag{1.8}$$

In 2017, Guo [15] discovered that the impulsive semi-linear fractional integro-differential equation using noncompact semigroup has a local as well as global mild solution. Afterward, there are a lot of writers investigated ordinary differential equations (ODE) by the use of Sadovskiis fixed point theorem in real Banach spaces just like Eq (1.4) in the postulation just like Eq (1.7), constants fulfill strong inequality analogous to Eq (1.8). For further information on these realities, please perceive [29] and their references. Inequality Eqs (1.3) and (1.8) are very strong prohibitive circumstances. Anyone can well notice to find out, in applications they are challenging contented. Thus to dispose of the solid limitations on constants in MNC condition as Eq (1.7) or (1.2). As postulation, writers studied the existence of global mild solutions for IVP of evolution equations in the real Banach space O.

$$\begin{cases} u'(s) + Au(s) = f(s, u(s)), \ s \in [0, b] \\ u(0) = u_0. \end{cases} \tag{1.9}$$

The non-linear function f is uniformly continuous on [0,b]×Y_E the writers suppose this and fulfill an appropriate condition of MNC just like Eq (1.2). We should notice that limitation condition analogous to Eq (1.3) have been deleted by writers in reference [38].

During the previous twenty years, fractional order semi linear evolution equations have been end up being significant instruments in the examination of numerous marvels in Chemistry, Economy, Physics, Engineering, Electrodynamics and Aerodynamics of complex channel. Fractional evolution equations has charm in growing thought lately and it has framed into a critical piece of FC and FDE. A monotone iterative strategy was presented in reference [4] for a class of semilinear evolution equations with nonlocal conditions. In reference [5], there is further information regarding approximation techniques for fractional evolution equations with nonlocal integral conditions. For further information about fractional calculus we refer [1, 13] and the references therein.

In any case, among the past investigates, a large portion of analysts center around the model that in autonomous considerations the differential operators independent of times. In reference [6], the fractional non-autonomous evolution equation with nonlocal circumstances was studied. A note on approximate controllability of the Hilfer fractional neutral differential inclusions with infinite delay was discussed in reference [23]. We find that when we use some parabolic evolution equations, partial differential operators that are dependent on time s are frequently seen in examples, according to popular belief. In reference [24] results on approximate controllability of Sobolev-type fractional neutral differential inclusions of Clarke subdifferential type. For a blowup alternative result for fractional non-autonomous evolution equation of Volterra type we refer [7]. As a consequence, locating the differential operators (D.O.) in significant areas of reasoned problems is critical and fascinating. Indeed, reference [10] check out continuous dependence and existence of basic solutions for a form of linear non-autonomous fractional evolution equations in the year 2004. Reference [11] provided a set of conditions to guarantee the presence of a resolvent operator for a group of fractional non-autonomous evolution equations with a conventional Cauchy initial condition in 2010.

The existence of mild solutions for the approaching Cauchy problems to nonlinear non-autonomous mixed type integro-differential fractional evolution equation is investigated in this work via MNC with nonlocal condition in Banach space O, motivated by the previously described characteristic.

$$\begin{cases} {}^{C}D_s^{\gamma}u(s) = A(s)u(s) + f(s, u(s), (Su)(s), (Tu)(s)), \ s \in \ Z \\ u(0) = u_0 + g(u), \ u'(0) = u_1, \end{cases} \tag{1.10}$$

where ${}^{C}D_s^{\gamma}$ is Caputo's fractional time derivative of order $1 < \gamma \leq 2$ Z= [0, b] where $b > 0$ is a constant. Writers verified that, f is said to be uniformly continuous on [0,b]× Y_E,if for any constant $E > 0$ and fulfills the condition of MNC family of closed linear operators defined on a dense domain δ(A) is A(s) in Banach space O into O such that δ(A) is independent of s, f : Z ×O×O×O⟶ O is a Carathodory type function, $u_0 \in$ O, Volterra integral operator is S defined by Eq (1.5) and Fredholm integral operator is T defined by Eq (1.6). The need and particular in this paper are as given below:

a. We offer three operators $\phi(s, \bar{\varrho})$,$\psi(s, t)$, and U(s)to give the suitable formulation of mild solution for the Cauchy problem to the mixed type non-linear time-fractional non-autonomous integro-differential evolution Eq (1.10).

b. We discovered that the nonlinear component f was supposed to be uniformly continuous by all authors in [30, 37], which is a really credible assumption. Indeed, if f (s, u) is Lipschitz continuous on Z×Y_E w.r.t. u, then the requirement (1.2) is satisfied, although f may not require uniformly continuous on Z×Y_E,a new approximation method of MNC (check Lemma 2.6) introduced by reference [4].

c. As we identify earlier, a very solid prohibitive circumstance is the inequality Eq (1.8). In the MNC conditions, instructions to disregard the limit on the constants are a substantial challenge. In this article, we firmly utilized new sort of fixed point theorem w.r.t. K-set-contractive operator (check Definition 2.5) and the strong limitations on the constants are fully formatted in conditions of MNC.

We have managed this article as following: Some common hypotheses on the linear operator-A(s), main proofs and their assumption will be provided in the follow-up of section 1. We give few notations, definitions and essential Basics on Kuratowski MNC, fixed point theorem w.r.t. K-set-contractive operator and fractional derivatives in section 2. Specifically, using integral impulsive Eq (1.10), the characteristics for the operators $\phi(s, \bar{\varrho})$, $\psi(s, t)$, and U(s) were included in the specification of a moderate solution for the Cauchy problem to fractional non-autonomous mixed type integro-differential evolution equation. In section 3 the evidences of the primary Theorem 3.1 are provided. We give an example of question where the proofs of the earlier sections will be applied and Theorem 3.2 in the last section 4. Consider L(O) be the Banach space where the operator norm describes the topology of all bounded and linear operators in O. We assume that the A(s) linear operator meets the following requirements throughout this essay:

(B1) For some μ with Re$\mu \geq 0$, the operator $[\mu I_d + A(s)]$ existing a bounded inverse operator$[\mu I_d + A(s)]^{-1}$ here C is a +ve constant independent of both s and μ;

$$\|[\mu\ I_d + A(s)]^{-1}\| \leq \frac{C}{|\mu| + I},$$

(B2) For some s, t, $\tau \in$ Z, there must exist a constant $\alpha \in$ (1,2] such that

$$\|[A(s) - A(\tau)]A^{-1}(t)\| \leq C|s - \tau|^{\alpha},$$

in which $\alpha > 0$ and C are independent of t, s and τ.

Remark 1.1. *By the virtue of [18, 34, 39], we can easily understand that the hypotheses (B1) shows that for every t ∈ Z, the operator A(t) produce an analytic semi-group $e^{-s}A(t)$ $(s > 0)$, and then exists $C> 0$ independent of both s and t such a way that*

$$\|A^m(t)e^{-sA(t)}\| \leq \frac{C}{s^m},$$

where $m = 1, 2,\ s > 0, t \in Z$.

Remark 1.2. *In hypotheses (B1), if we select μ= 0, s = 0, then there must exist a $C> 0$ independent of both s and μ given that*

$$\|A^{-1}(0)\| \leq C.$$

Definition 1.1. A function $\eta : [0, b] \times O \longrightarrow O$ is called Carathodory continuous if following postulates are satisfied:

(a) For all u ∈ O, $\eta(, u)$ is strongly measurable;

(b) For a.e. s ∈ [0, b], $\eta(s,)$ is continuous.

It is sufficient to impose MNC and some natural growth requirements on f, which is a nonlinear function, to illustrate the occurrence of mild solutions to the IVP for nonlinear fractional non-autonomous mixed type integro-differential evolution equation with integral impulse condition Eq (1.10).

(G1) There must exist constants $0 \leq \xi < \min\{ \gamma , \alpha\}, \bar{\varrho}_1 > 0$, For any $r > 0$ and functions $\psi_r \in L^{\frac{1}{\xi}}(Z, \mathbf{O}^+)$ such a way that for a.e. s ∈ Z and satisfying that $\|u\| \leq$ r, u ∈ O

$$\psi_r(s) \geq \|f(s, u, Su, Tu)\|,$$

and

$$\liminf_{r \longrightarrow +\infty} \frac{\|\psi_r\| L^{\frac{1}{\xi}}[0, b]}{r} = \bar{\varrho} < +\infty.$$

(G2) There must exist +ve constants $0 \leq \xi < min\{\gamma, \alpha\}, \bar{\varrho}_1 > 0$ L_1, L_2 and L_3 so much that for any countable and bounded sets $D_2, D_3, D_1 \subset O$ and a.e. s *in* Z,

$$\tau(f(s, D_1, D_2, D_3)) \leq L_1\tau(D_1) + L_2\tau(D_2) + L_3\tau(D_3).$$

For the sake of clarity, we signify

$$a_1 = \left(\frac{1-\xi}{\gamma-\xi}\right)^{1-\xi} + C\mathbf{B}(\gamma, \alpha)\, b^{\alpha} \left(\frac{1-\xi}{\gamma+\alpha-\xi}\right)^{1-\xi},$$

and

$$a_2 = 1 + A(0)C^2 b^{\gamma} \left(\frac{1}{\gamma} + b^{\alpha}\mathbf{B}(\gamma, \alpha+1)\right),$$

$$a_3 = \frac{1}{\gamma} + \frac{2C\mathbf{B}(\gamma, \alpha)\, b^{\alpha}}{\gamma+\alpha},$$

where

$$\mathbf{B}(\gamma, \alpha) = \int_1^2 (1-s)^{\alpha-1}\, s^{\gamma-1} ds,$$

is the Beta function.

