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Research article

Fixed points of non-linear set-valued (α_*, ϕ_M)-contraction mappings and related applications

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Abstract: The aim of this manuscript is to prove some fixed point results for non-linear set-valued maps with new approach of (α_*, ϕ_M)-contraction in complete M-metric space. Also, we prove some fixed point results in ordered M-metric space. As an presented work which are the extension and improves the current study of set-valued mappings. Finally, we also give an non-trivial extensive examples and application to homotopy theory and the existence solution of functional equations to show that our concepts are meaningful and to support our results.

Keywords: (α_*, ϕ_M)-contractions; M-metric space; homotopy fixed point results; functional equation
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1. Introduction and preliminaries

In 1922, S. Banach [15] provided the concept of Contraction theorem in the context of metric space. After, Nadler [28] introduced the concept of set-valued mapping in the module of Hausdroff metric space which is one of the potential generalizations of a Contraction theorem. Let (X, d) is a complete metric space and a mapping $T : X \to CB(X)$ satisfying

$$H(T(x), T(y)) \leq \gamma d(x, y)$$

for all $x, y \in X$, where $0 \leq \gamma < 1$, H is a Hausdorff with respect to metric d and $CB(X) = \{\mathbf{S} \subseteq X : S \text{ is closed and bounded subset of } X \text{ equipped with a metric } d\}$. Then T has a fixed point in X.

In the recent past, Matthews [26] initiate the concept of partial metric spaces which is the classical extension of a metric space. After that, many researchers generalized some related results in the frame of partial metric spaces. Recently, Asadi et al. [4] introduced the notion of an M-metric space which is the one of interesting generalizations of a partial metric space. Later on, Samet et al. [33] introduced the class of mappings which known as (α, ψ)-contractive mapping. The notion of (α, ψ)-contractive mapping has been generalized in metric spaces (see more [10, 12, 14, 17, 19, 25, 29, 30, 32]).

Throughout this manuscript, we denote the set of all positive integers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. Let us recall some basic concept of an M-metric space as follows:

Definition 1.1. [4] Let $m : X \times X \to \mathbb{R}^+$be a mapping on nonempty set X is said to be an M-metric if for any x, y, z in X, the following conditions hold:

(i) $m(x, x) = m(y, y) = m(x, y)$ if and only if $x = y$;
(ii) $m_{xy} \leq m(x, y)$;
(iii) $m(x, y) = m(y, x)$;
(iv) $m(x, y) - m_{xy} \leq (m(x, z) - m_{xz}) + (m(z, y) - m_{z,y})$ for all $x, y, z \in X$. Then a pair (X, m) is called M-metric space. Where

$$m_{xy} = \min\{m(x, x), m(y, y)\}$$

and

$$M_{xy} = \max\{m(x, x), m(y, y)\}.$$

Remark 1.2. [4] For any x, y, z in M-metric space X, we have

(i) $0 \leq M_{xy} + m_{xy} = m(x, x) + m(y, y)$;
(ii) $0 \leq M_{xy} - m_{xy} = |m(x, x) - m(y, y)|$;
(iii) $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + \left(M_{zy} - m_{zy}\right)$.

Example 1.3. [4] Let (X, m) be an M-metric space. Define m^w, $m^s : X \times X \to \mathbb{R}^+$ by:

(i)
$$m^w(x, y) = m(x, y) - 2m_{x,y} + M_{x,y},$$

(ii)
$$m^s = \begin{cases} m(x, y) - m_{x,y}, \text{ if } x \neq y \\ 0, \text{ if } x = y. \end{cases}$$

Then m^w and m^s are ordinary metrics. Note that, every metric is a partial metric and every partial metric is an M-metric. However, the converse does not hold in general. Clearly every M-metric on X generates a T_0 topology τ_m on X whose base is the family of open M-balls

$$\{B_m(x, \epsilon) : x \in X,\ \epsilon > 0\},$$

where

$$B_m(x, \epsilon) = \{y \in X : m(x, y) < m_{xy} + \epsilon\}$$

for all $x \in X$, $\varepsilon > 0$. (see more [3, 4, 23]).

Definition 1.4. [4] Let (X, m) be an M-metric space. Then,

(i) A sequence $\{x_n\}$ in (X, m) is said to be converges to a point x in X with respect to τ_m if and only if

$$\lim_{n\to\infty} (m(x_n, x) - m_{x_n x}) = 0.$$

(ii) Furthermore, $\{x_n\}$ is said to be an M-Cauchy sequence in (X, m) if and only if

$$\lim_{n,m\to\infty} (m(x_n, x_m) - m_{x_n x_m}), \text{ and } \lim_{n,m\to\infty} (M_{x_n, x_m} - m_{x_n x_m})$$

exist (and are finite).

(iii) An M-metric space (X, m) is said to be complete if every M-Cauchy sequence $\{x_n\}$ in (X, m) converges with respect to τ_m to a point $x \in X$ such that

$$\lim_{n\to\infty} m(x_n, x) - m_{x_n x} = 0, \text{ and } \lim_{n\to\infty} (M_{x_n, x} - m_{x_n x}) = 0.$$

Lemma 1.5. *[4] Let (X, m) be an M-metric space. Then:*

(i) $\{x_n\}$ is an M-Cauchy sequence in (X, m) if and only if $\{x_n\}$ is a Cauchy sequence in a metric space (X, m^w).

(ii) An M-metric space (X, m) is complete if and only if the metric space (X, m^w) is complete. Moreover,

$$\lim_{n\to\infty} m^w(x_n, x) = 0 \text{ if and only if } \left(\lim_{n\to\infty} (m(x_n, x) - m_{x_n x}) = 0, \ \lim_{n\to\infty} (M_{x_n x} - m_{x_n x}) = 0\right).$$

Lemma 1.6. *[4] Suppose that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y as n approaches to ∞ in M-metric space (X, m). Then we have*

$$\lim_{n\to\infty} \left(m(x_n, y_n) - m_{x_n y_n}\right) = m(x, y) - m_{xy}.$$

Lemma 1.7. *[4] Suppose that $\{x_n\}$ converges to x as n approaches to ∞ in M-metric space (X, m). Then we have*

$$\lim_{n\to\infty} \left(m(x_n, y) - m_{x_n y}\right) = m(x, y) - m_{xy} \text{ for all } y \in X.$$

Lemma 1.8. *[4] Suppose that $\{x_n\}$ converges to x and $\{x_n\}$ converges to y as n approaches to ∞ in M-metric space (X, m). Then $m(x, y) = m_{xy}$ moreover if $m(x, x) = m(y, y)$, then $x = y$.*

Definition 1.9. Let $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be an α-admissible mapping if for all $x, y \in X$

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(T(x), T(y)) \geq 1.$$

Let Ψ be the family of the (c)-comparison functions $\psi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ which satisfy the following properties:

(i) ψ is nondecreasing,

(ii) $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n-iterate of ψ (see [7, 8, 10, 11]).

