Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 8822–8846.
DOI: 10.3934/math.2022492
Received: 26 November 2021
Revised: 23 January 2022
Accepted: 28 January 2022
Published: 04 March 2022

Research article

Existence theory and numerical solution of leptospirosis disease model via exponential decay law

Amir Khan[1,5], Abdur Raouf[1], Rahat Zarin[2], Abdullahi Yusuf[3,4] and Usa Wannasingha Humphries[5,*]

[1] Department of Mathematics and Statistics, University of Swat, Khyber Pakhtunkhawa, Pakistan

[2] Department of Basic Sciences, University of Engineering and Technology Peshawar, Khyber Pakhtunkhwa, Pakistan

[3] Department of Computer Engineering, Biruni University, Istanbul, Turkey

[4] Department of Mathematics, Near East University TRNC, Mersin 10, Nicosia 99138, Turkey

[5] Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand

* **Correspondence:** Email: usa.wan@kmutt.ac.th.

Abstract: We investigated the leptospirosis epidemic model by using Caputo and Fabrizio fractional derivatives. Picard's successive iterative method and Sumudu transform are taken into consideration for developing the iterative solutions for the leptospirosis disease. Employing nonlinear functional analysis, the stability and uniqueness of the proposed model are established. Sensitivity analysis is taken into account to highlight the most sensitive parameters corresponding to the basic reproductive number. Various solutions to the proposed system have been interpolated by graphs with the application of Matlab software.

Keywords: mathematical model; exponential decay law; Sumudu transform; existence and uniqueness
Mathematics Subject Classification: 92B05, 37G05, 37G15, 39A28

1. Introduction

Leptospirosis is a disease caused by infection with leptospira bacteria. These bacteria can be found worldwide in soil and water. There are many strains of leptospira bacteria that can cause disease. Leptospirosis is a zoonotic disease, which means it can be spread from animals to people. Infection in people can cause flu-like symptoms and can cause liver or kidney disease. In the United States, most cases of human leptospirosis result from recreational activities involving water. Infection resulting from contact with an infected pet is much less common, but it is possible. Leptospirosis is more common in areas with warm climates and high annual rainfall but it can occur anywhere [1].

Signs of leptospirosis may include fever, shivering, muscle tenderness, reluctance to move, increased thirst, changes in the frequency or amount of urination, dehydration, vomiting, diarrhea, loss of appetite, lethargy, jaundice (yellowing of the skin and mucous membranes), or painful inflammation within the eyes. The disease can cause kidney failure with or without liver failure [2]. Leptospirosis is generally treated with antibiotics and supportive care. When treated early and aggressively, the chances for recovery are good but there is still a risk of permanent residual kidney or liver damage [3, 4].

There are so many models proposed for the dynamics of humans [5–11] as well as vector populations [12, 13]. To analyze the nature of leptospirosis, Pongsumpon [14] develop a mathematical model where they show the rate of change of both humans and rat populations. The population of humans is further split into two major groups: The juveniles and the adults. A dynamical model for leptospirosis transmission was considered by Triampo et al. [15]. In their work, they considered a number of leptospiroses infections in Thailand and worked on their numerical simulations. In [16], Zaman took the real data and studied the behavior of the theory optimal control.

The generalisation of the classical calculus is called fractional calculus. Mathematical models for the epidemic disease with integer order derivative was recently studies by the researchers [17–20]. But mathematical modelings with variable order derivatives give an advance knowledge of a system. Fractional order models are very capable to simulate the memory effects and crossover natures and gives a big degree of performance. Fractional mathematical modelings provides more insights about a epidemic under given circumstances [21–24]. Various non-classical type operators with non-singular and singular type kernels have been exemplified in the literature [25,26]. The successful applications of such non-classical operators can be seen in early literature and papers therein [27,28]. Early, only some leptospirosis models defined by variable order operators are given. The researchers in [29] proposed a mathematical structure for cancer and hepatitis co-dynamics in non-classical derivative and studied its outcomes.

Nowadays, the mathematical models involving fractional order derivative were given noticeable importance because they are more accurate and realistic as compared to the classical order models [30–35]. “Motivated by the advancement of fractional calculus, many researchers have focused to investigate the solutions of nonlinear differential equations with the fractional operator by developing quite a few analytical or numerical techniques to find approximate solutions [36–40]. These differential equations involves several fractional differential operators like Riemann-Liouville, Caputo, Hilfer etc.” [41–43].

However, “these operators possess a power law kernel and have limitations in modeling physical problems. To overcome this difficulty, recently an alternate fractional differential operator having a kernel with exponential decay has been introduced by Caputo and Fabrizio [44]. This novel approach of fractional derivative is known as the Caputo-Fabrizio (C-F) operator which has attracted many research scholars due to the fact that it has a non-singular kernel. Also the C-F operator is most appropriate for modeling some class of real-world problem which follows the exponential decay law” [44]. With the passage of time, developing a mathematical model using the C-F fractional order derivative became a remarkable field of research. In recent times, several mathematicians were busy in development and simulation of CFFDE.

We develop a vector-host epidemic model in which the total population of the host (human) at time (t) is divided by susceptible S_h, infected I_h and recovered R_h humans. The population of vectors at time (t) is classified by susceptible S_v and infected I_v classes. The total amount of host is exemplified by N_h and the total amount of vectors are denoted by N_v. Here, $N_h = S_h + I_h + R_h$ and $N_v = S_v + I_v$. The mathematical representation of the model is given by

$$\begin{cases} \dfrac{dS_h(t)}{dt} = b_1 - \mu_h S_h - \beta_1 S_h I_h - \beta_2 S_h I_v + \lambda_h R_h, \\ \dfrac{dI_h(t)}{dt} = \beta_2 S_h I_v + \beta_1 S_h I_h - \mu_h I_h - \delta_h I_h - \gamma_h I_h, \\ \dfrac{dR_h(t)}{dt} = \gamma_h I_h - \mu_h R_h - \lambda_h R_h, \\ \dfrac{dS_v(t)}{dt} = b_2 - \gamma_v S_v - \beta_3 S_v I_h, \\ \dfrac{dI_v(t)}{dt} = \beta_3 S_v I_h - \gamma_v I_v - \delta_v I_v. \end{cases} \tag{1.1}$$

Here, b_1 is the rate of recruitment in humans size, susceptible humans can get infection by two roots, one is direct transmission or via infectious individuals where β_1 and β_2 are the mediate transmission rates, μ_h is the natural mortality rate for hosts where γ_h is the rate of recovery of humans from the infection, δ_h disease death rate of infected humans due to disease, λ_h is the constant rate of humans immune, b_2 recruitment rate for human density, δ_v is the mortality rate of vectors due to disease, β_3 represent disease carrying due to susceptible vectors per host per unit time and γ_v is the natural mortality rate of vector.

Motivated by the above discussion, we modify the above model (1.1) by using Caputo-Fabrizio fractional operator, and the modified model is given as follow:

$$\begin{cases} {}^{CF}_{0}D^{\alpha}_{t} S_h(t) = b_1 - \mu_h S_h - \beta_1 S_h I_h - \beta_2 S_h I_v + \lambda_h R_h, \\ {}^{CF}_{0}D^{\alpha}_{t} I_h(t) = \beta_2 S_h I_v + \beta_1 S_h I_h - \mu_h I_h - \delta_h I_h - \gamma_h I_h, \\ {}^{CF}_{0}D^{\alpha}_{t} R_h(t) = \gamma_h I_h - \mu_h R_h - \lambda_h R_h, \\ {}^{CF}_{0}D^{\alpha}_{t} S_v(t) = b_2 - \gamma_v S_v - \beta_3 S_v I_h, \\ {}^{CF}_{0}D^{\alpha}_{t} I_v(t) = \beta_3 S_v I_h - \gamma_v I_v - \delta_v I_v. \end{cases} \tag{1.2}$$

To exemplify the existence and uniqueness of equilibrium solutions of the given system, the number of novel results are utilized with the help of Sumudu transform. By equilibrium solutions existence, we are concluding the well-posedness of the given system.