(G3) There is a nondecreasing continuous function $\Phi : E^+ \longrightarrow E^+$ and a constant $\bar{\varrho}_g > 0$ such that for some $r > 0$ and all u ∈ $G_r = u \in C(Z, O) : \|u(s) \leq r, \forall s \in Z$,

$$\|g(u)\| \leq \Phi(r) and \liminf_{r \longrightarrow +\infty} \frac{\Phi(r)}{r} = \bar{\varrho}_g < +\infty.$$

(G4) There exist +ve constant L_g such that for some countable and bounded sets $\delta \subset$ C(Z, O),

$$\tau(g(\delta)) \leq L_g\tau_c(\delta).$$

2. Preliminaries

In this part we present a few documentations, definitions, and essential idea including fractional derivatives and integrals, fixed point theorem as for K-set-contractive operator, Kuratowski MNC and the operators ϕ(s,$\bar{\varrho}$), ψ(s,t) and U(s), which are used end-to-end in this article. End-to-end in this article, we adjust Z = [0, b] indicate a compact interval in O, where $b > 0$ is a constant. Let Banach space O with norm ||.|| We designate it by C(Z, O) the Banach space of all continuous functions from interval Z into O provided with the supremum norm

$$sup\{\|u\|, s \in\ Z\}, \forall\ u \in C(\ Z,\ O) = \|u\|_c,$$

and by L(O) the Banach space of all linear and bounded operators in O equipped with the topology defined by the operator norm. Let L^1(Z,O) be the Banach space of all O value Bochner integrable functions defined on Z with the norm.

$$\int_0^b \|u\|ds = \|u\|_1.$$

Denote G_r = {u ∈ C(Z, O): r ≥ $\|u(s)\|$, s ∈ Z} for any $0 < r$, then in C(Z, O), G_r is closed ball with radius r and center ϑ . As a matter of first importance, we review the meaning of the RiemannLiouville Integral and Caputo Derivative of Fractional Order.

Definition 2.1. [8] The fractional integral having order $\gamma > 0$ having zero lower limit for a function $f \in L^1([0, +\infty), E)$ is defined below

$$I_0^\gamma f(s) = \frac{1}{\Gamma(\gamma)} \int_0^s f(t)(s-t)^{\gamma-1} dt.$$

Here and elsewhere

$$\Gamma(\gamma) = \int_0^\infty (s)^{\gamma-1}(e)^{-s} ds,$$

indicates the Gamma function.

Definition 2.2. [8] The Caputo fractional derivative having order γ having the lower limit 0 for a function $f : [0, +\infty) \longrightarrow E$, which is at least m-times differentiable is defined as Lemma 2.8

$${}^C D_s^\gamma f(s) = \frac{1}{\Gamma(m-\gamma)} \int_0^s (s-t)^{m-\gamma-1} f^{(m)}(t) dt = \ I_s^{m-\gamma} f^{(m)}(s),$$

where $m > \gamma > m-1$, $m \in$ N.

Definition 2.3. If a function u ∈ C(Z, O) meets the following conditions, it is considered a mild solution of Eq (1.10):

$$\begin{aligned} u(s) &= u_0 + g(u) + u_1 s + \int_0^s \psi(s-\bar{\varrho}, \bar{\varrho}) U(\bar{\varrho}) u_0 d\bar{\varrho} + \int_0^s \psi(s-\bar{\varrho}, \bar{\varrho}) U(\bar{\varrho}) u_1 d\bar{\varrho} \\ &+ \int_0^s \psi(s-\bar{\varrho}, \bar{\varrho}) f(\bar{\varrho}, u(\bar{\varrho}), (Su)(\bar{\varrho}), (Tu)(\bar{\varrho})) d\bar{\varrho} \end{aligned}$$

$$+ \int_0^s \int_0^{\bar{\varrho}} \psi(t-\bar{\varrho},\bar{\varrho})\phi(\bar{\varrho},s)f(s,u(t),(Su)(t),(Tu)(t))\,dt\,d\bar{\varrho}$$
$$+ \sum_{0<s_i<s} \psi(s_i-\bar{\varrho},\bar{\varrho})I_i\Big(\int_{s_i-\tau_i}^{s_i-\vartheta_i} N(t,u(t))\Big),$$

where the operators ϕ(s, $\bar{\varrho}$), ψ(s, t) and U(s) are defined by

$$\begin{aligned} \alpha\int_0^\infty \vartheta s^{\gamma-1}\zeta_\gamma(\vartheta)e^{-s^\gamma\vartheta A(t)}d\vartheta &= \psi(s,t),\\ \sum_{k=1}^{\infty}\phi_k(s,\bar{\varrho}) &= \phi(s,\bar{\varrho}),\\ -A(s)A^{-1}(0)-\int_0^s \phi(s,t)A(t)A^{-1}(0)dt &= U(s), \end{aligned}$$

probability density function ζ_γ is defined on [0,∞] and Laplace transform of probability density function ζ_γ is given below

$$\begin{aligned} \int_0^\infty e^{\vartheta x}\zeta_\gamma(\vartheta)d\vartheta &= \sum_{i=1}^{\infty}\frac{(-x)^{\ i}}{\Gamma(1+\gamma\, i)}, 1<\gamma\le 2, x>0\\ \phi_1(s,\bar{\varrho}) &= [A(s)-A(\bar{\varrho})]\psi(s-\bar{\varrho},\bar{\varrho})\\ \phi_{i+1}(s,\bar{\varrho}) &= \int_0^s \phi_i(s,t)\phi_1(t,\bar{\varrho})d\bar{\varrho}, i=1,2,... \end{aligned}$$

Lemma 2.1. *[10] In uniform topology the operator $\psi(s-\bar{\varrho},\bar{\varrho})$ and $A(s)\psi(s-\bar{\varrho},\bar{\varrho})$ are continuous about the variables s and $\bar{\varrho}$, where $s\in$ Z, $0\le\bar{\varrho}\le$ s-ϵ for any $\epsilon>0$, and*

$$\|\psi(s-\bar{\varrho},\bar{\varrho})\|\le C(s-\bar{\varrho})^{\gamma-1},$$

where +ve constant C is independent from both $\bar{\varrho}$ and s. Moreover

$$C(s-\bar{\varrho})^{\alpha-1}\ge\|\phi(s,\bar{\varrho})\|,$$

and

$$C(1+s^\alpha)\ge\|U(s)\|.$$

Using Lemma 2.1 and suitable computation, we have obtained the coming solution.

Lemma 2.2. *In the operator norm L(O) the integral $\int_0^s \psi(s-\bar{\varrho},\bar{\varrho})U(\bar{\varrho})d\bar{\varrho}$ is uniformly continuous for some $s\in Z$, and*

$$\|\int_0^s \psi(s-\bar{\varrho},\bar{\varrho})U(\bar{\varrho})d\bar{\varrho}\|\le C^2 s^\gamma\left(\frac{1}{\gamma}+t^\alpha\boldsymbol{B}(\gamma,\alpha+1)\right),\forall s\in\ Z.$$

Using proper integral transformation and RiemannLiouville integral's properties having fractional order , we have obtained the coming result.

Lemma 2.3. *For some $g \in L^1[0, b]$ and $s \in Z$, we have*

$$\int_0^s \int_0^{\bar{\varrho}} (s - \bar{\varrho})^{\gamma-1} (\bar{\varrho} - t)^{\alpha-1} g(t)\, dt\, d\bar{\varrho} = \boldsymbol{B}(\gamma, \alpha) \int_0^s (s - \bar{\varrho})^{\gamma+\alpha-1} g(\bar{\varrho}) d(\bar{\varrho}).$$

Next, we present the definition of Kuratowski MNC , we use it in our proof.

Definition 2.4. [9] The Kuratowski MNC $\tau()$ defined on a bounded set G in Banach space O

$$\tau(G) := inf\{0 < a : G = \cup_{i=1}^{m} G_i \text{ and } diam(G_i) \leq a \text{ for } i = 1, 2, 3, ...m\}.$$

Well known properties about the Kuratowski MNC are as follows.

Lemma 2.4. *[9] Let ν, U are subset of O are bounded and O be a Banach space. The coming attributions are satisfied:*

(a) $\tau(\nu) \leq \tau(U)$ if $\nu \subset U$.