Definition 1.10. [33] Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is called (α, ψ)-contractive mapping if for all $x, y \in X$, we have

$$\alpha(x, y) d(T(x), T(x)) \leq \psi(d(x, y)),$$

where $\psi \in \Psi$.

A subset K of an M-metric space X is called bounded if for all $x \in K$, there exist $y \in X$ and $r > 0$ such that $x \in B_m(y, r)$. Let $\overline{K}$ denote the closure of K. The set K is closed in X if and only if $\overline{K} = K$.

Definition 1.11. [31] Define $H_m : CB_m(X) \times CB_m(X) \to [0, \infty)$ by

$$H_m(K, L) = \max\{\nabla_m(K, L), \nabla_m(L, K)\},$$

where

$$\begin{aligned} m(x, L) &= \inf\{m(x, y) : y \in L\} \text{ and} \\ \nabla_m(L, K) &= \sup\{m(x, L) : x \in K\}. \end{aligned}$$

Lemma 1.12. *[31] Let F be any nonempty set in M-metric space (X, m), then*

$$x \in \overline{F} \text{ if and only if } m(x, F) = \sup_{a \in F}\{m_{xa}\}.$$

Proposition 1.13. *[31] Let $A, B, C \in CB_m(X)$, then*

(i) $\nabla_m(A, A) = \sup_{x \in A}\left\{\sup_{y \in A} m_{xy}\right\}$,
(ii) $\left(\nabla_m(A, B) - \sup_{x \in A}\sup_{y \in B} m_{xy}\right) \leq (\nabla_m(A, C) - \inf_{x \in A}\inf_{z \in C} m_{xz}) +$
$\left(\nabla_m(C, B) - \inf_{z \in C}\inf_{y \in B} m_{zy}\right)$.

Proposition 1.14. *[31] Let $A, B, C \in CB_m(X)$ following are hold*

(i) $H_m(A, A) = \nabla_m(A, A) = \sup_{x \in A}\left\{\sup_{y \in A} m_{xy}\right\}$,
(ii) $H_m(A, B) = H_m(B, A)$,
(iii) $H_m(A, B) - \sup_{x \in A}\sup_{y \in A} m_{xy}) \leq H_m(A, C) + H_m(B, C) - \inf_{x \in A}\inf_{z \in C} m_{xz} - \inf_{z \in C}\inf_{y \in B} m_{zy}$.

Lemma 1.15. *[31] Let $A, B \in CB_m(X)$ and $h > 1$. Then for each $x \in A$, there exist at the least one $y \in B$ such that*

$$m(x, y) \leq hH_m(A, B).$$

Lemma 1.16. *[31] Let $A, B \in CB_m(X)$ and $l > 0$. Then for each $x \in A$, there exist at least one $y \in B$ such that*

$$m(x, y) \leq H_m(A, B) + l.$$

Theorem 1.17. *[31] Let (X, m) be a complete M-metric space and $T : X \to CB_m(X)$. Assume that there exist $h \in (0, 1)$ such that*

$$H_m(T(x), T(y)) \leq hm(x, y), \tag{1.1}$$

for all $x, y \in X$. Then T has a fixed point.

Proposition 1.18. *[31] Let $T : X \to CB_m(X)$ be a set-valued mapping satisfying (1.1) for all x, y in an M-metric space X. If $z \in T(z)$ for some z in X such that $m(x, x) = 0$ for $x \in T(z)$.*

2. Main results

We start with the following definition:

Definition 2.1. Assume that Ψ is a family of non-decreasing functions $\phi_M : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(i) $\sum_n^{+\infty} \phi_M^n(x) < \infty$ for every $x > 0$ where ϕ_M^n is a n^{th}-iterate of ϕ_M,
(ii) $\phi_M(x+y) \leq \phi_M(x) + \phi_M(y)$ for all $x, y \in \mathbb{R}^+$,
(iii) $\phi_M(x) < x$, for each $x > 0$.

Remark 2.2. If $\sum \alpha_n|_{n=\infty} = 0$ is a convergent series with positive terms then there exists a monotonic sequence $(\beta_n)|_{n=\infty}$ such that $\beta_n|_{n=\infty} = \infty$ and $\sum \alpha_n\beta_n|_{n=\infty} = 0$ converges.

Definition 2.3. Let (X, m) be an M-metric pace. A self mapping $T : X \to X$ is called (α_*, ϕ_M)-contraction if there exist two functions $\alpha_* : X \times X \to [0, \infty)$ and $\phi_M \in \Psi$ such that

$$\alpha_*(x, y)\, m(T(x), T(y)) \leq \phi_M(m(x, y)),$$

for all $x, y \in X$.

Definition 2.4. Let (X, m) be an M-metric space. A set-valued mapping $T : X \to CB_m(X)$ is said to be (α_*, ϕ_M)-contraction if for all $x, y \in X$, we have

$$\alpha_*(x, y)\, H_m(T(x), T(x)) \leq \phi_M(m(x, y)), \tag{2.1}$$

where $\phi_M \in \Psi$ and $\alpha_* : X \times X \to [0, \infty)$.
A mapping T is called α_*-admissible if

$$\alpha_*(x, y) \geq 1 \Rightarrow \alpha_*(a_1, b_1) \geq 1$$

for each $a_1 \in T(x)$ and $b_1 \in T(y)$.

Theorem 2.5. *Let (X, m) be a complete M-metric space. Suppose that (α_*, ϕ_M) contraction and α_*-admissible mapping $T : X \to CB_m(X)$ satisfies the following conditions:*

(i) there exist $x_0 \in X$ such that $\alpha_*(x_0, a_1) \geq 1$ for each $a_1 \in T(x_0)$,
(ii) if $\{x_n\} \in X$ is a sequence such that $\alpha_*(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to x \in X$ as $n \to \infty$, then $\alpha_*(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Then T has a fixed point.