Section 2 is devoted to some basic definitions regarding fractional operators. In section 3, the uniqueness of the solution is derived. Stability analysis of the given algorithm by the application of fixed point theorem is carried out in Section 4. Section 5 deals with the derivation of special solutions and numerical results for the model. The sensitivity of different parameters corresponding to basic reproduction is discussed in Section 6. Section 7 deals with the conclusion and future work.

2. Preliminaries

Here we recall some definitions alongwith Caputo-Fabrizio operator of non-classical order which is defined with exponentially decay type kernel [44, 45].

Definition 2.1. *Assume $\phi \in G^1(a,b)$, $b > a$, $\alpha \in (0,1)$, then the definition of Caputo-Fabrizio derivative of variable order is given by*

$$^{C}D_t^{\alpha}(\phi(t)) = \frac{N(\alpha)}{1-\alpha}\int_a^t \phi'(x)\exp[-\alpha\frac{t-x}{1-\alpha}]dx,$$

where $N(\alpha)$ is the normalization function with $N(0) = N(1) = 1$. Also, when the function does not exist in $G^1(a,b)$, then the given operator can be defined by

$$^{C}D_t^{\alpha}(\phi(t)) = \frac{\alpha N(\alpha)}{1-\alpha}\int_a^t (\phi(t)-\phi(x))\exp[-\alpha\frac{t-x}{1-\alpha}]dx.$$

Remark 2.1. *If we fix $\epsilon = \frac{1-\alpha}{\alpha} \in [0,\infty)$, $\alpha = \frac{1}{1+\alpha} \in [0,1]$, then the definition of given Caputo-Fabrizio variable order derivative is redefined by*

$$^{C}D_t^{\alpha}(\phi(t)) = \frac{N(\epsilon)}{\epsilon}\int_a^t \phi'(x)\exp[-\frac{t-x}{\epsilon}]dx, \quad N(0) = N(\infty) = 1.$$

In addition,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\exp[-\frac{t-x}{\epsilon}] = \beta(x-t).$$

The related fractional Caputo-Fabrizio integral of the given derivative is simulated by Nieto and Losada [46] which is as follow:

Definition 2.2. *For $\alpha \in (0,1)$, the integral of fractional order α of the function f is expressed by*

$$^{CF}I_t^{\alpha}(\phi(t)) = \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} + \frac{2\alpha}{(2-\alpha)N(r)}\int_0^t \phi(s)ds, \quad t \geq 0. \tag{2.1}$$

Remark 2.2. *It is to be noted that, according to the above definition, the function of order $0 < r \leq 1$, which is fractional integral of Caputo type is just average among mapping ψ and the integral of it of order 1. Therefore, that can be written as*

$$\frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} + \frac{2\alpha}{(2-\alpha)N(\alpha)} = 1. \tag{2.2}$$

The above expression will assumes the form as

$$N(\alpha) = \frac{2}{(2-\alpha)}, \quad 0 \leq \alpha \leq 1. \tag{2.3}$$

Nieto and Losada [46] rewrote that this Caputo-Fabrizio derivative of order $0 < \alpha < 1$ can be reformulated as

$$^{C}D_t^{\alpha}(\phi(t)) = \frac{1}{1-\alpha}\int_a^t \phi'(x)\exp[-\alpha\frac{t-x}{1-\alpha}]dx. \tag{2.4}$$

Definition 2.3. *The Sumudu transform of a mapping $\phi(t)$ defined on a set specified by*

$$\mathbf{B} = \{\phi(t) : \textit{there exist } \Lambda, \omega_1, \omega_2 > 0,\ |\phi(t)| < \Lambda \exp\left(\frac{|t|}{\omega_i}\right),\ \textit{if } t \in (-1)^j \times [0, \infty)\}, \tag{2.5}$$

is given by

$$H(u) = \mathcal{S}[\phi(t)] = \int_0^\infty \exp(-t)\phi(ut)dt,\ u \in (-\tau_1, \tau_2). \tag{2.6}$$

Definition 2.4. *For the Caputo-type fractional operator of any function $\phi(t)$, the Sumudu transform is defined by*

$$\mathcal{S}[{}^C D^\alpha \phi(t)] = u^{-\alpha}\Big[H(u) - \sum_{i=0}^{n} u^{\alpha-i}[{}^C D^{\alpha-i}(\phi(t))]_{t=0}\Big],\ n-1 < \alpha \le n. \tag{2.7}$$

Similarly, the Sumudu transform for any function $\phi(t)$ in the sense of Caputo-Fabrizio operator is given as follow:

Definition 2.5. *For any function $\phi(t)$ where the C-F derivative is defined, the Sumudu transform in a sense of C-F is expressed by*

$$\mathcal{S}({}_0^{CF}D_t^\alpha)(\phi(t)) = N(\alpha)\frac{\mathcal{S}(\phi(t)) - \phi(0)}{1 - \alpha + \alpha u}. \tag{2.8}$$

3. Special solution derivation

In this section, we apply Sumudu transform to both sides of the model (1.2) using Picard's iterative mathod [22]:

$$\begin{aligned}
N(\alpha)\frac{\mathcal{S}(S_h(t)) - S_h(0)}{1-\alpha+\alpha u} &= \mathcal{S}[b_1 - \mu_h S_h - \beta_1 S_h I_h - \beta_2 S_h I_v + \lambda_h R_h],\\
N(\alpha)\frac{\mathcal{S}(I_h(t)) - I_h(0)}{1-\alpha+\alpha u} &= \mathcal{S}[\beta_2 S_h I_v + \beta_1 S_h I_h - \mu_h I_h - \delta_h I_h - \gamma_h I_h],\\
N(\alpha)\frac{\mathcal{S}(R_h(t)) - R_h(0)}{1-\alpha+\alpha u} &= \mathcal{S}[\gamma_h I_h - \mu_h R_h - \lambda_h R_h],\\
N(\alpha)\frac{\mathcal{S}(S_v(t)) - S_v(0)}{1-\alpha+\alpha u} &= \mathcal{S}[b_2 - \gamma_v S_v - \beta_3 S_v I_h],\\
N(\alpha)\frac{\mathcal{S}(I_v(t)) - I_v(0)}{1-\alpha+\alpha u} &= \mathcal{S}[\beta_3 S_v I_h - \gamma_v I_v - \delta_v I_v].
\end{aligned} \tag{3.1}$$

Rearranging, we get

$$\begin{aligned}
\mathcal{S}(S_h(t)) &= S_h(0) + \frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_1 - \mu_h S_h - \beta_1 S_h I_h - \beta_2 S_h I_v + \lambda_h R_h],\\
\mathcal{S}(I_h(t)) &= I_h(0) + \frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_2 S_h I_v + \beta_1 S_h I_h - \mu_h I_h - \delta_h I_h - \gamma_h I_h],\\
\mathcal{S}(R_h(t)) &= R_h(0) + \frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\gamma_h I_h - \mu_h R_h - \lambda_h R_h],\\
\mathcal{S}(S_v(t)) &= S_v(0) + \frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_2 - \gamma_v S_v - \beta_3 S_v I_h],\\
\mathcal{S}(I_v(t)) &= I_v(0) + \frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_3 S_v I_h - \gamma_v I_v - \delta_v I_v].
\end{aligned} \tag{3.2}$$

By operating every side of Eq (3.2) by the inverse of Sumudu transform, we obtain

$$\begin{aligned}
S_h(t) &= S_h(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_1 - \mu_h S_h - \beta_1 S_h I_h - \beta_2 S_h I_v + \lambda_h R_h],\\
I_h(t) &= I_h(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_2 S_h I_v + \beta_1 S_h I_h - \mu_h I_h - \delta_h I_h - \gamma_h I_h],\\
R_h(t) &= R_h(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\gamma_h I_h - \mu_h R_h - \lambda_h R_h],\\
S_v(t) &= S_v(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_2 - \gamma_v S_v - \beta_3 S_v I_h],\\
I_v(t) &= I_v(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_3 S_v I_h - \gamma_v I_v - \delta_v I_v].
\end{aligned} \tag{3.3}$$