(b) $\tau(\bar{\nu}) = \tau(\nu) = \tau(conv\ \nu)$, where convex hull of ν is conv ν.

(c) $\tau(\nu) = 0$ iff ν is compact, where closure hull of ν is $\bar{\nu}$.

(d) $\tau(\ \mu\ \nu) =$ — μ —$\tau(\nu)$, where $\mu \in R$.

(e) $\tau(\nu \cup U) = max\ \{\tau(\nu), \tau(U)\}$.

(f) $\tau(\nu) + \tau(U) \geq \tau(\nu + U)$, where $\{y$ — $y = x + z,\ y \in \nu, z \in U\} = \nu + U$.

(g) $\tau(\nu + y) = \tau(\nu)$, for any $y \in O$.

(h) Y is another Banach space.

If the mapping P: $\delta(P) \subset O \longrightarrow Y$ is Lipschitz continuous having constant i, then $i\tau(G) \geq \tau(P(G))$ for any bounded subset $G \subset \delta(P)$, In this paper, we indicate $\tau()$ and $\tau_C()$ by the Kuratowski MNC on the bounded set of O and C(Z, O) respectively. For some $\delta \subset C(Z, O)$ and s $\in$J, set $\delta(s) = u(s)$ — $u \in \delta$ then $\delta(s) \subset O$. If $\delta \subset C(Z, O)$ is bounded, then $\delta(s)$ is bounded in O and $\tau_C(\delta) \geq \tau(\delta(s))$. For further information about the properties of the Kuratowski MNC, we suggest to [9]. The coming lemmas will be used in our evidence.

Lemma 2.5. *[9] Let $\delta \subset C(Z, O)$ be bounded and equicontinuous. in Banach space O Then $\tau(\delta(s))$ is continuous on [0, b] and $max_{s\in[0,b]}\ \tau(\delta(s)) = \tau_c(\delta)$.*

Lemma 2.6. *[4] let $\delta \subset O$ be bounded. Let O be a Banach space, Then there exists a countable set $\delta \supset D_0$, such that $2\tau(D_0) \geq \tau(\delta)$.*

Lemma 2.7. *[17] If $\delta = \{u_m\}_{m=1}^{\infty} \subset C([0, b], O)$ is a countable set. Let O be a Banach space. And there exists a function $n \in L^1([0, b], \boldsymbol{E}^+)$ such a way that for every $m \in \boldsymbol{N}$*

$$\|u_m(s)\| \leq m(s), a.e.\ s \in [0, b].$$

Then $\tau(\delta(s))$ is said to be Lebesgue integral on [0, b], and

$$2 \int_0^b \tau(\delta(s)) ds \geq \tau\left(\{ \int_0^b u_m(s) ds\ m \in \boldsymbol{N}\}\right).$$

Definition 2.5. let nonempty subset of O is G then continuous mapping $P : G \longrightarrow O$ in Banach space O is said to be K-set-contractive if there is existing a constant i $\in$ [1, 2) such that, for every bounded set δ is subset of G,

$$i\tau(\delta) \geq \tau(P(\delta)). \tag{2.1}$$

Lemma 2.8. *Let $\delta \subset O$ be a bounded in Banach space O. Then there is existing a countable set $\delta \supset D_0$, such a way that $2\tau(D_0) \geq \tau(\delta)$.*

Lemma 2.9. *[9] Suppose that $\delta \subset O$ be a bounded closed and convex set on O and O is a Banach space, the operator $P : \delta \longrightarrow \delta$ is called K-set-contractive. Then P having one minimum fixed point in δ.*

3. Proof of the main results

We give the proof of Theorem 3.1 in this section.

Theorem 3.1. *Suppose nonlinear function $f : Z \times O \times O \times O \longrightarrow O$ is said to be Carathodory continuous. If the hypotheses (G1) to (G4) are mitigated, then minimum one mild solution of problem Eq (1.10) will exist in C(Z, O) is given*

$$C\bar{\varrho}b^{\gamma-\xi}a_1 + \bar{\varrho}_g a_2 < 1, \tag{3.1}$$

and

$$2\Big[\frac{L_g}{2} + 2CM\,(a_3)\,b^{\gamma}\Big] < 1. \tag{3.2}$$

Proof of Theorem 3.1. As shown below, define an operator P on the space of continuous functions C(Z, O)

$$\begin{aligned} (Pu)(s) &= u_0 + g(u) + u_1 s + \int_0^s \psi(s-\bar{\varrho},\bar{\varrho})U(\bar{\varrho})A(0)(u_0 + g(u))d\bar{\varrho} \\ &+ \int_0^s \psi(s-\bar{\varrho},\bar{\varrho})U(\bar{\varrho})A(0)u_1 d\bar{\varrho} + \int_0^s \psi(s-\bar{\varrho},\bar{\varrho}) f(\bar{\varrho}, u(\bar{\varrho}), (Su)(\bar{\varrho}), (Tu)(\bar{\varrho}))d\bar{\varrho} \\ &+ \int_0^s \int_0^{\bar{\varrho}} \psi(s-\bar{\varrho},\bar{\varrho})\phi(\bar{\varrho}, t) f(t, u(t), (Su)(t), (Tu)(t))\, dt\, d\bar{\varrho}. \end{aligned} \tag{3.3}$$

By directly computation and by the properties of operators ϕ(s, $\bar{\varrho}$), ψ(s, t) and U(s). We see that the operator P is mapping from C(Z, O) to C(Z, O) and it is clear-cut. From Definition 1.1, simply check that the mild solution of IVP equation (1.10) is comparable to the fixed point of the operator P described by Eq (3.3). By applying Lemma 2.8, we'll show that P does have at least one fixed point. First of all, we will prove that there is existing a +ve constant E in such a way that the operator P defined by Eq (3.3) mapping the set G_E to G_E. Assuming this isn't correct, there would exist $s_r \in$ Z and $u_r \in G_r$ in such a way that for every $r > 0$, $\|(Pu_r)(s_r)\| > r$. Combine with supposition (G1), (G2) Hlder inequality and Lemmas 2.1–2.3. We are noticing

$$\begin{aligned} r &< \|(Pu_r)(s_r)\| \leq \|u_0\| + \|g(u_r)\| + \|u_1 s_r\| + \|\int_0^{s_r} \psi(s_r-\bar{\varrho},\bar{\varrho})U(\bar{\varrho})A(0)(u_0 + g(u_r))d\bar{\varrho}\| \\ &+ \|\int_0^{s_r} \psi(s_r-\bar{\varrho},\bar{\varrho})U(\bar{\varrho})A(0)(u_1)d\bar{\varrho}\| + \|\int_0^{s_r} \psi(s_r-\bar{\varrho},\bar{\varrho}) f(\bar{\varrho}, u_r(\bar{\varrho}), (Su_r)(\bar{\varrho}), (Tu_r)(\bar{\varrho}))d\bar{\varrho}\| \\ &+ \|\int_0^{s_r} \int_0^{\bar{\varrho}} \psi(s_r-\bar{\varrho},\bar{\varrho})\phi(\bar{\varrho}, t) f(t, u_r(t), (Su_r)(t), (Tu_r)(t))\, dt\, d\bar{\varrho}\| \\ &\leq \|u_0\| + \Phi(r) + \|u_1 s_r\| + C^2 \int_0^{s_r} (s_r-\bar{\varrho})^{\gamma-1}(1+\bar{\varrho}^{\alpha})\|u_0 A(0)\| d\bar{\varrho} \end{aligned}$$