Proof. Let $x_1 \in T(x_0)$ then by the hypothesis (i) $\alpha_*(x_0, x_1) \geq 1$. From Lemma 1.16, there exist $x_2 \in T(x_1)$ such that

$$m(x_1, x_2) \leq H_m(T(x_0), T(x_1)) + \phi_M(m(x_0, x_1)).$$

Similarly, there exist $x_3 \in T(x_2)$ such that

$$m(x_2, x_3) \leq H_m(T(x_1), T(x_2)) + \phi_M^2(m(x_0, x_1)).$$

Following the similar arguments, we obtain a sequence $\{x_n\} \in X$ such that there exist $x_{n+1} \in T(x_n)$ satisfying the following inequality

$$m(x_n, x_{n+1}) \leq H_m(T(x_{n-1}), T(x_n)) + \phi_M^n(m(x_0, x_1)).$$

Since T is α_*-admissible, therefore $\alpha_*(x_0, x_1) \geq 1 \Rightarrow \alpha_*(x_1, x_2) \geq 1$. Using mathematical induction, we get

$$\alpha_*(x_n, x_{n+1}) \geq 1. \tag{2.2}$$

By (2.1) and (2.2), we have

$$\begin{aligned}
m(x_n, x_{n+1}) &\leq H_m(T(x_{n-1}), T(x_n)) + \phi_M^n(m(x_0, x_1)) \\
&\leq \alpha_*(x_n, x_{n+1}) H_m(T(x_{n-1}), T(x_n)) \\
&\quad + \phi_M^n(m(x_0, x_1)) \\
&\leq \phi_M(m(x_{n-1}, x_n)) + \phi_M^n(m(x_0, x_1)) \\
&= \phi_M\left[(m(x_{n-1}, x_n)) + \phi_M^{n-1}(m(x_0, x_1))\right] \\
&\leq \phi_M\left[H_m(T(x_{n-2}), T(x_{n-1})) + \phi_M^{n-1}(m(x_0, x_1))\right] \\
&\leq \phi_M\left[\alpha_*(x_{n-1}, x_n) H_m(T(x_{n-1}), T(x_n)) + \phi_M^{n-1}(m(x_0, x_1))\right] \\
&\leq \phi_M\left[\phi_M(m(x_{n-2}, x_{n-1})) + \phi_M^{n-1}(m(x_0, x_1)) + \phi_M^{n-1}(m(x_0, x_1))\right] \\
&\leq \phi_M^2(m(x_{n-2}, x_{n-1})) + 2\phi_M^n(m(x_0, x_1)) \\
&\quad \ldots.
\end{aligned}$$

$$\begin{aligned}
m(x_n, x_{n+1}) &\leq \phi_M^n(m(x_0, x_1)) + n\phi_M^n(m(x_0, x_1)) \\
m(x_n, x_{n+1}) &\leq (n+1)\phi_M^n(m(x_0, x_1)).
\end{aligned}$$

Let us assume that $\epsilon > 0$, then there exist $n_0 \in N$ such that

$$\sum_{n \geq n_0} (n+1)\phi_M^n(m(x_0, x_1)) < \epsilon.$$

By the Remarks (1.2) and (2.2), we get

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0.$$

Using the above inequality and (m_2), we deduce that

$$\begin{aligned}
\lim_{n\to\infty} m(x_n, x_n) &= \lim_{n\to\infty} \min\{m(x_n, x_n), m(x_{n+1}, x_{n+1})\} \\
&= \lim_{n\to\infty} m_{x_n x_{n+1}} \\
&\leq \lim_{n\to\infty} m(x_n, x_{n+1}) = 0.
\end{aligned}$$

Owing to limit, we have $\lim_{n\to\infty} m(x_n, x_n) = 0$,

$$\lim_{n,m\to\infty} m_{x_n x_m} = 0.$$

Now, we prove that $\{x_n\}$ is M-Cauchy in X. For m, n in N with $m > n$ and using the triangle inequality of an M-metric we get

$$
\begin{aligned}
m(x_n, x_m) - m_{x_n x_m} &\leq m(x_n, x_{n+1}) - m_{x_n x_{n+1}} + m(x_{n+1}, x_m) - m_{x_{n+1} x_m} \\
&\leq m(x_n, x_{n+1}) - m_{x_n x_{n+1}} + m(x_{n+1}, x_{n+2}) - m_{x_{n+1} x_{n+1}} \\
&\quad + m(x_{n+2}, x_m) - m_{x_{n+2} x_m} \\
&\leq m(x_n, x_{n+1}) - m_{x_n x_{n+1}} + m(x_{n+1}, x_{n+2}) - m_{x_{n+1} x_{n+2}} \\
&\quad + \cdots + m(x_{m-1}, x_m) - m_{x_{m-1} x_m} \\
&\leq m(x_n, x_{n+1}) + m(x_{n+1}, x_{n+2}) + \cdots + m(x_{m-1}, x_m) \\
&= \sum_{r=n}^{m-1} m(x_r, x_{r+1}) \\
&\leq \sum_{r=n}^{m-1} (r+1)\, \phi_M^r (m(x_0, x_1)) \\
&\leq \sum_{r \geq n_0}^{m-1} (r+1)\, \phi_M^r (m(x_0, x_1)) \\
&\leq \sum_{r \geq n_0}^{m-1} (r+1)\, \phi_M^r (m(x_0, x_1)) < \epsilon.
\end{aligned}
$$

$m(x_n, x_m) - m_{x_n x_m} \to 0$, as $n \to \infty$, we obtain $\lim_{m,n\to\infty} (M_{x_n x_m} - m_{x_n x_m}) = 0$. Thus $\{x_n\}$ is a M-Cauchy sequence in X. Since (X, m) is M-complete, there exist $x^\star \in X$ such that

$$
\begin{aligned}
\lim_{n\to\infty} (m(x_n, x^\star) - m_{x_n x^\star}) &= 0 \text{ and} \\
\lim_{n\to\infty} (M_{x_n x^\star} - m_{x_n x^\star}) &= 0.
\end{aligned}
$$