The recursive formula for given model can be exemplified by

$$\begin{aligned}
S_{h(n+1)}(t) &= S_{h(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_1 - \mu_h S_{h(n)} - \beta_1 S_{h(n)} I_{h(n)} - \beta_2 S_{h(n)} I_{v(n)} + \lambda_h R_{h(n)}],\\
I_{h(n+1)}(t) &= I_{h(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_2 S_{h(n)} I_{v(n)} + \beta_1 S_{h(n)} I_{h(n)} - \mu_h I_{h(n)} - \delta_h I_{h(n)} - \gamma_h I_{h(n)}],\\
R_{h(n+1)}(t) &= R_{h(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\gamma_h I_{h(n)} - \mu_h R_{h(n)} - \lambda_h R_{h(n)}],\\
S_{v(n+1)}(t) &= S_{v(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_2 - \gamma_v S_{v(n)} - \beta_3 S_{v(n)} I_{h(n)}],\\
I_{v(n+1)}(t) &= I_{v(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_3 S_{v(n)} I_{h(n)} - \gamma_v I_{v(n)} - \delta_v I_{v(n)}].
\end{aligned} \tag{3.4}$$

The solution of (3.4) is expressed as

$$\begin{aligned}
S_h(t) &= \lim_{n\to\infty} S_{h(n)}(t),\\
I_h(t) &= \lim_{n\to\infty} I_{h(n)}(t),\\
R_h(t) &= \lim_{n\to\infty} R_{h(n)}(t),\\
S_v(t) &= \lim_{n\to\infty} S_{h(n)}(t),\\
I_v(t) &= \lim_{n\to\infty} I_{h(n)}(t).
\end{aligned}$$

For simulating the given model (3.4), we use following values $S_h = 1.2$, $I_h = 2.8$, $R_h = 1.3$, $S_v = 2.8$, $I_v = 1.15$, $b_1 = 0.121$, $\mu_h = 0.0121$, $\beta_1 = 0.14$, $\beta_2 = 0.02$, $\beta_3 = 0.10$, $\lambda_h = 1$, $\delta_h = 0.8$, $\gamma_h = 0.25$, $b_2 = 0.002$, $\gamma_v = 0.23$, $\delta_v = 0.001$. Then

$$S_{h1}(t) = 1.2 - \frac{3.04}{N(\alpha)}\Big((1-\alpha) + \alpha t\Big),$$

$$I_{h1}(t) = 2.8 + \frac{5.65}{N(\alpha)}\Big((1-\alpha) + \alpha t\Big),$$

$$R_{h1}(t) = 1.3 - \frac{1.23}{N(\alpha)}\Big((1-\alpha) + \alpha t\Big),$$

$$S_{v1}(t) = 2.8 - \frac{5.65}{N(\alpha)}\Big((1-\alpha) + \alpha t\Big),$$

$$I_{v1}(t) = 1.15 + \frac{0.043}{N(\alpha)}\Big((1-\alpha) + \alpha t\Big).$$

$$\begin{cases} S_{h2}(t) = 1.2 - \frac{3.04}{N(\alpha)}(1-\alpha) - (1-\alpha)\Big[\frac{3.04}{N(\alpha))^2} - \frac{18.64}{N(\alpha))^3}(1-\alpha) + \frac{2.25}{N(\alpha))^3}(1-\alpha)^2\Big]t \\ -\Big[\frac{3.04\alpha}{2N(\alpha))^2} - \frac{18.64}{N(\alpha))^3}\Big]t^2, \\ I_{h2}(t) = 2.8 + \frac{5.65}{N(\alpha)}((1-\alpha) + (1-\alpha)\Big[\frac{5.97}{N(\alpha))} - \frac{6.83}{N(\alpha))^2}(1-\alpha) + \frac{7.25}{N(\alpha))^3}(1-2\alpha)\Big]t \\ +\alpha\Big[\frac{5.97}{2N(\alpha))} - \frac{6.83}{N(\alpha))^2}(1-\alpha) + \frac{2.125}{N(\alpha))^3}(1-\alpha)(2-\alpha)\Big]t^2, \\ R_{h2}(t) = 1.3 - \frac{1.23}{N(\alpha)}((1-\alpha) - (1-\alpha)\Big[\frac{1.23}{N(\alpha))} - \frac{1.82}{N(\alpha)}(1-\alpha) + \frac{0.5983}{N(\alpha)}(1-\alpha)^2\Big]t \\ -\alpha\Big[\frac{1.32}{N(\alpha))} - \frac{1.412}{N(\alpha))}(1-\alpha) + \frac{0.8974}{N(\alpha))^2}(1-\alpha)^2\Big]t^2, \\ S_{v2}(t) = 2.8 + \frac{5.65}{N(\alpha)}(1-\alpha) + (1-\alpha)\Big[\frac{5.65}{N(\alpha)} - \frac{1.22}{N(\alpha))^2}(1-\alpha) - \frac{0.5984}{N(\alpha))^2}(1+\alpha^2-2\alpha)^2\Big]t \\ +\Big[\frac{1.521\alpha}{N(\alpha))} - \frac{1.51}{N(\alpha))^2} - \frac{0.2992}{N(\alpha))^2}(1-\alpha)\Big]t^2, \\ I_{v2}(t) = 1.15 + \frac{0.043}{N(\alpha)}((1-\alpha) + (1-\alpha)\Big[\frac{0.043}{N(\alpha))} - \frac{0.03763}{N(\alpha))^2}(1-\alpha)\Big]t \\ +\alpha\Big[\frac{0.0215}{N(\alpha))} - \frac{0.0372}{N(\alpha))^2}(1-\alpha) - \frac{0.00086}{N(\alpha))^3}(1-\alpha)\Big]t^2. \end{cases} \tag{3.5}$$

4. Stability analysis of given algorithm by the application of fixed-point theorem

Suppose a Banach space $(X, ||\cdot||)$ and a self-map K^* of X. Now consider a random recursive algorithm of the formate, $x_{n+1} = g(K^*, x_n)$. Assume $F(K^*)$ be the set of fixed-point of K^* which is non-empty and a point $k^* \in F(K^*)$ where x_n converges. Adopt $\{y_n^* \subseteq X\}$ and derive $e_n = ||y_{n+1}^* - g(K^*, y_n^*)||$. The numerical algorithm $x_{n+1} = g(K^*, x_n)$ is called K^*-stable if $\lim_{n\to\infty} e^n = 0$ implies that $lim_{n\to\infty} y_n^* = k^*$. Analogously, we consider that the sequence $\{y_n^*\}$ exists an upper bound, else the sequence will be not convergent. If all such constraints are satisfied for $x_{n+1} = K^* x_n$ which is known as Picard's iteration as given in [47], consequently the iteration will be K^*-stable. We then state the following theorem.

Theorem 4.1. *Consider a Banach space $(X, \|\cdot\|)$ and self-map K^* of X agreeing*

$$\|K_x^* - K_y^*\| \leq c\|x - y\| + C\|x - K_x^*\|,$$

$\forall x, y \in X$, here, $0 \leq C, 0 \leq c < 1$. Assume that K^* is Picard K^*-stable. Now consider the recursive algorithm from (3.4) related to (1.2).

$$\begin{aligned}
S_{h(1+n)}(t) &= S_{h(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_1 - \mu_h S_{h(n)} - \beta_1 S_{h(n)} I_{h(n)} - \beta_2 S_{h(n)} I_{v(n)} + \lambda_h R_{h(n)}],\\
I_{h(1+n)}(t) &= I_{h(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_2 S_{h(n)} I_{v(n)} + \beta_1 S_{h(n)} I_{h(n)} - \mu_h I_{h(n)} - \delta_h I_{h(n)} - \gamma_h I_{h(n)}],\\
R_{h(1+n)}(t) &= R_{h(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\gamma_h I_{h(n)} - \mu_h R_{h(n)} - \lambda_h R_{h(n)}],\\
S_{v(1+n)}(t) &= S_{v(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_2 - \gamma_v S_{v(n)} - \beta_3 S_{v(n)} I_{h(n)}],\\
I_{v(1+n)}(t) &= I_{v(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_3 S_{v(n)} I_{h(n)} - \gamma_v I_{v(n)} - \delta_v I_{v(n)}],
\end{aligned}$$

where $\frac{(1-\alpha+\alpha u)}{N(\alpha)}$ is the non-classical Lagrange multiplier.