$$
\begin{aligned}
&+ \quad C^2\|A(0)\|(\Phi(r))\int_0^{s_r}(s_r-\bar{\varrho})^{\gamma-1}(1+\bar{\varrho}^{\alpha})d\bar{\varrho}+C^2\int_0^{s_r}(s_r-\bar{\varrho})^{\gamma-1}(1+\bar{\varrho}^{\alpha})\|A(0)u_1\|d\bar{\varrho}\\
&+ \quad C\int_0^{s_r}(s_r-\bar{\varrho})^{\gamma-1}\psi_r(\bar{\varrho})d\bar{\varrho}+C^2\int_0^{s_r}\int_0^{\bar{\varrho}}(s_r-\bar{\varrho})^{\gamma-1}(\bar{\varrho}-s)^{\alpha-1}\psi_r(\bar{\varrho})dtd\bar{\varrho}\\
&\leq \quad \|u_0\|+\Phi(r)+\|u_1s_r\|+C^2\|A(0)u_0\|(s_r^{\gamma})\left(\frac{1}{\gamma}+(s_r)^{\alpha}\mathbf{B}(\gamma,\alpha+1)\right)\\
&+ \quad C^2\|A(0)\|\Phi(r)(s_r^{\gamma})\left(\frac{1}{\gamma}+(s_r)^{\alpha}\mathbf{B}(\gamma,\alpha+1)\right)+C^2\|A(0)u_1\|(s_r^{\gamma})\left(\frac{1}{\gamma}+(s_r)^{\alpha}\mathbf{B}(\gamma,\alpha+1)\right)\\
&+ \quad C\int_0^{s_r}(s_r-\bar{\varrho})^{\gamma-1}\psi_r(\bar{\varrho})d\bar{\varrho}+C^2\mathbf{B}(\gamma,\alpha)\int_0^{s_r}(s_r-\bar{\varrho})^{\gamma+\alpha-1}\psi_r(\bar{\varrho})d\bar{\varrho}\\
&\leq \quad \|u_0\|+\Phi(r)+\|u_1s_r\|+C^2\|A(0)u_0\|(b^{\gamma})\left(\frac{1}{\gamma}+(s_r)^{\alpha}\mathbf{B}(\gamma,\alpha+1)\right)\\
&+ \quad C^2\Phi(r)\|A(0)\|(b^{\gamma})\left(\frac{1}{\gamma}+(s_r)^{\gamma}\mathbf{B}(\gamma,\alpha+1)\right)+C^2\|A(0)u_1\|(b^{\gamma})\left(\frac{1}{\gamma}+(s_r)^{\alpha}\mathbf{B}(\gamma,\alpha+1)\right).\\
&+ \quad C\left(\int_0^{s_r}(s_r-\bar{\varrho})^{\frac{\gamma-1}{1-\xi}}d\bar{\varrho}\right)^{1-\xi}\left(\int_0^{s_r}\psi_r^{\frac{1}{\xi}}(\bar{\varrho})d\bar{\varrho}\right)^{\xi}\\
&+ \quad C^2\mathbf{B}(\gamma,\alpha)\left(\int_0^{s_r}(s_r-\bar{\varrho})^{\frac{\gamma+\alpha-1}{1-\xi}}d\bar{\varrho}\right)^{1-\xi}\left(\int_0^{s_r}\psi_r^{\frac{1}{\xi}}(\bar{\varrho})d\bar{\varrho}\right)^{\xi}\\
&\leq \quad \|u_0\|+\Phi(r)+\|u_1s_r\|+C^2\|A(0)u_0\|(b^{\gamma})\left(\frac{1}{\gamma}+(s_r)^{\alpha}\mathbf{B}(\gamma,\alpha+1)\right)\\
&+ \quad C^2\|A(0)\|\Phi(r)(b^{\gamma})\left(\frac{1}{\gamma}+(s_r)^{\alpha}\mathbf{B}(\gamma,\alpha+1)\right)+C^2\|A(0)u_1\|(b^{\gamma})\left(\frac{1}{\gamma}+(s_r)^{\alpha}\mathbf{B}(\gamma,\alpha+1)\right)\\
&+ \quad Cb^{\gamma-\xi}\left(\frac{1-\xi}{\gamma-\xi}\right)^{1-\xi}\|\psi_r\|_{L^{\frac{1}{\xi}}[0,b]}+C^2\mathbf{B}(\gamma,\alpha)b^{\gamma+\alpha-\xi}\left(\frac{1-\xi}{\gamma+\alpha-1}\right)^{1-\xi}\|\psi_r\|_{L^{\frac{1}{\xi}}[0,b]}. \qquad (3.4)
\end{aligned}
$$

Dividing both side of Eq (3.4) by r and taking the lower limit as r $\longrightarrow +\infty$, combined with the assumption Eq (3.1) we get that

$$
1\leq\bar{\varrho}_g a_2+b^{\gamma-\xi}\bar{\varrho}\left[\left(\frac{1-\xi}{\gamma-\xi}\right)^{1-\xi}+C\mathbf{B}(\gamma,\alpha)b^{\alpha}\left(\frac{1-\xi}{\gamma+\alpha-1}\right)^{1-\xi}\right]=C\bar{\varrho}b^{\gamma-\xi}a_1+\bar{\varrho}_g a_2<1. \qquad (3.5)
$$

One can simply see that Eq (3.5) is a contradiction. Hence, we have verified that P : $G_E \longrightarrow G_E$. Second, we are going to prove that operator P : $G_E \longrightarrow G_E$ is continuous. let $\{u_m\}_{m=1}^{\infty}=1 \subset G_E$ be a sequence such that $\lim_{m\longrightarrow\infty}\{u_m\}=$ u in G_E. By the Carathodory continuity of the nonlinear function f, we see that

$$
\lim_{m\longrightarrow+\infty}\|f(s,u_m(s),(Tu_m)(s),(Su_m)(s))-f(s,u(s),(Tu)(s),(Su)(s))\|=0, \qquad (3.6)
$$

for a.e. s ∈ Z. By Eq (3.3) and Lemmas 2.1–2.3 combining it with the analogous calculus method which is used in Eq (3.4), we have

$$\begin{aligned}
\|(Pu_m(s)) &- (Pu(s))\| \leq \| \int_0^s \psi(s-\bar{\varrho},\bar{\varrho}) f(\bar{\varrho}, u_m(\bar{\varrho}), (Su_m)(\bar{\varrho}), (Tu_m)(\bar{\varrho})) d\bar{\varrho} \\
&- \int_0^s \psi(s-\bar{\varrho},\bar{\varrho}) f(\bar{\varrho}, u(\bar{\varrho}), (Su)(\bar{\varrho}), (Tu)(\bar{\varrho})) d\bar{\varrho}\| \\
&+ \| \int_0^s \int_0^{\bar{\varrho}} \psi(s-\bar{\varrho},\bar{\varrho}) \phi(\bar{\varrho}, t) f(t, u_m(t), (Su_m)(t), (Tu_m)(t))\, dt\, d\bar{\varrho} \\
&- \int_0^s \int_0^{\bar{\varrho}} \psi(s-\bar{\varrho},\bar{\varrho}) \phi(\bar{\varrho}, t) f(t, u(t), (Su)(t), (Tu)(t))\, dt\, d\bar{\varrho}\| \\
&\leq \| \int_0^s \psi(s-\bar{\varrho},\bar{\varrho}) \left(f(\bar{\varrho}, u_m(\bar{\varrho}), (Su_m)(\bar{\varrho}), (Tu_m)(\bar{\varrho})) - f(\bar{\varrho}, u(\bar{\varrho}), (Su)(\bar{\varrho}), (Tu)(\bar{\varrho})) \bar{\varrho} \right) \| \\
&+ \| \int_0^s \int_0^{\bar{\varrho}} \psi(s-\bar{\varrho},\bar{\varrho}) \phi(\bar{\varrho}, t) (f(t, u_m(t), (Su_m)(t), (Tu_m)(t)) \\
&- f(t, u(t), (Su)(t), (Tu)(t)))\, dt\, d\bar{\varrho}\| \\
&\leq C \int_0^s (s-\bar{\varrho})^{\gamma-1} \|(f(\bar{\varrho}, u_m(\bar{\varrho}), (Su_m)(\bar{\varrho}), (Tu_m)(\bar{\varrho})) - f(\bar{\varrho}, u(\bar{\varrho}), (Su)(\bar{\varrho}), (Tu)(\bar{\varrho}))\| d\bar{\varrho}. \\
&+ C^2 \int_0^s \int_0^{\bar{\varrho}} (s-\bar{\varrho})^{\gamma-1} (\bar{\varrho}-t)^{\alpha-1} \|f(\bar{\varrho}, u_m(\bar{\varrho}), (Su_m)(\bar{\varrho}), (Tu_m)(\bar{\varrho})) \\
&- f(\bar{\varrho}, u(\bar{\varrho}), (Su)(\bar{\varrho}), (Tu)(\bar{\varrho}))\| dt d\bar{\varrho}. \qquad (3.7)
\end{aligned}$$

By the supposition (G1) , we can understand that for every s ∈ Z,

$$(s-\bar{\varrho})^{\gamma-1} \|(f(\bar{\varrho}, u_m(\bar{\varrho}), (Su_m)(\bar{\varrho}), (Tu_m)(\bar{\varrho})) - f(\bar{\varrho}, u(\bar{\varrho}), (Su)(\bar{\varrho}), (Tu)(\bar{\varrho})))\| \leq 2(s-\bar{\varrho})^{\gamma-1} \psi_E(\bar{\varrho}), \quad (3.8)$$

for a.e. $\bar{\varrho} \in$ [0,s]. By once again the combining Lemma 2.3 with supposition (G1), we acquired for any $s \in Z$, a.e. t ∈ [0, $\bar{\varrho}$] and $0 \leq \bar{\varrho} \leq s$,

$$\begin{aligned}
&\int_0^s \int_0^{\bar{\varrho}} (s-\bar{\varrho})^{\gamma-1} (\bar{\varrho}-t)^{\alpha-1} \|(f(\bar{\varrho}, u_m(\bar{\varrho}), (Su_m)(\bar{\varrho}), (Tu_m)(\bar{\varrho})) \\
&- f(\bar{\varrho}, u(\bar{\varrho}), (Su)(\bar{\varrho}), (Tu)(\bar{\varrho}))\| dt d\bar{\varrho} \leq 2 \int_0^s \int_0^{\bar{\varrho}} (s-\bar{\varrho})^{\gamma-1} (\bar{\varrho}-t)^{\alpha-1} \psi_E(\bar{\varrho}) d\bar{\varrho} \\
&= 2\mathbf{B}(\gamma, \alpha) \int_0^s (s-\bar{\varrho})^{\gamma+\alpha-1} \psi_E(\bar{\varrho}) d\bar{\varrho}. \qquad (3.9)
\end{aligned}$$