Also, $\lim_{n\to\infty} m(x_n, x_n) = 0$ gives that

$$\lim_{n\to\infty} m(x_n, x^\star) = 0 \text{ and } \lim_{n\to\infty} M_{x_n x^\star} = 0, \tag{2.3}$$

$$\lim_{n\to\infty} \{\max (m(x_n, x^\star), m(x^\star, x^\star))\} = 0,$$

which implies that $m(x^\star, x^\star) = 0$ and hence we obtain $m_{x^\star T(x^\star)} = 0$. By using (2.1) and (2.3) with

$$\lim_{n\to\infty} \alpha_* (x_n, x^\star) \geq 1.$$

Thus,

$$\lim_{n\to\infty} H_m (T(x_n), T(x^\star)) \leq \lim_{n\to\infty} \phi_M (m(x_n, x^\star)) \leq \lim_{n\to\infty} m(x_n, x^\star).$$

$$\lim_{n\to\infty} H_m (T(x_n), T(x^\star)) = 0. \tag{2.4}$$

Now from (2.3), (2.4), and $x_{n+1} \in T(x_n)$, we have

$$m(x_{n+1}, T(x^\star)) \leq H_m (T(x_n), T(x^\star)) = 0.$$

Taking limit as $n \to \infty$ and using (2.4), we obtain that

$$\lim_{n\to\infty} m\left(x_{n+1}, T\left(x^{\star}\right)\right) = 0. \tag{2.5}$$

Since $m_{x_{n+1}T(x^{\star})} \leq m\left(x_{n+1}, T\left(x^{\star}\right)\right)$ which gives

$$\lim_{n\to\infty} m_{x_{n+1}T(x^{\star})} = 0. \tag{2.6}$$

Using the condition (m_4), we obtain

$$\begin{aligned} m\left(x^{\star}, T\left(x^{\star}\right)\right) - \sup_{y\in T(x^{\star})} m_{x^{\star}y} &\leq m\left(x^{\star}, T\left(x^{\star}\right)\right) - m_{x^{\star},T(x^{\star})} \\ &\leq m\left(x^{\star}, x_{n+1}\right) - m_{x^{\star}x_{n+1}} \\ &\quad + m\left(x_{n+1}, T\left(\left(x^{\star}\right)\right) - m_{x_{n+1}T(x^{\star})}. \end{aligned}$$

Applying limit as $n \to \infty$ and using (2.3) and (2.6), we have

$$m\left(x^{\star}, T\left(x^{\star}\right)\right) \leq \sup_{y\in T(x^{\star})} m_{x^{\star}y}. \tag{2.7}$$

From (m_2), $m_{x^{\star}y} \leq m\left(x^{\star}y\right)$ for each $y \in T\left(x^{\star}\right)$ which implies that

$$m_{x^{\star}y} - m\left(x^{\star}, y\right) \leq 0.$$

Hence,

$$\sup\left\{m_{x^{\star}y} - m\left(x^{\star}, y\right) : y \in T\left(x^{\star}\right)\right\} \leq 0.$$

Then

$$\sup_{y\in T(x^{\star})} m_{x^{\star}y} - \inf_{y\in T(x^{\star})} m\left(x^{\star}, y\right) \leq 0.$$

Thus

$$\sup_{y\in T(x^{\star})} m_{x^{\star}y} \leq m\left(x^{\star}, T\left(x^{\star}\right)\right). \tag{2.8}$$

Now, from (2.7) and (2.8), we obtain

$$m\left(T\left(x^{\star}\right), x^{\star}\right) = \sup_{y\in T(x^{\star})} m_{x^{\star}y}.$$

Consequently, owing to Lemma (1.12), we have $x^{\star} \in \overline{T\left(x^{\star}\right)} = T\left(x^{\star}\right)$. □

Corollary 2.6. *Let (X, m) be a complete M-metric space and an self mapping $T : X \to X$ an α_*-admissible and (α_*, ϕ_M)-contraction mapping. Assume that the following properties hold:*

(i) there exists $x_0 \in X$ such that $\alpha_*\left(x_0, T\left(x_0\right)\right) \geq 1$,

(ii) either T is continuous or for any sequence $\{x_n\} \in X$ with $\alpha_*\left(x_n, x_{n+1}\right) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\} \to x$ as $n \to \infty$, we have $\alpha_*\left(x_n, x\right) \geq 1$ for all $n \in \mathbb{N}$. Then T has a fixed point.

Some fixed point results in ordered M-metric space.

Definition 2.7. Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be non-decreasing if $x_n \preceq x_{n+1}$ for all n.

Definition 2.8. [16] Let F and G be two nonempty subsets of partially ordered set $(X, \preceq)$. The relation between F and G is defined as follows: $F \prec_1 G$ if for every $x \in F$, there exists $y \in G$ such that $x \preceq y$.

Definition 2.9. Let $(X, m, \preceq)$ be a partially ordered set on M-metric. A set-valued mapping $T : X \to CB_m(X)$ is said to be ordered (α_*, ϕ_M)-contraction if for all $x, y \in X$, with $x \preceq y$ we have

$$H_m(T(x), T(y)) \leq \phi_M(m(x, y))$$

where $\phi_M \in \Psi$. Suppose that $\alpha_* : X \times X \to [0, \infty)$ is defined by

$$\alpha_*(x, y) = \begin{cases} 1 & \text{if } Tx \prec_1 Ty \\ 0 & \text{otherwise.} \end{cases}$$

A mapping T is called α_*-admissible if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha_*(a_1, b_1) \geq 1,$$

for each $a_1 \in T(x)$ and $b_1 \in T(y)$.

Theorem 2.10. *Let $(X, m, \preceq)$ be a partially ordered complete M-metric space and $T : X \to CB_m(X)$ an α_*-admissible ordered (α_*, ϕ_M)-contraction mapping satisfying the following conditions:*

(i) there exist $x_0 \in X$ such that $\{x_0\} \prec_1 \{T(x_0)\}$, $\alpha_*(x_0, a_1) \geq 1$ for each $a_1 \in T(x_0)$,
(ii) for every $x, y \in X$, $x \preceq y$ implies $T(x) \prec_1 T(y)$,
(iii) If $\{x_n\} \in X$ is a non-decreasing sequence such that $x_n \preceq x_{n+1}$ for all n and $\{x_n\} \to x \in X$ as $n \to \infty$ gives $x_n \preceq x$ for all $n \in \mathbb{N}$. Then T has a fixed point.