Theorem 4.2. *Consider the self-map H given by*

$$\begin{aligned}
H(S_{h(n)}(t)) &= S_{h(1+n)}(t) = S_{h(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\Lambda - \beta_1 P_n S_n - \mu P_n],\\
H(I_{h(n)}(t)) &= I_{h(1+n)}(t) = I_{h(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_1 P_n S_n - \alpha_1 Q_n - \mu Q_n],\\
H(R_{h(n)}(t)) &= R_{h(1+n)}(t) = R_{h(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\alpha_1 Q_n + \alpha_2 S_n R_n - (\mu+\gamma) S_n],\\
H(S_{v(n)}(t)) &= S_{v(1+n)}(t) = S_{v(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[-\alpha_2 S_n R_n - \mu R_n + \gamma(1-\delta) S_n],\\
H(I_{v(n)}(t)) &= I_{v(1+n)}(t) = I_{h(n)}(0) + \mathcal{S}^{-1}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\delta\gamma S_n - \mu U_n],
\end{aligned}$$

is H-stable in $L^1(a, b)$ if

$$\begin{cases}
\{1 - \beta_1(P+J)f(\gamma) - \beta_2(J_1+J)f_1(\gamma) - (\mu_h+\lambda_h)g(\gamma)\} < 1,\\
\{1 - (\mu_h+\delta_h+\gamma_h)g_1(\gamma) - \beta_1(P+J)f_2(\gamma) - \beta_2(J_1+J)f_3(\gamma)\} < 1,\\
\{1 - (\gamma_h-\mu_h-\lambda_h)g_3(\gamma)\} < 1,\\
\{1 - \gamma_v g_4(\gamma) - \beta_3(Q+P_1)f_4(\gamma)\} < 1,\\
\{1 - \beta_3(Q+P_1)f_5(\gamma) - (\gamma_v+\delta_v)g_5(\gamma)\} < 1.
\end{cases} \tag{4.1}$$

Proof. Initially we start to proving that H exists a fixed-point. For this achievement, we simulate the given below terms for all $(m, n) \in M \times M$.

$$\begin{aligned}
H(S_{h(n)}(t)) - H(S_{h(m)}(t)) =& S_{h(n)}(t) - S_{h(m)}(t) \\
&+ \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_1 - \mu_h S_{hn} - \beta_1 S_{hn} I_{hn} - \beta_2 S_{hn} I_{vn} + \lambda_h R_{hn}]\Big] \\
&- \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_1 - \mu_h S_{hm} - \beta_1 S_{hm} I_{hm} - \beta_2 S_{hm} I_{vm} + \lambda_h R_{hm}]\Big], \\
H(I_{h(n)}(t)) - H(I_{h(m)}(t)) =& I_{h(n)}(t) - I_{h(m)}(t) \\
&+ \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_2 S_{hn} I_{vn} + \beta_1 S_{hn} I_{hn} - \mu_h I_{hn} - \delta_h I_{hn} - \gamma_h I_{hn}]\Big] \\
&- \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_2 S_{hm} I_{vm} + \beta_1 S_{hm} I_{hm} - \mu_h I_{hm} - \delta_h I_{hm} - \gamma_h I_{hm}]\Big], \\
H(R_{h(n)}(t)) - H(R_{h(m)}(t)) =& R_{h(n)}(t) - R_{h(m)}(t) \\
&+ \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\gamma_h I_{hn} - \mu_h R_{hn} - \lambda_h R_{hn}]\Big] \\
&- \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\gamma_h I_{hm} - \mu_h R_{hm} - \lambda_h R_{hm}]\Big], \\
H(S_{v(n)}(t)) - H(S_{v(m)}(t)) =& S_{v(n)}(t) - S_{v(m)}(t) \\
&+ \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_2 - \gamma_v S_{vn} - \beta_3 S_{vn} I_{hn}]\Big] \\
&- \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_2 - \gamma_v S_{vm} - \beta_3 S_{vm} I_{hm}]\Big], \\
H(I_{v(n)}(t)) - H(I_{v(m)}(t)) =& I_{v(n)}(t) - I_{v(m)}(t) \\
&+ \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_3 S_{vn} I_{hn} - \gamma_v I_{vn} - \delta_v I_{vn}]\Big] \\
&- \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[\beta_3 S_{vm} I_{hm} - \gamma_v I_{vm} - \delta_v I_{vm}]\Big].
\end{aligned} \tag{4.2}$$

Analysing the initial equation of (4.2) and take norm on both-sides,

$$\begin{aligned}
\|H(S_{hn}(t)) - H(S_{hm}(t))\| =& \|S_{hn}(t) - S_{hm}(t) \\
&+ \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_1 - \mu_h S_{hn} - \beta_1 S_{hn} I_{hn} - \beta_2 S_{hn} I_{vn} + \lambda_h R_{hn}]\Big] \\
&- \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_1 - \mu_h S_{hm} - \beta_1 S_{hm} I_{hm} - \beta_2 S_{hm} I_{vm} + \lambda_h R_{hm}]\Big]\|.
\end{aligned} \tag{4.3}$$

Applying the triangular inequality, we receive

$$\begin{aligned}
\|H(S_{hn}(t)) - H(S_{hm}(t))\| \leq& \|S_{hn}(t) - S_{hm}(t)\| \\
&+ \Big\|\mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_1 - \mu_h S_{hn} - \beta_1 S_{hn} I_{hn} - \beta_2 S_{hn} I_{vn} + \lambda_h R_{hn}\Big] \\
&- \mathcal{S}^{-1}\Big[\frac{(1-\alpha+\alpha u)}{N(\alpha)}\mathcal{S}[b_1 - \mu_h S_{hm} - \beta_1 S_{hm} I_{hm} - \beta_2 S_{hm} I_{vm} + \lambda_h R_{hm}\Big]\Big\|.
\end{aligned} \tag{4.4}$$

By further simplification, (4.4) becomes as

$$\begin{aligned}\|H(S_{h(n)}(t)) - H(S_{h(m)}(t))\| \le & \|S_{h(n)}(t) - S_{h(m)}(t)\| \\ & + \mathcal{S}^{-1}\Big[\mathcal{S}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\Big(\|-\beta_1 I_{hn}(S_{hn} - S_{hm})\| + \|-\beta_1 S_{hm}(I_{hn} - I_{hm})\| \\ & + (\|-\beta_2 I_{vn}(S_{hn} - S_{hm})\| + \|-\beta_2 S_{hm}(I_{vn} - I_{vm})\| + \|-\mu_h(S_{hn} - S_{hm})\| \\ & + \|\lambda_h(R_{hn} - R_{hm})\|\Big)\Big].\end{aligned} \tag{4.5}$$

Since the same role is forming by both solutions, so we adopt in that case

$$\begin{aligned}\|S_{h(n)}(t) - S_{h(m)}(t)\| &\cong \|I_{h(n)}(t) - I_{h(m)}(t)\|, \\ \|S_{h(n)}(t) - S_{h(m)}(t)\| &\cong \|R_{h(n)}(t) - R_{h(m)}(t)\|, \\ \|S_{h(n)}(t) - S_{h(m)}(t)\| &\cong \|S_{v(n)}(t) - S_{v(m)}(t)\|, \\ \|S_{h(n)}(t) - S_{h(m)}(t)\| &\cong \|I_{v(n)}(t) - I_{v(m)}(t)\|.\end{aligned}$$