From the fact that the functions $\bar{\varrho} \longrightarrow 2(\gamma, \alpha) \int_0^s (s-\bar{\varrho})^{\gamma+\alpha-1} \psi_E(\bar{\varrho})$ and $\bar{\varrho} \longrightarrow 2\mathbf{B}(\gamma, \alpha) \int_0^s (s-\bar{\varrho})^{\gamma+\alpha-1} \psi_E(\bar{\varrho})$ are Lebesgue integrable for a.e. $\bar{\varrho} \in$ [0, s] and every s ∈ Z, we observe the following norm by combining Eqs (3.6)–(3.9), as well as the Lebesgue dominated convergence theorem,

$$\|(Pu_m(s)) - (Pu(s))\| \longrightarrow 0 \; as \; \bar{\varrho} \longrightarrow \infty,$$

for any s ∈ Z. Therefore, we get that

$$\|Pu_m - Pu\|_c \longrightarrow 0 \; as \; (\bar{\varrho} \longrightarrow \infty),$$

which means that P : $G_E \longrightarrow G_E$ is a continuous operator. We now have the ability to prove that P : $G_E \longrightarrow G_E$ is an equi-continuous operator. For any u $\in G_E$ and $0 \leq s' < s'' \leq$ b, by Eq (3.3) and the assumption (G1), we know that

$$\begin{aligned}
||(Pu(s'')) - (Pu(s'))|| &\leq ||\int_{s'}^{s''} \psi(s'' - \bar{\varrho}, \bar{\varrho}) U(\bar{\varrho}) A(0) u_0 d\bar{\varrho}|| + ||\int_{s'}^{s''} \psi(s'' - \bar{\varrho}, \bar{\varrho}) U(\bar{\varrho}) A(0) g(u) d\bar{\varrho}|| \\
&+ ||\int_{s'}^{s''} \psi(s'' - \bar{\varrho}, \bar{\varrho}) U(\bar{\varrho}) A(0) u_1 d\bar{\varrho}|| \\
&+ ||\int_{0}^{s'} [\psi(s'' - \bar{\varrho}, \bar{\varrho}) - \psi(s' - \bar{\varrho}, \bar{\varrho})] U(\bar{\varrho}) A(0) u_0 d\bar{\varrho}|| \\
&+ ||\int_{0}^{s'} [\psi(s'' - \bar{\varrho}, \bar{\varrho}) - \psi(s' - \bar{\varrho}, \bar{\varrho})] U(\bar{\varrho}) A(0) g(u) d\bar{\varrho}|| \\
&+ ||\int_{0}^{s'} [\psi(s'' - \bar{\varrho}, \bar{\varrho}) - \psi(s' - \bar{\varrho}, \bar{\varrho})] U(\bar{\varrho}) A(0) u_1 d\bar{\varrho}|| \\
&+ ||\int_{s'}^{s''} \psi(s'' - \bar{\varrho}, \bar{\varrho}) f(\bar{\varrho}, u(\bar{\varrho}), (Tu)(\bar{\varrho}), (Su)(\bar{\varrho}))|| d\bar{\varrho} \\
&+ ||\int_{0}^{s'} [\psi(s'' - \bar{\varrho}, \bar{\varrho} - \psi(s' - \bar{\varrho}, \bar{\varrho}))] f(\bar{\varrho}, u(\bar{\varrho}), (Su)(\bar{\varrho}), (Tu)(\bar{\varrho}))|| d\bar{\varrho} \\
&+ ||\int_{s'}^{s''} \int_{0}^{\bar{\varrho}} [\psi(s'' - \bar{\varrho}\bar{\varrho}) - \psi(s' - \bar{\varrho}, \bar{\varrho})] \phi(\bar{\varrho}, t) f(\bar{\varrho}, u(\bar{\varrho}), (Su)(\bar{\varrho}), Tu)(\bar{\varrho}))|| d\bar{\varrho} \\
&+ ||\int_{0}^{s'} \int_{0}^{\bar{\varrho}} \psi(s'' - \bar{\varrho}, \bar{\varrho}) \phi(\bar{\varrho}, s) f(\bar{\varrho}, u(\bar{\varrho}), (Su)(\bar{\varrho}), (Tu)(\bar{\varrho}))|| d\bar{\varrho} \\
&\leq I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10},
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_{s'}^{s''} ||\psi(s'' - \bar{\varrho}, \bar{\varrho}) U(\bar{\varrho})||.||A(0) u_0|| d\bar{\varrho} \\
I_2 &= \int_{s'}^{s''} ||\psi(s'' - \bar{\varrho}, \bar{\varrho}) U(\bar{\varrho}) A(0)|| \Phi(E) d\bar{\varrho} \\
I_3 &= \int_{s'}^{s''} ||\psi(s'' - \bar{\varrho}, \bar{\varrho}) U(\bar{\varrho})||.||A(0) u_1|| d\bar{\varrho} \\
I_4 &= \int_{0}^{s'} ||[\psi(s'' - \bar{\varrho}, \bar{\varrho}) - \psi(s' - \bar{\varrho}, \bar{\varrho})] U(\bar{\varrho})||.||A(0) u_0|| d\bar{\varrho} \\
I_5 &= \int_{0}^{s'} [\psi(s'' - \bar{\varrho}, \bar{\varrho}) - \psi(s' - \bar{\varrho}, \bar{\varrho})] U(\bar{\varrho}) A(0)|| \Phi(E) d\bar{\varrho} \\
I_6 &= \int_{0}^{s'} ||[\psi(s'' - \bar{\varrho}, \bar{\varrho}) - \psi(s' - \bar{\varrho}, \bar{\varrho})] U(\bar{\varrho})||.||u_1|| d\bar{\varrho} \\
I_7 &= \int_{s'}^{s''} ||\psi(s'' - \bar{\varrho}, \bar{\varrho})|| \psi_E(t) dt d\bar{\varrho} \\
I_8 &= \int_{0}^{s'} ||\psi(s'' - \bar{\varrho}, \bar{\varrho} - \psi(s' - \bar{\varrho}, \bar{\varrho})|| \psi_E(t) dt d\bar{\varrho}
\end{aligned}$$

$$I_9 = \int_{s'}^{s''}\int_0^{\bar{\varrho}} \|\psi(s''-\bar{\varrho},\bar{\varrho})\phi(\bar{\varrho},t)\|\psi_E(t)dtd\bar{\varrho}$$

$$I_{10} = \int_0^{s'}\int_0^{\bar{\varrho}} [\psi(s''-\bar{\varrho},\bar{\varrho})-\psi(s'-\bar{\varrho},\bar{\varrho})]\phi(\bar{\varrho},t)\|\psi_E(t)dtd\bar{\varrho}.$$

As a result, we just need to show that $I_i \longrightarrow 0$ holds regardless of u $\in G_E$ as $s'' \longrightarrow s' \longrightarrow 0$ for i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. For I_1, I_2 and I_3 by Lemma 2.1 we know that

$$I_1 \leq C^2\|A(0)u_0\|\int_{s'}^{s''}(s''-\bar{\varrho})^{\gamma-1}(1+\bar{\varrho}^{\alpha})d\bar{\varrho} \longrightarrow 0 \; as \; s'' \longrightarrow s' \longrightarrow 0,$$

$$I_2 \leq C^2\Phi_E\|A(0)\|\int_{s'}^{s''}(s''-\bar{\varrho})^{\gamma-1}(1+\bar{\varrho}^{\alpha})d\bar{\varrho} \longrightarrow 0 \; as \; s'' \longrightarrow s' \longrightarrow 0,$$

$$I_3 \leq C^2\|A(0)u_1\|\int_{s'}^{s''}(s''-\bar{\varrho})^{\gamma-1}(1+\bar{\varrho}^{\alpha})d\bar{\varrho} \longrightarrow 0 \; as \; s'' \longrightarrow s' \longrightarrow 0.$$