Proof. By assumption (i) there exist $x_1 \in T(x_0)$ such that $x_0 \preceq x_1$ and $\alpha_*(x_0, x_1) \geq 1$. By hypothesis (ii), $T(x_0) \prec_1 T(x_1)$. Let us assume that there exist $x_2 \in T(x_1)$ such that $x_1 \preceq x_2$ and we have the following

$$m(x_1, x_2) \leq H_m(T(x_0), T(x_1)) + \phi_M(m(x_0, x_1)).$$

In the same way, there exist $x_3 \in T(x_2)$ such that $x_2 \preceq x_3$ and

$$m(x_2, x_3) \leq H_m(T(x_1), T(x_2)) + \phi_M^2(m(x_0, x_1)).$$

Following the similar arguments, we have a sequence $\{x_n\} \in X$ and $x_{n+1} \in T(x_n)$ for all $n \geq 0$ satisfying $x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq ...x_n \preceq x_{n+1}$. The proof is complete follows the arguments given in Theorem 2.5. □

Example 2.11. Let $X = \left[\frac{1}{6}, 1\right]$ be endowed with an M-metric given by $m(x, y) = \frac{x+y}{2}$. Define $T : X \to CB_m(X)$ by

$$T(x) = \begin{cases} \left\{\frac{1}{2}x + \frac{1}{6}, \frac{1}{4}\right\}, & \text{if } x = \frac{1}{6} \\ \left\{\frac{x}{2}, \frac{x}{3}\right\}, & \text{if } \frac{1}{4} \leq x \leq \frac{1}{3} \\ \left\{\frac{2}{3}, \frac{5}{6}\right\}, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Define a mapping $\alpha_* : X \times X \to [0, \infty)$ by

$$\alpha_*(x, y) = \begin{cases} 1 & \text{if } x, y \in \left[\frac{1}{4}, \frac{1}{3}\right] \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi_M : \mathbb{R}^+ \to \mathbb{R}^+$ be given by $\phi_M(t) = \frac{17}{10}$ where $\phi_M \in \Psi$, for $x, y \in X$. If $x = \frac{1}{6}$, $y = \frac{1}{4}$ then $m(x, y) = \frac{5}{24}$, and

$$\begin{aligned} H_m(T(x), T(y)) &= H_m\left(\left\{\frac{3}{12}, \frac{1}{4}\right\}, \left\{\frac{1}{8}, \frac{1}{12}\right\}\right) \\ &= \max\left(\nabla_m\left(\left\{\frac{3}{12}, \frac{1}{4}\right\}, \left\{\frac{1}{8}, \frac{1}{12}\right\}\right), \nabla_m\left(\left\{\frac{1}{8}, \frac{1}{12}\right\}, \left\{\frac{3}{12}, \frac{1}{4}\right\}\right)\right) \\ &= \max\left\{\frac{3}{16}, \frac{2}{12}\right\} = \frac{3}{16} \\ &\leq \phi_M(t)\, m(x, y). \end{aligned}$$

If $x = \frac{1}{3}$, $y = \frac{1}{2}$ then $m(x, y) = \frac{5}{12}$, and

$$\begin{aligned} H_m(T(x), T(y)) &= H_m\left(\left\{\frac{1}{6}, \frac{1}{9}\right\}, \left\{\frac{2}{3}, 1\right\}\right) \\ &= \max\left(\nabla_m\left(\left\{\frac{1}{6}, \frac{1}{9}\right\}, \left\{\frac{2}{3}, 1\right\}\right), \nabla_m\left(\left\{\frac{2}{3}, 1\right\}, \left\{\frac{1}{6}, \frac{1}{9}\right\}\right)\right) \\ &= \max\left\{\frac{17}{36}, \frac{7}{18}\right\} = \frac{17}{36} \\ &\leq \phi_M(t)\, m(x, y). \end{aligned}$$

If $x = \frac{1}{6}$, $y = 1$, then $m(x, y) = \frac{7}{12}$ and

$$\begin{aligned} H_m(T(x), T(y)) &= H_m\left(\left\{\frac{3}{12}, \frac{1}{4}\right\}, \left\{\frac{2}{3}, \frac{5}{6}\right\}\right) \\ &= \max\left(\nabla_m\left(\left\{\frac{3}{12}, \frac{1}{4}\right\}, \left\{\frac{2}{3}, \frac{5}{6}\right\}\right), \nabla_m\left(\left\{\frac{2}{3}, \frac{5}{6}\right\}, \left\{\frac{3}{12}, \frac{1}{4}\right\}\right)\right) \\ &= \max\left\{\frac{11}{24}, \frac{13}{24}\right\} = \frac{13}{24} \\ &\leq \phi_M(t)\, m(x, y). \end{aligned}$$

In all cases, T is (α_*, ϕ_M)-contraction mapping. If $x_0 = \frac{1}{3}$, then $T(x_0) = \left\{\frac{x}{2}, \frac{x}{3}\right\}$.Therefore $\alpha_*(x_0, a_1) \geq 1$ for every $a_1 \in T(x_0)$. Let $x, y \in X$ be such that $\alpha_*(x, y) \geq 1$, then $x, y \in \left[\frac{x}{2}, \frac{x}{3}\right]$ and $T(x) = \left\{\frac{x}{2}, \frac{x}{3}\right\}$ and $T(y) = \left\{\frac{x}{2}, \frac{x}{3}\right\}$ which implies that $\alpha_*(a_1, b_1) \geq 1$ for every $a_1 \in T(x)$ and $b_1 \in T(x)$. Hence T is α_*-admissble.

Let $\{x_n\} \in X$ be a sequence such that $\alpha_*(x_n, x_{n+1}) \geq 1$ for all n in $\mathbb{N}$ and x_n converges to x as n converges to ∞, then $x_n \in \left[\frac{x}{2}, \frac{x}{3}\right]$. By definition of α_*-admissblity, therefore $x \in \left[\frac{x}{2}, \frac{x}{3}\right]$ and hence $\alpha_*(x_n, x) \geq 1$. Thus all the conditions of Theorem 2.3 are satisfied. Moreover, T has a fixed point.