By fixing it in Eq (4.5), we will get the following result:

$$\begin{aligned}\|H(S_{h(n)}(t)) - H(S_{h(m)}(t))\| \le & \|S_{h(n)}(t) - S_{h(m)}(t)\| \\ & + \mathcal{S}^{-1}\Big[\mathcal{S}\frac{(1-\alpha+\alpha u)}{N(\alpha)}\Big(\|-\beta_1 I_{hn}(S_{hn} - S_{hm})\| + \|-\beta_1 S_{hm}(S_{hn} - S_{hm})\| \\ & + (\|-\beta_2 I_{vn}(S_{hn} - S_{hm})\| + \|-\beta_2 S_{hm}(S_{vn} - S_{vm})\| + \|-\mu_h(S_{hn} - S_{hm})\| \\ & + \|\lambda_h(S_{hn} - S_{hm})\|\Big)\Big].\end{aligned} \tag{4.6}$$

Since $S_{hm}, S_{vn}, I_{hm}, I_{hn}, I_{vn}$ are convergent sequence, so all are bounded, then, there exist five different positive constants: J, Q, P_1 , P and J_1 for all t such that

$$\|S_{hm}\| < J,\ \|S_{vn}\| < Q,\ \|I_{hm}\| < P_1,\ \|I_{hn}\| < P,\ \|I_{vn}\| < J_1,\ (m,n) \in \mathbb{N} \times \mathbb{N}. \tag{4.7}$$

Now by Eqs (4.6) and (4.7), we get

$$\begin{aligned}\|H(S_{h(n)}(t)) - H(S_{h(m)}(t))\| \le & \Big[1 - \beta_1(P+J)f(\gamma) - \beta_2(J_1+J)f_1(\gamma) - (\mu_h + \lambda_h)g(\gamma)\Big] \\ & \|(S_{h(n)} - S_{h(m)})\|,\end{aligned} \tag{4.8}$$

where f and g are functions from $\mathcal{S}^{-1}\Big[\mathcal{S}(\frac{1-\alpha+\alpha u}{N(\alpha)})\Big]$.

Similarly, we receive

$$\begin{aligned}\|H(I_{h(n)}(t)) - H(I_{h(m)}(t))\| \le & \{1 - (\mu_h + \delta_h + \gamma_h)g_1(\gamma) - \beta_1(P+J)f_2(\gamma) - \beta_2(J_1+J)f_3(\gamma)\} \\ & \|(I_{h(n)}(t)) - (I_{h(m)}(t))\|, \\ \|H(R_{h(n)}(t)) - H(R_{h(m)}(t))\| \le & \{1 - (\gamma_h - \mu_h - \lambda_h)g_3(\gamma)\}\|(R_{h(n)}(t)) - (R_{h(m)}(t))\|, \\ \|H(S_{v(n)}(t)) - H(S_{v(m)}(t))\| \le & \{1 - \gamma_v g_4(\gamma) - \beta_3(Q+P_1)f_4(\gamma)\} \\ & \|(S_{v(n)}(t)) - (S_{v(m)}(t))\|, \\ \|H(I_{v(n)}(t)) - H(I_{v(m)}(t))\| \le & \{1 - \beta_3(Q+P_1)f_5(\gamma) - (\gamma_v + \delta_v)g_5(\gamma)\}\|(I_{v(n)}(t)) - (I_{v(m)}(t))\|,\end{aligned} \tag{4.9}$$

where

$$\{1-\beta_1(P+J)f(\gamma)-\beta_2(J_1+J)f_1(\gamma)-(\mu_h+\lambda_h)g(\gamma)\}<1,$$
$$\{1-(\mu_h+\delta_h+\gamma_h)g_1(\gamma)-\beta_1(P+J)f_2(\gamma)-\beta_2(J_1+J)f_3(\gamma)\}<1,$$
$$\{1-(\gamma_h-\mu_h-\lambda_h)g_3(\gamma)\}<1,$$
$$\{1-\gamma_v g_4(\gamma)-\beta_3(Q+P_1)f_4(\gamma)\}<1,$$
$$\{1-\beta_3(Q+P_1)f_5(\gamma)-(\gamma_v+\delta_v)g_5(\gamma)\}<1,$$

which gives that the non-linear H-self map exists a fixed point. Now we prove that H agrees with all the restrictions of Theorem 4.1. Let (4.8) and (4.9) hold, so using

$$c=(0,0,0,0,0),\quad C=\begin{cases}\{1-\beta_1(P+J)f(\gamma)-\beta_2(J_1+J)f_1(\gamma)-(\mu_h+\lambda_h)g(\gamma)\}<1,\\ \{1-(\mu_h+\delta_h+\gamma_h)g_1(\gamma)-\beta_1(P+J)f_2(\gamma)-\beta_2(J_1+J)f_3(\gamma)\}<1,\\ \{1-(\gamma_h-\mu_h-\lambda_h)g_3(\gamma)\}<1,\\ \{1-\gamma_v g_4(\gamma)-\beta_3(Q+P_1)f_4(\gamma)\}<1,\\ \{1-\beta_3(Q+P_1)f_5(\gamma)-(\gamma_v+\delta_v)g_5(\gamma)\}<1.\end{cases}$$

The above expression shows that all conditions of Theorem 4.1 exist for the nonlinear map-H. So every restrictions mentioned in Theorem 4.1 are agreed for the given non-linear map-H. Hence H is Picard H-stable, which finishes the proof of Theorem 4.2.

5. Special solution uniqueness

In this portion, we are going to show the uniqueness of the special solution of Eq (1.2) applying the numerical algorithm. Here first we assume an exact solution for Eq (1.2), for which the special solution converges for a large quantity m. Here we take the Hilbert space $H=L((a,b)\times(0,T))$ which can be formulated as follow:

$$f:(a,b)\times(0,T)\longrightarrow\mathbb{R},\ \text{s.t.}\ \int\int uydudy<\infty.$$

Now, we adopt the given operator

$$G(S_h,I_h,R_h,S_v,I_v)=\begin{cases}b_1-\mu_h S_h-\beta_1 S_h I_h-\beta_2 S_h I_v+\lambda_h R_h,\\ \beta_2 S_h I_v+\beta_1 S_h I_h-\mu_h I_h-\delta_h I_h-\gamma_h I_h,\\ \gamma_h I_h-\mu_h R_h-\lambda_h R_h,\\ b_2-\gamma_v S_v-\beta_3 S_v I_h,\\ \beta_3 S_v I_h-\gamma_v I_v-\delta_v I_v.\end{cases}$$

The target of this portion is to prove that the inner product of

$$G\Big((Z_{11}-Z_{12},Z_{21}-Z_{22},Z_{31}-Z_{32},Z_{41}-Z_{42},Z_{51}-Z_{52}),(X_1,X_2,X_3,X_4,X_5)\Big),$$

where $(Z_{11} - Z_{12})$, $(Z_{21} - Z_{22})$, $(Z_{31} - Z_{32})$, $(Z_{41} - Z_{42})$ and $(Z_{51} - Z_{52})$ are special solution of system. However,

$$G\Big((Z_{11} - Z_{12}, Z_{21} - Z_{22}, Z_{31} - Z_{32}, Z_{41} - Z_{42}, Z_{51} - Z_{52}), (X_1, X_2, X_3, X_4, X_5)\Big)$$
$$= \begin{cases} (-\beta_1(Z_{11} - Z_{12})(Z_{21} - Z_{22}) - \mu_h(Z_{11} - Z_{12}) - \beta_2(Z_{11} - Z_{12})(Z_{51} - Z_{52}) + \lambda_h(Z_{31} - Z_{32}), X_1), \\ (\beta_1(Z_{11} - Z_{12})(Z_{21} - Z_{22}) + \beta_2(Z_{11} - Z_{12})(Z_{51} - Z_{52}) - \mu_h(Z_{21} - Z_{22}) - \delta_h(Z_{21} - Z_{22}) \\ - \gamma_h(Z_{21} - Z_{22}), X_2), \\ (\gamma_h(Z_{21} - Z_{22}) - \mu_h(Z_{31} - Z_{32}) - (\lambda_h)(Z_{31} - Z_{32}), X_3), \\ (-\gamma_v(Z_{41} - Z_{42}) - \beta_3(Z_{41} - Z_{42})(Z_{21} - Z_{22}), X_4).(\beta_3(Z_{41} - Z_{42})(Z_{51} - Z_{52}), X_5). \end{cases} \tag{5.1}$$