For $s' = 0$ and $0 < s'' \leq$ b, it is easy to see that $I_4 = 0$. For $s' > 0$ and $\epsilon > 0$ small enough, by Lemma 2.1 and the fact that operator-valued function ψ(s-$\bar{\varrho}$,$\bar{\varrho}$) is continuous in uniform topology about the variables s and $\bar{\varrho}$ for $0 \leq$ s $\leq$ b and $0 \leq \bar{\varrho} \leq$ s-ϵ, we have

$$\begin{aligned}
I_4 \leq & \sup_{m\in[0,s'-\epsilon]}\|[\psi(s''-\bar{\varrho},\bar{\varrho})-\psi(s'-\bar{\varrho},\bar{\varrho})]\|.C\|A(0)u_0\|\int_0^{s'-\epsilon}(1+\bar{\varrho}^{\alpha})d\bar{\varrho} \\
+ & C^2\|A(0)u_0\|\int_{s'-\epsilon}^{s'}[(s''-\bar{\varrho})^{\gamma-1}+(s'-\bar{\varrho})^{\gamma-1}](1+\bar{\varrho}^{\alpha})d\bar{\varrho} \longrightarrow 0 \\
& as \; s'' \longrightarrow s' \longrightarrow 0 \; and \; \epsilon \longrightarrow 0, \\
I_5 \leq & sup \lim_{m\in[0,s'-\epsilon]}\|[\psi(s''-\bar{\varrho},\bar{\varrho})-\psi(s'-\bar{\varrho},\bar{\varrho})]\|.C\Phi(E)\|A(0)\|\int_0^{s'-\epsilon}(1+\bar{\varrho}^{\alpha})d\bar{\varrho} \\
+ & C^2\Phi(E)\|A(0)\|\int_{s'-\epsilon}^{s'}[(s''-\bar{\varrho})^{\gamma-1}+(s'-\bar{\varrho})^{\gamma-1}](1+\bar{\varrho}^{\alpha})d\bar{\varrho} \longrightarrow 0 \\
& as \; s'' \longrightarrow s' \longrightarrow 0 \; and \; \epsilon \longrightarrow 0, \\
I_6 \leq & \sup_{m\in[0,s'-\epsilon]}\|[\psi(s''-\bar{\varrho},\bar{\varrho})-\psi(s'-\bar{\varrho},\bar{\varrho})]\|.C\|A(0)u_1\|\int_0^{s'-\epsilon}(1+\bar{\varrho}^{\alpha})d\bar{\varrho} \\
+ & C^2\|A(0)u_1\|\int_{s'-\epsilon}^{s'}[(s''-\bar{\varrho})^{\gamma-1}+(s'-\bar{\varrho})^{\gamma-1}](1+\bar{\varrho}^{\alpha})d\bar{\varrho} \longrightarrow 0 \\
& as \; s'' \longrightarrow s' \longrightarrow 0 \; and \; \epsilon \longrightarrow 0.
\end{aligned}$$

For I_7, by Lemma 2.1, the assumption (G1) and Holder inequality, we get that

$$\begin{aligned}
I_7 \leq & C\int_{s'}^{s''}(s''-\bar{\varrho})^{\gamma-1}\psi_E(\bar{\varrho})d\bar{\varrho} \leq C\left(\int_{s'}^{s''}(s''-\bar{\varrho})^{\frac{\gamma-1}{1-\xi}}d\bar{\varrho}\right)^{1-\xi}\left(\int_{s'}^{s''}\psi_r^{\frac{1}{\xi}}(\bar{\varrho})d\bar{\varrho}\right)^{\xi} \\
\leq & C\left(\frac{1-\xi}{\gamma-\xi}\right)^{1-\xi}\|\psi_r\|_{L^{\frac{1}{\xi}}[0,b]}(s''-s')^{\gamma-\xi} \longrightarrow 0 \; as \; s'' \longrightarrow s' \longrightarrow 0.
\end{aligned}$$

For $s' = 0$ and $0 < s'' \leq$ b, it is easy to see that $I_8 = 0$. For $s' > 0$ and $\epsilon > 0$ small enough, by Lemma 2.1 and the fact that operator-valued function ψ(s-$\bar{\varrho}$,$\bar{\varrho}$) is continuous in uniform topology about the variables s and $\bar{\varrho}$ for 0 ≤ s ≤ b and 0 ≤ $\bar{\varrho}$ ≤ s-ϵ, we have

$$\begin{aligned} I_8 &\leq \sup_{m\in[0,s'-\epsilon]} \|[\psi(s''-\bar{\varrho},\bar{\varrho})-\psi(s'-\bar{\varrho},\bar{\varrho})]\| \\ &+ C\int_{s'-\epsilon}^{s'}[(s''-\bar{\varrho})^{\gamma-1}+(s''-\bar{\varrho})^{\gamma-1}]\psi_E(\bar{\varrho})d\bar{\varrho} \longrightarrow 0 \; as \; s'' \longrightarrow s' \longrightarrow 0 \; and \; \epsilon \longrightarrow 0. \end{aligned}$$

For I_9, by Lemma 2.1, the assumption (G1) and the fact the function $\bar{\varrho} \longrightarrow (s''-\bar{\varrho})^{\gamma-1} I_{\bar{\varrho}}^{\alpha}\psi_E(\bar{\varrho})$ is Lebsegue integrable, we have

$$\begin{aligned} I_9 &\leq C^2\int_{s'}^{s''}\int_0^{\bar{\varrho}}(s''-\bar{\varrho})^{\gamma-1}(\bar{\varrho}-t)^{\alpha-1}\psi_E(t)dtd\bar{\varrho} \\ &\leq C^2\Gamma(\alpha)\int_{s'}^{s''}(s''-\bar{\varrho})^{\gamma-1} I_{\gamma}^{\alpha}\psi_E(t)dtd\bar{\varrho} \longrightarrow 0 \; as \; s'' \longrightarrow s' \longrightarrow 0. \end{aligned}$$

For $s' = 0$ and $0 < s'' \leq$ b, it is easy to see that $I_{10} = 0$. For $s' > 0$ and $\epsilon > 0$ small enough, by Lemma 2.1 and the fact that operator-valued function $\psi(s-\bar{\varrho},\bar{\varrho})$ is continuous in uniform topology about the variables s and $\bar{\varrho}$, the assumption (G1), the facts that the functions$\bar{\varrho} \longrightarrow (s''-\bar{\varrho})^{\gamma-1} I_{\bar{\varrho}}^{\alpha}\psi_E(\bar{\varrho})$ is Lebesgue integrable, we have and $\bar{\varrho} \longrightarrow (s'-\bar{\varrho})^{\gamma-1} I_{\bar{\varrho}}^{\alpha}\psi_E(\bar{\varrho})$ are Lebesgue integrable as well as the operator-valued function ψ(s - $\bar{\varrho}$, $\bar{\varrho}$) is continuous in uniform topology about the variables s and $\bar{\varrho}$ for 0 ≤ s ≤ b and 0 ≤ $\bar{\varrho}$ ≤ s - ϵ, we know that

$$\begin{aligned} I_{10} &\leq sup \lim_{m\in[0,s'-\epsilon]} \|\psi(s''-\bar{\varrho},\bar{\varrho})-\psi(s'-\bar{\varrho},\bar{\varrho})\| \\ &+ C\int_0^{s'-\epsilon}\int_0^{\bar{\varrho}}(\bar{\varrho}-t)^{\alpha-1}\psi_E(t)dtd\bar{\varrho} + C^2\int_{s'-\epsilon}^{s'}\int_0^{\bar{\varrho}}[(s''-\bar{\varrho})^{\gamma-1}+(s'-\bar{\varrho})^{\gamma-1}](\bar{\varrho}-t)^{\alpha-1}\psi_E(t)dtd\bar{\varrho} \\ &+ \left(\frac{1-\xi}{\alpha-\xi}\right)^{1-\xi}\frac{C(s')^{\alpha}\|\psi_r\|_{L^{\frac{1}{\xi}}[0,b]}}{\alpha} sup \lim_{m\in[0,s'-\epsilon]} \|\psi(s''-\bar{\varrho},\bar{\varrho})-\psi(s'-\bar{\varrho},\bar{\varrho})\| \\ &+ C^2\Gamma(\alpha)\int_{s'-\epsilon}^{s'}[(s''-\bar{\varrho})^{\gamma-1} I_{\bar{\varrho}}^{\alpha}\psi_E(\bar{\varrho})]d\bar{\varrho} \longrightarrow 0 \; as \; s'' \longrightarrow s' \longrightarrow 0 \; and \; \epsilon \longrightarrow 0. \end{aligned}$$