Example 2.12. Let $X = \left\{(0,0), \left(0,-\frac{1}{5}\right), \left(-\frac{1}{8},0\right)\right\}$ be the subset of $\mathbb{R}^2$ with order $\preceq$ defined as: For $(x_1,y_1),(x_2,y_2) \in X$, $(x_1,y_1) \preceq (x_2,y_2)$ if and only if $x_1 \leq x_2$, $y_1 \leq y_2$. Let $m : X \times X \to \mathbb{R}^+$ be defined by

$$m((x_1,y_1),(x_2,y_2)) = \left|\frac{x_1+x_2}{2}\right| + \left|\frac{y_1+y_2}{2}\right|, \text{ for } x = (x_1,y_1),\ y = (x_2,y_2) \in X.$$

Then (X,m) is a complete M-metric space. Let $T : X \to CB_m(X)$ be defined by

$$T(x) = \begin{cases} \{(0,0)\}, \text{ if } x = (0,0), \\ \left\{(0,0), \left(-\frac{1}{8},0\right)\right\}, \quad \text{if } x \in \left(0,-\frac{1}{5}\right) \\ \{(0,0)\}, \quad \text{if } x \in \left(-\frac{1}{8},0\right). \end{cases}$$

Define a mapping $\alpha_* : X \times X \to [0,\infty)$ by

$$\alpha_*(x,y) = \begin{cases} 1 & \text{if } x,y \in X \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi_M : \mathbb{R}^+ \to \mathbb{R}^+$ be given by $\phi_M(t) = \frac{1}{2}$. Obviously, $\phi_M \in \Psi$. For $x,y \in X$,
if $x = \left(0,-\frac{1}{5}\right)$ and $y = (0,0)$, then $H_m(T(x),T(y)) = 0$ and $m(x,y) = \frac{1}{10}$ gives that

$$\begin{aligned} H_m(T(x),T(y)) &= H_m\left(\left\{(0,0), \left(-\frac{1}{8},0\right)\right\}, \{(0,0)\}\right) \\ &= \max\begin{pmatrix} \nabla_m\left(\left\{(0,0), \left(-\frac{1}{8},0\right)\right\}, \{(0,0)\}\right), \\ \nabla_m\left(\{(0,0)\}, \left\{(0,0), \left(-\frac{1}{8},0\right)\right\}\right) \end{pmatrix} \\ &= \max\{0,0\} = 0 \\ &\leq \phi_M(t)\, m(x,y). \end{aligned}$$

If $x = \left(-\frac{1}{8},0\right)$ and $y = (0,0)$ then $H_m(T(x),T(y)) = 0$, and $m(x,y) = \frac{1}{16}$ implies that

$$H_m(T(x),T(y)) \leq \phi_M(t)\, m(x,y).$$

If $x = (0,0)$ and $y = (0,0)$ then $H_m(T(x),T(y)) = 0$, and $m(x,y) = 0$ gives

$$H_m(T(x),T(y)) \leq \phi_M(t)\, m(x,y).$$

If $x = \left(0,-\frac{1}{5}\right)$ and $y = \left(0,-\frac{1}{5}\right)$ then $H_m(T(x),T(y)) = 0$, and $m(x,y) = \frac{1}{5}$ implies that

$$H_m(T(x),T(y)) \leq \phi_M(t)\, m(x,y).$$

If $x = \left(0,-\frac{1}{8}\right)$ and $y = \left(0,-\frac{1}{8}\right)$ then $H_m(T(x),T(y)) = 0$, and $m(x,y) = \frac{1}{8}$ gives that

$$H_m(T(x),T(y)) \leq \phi_M(t)\, m(x,y).$$

Thus all the condition of Theorem 2.10 satisfied. Moreover, $(0,0)$ is the fixed point of T.

3. Application

In this section, we present an application of our result in homotopy theory. We use the fixed point theorem proved for set-valued (α_*, ϕ_M)-contraction mapping in the previous section, to establish the result in homotopy theory. For further study in this direction, we refer to [6, 35] .

Theorem 3.1. *Suppose that (X, m) is a complete M-metric space and A and B are closed and open subsets of X respectively, such that $A \subset B$. For $a, b \in \mathbb{R}$, let $T : B \times [a, b] \to CB_m(X)$ be a set-valued mapping satisfying the following conditions:*

(i) *$x \notin T(y, t)$ for each $y \in B/A$ and $t \in [a, b]$,*

(ii) *there exist $\phi_M \in \Psi$ and $\alpha_* : X \times X \to [0, \infty)$ such that*

$$\alpha_*(x, y) H_m(T(x, t), T(y, t)) \leq \phi_M(m(x, y)),$$

for each pair $(x, y) \in B \times B$ and $t \in [a, b]$,

(iii) *there exist a continuous function $\Omega : [a, b] \to \mathbb{R}$ such that for each $s, t \in [a, b]$ and $x \in B$, we get*

$$H_m(T(x, s), T(y, t)) \leq \phi_M |\Omega(s) - \Omega(t)|,$$

(iv) *if $x^\star \in T(x^\star, t)$, then $T(x^\star, t) = \{x^\star\}$,*

(v) *there exist x_0 in X such that $x_0 \in T(x_0, t)$,*

(vi) *a function $\mathfrak{R} : [0, \infty) \to [0, \infty)$ defined by $\mathfrak{R}(x) = x - \phi_M(x)$ is strictly increasing and continuous if $T(., t^\top)$ has a fixed point in B for some $t^\top \in [a, b]$, then $T(., t)$ has a fixed point in A for all $t \in [a, b]$. Moreover, for a fixed $t \in [a, b]$, fixed point is unique provided that $\phi_M(t) = \frac{1}{2}t$ where $t > 0$.*

Proof. Define a mapping $\alpha_* : X \times X \to [0, \infty)$ by

$$\alpha_*(x, y) = \begin{cases} 1 & \text{if } x \in T(x, t),\ y \in T(y, t) \\ \\ 0 & \text{otherwise.} \end{cases}$$

We show that T is α_*-admissible. Note that $\alpha_*(x, y) \geq 1$ implies that $x \in T(x, t)$ and $y \in T(y, t)$ for all $t \in [a, b]$. By hypothesis (iv), $T(x, t) = \{x\}$ and $T(y, t) = \{y\}$. It follows that T is α_*-admissible. By hypothesis (v), there exist $x_0 \in X$ such that $x_0 \in (x_0, t)$ for all t, that is $\alpha_*(x_0, x_0) \geq 1$. Suppose that $\alpha_*(x_n, x_{n+1}) \geq 1$ for all n and x_n converges to q as n approaches to ∞ and $x_n \in T(x_n, t)$ and $x_{n+1} \in T(x_{n+1}, t)$ for all n and $t \in [a, b]$ which implies that $q \in T(q, t)$ and thus $\alpha_*(x_n, q) \geq 1$. Set

$$D = \{t \in [a, b] : \ x \in T(x, t) \ \text{ for } x \in A\}.$$

So $T(., t^\top)$ has a fixed point in B for some $t^\top \in [a, b]$, there exist $x \in B$ such that $x \in T(x, t)$. By hypothesis (i) $x \in T(x, t)$ for $t \in [a, b]$ and $x \in A$ so $D \neq \phi$. Now we now prove that D is open and close in $[a, b]$. Let $t_0 \in D$ and $x_0 \in A$ with $x_0 \in T(x_0, t_0)$. Since A is open subset of X, $\overline{B_m(x_0, r)} \subseteq A$ for some $r > 0$. For $\epsilon = r + m_{xx_0} - \phi(r + m_{xx_0})$ and a continuous function Ω on $[a, b]$, there exist $\delta > 0$ such that

$$\phi_M |\Omega(t) - \Omega(t_0)| < \epsilon \text{ for all } t \in (t_0 - \delta, t_0 + \delta).$$