Here, we calculate the initial equation showing in the model without forgetting the general procedure:

$$\begin{aligned} &(-\beta_1(Z_{11} - Z_{12})(Z_{21} - Z_{22}) - \mu_h(Z_{11} - Z_{12}) - \beta_2(Z_{11} - Z_{12})(Z_{51} - Z_{52}) + \lambda_h(Z_{31} - Z_{32}), X_1) \\ \cong &(-\beta_1(Z_{11} - Z_{12})(Z_{21} - Z_{22}), X_1) + (-\mu_h(Z_{11} - Z_{12}), X_1) + (-\beta_2(Z_{11} - Z_{12})(Z_{51} - Z_{52}), X_1) \\ &+ (\lambda_h(Z_{31} - Z_{32}), X_1). \end{aligned} \tag{5.2}$$

Because the play of both solutions is nearly same, so we can consider that

$$(Z_{11} - Z_{12}) \cong (Z_{21} - Z_{22}) \cong (Z_{31} - Z_{32}) \cong (Z_{41} - Z_{42}) \cong (Z_{51} - Z_{52}).$$

Then Eq (5.2) becomes

$$[-\beta_1(Z_{11} - Z_{12})^2 - \mu(Z_{11} - Z_{12}) - \beta_2(Z_{11} - Z_{12}^2) + \lambda_h(Z_{11} - Z_{12}), X_1].$$

By the relation of inner product and norm, we receive the given results:

$$\begin{aligned} &[-\beta_1(Z_{11} - Z_{12})^2 - \mu(Z_{11} - Z_{12}) - \beta_2(Z_{11} - Z_{12})_2 + \lambda_h(Z_{11} - Z_{12}), X_1] \\ \cong &[-\beta_1(Z_{11} - Z_{12})^2, X_1) + (-\mu_h(Z_{11} - Z_{12}), X_1) + (-\beta_2(Z_{11} - Z_{12})^2, X_1) + (\lambda_h(Z_{11} - Z_{12}), X_1] \\ \leq &\beta_1\|(Z_{11} - Z_{12})^2\|\|X_1\| - \mu_h\|(Z_{11} - Z_{12})\|\|X_1\| - \beta_2\|(Z_{11} - Z_{12})^2\|, \|X_1\| + \lambda_h\|(Z_{11} - Z_{12})\|, \|X_1\| \\ = &(\beta_1\overline{v_1} + \mu_h + \beta_2\bar{v}_1 + \lambda_h)\|(Z_{11} - Z_{12})\|\|X_1\|. \end{aligned} \tag{5.3}$$

Again repeating the similar manner, from the second equation of the model (5.1), we get the outputs

$$\begin{aligned} &[\beta_1(Z_{11} - Z_{12})(Z_{21} - Z_{22}) + \beta_2(Z_{11} - Z_{12})(Z_{51} - Z_{52}) - \mu_h(Z_{21} - Z_{22}) - \delta(Z_{21} - Z_{22}) - \gamma_h(Z_{21} - Z_{22}), X_2] \\ &\leq \beta_1\|(Z_{21} - Z_{22})^2\|\|X_2\| - \beta_2\|(Z_{21} - Z_{22})^2\|\|X_2\| + (\mu_h + \delta_h + \gamma_h)\|(Z_{21} - Z_{22})\|\|X_2\| \\ &= (\beta_1\overline{v_2} + \beta_2\overline{v_2} + \mu_h + \delta_h + \gamma_h)\|(Z_{21} - Z_{22})\|\|X_2\|. \end{aligned} \tag{5.4}$$

Following similar manner, the third equation of the model (5.1) gives that

$$(\gamma_h(Z_{21} - Z_{22}) - \mu_h(Z_{31} - Z_{32}) - \lambda_h(Z_{31} - Z_{32}), X_3) \leq (\gamma_h + \mu_h + \lambda_h)\|(Z_{31} - Z_{32})\|\|X_3\|. \tag{5.5}$$

The fourth one is

$$(-\gamma_v(Z_{41} - Z_{42}) - \beta_3(Z_{41} - Z_{42})(Z_{21} - Z_{22})), X_4) \leq (\gamma_v + \beta_3\overline{v_4})\|(Z_{41} - Z_{42})\|\|X_4\|. \tag{5.6}$$

And the fifth one is

$$\begin{aligned}&(\beta_3(Z_{41}-Z_{42})(Z_{21}-Z_{22})-\gamma_v(Z_{51}-Z_{52})-\delta_v(Z_{51}-Z_{52}),X_5)\\ \leq&(\beta_3\overline{v_5}+\gamma_v+\delta_v)\|(Z_{51}-Z_{52})\|\|X_5\|.\end{aligned} \tag{5.7}$$

Upon putting Eqs (5.3)–(5.7) in Eq (5.1), we get

$$\begin{aligned}&G\Big((Z_{11}-Z_{12},Z_{21}-Z_{22},Z_{31}-Z_{32},Z_{41}-Z_{42},Z_{51}-Z_{52}),(X_1,X_2,X_3,X_4,X_5)\Big)\\ &\leq\begin{cases}(\beta_1\overline{v_1}+\mu_h+\beta_2\overline{v}_1+\lambda_h)\|(Z_{11}-Z_{12})\|\|X_1\|,\\ (\beta_1\overline{v_2}+\beta_2\overline{v_2}+\mu_h+\delta_h+\gamma_h)\|(Z_{21}-Z_{22})\|\|X_2\|,\\ (\gamma_h+\mu_h+\lambda_h)\|(Z_{31}-Z_{32})\|\|X_3\|,\\ (\gamma_v+\beta_3\overline{v_4})\|(Z_{41}-Z_{42})\|\|X_4\|,\\ (\beta_3\overline{v_5}+\gamma_v+\delta_v)\|(Z_{51}-Z_{52})\|\|X_5\|.\end{cases}\end{aligned} \tag{5.8}$$

But, for sufficiently bigger value of m_i, where $i = 1,2,3,4,5$ both solutions converge to the exact solution, applying the consequences of topological results, here three very tiny positive parameters exist $l_{m_1}, l_{m_2}, l_{m_3}, l_{m_4}$and l_{m_5} s.t.

$$\begin{aligned}&\|S_h-Z_{11}\|,\|S_h-Z_{12}\|<\frac{l_{m_1}}{3(\beta_1\overline{v_1}+\mu_h+\beta_2\overline{v}_1+\lambda_h)\|X_1\|},\\ &\|I_h-Z_{21}\|,\|I_h-Z_{22}\|<\frac{l_{m_2}}{3(\beta_1\overline{v_2}+\beta_2\overline{v_2}+\mu_h+\delta_h+\gamma_h)\|X_2\|},\\ &\|R_h-Z_{31}\|,\|R_h-Z_{32}\|<\frac{l_{m_3}}{3(\gamma_h+\mu_h+\lambda_h)\|(Z_{31}-Z_{32})\|X_3\|},\\ &\|S_v-Z_{41}\|,\|S_v-Z_{42}\|<\frac{l_{m_4}}{3(\gamma_v+\beta_3\overline{v_4})\|X_4\|},\\ &\|I_v-Z_{51}\|,\|I_v-Z_{52}\|<\frac{l_{m_5}}{3(\beta_3\overline{v_5}+\gamma_v+\delta_v)\|X_5\|}.\end{aligned}$$