As a result, $\|(Pu(s'')) - (Pu(s'))\| \longrightarrow 0$ independently of u ∈ G_R as s'' - s' ⟶ 0, which means that the operator P : $G \longrightarrow G_E$ is equicontinuous. Now we show that P : Y ⟶ Y is a K-set-contractive operator, with Y = $\bar{co}$P(G_E) and $\bar{co}$ denoting convex hull closure. The operator P may thus be easily verified to map Y into itself, and Y ⊂ C(Z, O) is equicontinuous. Let's say m_0 ∈ P, in the following, we shall show that for any bounded and nonprecompact subset δ ⊂ Y, there exists a positive integer m_0. We know there is a countable set $D_0 = u_m \subset \delta$ because of Lemma 2.8 such that

$$\tau_c(P(\delta)) \leq 2\tau_c(P(D_0)). \tag{3.10}$$

By the definition of operator Y and the equicontinuity Y, we know that $D_0 \subset Y$ is also equi continuous. Therefore by Eq (3.3), Lemmas 2.1, 2.3 and 2.7 the supposition (G2) and (G4) we have

$$
\begin{aligned}
\tau\,(P(D_0)(s)) &\leq \tau(u_0) + \tau(g(u)) + \tau(u_1 s) + \tau \int_0^s \psi(s-\bar{\varrho},\bar{\varrho})U(\bar{\varrho})(A(0)u_0)d\bar{\varrho} \\
&+ \tau \int_0^s \psi(s-\bar{\varrho},\bar{\varrho})U(\bar{\varrho})(A(0)g(u))d\bar{\varrho} + \tau \int_0^s \psi(s-\bar{\varrho},\bar{\varrho})U(\bar{\varrho})(A(0)u_1)d\bar{\varrho} \\
&+ \tau \int_0^s \psi(s-\bar{\varrho},\bar{\varrho})f(\bar{\varrho},u(\bar{\varrho}),(Su)(\bar{\varrho}),(Tu)(\bar{\varrho}))d\bar{\varrho} \\
&+ \tau \int_0^s \int_0^{\bar{\varrho}} \psi(s-\bar{\varrho},\bar{\varrho})\phi(\bar{\varrho},t)f(t,u(t),(Su)(t),(Tu)(t))\,dt\,d\bar{\varrho} \\
&\leq \tau(u_0) + \tau(g(u_m)) + \tau(u_1 s) + 2C^2 \int_0^s (s-\bar{\varrho})^{\gamma-1}(1+\bar{\varrho}^{\alpha})\tau(A(0)u_0)d\bar{\varrho} \\
&+ 2C^2 \int_0^s (s-\bar{\varrho})^{\gamma-1}(1+\bar{\varrho}^{\alpha})(\tau A(0)(g(u_m)))d\bar{\varrho} \\
&+ 2C^2 \int_0^s (s-\bar{\varrho})^{\gamma-1}(1+\bar{\varrho}^{\alpha})\tau(A(0)u_1)d\bar{\varrho} \\
&+ 4C \int_0^s (s-\bar{\varrho})^{\gamma-1}\left([L_1\tau(D_1(\bar{\varrho})) + L_2\tau(SD_1(\bar{\varrho})) + L_3\tau(TD_1(\bar{\varrho}))]\,.\right)d\bar{\varrho} \\
&+ 8C^2 \int_0^s \int_0^{\bar{\varrho}} (s-\bar{\varrho})^{\gamma-1}(\bar{\varrho}-t)^{\alpha-1}\left([L_1\tau(D_1(t)) + L_2\tau(D_1(t)) + L_3\tau(D_1(t))]\right)d\bar{\varrho} \\
&\leq L_g\tau_c(\delta) + 4C \int_0^s (s-\bar{\varrho})^{\gamma-1}\left[L_1 + bK_0L_2 + bH_0L_3\right]\tau(D_1(\bar{\varrho}))d\bar{\varrho} \\
&+ 8C^2\mathbf{B}(\gamma,\alpha) \int_0^s (s-\bar{\varrho})^{\gamma+\alpha-1}\left[L_1 + bK_0L_2 + bH_0L_3\right]\tau(D_1(\bar{\varrho}))d\bar{\varrho} \\
&\leq L_g\tau_c(\delta) + \tau_c(\delta)\frac{4CMb^{\gamma}}{\gamma} + \tau_c(\delta)\frac{8C^2M\mathbf{B}(\gamma,\alpha)b^{\gamma+\alpha}}{\gamma+\alpha}, \qquad (3.11)
\end{aligned}
$$

where

$$M = L_1 + bK_0L_2 + bH_0L_3.$$

We know from Lemma 2.5 that $P(D_0) \subset$ Y is bounded and equicontinuous since it is bounded and equicontinuous, therefore

$$\tau_c(P(D_0)) = \max_{s\in[0,b]} \tau(P(D_0)(s)). \qquad (3.12)$$

Therefore, from Eqs (3.10)–(3.12) one gets that

$$\tau_c(P(\delta)) \leq 2\Big[\frac{L_g}{2} + 2CM\,(a_3)\,b^{\gamma}\Big]\tau_c(\delta). \qquad (3.13)$$

As a result, we may deduce from Eqs (3.2), (3.13), and Definition 2.5 that $P : Y \longrightarrow Y$ is a K-set-contractive operator. According to Lemma 2.9, the operator P given by Eq (3.3) has at least one fixed point u $\in$ Y on the interval $[0, b]$, which is merely a mild solution of the time fractional non-autonomous evolution equation with nonlocal conditions Eq (1.10). The proof of Theorem 3.1 is now complete.

Theorem 3.2. *Assume that the nonlinear function $f : O \times O \times O \longrightarrow O$ and the nonlocal function $g : C(Z, O) \longrightarrow O$ are continuous. If the supposition (G2) and (G4) and the following supposition:*

(G1) There exist a function* $h \in L^{\frac{1}{\xi}}(Z, E^+) for\ 0 \leq \xi < min\{\gamma, \alpha\}$ *and a nondecreasing continuous function* $\Upsilon : E^+ \longrightarrow E^+$ *such that*

$$\|f(s, u, Su, Tu)\| \leq h(s)\Upsilon(\|u\|),$$

for every $s \in Z$ *and* $u \in Z$.

(G2) There exist a constant* $0 < \zeta < \frac{1}{Ca_2}$ *such that* $\|g(u)\| \leq \zeta\|u\|_c$ *for all u∈C(Z,O), hold then time fractional non autonomous evolution equation with nonlocal condition 1.10 exists at least one mild solution on interval [0, b] provided that*

$$\lim \inf_{r \longrightarrow +\infty} \frac{\Upsilon(r)}{r} < \frac{1 - C\zeta a_2}{Ca_1 b^{\gamma-\xi}} \|h\|_{L^{\frac{1}{\xi}}}[0, b], \tag{3.14}$$

and inequality Eq (3.2) are satisfied.

Proof. From the proof of Theorem 3.1, we observe that the mild solution of the time fractional non-autonomous evolution equation with nonlocal conditions, Eq (1.10) is the same as the fixed point of the operator P, provided by Eq (3.3). We will show that there is a positive constant E such that the operator P transfers the set G_E to G_E in the next section. We know that a positive constant E exists because of Eq (3.14) is

$$\frac{Ca_1\Upsilon(E)b^{\gamma-\xi}\|h\|_{L^{\frac{1}{\xi}}}[0, b]}{1 - C\zeta a_2} \leq \mathbf{E}. \tag{3.15}$$

Therefore for any u $\in G_E$ and s $\in$ Z by Eqs (3.3) and (3.15). Lemmas 2.1–2.3 the assumptions (G1*) and (G2*) and Hölder inequality, we know that

$$\begin{aligned}
\|(Pu)(s)\| &= \|u_0\| + \|g(u)\| + \|u_1 s\| + \|\int_0^s \psi(s - \bar{\varrho}, \bar{\varrho})U(\bar{\varrho})(A(0)u_0)d\bar{\varrho}\| \\
&+ \|\int_0^s \psi(s - \bar{\varrho}, \bar{\varrho})U(\bar{\varrho})(A(0)g(u))d\bar{\varrho}\| + \|\int_0^s \psi(s - \bar{\varrho}, \bar{\varrho})U(\bar{\varrho})(A(0)u_1)d\bar{\varrho}\| \\
&+ \|\int_0^s \psi(s - \bar{\varrho}, \bar{\varrho})f(\bar{\varrho}, u(\bar{\varrho}), (Su)(\bar{\varrho}), (Tu)(\bar{\varrho}))d\bar{\varrho}\| \\
&+ \|\int_0^s \int_0^{\bar{\varrho}} \psi(s - \bar{\varrho}, \bar{\varrho})\phi(\bar{\varrho}, t)f(t, u(t), (Su)(t), (Tu)(t))\, dt\, d\bar{\varrho}\|(Pu)(s)\| \\
&\leq \|u_0\| + \zeta\|u\| + \|u_1 s\| + C^2 \int_0^s (s - \bar{\varrho})^{\gamma-1}(1 + \bar{\varrho}^{\alpha})\|A(0)u_0\|d\bar{\varrho} \\
&+ C^2 \int_0^s (s - \bar{\varrho})^{\gamma-1}(1 + \bar{\varrho}^{\alpha})\|A(0)\|\zeta\|u\|d\bar{\varrho} + C^2 \int_0^s (s - \bar{\varrho})^{\gamma-1}(1 + \bar{\varrho}^{\alpha})\|A(0)u_1\|d\bar{\varrho} \\
&+ C \int_0^s (s - \bar{\varrho})^{\gamma-1}h(\bar{\varrho})\Upsilon(E)d\bar{\varrho} + C^2 \int_0^s \int_0^{\bar{\varrho}} (s - \bar{\varrho})^{\gamma-1}(\bar{\varrho} - t)^{\alpha-1}h(t)\Upsilon(E)dtd\bar{\varrho} \qquad (3.16) \\
&\leq \|u_0\| + \zeta E + \|u_1 s\| + C^2\|A(0)u_0\|(b^{\gamma})\left(\frac{1}{\gamma} + (s_r)^{\alpha}\mathbf{B}(\gamma, \alpha + 1)\right) \\
&+ C^2\|A(0)\|\zeta E(b^{\gamma})\left(\frac{1}{\gamma} + (s_r)^{\alpha}\mathbf{B}(\gamma, \alpha + 1)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\quad C^2\|A(0)u_1\|(b^\gamma)\left(\frac{1}{\gamma}+(b)^\alpha\mathbf{B}(\gamma,\alpha+1)\right)\\
&+\quad C\Upsilon(E)\left(\int_0^s(s-\bar{\varrho})^{\frac{\gamma-1}{1-\xi}}d\bar{\varrho}\right)^{1-\xi}\left(\int_0^{s_r}h^{\frac{1}{\xi}}(\bar{\varrho})d\bar{\varrho}\right)^{\xi}\\
&+\quad C^2\Upsilon(E)\mathbf{B}(\gamma,\alpha)\left(\int_0^{s_r}(s_r-\bar{\varrho})^{\frac{\gamma+\alpha-1}{1-\xi}}d\bar{\varrho}\right)^{1-\xi}\left(\int_0^{s}h^{\frac{1}{\xi}}(\bar{\varrho})d\bar{\varrho}\right)^{\xi}\\
&\leq\quad (\zeta E+\|u_0\|+\|u_1\|)a_2+\|u_1 s\|-\|u_1\|+Ca_1\Upsilon(E)\,(b)^{\gamma-\xi}\,\|h\|_{L^{\frac{1}{\xi}}[0,b]}\\
&\leq\quad E.
\end{aligned}
$$