If $t \in (t_0 - \delta, t_0 + \delta)$ for $x \in B_m(x_0, r) = \{x \in X : m(x_0, x) \leq m_{x_0 x} + r\}$ and $l \in T(x, t)$, we obtain

$$\begin{aligned} m(l, x_0) &= m(T(x,t), x_0) \\ &= H_m(T(x,t), T(x_0, t_0)). \end{aligned}$$

Using the condition (*iii*) of Proposition 1.13 and Proposition 1.18, we have

$$m(l, x_0) \leq H_m(T(x,t), T(x_0,t_0)) + H_m(T(x,t), T(x_0,t_0)) \tag{2.9}$$

as $x \in T(x_0, t_0)$ and $x \in B_m(x_0, r) \subseteq A \subseteq B$, $t_0 \in [a, b]$ with $\alpha_*(x_0, x_0) \geq 1$. By hypothesis (*ii*), (*iii*) and (2.9)

$$\begin{aligned} m(l, x_0) &\leq \phi_M |\Omega(t) - \Omega(t_0)| + \alpha_*(x_0, x_0) H_m(T(x,t), T(x_0,t_0)) \\ &\leq \phi_M |\Omega(t) - \Omega(t_0)| + \phi_M(m(x, x_0)) \\ &\leq \phi_M(\epsilon) + \phi_M(m_{xx_0} + r) \\ &\leq \phi_M(r + m_{xx_0} - \phi_M(r + m_{xx_0})) + \phi_M(m_{xx_0} + r) \\ &< r + m_{xx_0} - \phi_M(r + m_{xx_0}) + \phi_M(m_{xx_0} + r) = r + m_{xx_0}. \end{aligned}$$

Hence $l \in \overline{B_m(x_0, r)}$ and thus for each fixed $t \in (t_0 - \delta, t_0 + \delta)$, we obtain $T(x,t) \subset \overline{B_m(x_0, r)}$ therefore $T : \overline{B_m(x_0, r)} \to CB_m\left(\overline{B_m(x_0, r)}\right)$ satisfies all the assumption of Theorem (3.1) and $T(., t)$ has a fixed point $\overline{B_m(x_0, r)} = B_m(x_0, r) \subset B$. But by assumption of (*i*) this fixed point belongs to A. So $(t_0 - \delta, t_0 + \delta) \subseteq D$, thus D is open in $[a, b]$. Next we prove that D is closed. Let a sequence $\{t_n\} \in D$ with t_n converges to $t_0 \in [a, b]$ as n approaches to ∞. We will prove that t_0 is in D.

Using the definition of D, there exist $\{t_n\}$ in A such that $x_n \in T(x_n, t_n)$ for all n. Using Assumption (*iii*)–(*v*), and the condition (*iii*) of Proposition 1.13, and an outcome of the Proposition 1.18, we have

$$\begin{aligned} m(x_n, x_m) &\leq H_m(T(x_n, t_n), T(x_m, t_m)) \\ &\leq H_m(T(x_n, t_n), T(x_n, t_m)) + H_m(T(x_n, t_m), T(x_m, t_m)) \\ &\leq \phi_M |\Omega(t_n) - \Omega(t_m)| + \alpha_*(x_n, x_m) H_m(T(x_n, t_m), T(x_m, t_m)) \\ &\leq \phi_M |\Omega(t_n) - \Omega(t_m)| + \phi_M(m(x_n, x_m)) \\ &\Rightarrow \\ m(x_n, x_m) - \phi_M(m(x_n, x_m)) &\leq \phi_M |\Omega(t_n) - \Omega(t_m)| \\ &\Rightarrow \\ \Re(m(x_n, x_m)) &\leq \phi_M |\Omega(t_n) - \Omega(t_m)| \\ \Re(m(x_n, x_m)) &< |\Omega(t_n) - \Omega(t_m)| \\ m(x_n, x_m) &< \frac{1}{\Re} |\Omega(t_n) - \Omega(t_m)|. \end{aligned}$$

So, continuity of $\frac{1}{\Re}$, $\Re$ and convergence of $\{t_n\}$, taking the limit as $m, n \to \infty$ in the last inequality, we obtain that