Now putting the exact solution to the right-side of Eq (5.8) and using the triangular inequality with taking $M = \max(m_1, m_2, m_3, m_4, m_5)$, $l = \max(l_{m_5}, l_{m_4}, l_{m_3}, l_{m_2}, l_{m_1})$, we get

$$\begin{cases}(\beta_1\overline{v_1}+\mu_h+\beta_2\overline{v}_1+\lambda_h)\|(Z_{11}-Z_{12})\|\|X_1\|,\\ (\beta_1\overline{v_2}+\beta_2\overline{v_2}+\mu_h+\delta_h+\gamma_h)\|(Z_{21}-Z_{22})\|\|X_2\|,\\ (\gamma_h+\mu_h+\lambda_h)\|(Z_{31}-Z_{32})\|\|X_3\|,\\ (\gamma_v+\beta_3\overline{v_4})\|(Z_{41}-Z_{42})\|\|X_4\|,\\ (\beta_3\overline{v_5}+\gamma_v+\delta_v)\|(Z_{51}-Z_{52})\|\|X_5\|.\end{cases} < \begin{cases}l,\\ l,\\ l,\\ l,\\ l.\end{cases}$$

As l is very very small positive value, so on the basis of topological result, we get

$$\begin{cases}(\beta_1\overline{v_1}+\mu_h+\beta_2\overline{v}_1+\lambda_h)\|(Z_{11}-Z_{12})\|\|X_1\|,\\ (\beta_1\overline{v_2}+\beta_2\overline{v_2}+\mu_h+\delta_h+\gamma_h)\|(Z_{21}-Z_{22})\|\|X_2\|,\\ (\gamma_h+\mu_h+\lambda_h)\|(Z_{31}-Z_{32})\|\|X_3\|,\\ (\gamma_v+\beta_3\overline{v_4})\|(Z_{41}-Z_{42})\|\|X_4\|,\\ (\beta_3\overline{v_5}+\gamma_v+\delta_v)\|(Z_{51}-Z_{52})\|\|X_5\|.\end{cases} < \begin{cases}0,\\ 0,\\ 0,\\ 0,\\ 0.\end{cases} \tag{5.9}$$

But, it is obvious that

$$
\begin{aligned}
&(\beta_1\overline{v_1} + \mu_h + \beta_2\bar{v}_1 + \lambda_h) \neq 0,\\
&(\beta_1\overline{v_2} + \beta_2\overline{v_2} + \mu_h + \delta_h + \gamma_h) \neq 0,\\
&(\gamma_h + \mu_h + \lambda_h) \neq 0,\\
&(\gamma_v + \beta_3\overline{v_4}) \neq 0,\\
&(\beta_3\overline{v_5} + \gamma_v + \delta_v) \neq 0.
\end{aligned}
\tag{5.10}
$$

Therefore, we have

$$
\begin{aligned}
&\|Z_{11} - Z_{12}\| = 0,\\
&\|Z_{21} - Z_{22}\| = 0,\\
&\|Z_{31} - Z_{32}\| = 0,\\
&\|Z_{41} - Z_{42}\| = 0,\\
&\|Z_{51} - Z_{52}\| = 0,
\end{aligned}
\tag{5.11}
$$

which yields that

$$
Z_{11} = Z_{12},\ Z_{21} = Z_{22},\ Z_{31} = Z_{32},\ Z_{41} = Z_{42},\ Z_{51} = Z_{52}. \tag{5.12}
$$

This completes the proof of uniqueness.

6. Numerical simulations and discussions

This portion gives the numerical simulation results based on the numerical algorithm as Eq (3.5) for leptospirosis disease model (1.2). The numerical method employed on Eq (1.2) hinged with (3.5). The numerical simulations are performed by adopting the parameter numerical values, $b_1 = 0.121$, $\mu_h = 0.0121$, $\beta_1 = 0.14$, $\beta_2 = 0.02$, $\beta_3 = 0.10$, $\lambda_h = 1$, $\delta_h = 0.8$, $\gamma_h = 0.25$, $b_2 = 0.002$, $\gamma_v = 0.23$, $\delta_v = 0.001$. In this practical work, the integer order cases are plotted in the company of Figure 1. Here we have just checked that how the given model is behaving at the classical sense. After following it, in the assembly of Figure 2, we plotted the model dynamics at distinct derivative operator orders α. Here the disease shows almost same nature at each values but just the amount of the given population varies at distinct values. The assembly of Figures 3 and 4 is dedicated to the model dynamics at the values of parameters β_1, β_2, β_3. From the given plots we can see that the impact of fractional order is measurable and every value demonstrates a unique nature of the given disease. The given parameter range is the best fit for the given model structure.

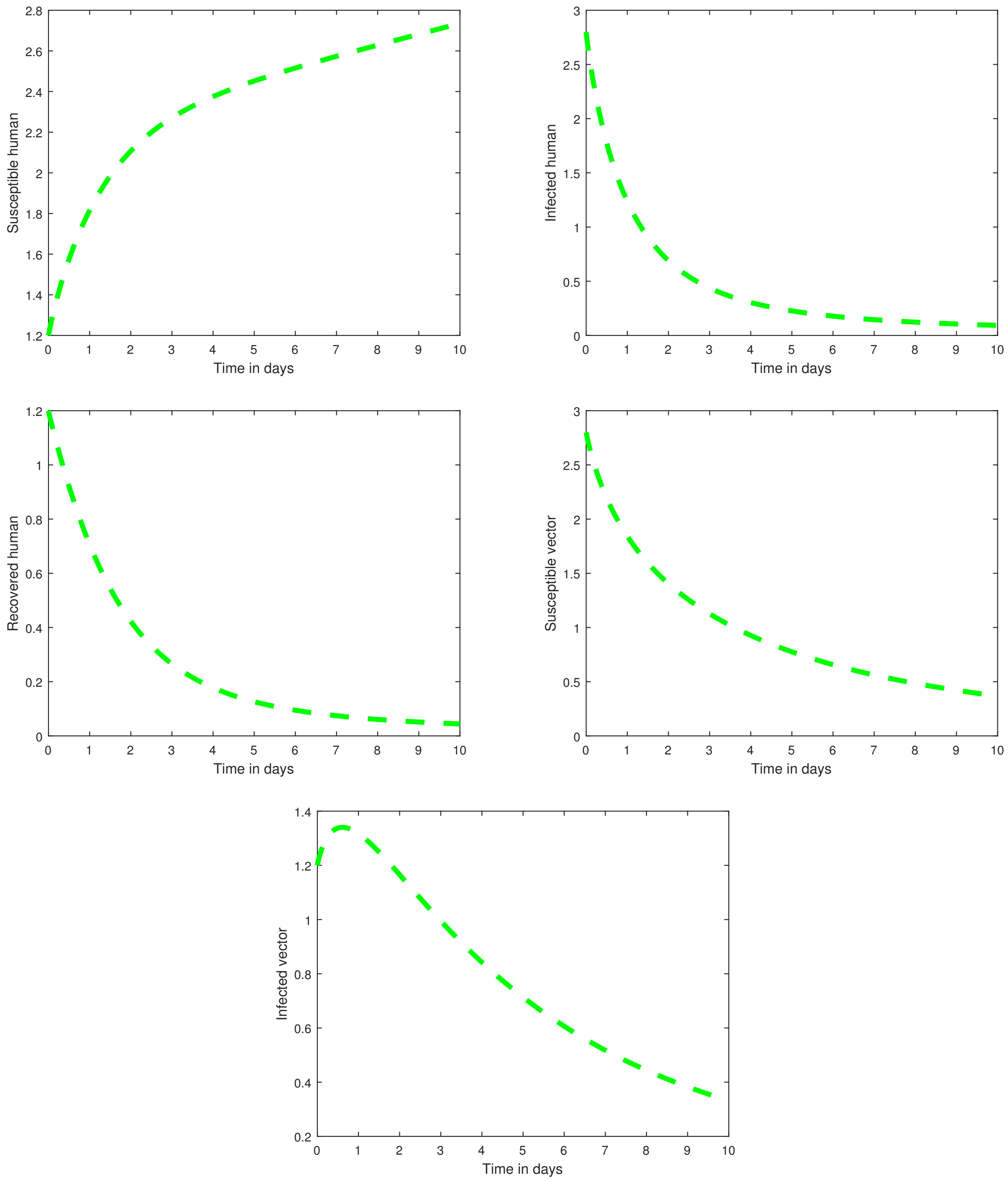

Figure 1. Dynamics of given model classes in the sense of Caputo-Fabrizio operator by utilizing the mentioned parameter values.