Therefore, from the inequality (3.16), Eq (3.3) defines the operator P, which maps G_E to G_E. We can establish that P: $GE \longrightarrow G_E$ is a continuous and equicontinuous operator, and that P: $\bar{co}$P(GE) is a continuous and equicontinuous operator, using an approach very similar to that employed in the proof of Theorem 3.1. A K-set-contractive operator is $\longrightarrow \bar{co}$P(GE). On the interval $[0, b]$, we know that the operator P described by Eq (3.3) has at least one fixed point u $\in \bar{co}$P(GE), which is really a mild solution of the time fractional non-autonomous evolution equation with nonlocal conditions (1.10), thanks to Lemma 2.9. The proof of Theorem 3.2 is now complete.

4. Application

In this section, we give a model that, although not focusing on our theoretical solution for the most part, demonstrates how it might be applied to a real-world situation. Consider the fractional non-autonomous partial differential equation with impulsive integral conditions and homogeneous Dirichlet boundary conditions, which will be discussed shortly.

$$
\begin{cases}
\dfrac{\partial^\gamma}{\partial s^\gamma}u(\chi,s)-\Omega(\chi,s)\dfrac{\partial^2}{\partial\chi^2}u(\chi,s)=\dfrac{sin(\pi s)}{1+|u(\chi,s)|}+e^{-s}Sin(\int_0^s(s-t)u(\chi,t)dt)\\
\qquad\qquad +e^{-s}Cos(\int_1^2(-|s-t|)u(\chi,t)dt),\ \ \chi\in G,\quad s\in Z,\\
u(\chi,s)=0,\\
u'(\chi,0)=u_1,\\
u(\chi,0)=\Lambda(\chi)+\sum\limits_{p}^{i=1}\int_0^s\Theta(\chi,y)\dfrac{|u(y,s_i)|}{|1+u(y,s_i)|}dy,\ \ \chi\in G,
\end{cases}
\tag{4.1}
$$

where $\dfrac{\partial^\gamma}{\partial s^\gamma}$ is the Caputo fractional order partial derivative of order γ, $1 < \gamma \leq 2$, Z = [1,2], the coefficient of heat conductivity $\Omega(\chi$, s) is continuous on $G\times[1,2]$ and it is is uniformly Hlder continuous in s, which shows that for any s_1, $s_2 \in$ Z, there remain a constant $1 < \alpha \leq 2$ and a +ve constant C independent of s_1 and s_2, such that

$$
|\Omega(\chi,s_2)-\Omega(\chi,s_1)\leq C|s_2-s_1|^\alpha,\chi\in G,
\tag{4.2}
$$

$p > 0$ is a positive integer number , the function $\Theta(.,.)$ is measureable on $G\times G$ and

$$
\int_0^t\int_0^t\Theta^2(x,y)dxdy:\tau<+\infty,
\tag{4.3}
$$

and $G \in L^2(G)$.

Let O=$L^2(G)$ is a Banach space with the L^2-norm $\|.\|_2$. In Banach space O, we define an operator A(s) by

$$H^2(G) \cap H_1^2(G), A(s)u = -\Omega(\chi, s)\Delta u = \delta(A), \tag{4.4}$$

then it is well known from [13] that-A(s) generates an analytic semigroup $e^{-sA(t)}$ in O

$$u(s) = u(., s),\ K(s,t) = s - t\ for\ 0 \leq t \leq s \leq 1,$$

$$u = 1,\ H(s,t) = e^{-|s-t|} for\ 0 \leq t, s \leq 1,$$

$$(Su)(s) = \int_0^s K(s,t)u(.,t)dt,\ (Tu)(s) = \int_0^1 H(s,t)u(.,t)dt,$$

$$f(s, u(s), (Tu)(s), (Su)(s)) = \frac{sin(\pi s)}{1 + |u(., s)|} + e^{-s}Sin((Tu)(s)) + e^{-s}Cos((Su)(s)),$$

$$g(u) = \sum_{p}^{i=1} \int_0^s \Theta(\chi, y)\frac{|u(y, s_i)|}{|1 + u(y, s_i)|}dy,\ \Lambda(.) = u_0.$$

The fact that A(t) creates an analytic semi-group $e^{-s}A(t)$ in banach space O is generally understood from [13]. Equations (3.15) and (4.1) make it simple to check that the linear operator A(s) meets the hypotheses (G1) and (G2). Further, for any s $\in$ [1,2], we demonstrate how the time fractional non-autonomous partial differential equation with nonlocal initial condition (4.1) and homogeneous Dirichlet boundary condition may be transformed into the time fractional non-autonomous evolution equation with nonlocal conditions (1.10) in its abstract form.

Theorem 4.1. *There is at least one mild solution u $\in$ CG×[1,2]) to the time fractional non-autonomous partial differential equation with nonlocal conditions (4.1) and homogeneous Dirichlet boundary condition, provided that*

$$4Cb^{\gamma}a_3 < 1. \tag{4.5}$$

Proof. By the definitions of nonlinear term f and nonlocal function g, combined with Eq (4.3) one can easily to verify that the assumptions (G1) and (G3) are satisfied with

$$h_r(s) = \sqrt{\pi}(sin(\pi)s) + 2e^{-1}), \Phi(r) = p\sqrt{\pi\tau},$$

and

$$\xi = \bar{\varrho} = \bar{\varrho}_g = 0.$$

We know that f (s,u) is Lipschitz continuous about the variables u with Lipschitz constants 1 from the definition of nonlinear term f. As a result of Lemma 2.4 [9], the assumption (G3) is satisfied given a positive constant $L_f = 1$. The fact that

$$\bar{\varrho} = \bar{\varrho}_g = 0.$$

It is simple to check if the condition Eq (3.1) is valid. Furthermore, we know that the nonlocal term g is a compact operator thanks to [28] and some simple analysis, which indicates that the constant L_g = 0. The condition Eq (3.2) is met when the facts $Lf = 1$ and $Lg = 0$ are paired with Eq (4.5). As a result, all of Theorem 3.1 hypotheses are met. As a result of Theorem 3.1, for the time fractional non-autonomous partial differential equation with nonlocal conditions (4.1) and homogeneous Dirichlet boundary condition, there is at least one mild solution u $\in C(G\times[1,2])$. This brings the proof of Theorem 4.1 to a close.

5. Conclusions

In this paper, mild solution of non-autonomous fractional evolution equations is find out with modifications and generalization of existing relevant literature. The following have been conceptualized and characterized:

(a) We find an appropriate definition of mild solution.
(b) The existence and uniqueness of a mild solution for a system whose probability density function and evolution families are correlated.
(c) His research introduces a new operator that differs from the previous one. It's worth noting that this operator is continuous, linearly bounded, and compact.
(d) Some relevant elementary definition and basic terminologies are recalled with important results and properties.
(e) We also showed that the problem is slightly solvable, with an unique mild solution.
(f) We prove the continuous dependence of mild solution and last we find final result the existence of integral impulsive fractional non autonomous evolution equations for the Cauchy problem with non-local conditions. These proved results will participate a share in the field of study.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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