$$\lim_{m,n \to \infty} m(x_n, x_m) = 0.$$

Sine $m_{x_n x_m} \leq m(x_n, x_m)$, therefore
$$\lim_{m,n\to\infty} m_{x_n x_m} = 0.$$
Thus, we have $\lim_{n\to\infty} m(x_n, x_n) = 0 = \lim_{m\to\infty} m(x_m, x_m)$. Also,
$$\lim_{m,n\to\infty} (m(x_n, x_m) - m_{x_n x_m}) = 0, \quad \lim_{m,n\to\infty} (M_{x_n x_m} - m_{x_n x_m}).$$
Hence $\{x_n\}$ is an M-Cauchy sequence. Using Definition 1.4, there exist x^* in X such that
$$\lim_{n\to\infty} (m(x_n, x^*) - m_{x_n x^*}) = 0 \text{ and } \lim_{n\to\infty} (M_{x_n x^*} - m_{x_n x^*}) = 0.$$
As $\lim_{n\to\infty} m(x_n, x_n) = 0$, therefore
$$\lim_{n\to\infty} m(x_n, x^*) = 0 \text{ and } \lim_{n\to\infty} M_{x_n x^*} = 0.$$
Thus, we have $m(x, x^*) = 0$. We now show that $x^* \in T\left(x^*, t^*\right)$. Note that
$$\begin{aligned} m\left(x_n, T\left(x^*, t^*\right)\right) &\leq H_m\left(T(x_n, t_n), T\left(x^*, t^*\right)\right) \\ &\leq H_m\left(T(x_n, t_n), T\left(x_n, t^*\right)\right) + H_m\left(T\left(x_n, t^*\right), T\left(x^*, t^*\right)\right) \\ &\leq \phi_M \left|\Omega(t_n) - \Omega\left(t^*\right)\right| + \alpha_*\left(x_n, t^*\right) H_m\left(T\left(x_n, t^*\right), T\left(x^*, t^*\right)\right) \\ &\leq \phi_M \left|\Omega(t_n) - \Omega\left(t^*\right)\right| + \phi_M\left(m\left(x_n, t^*\right)\right). \end{aligned}$$
Applying the limit $n \to \infty$ in the above inequality, we have
$$\lim_{n\to\infty} m\left(x_n, T\left(x^*, t^*\right)\right) = 0.$$
Hence
$$\lim_{n\to\infty} m\left(x_n, T\left(x^*, t^*\right)\right) = 0. \tag{2.10}$$
Since $m(x^*, x^*) = 0$, we obtain
$$\sup_{y\in T\left(x^*, t^*\right)} m_{x^* y} = \sup_{y\in T\left(x^*, t^*\right)} \min\{m(x^*, x^*), m(y, y)\} = 0. \tag{2.11}$$
From above two inequalities, we get
$$m\left(x^*, T\left(x^*, t^*\right)\right) = \sup_{y\in T\left(x^*, t^*\right)} m_{x^* y}.$$
Thus using Lemma 1.12 we get $x^* \in T\left(x^*, t^*\right)$. Hence $x^* \in A$. Thus $x^* \in D$ and D is closed in $[a, b]$, $D = [a, b]$ and D is open and close in $[a, b]$. Thus $T(., t)$ has a fixed point in A for all $t \in [a, b]$. For uniqueness, $t \in [a, b]$ is arbitrary fixed point, then there exist $x \in A$ such that $x \in T(x, t)$. Assume that y is an other point of $T(x, t)$, then by applying condition 4, we obtain
$$\begin{aligned} m(x, y) &= H_m(T(x, t), T(y, t)) \\ &\leq \alpha_M(x, y) H_m(T(x, t), T(y, t)) \leq \phi_M(m(x, y)). \end{aligned}$$
For$\phi_M(t) = \frac{1}{2}t$, where $t > 0$, the uniqueness follows. □

4. Application to integral equation

In this section we will apply the previous theoretical results to show the existence of solution for some integral equation. For related results (see [13, 20]). We see for non-negative solution of (3.1) in $X = C([0,\delta],\mathbb{R})$. Let $X = C([0,\delta],\mathbb{R})$ be a set of continuous real valued functions defined on $[0,\delta]$ which is endowed with a complete M-metric given by

$$m(x,y) = \sup_{t\in[0,\delta]}\left(\left|\frac{x(t)+x(t)}{2}\right|\right) \text{ for all } x,y\in X.$$

Consider an integral equation

$$v_1(t) = \rho(t) + \int_0^\delta h(t,s)\, J(s, v_1(s))\, ds \text{ for all } 0\le t\le\delta. \tag{3.1}$$

Define $g : X \to X$ by

$$g(x)(t) = \rho(t) + \int_0^\delta h(t,s)\, J(s,x(s))\, ds$$

where

(i) for $\delta > 0$, (a) $J : [0,\delta]\times\mathbb{R}\to\mathbb{R}$, ($b$) $h : [0,\delta]\times[0,\delta]\to[0,\infty)$, ($c$) $\rho : [0,\delta]\to\mathbb{R}$ are all continuous functions

(ii) Assume that $\sigma : X\times X\to\mathbb{R}$ is a function with the following properties,

(iii) $\sigma(x,y)\ge 0$ implies that $\sigma(T(x),T(y))\ge 0$,

(iv) there exist $x_0\in X$ such that $\sigma(x_0,T(x_0))\ge 0$,

(v) if $\{x_n\}\in X$ is a sequence such that $\sigma(x_n,x_{n+1})\ge 0$ for all $n\in\mathbb{N}$ and $x_n\to x$ as $n\to\infty$, then $\sigma(x,T(x))\ge 0$

(vi)

$$\sup_{t\in[0,\delta]}\int_0^\delta h(t,s)\, ds \le 1$$

where $t\in[0,\delta]$, $s\in\mathbb{R}$,

(vii) there exist $\phi_M\in\Psi$, $\sigma(y,T(y))\ge 1$ and $\sigma(x,T(x))\ge 1$ such that for each $t\in[0,\delta]$, we have

$$|J(s,x(t)) + J(s,y(t))| \le \phi_M(|x+y|). \tag{3.3}$$

Theorem 4.1. *Under the assumptions* (i) − (vii) *the integral Eq* (3.1) *has a solution in* $\{X = C([0,\delta],\mathbb{R})$ *for all* $t\in[0,\delta]\}$.

Proof. Using the condition (vii), we obtain that

$$\begin{aligned} m(g(x),g(y)) &= \left|\frac{g(x)(t)+g(y)(t)}{2}\right| = \left|\int_0^\delta h(t,s)\left[\frac{J(s,x(s))+J(s,y(s))}{2}\right]ds\right| \\ &\le \int_0^\delta h(t,s)\left|\frac{J(s,x(s))+J(s,y(s))}{2}\right|ds \\ &\le \int_0^\delta h(t,s)\left[\phi_M\left|\frac{x(s)+y(s)}{2}\right|\right]ds \end{aligned}$$

$$\leq \left(\sup_{t\in[0,\delta]} \int_0^\delta h(t,s)\,ds\right)\left(\phi_M \left|\frac{x(s)+y(s)}{2}\right|\right)$$
$$\leq \phi_M\left(\left|\frac{x(s)+y(s)}{2}\right|\right)$$

$$m(g(x), g(y)) \leq \phi(m(x,y))$$

Define $\alpha_* : X \times X \to [0, +\infty)$ by

$$\alpha_*(x,y) = \begin{cases} 1 & \text{if } \sigma(x,y) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

which implies that

$$m(g(x), g(y)) \leq \phi_M(m(x,y)).$$

Hence all the assumption of the Corollary 2.6 are satisfied, the mapping g has a fixed point in $X = C([0,\delta], \mathbb{R})$ which is the solution of integral Eq (3.1). □

5. Conclusions

In this study we develop some set-valued fixed point results based on (α_*, ϕ_M)-contraction mappings in the context of M-metric space and ordered M-metric space. Also, we give examples and applications to the existence of solution of functional equations and homotopy theory.

Conflict of interest

The authors declare that they have no competing interests.

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