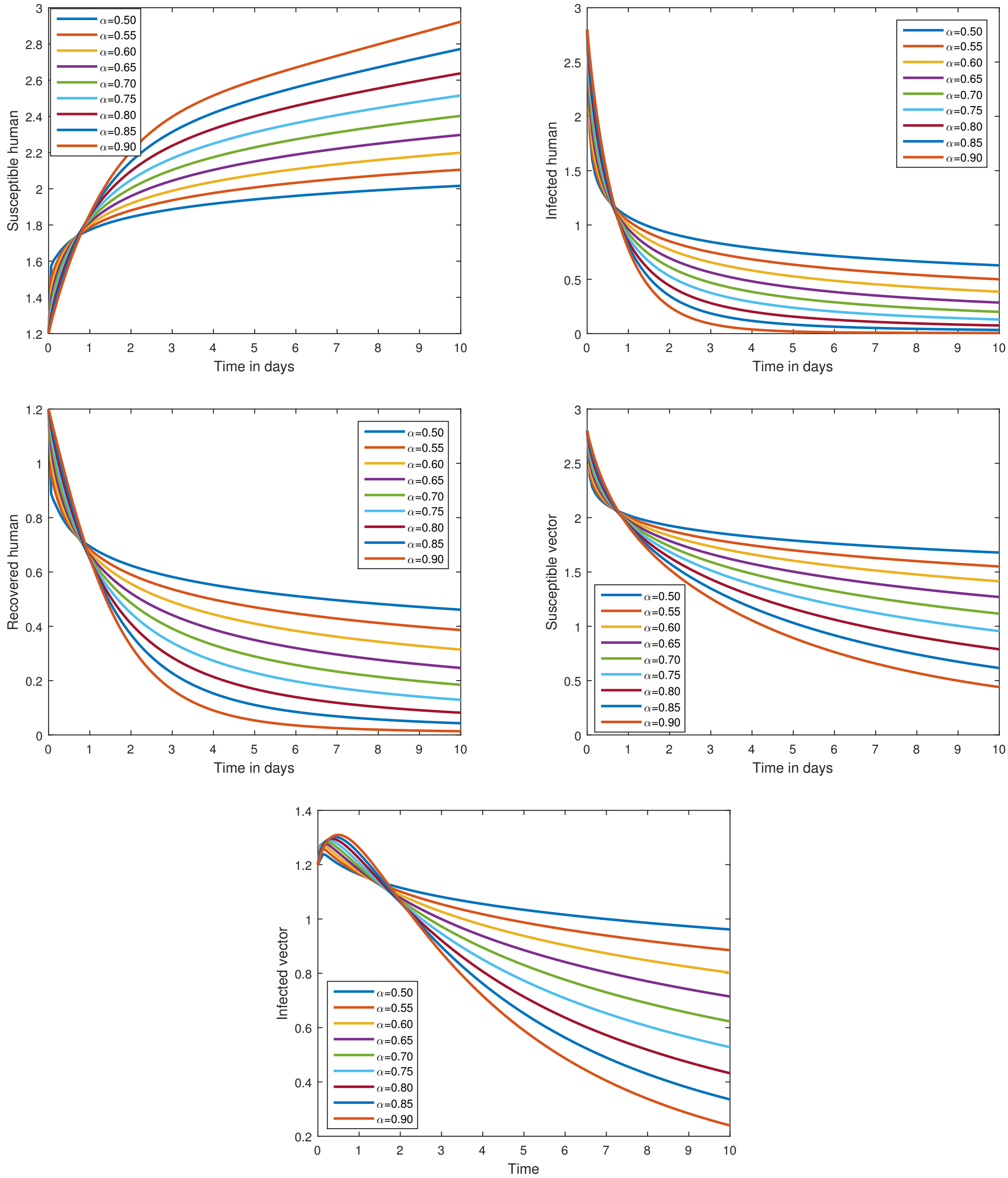

Figure 2. The dynamics of each state variable for different α values, i.e., $\alpha = 0.50, 0.55, 0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90$.

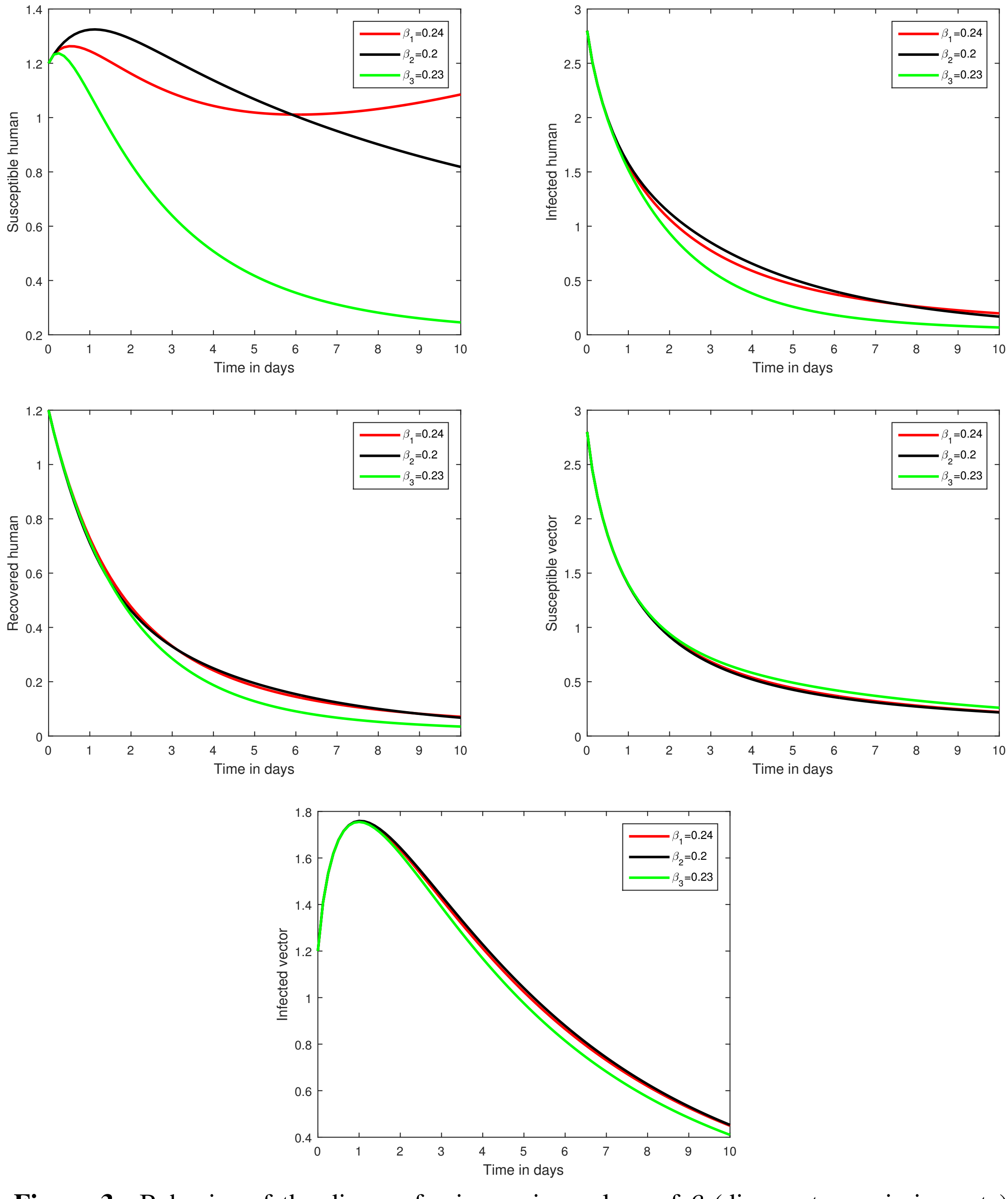

Figure 3. Behavior of the disease for increasing values of β (disease transmission rate), $\beta_1 = 0.24$, $\beta_2 = 0.2$, $\beta_3 = 0.21$.

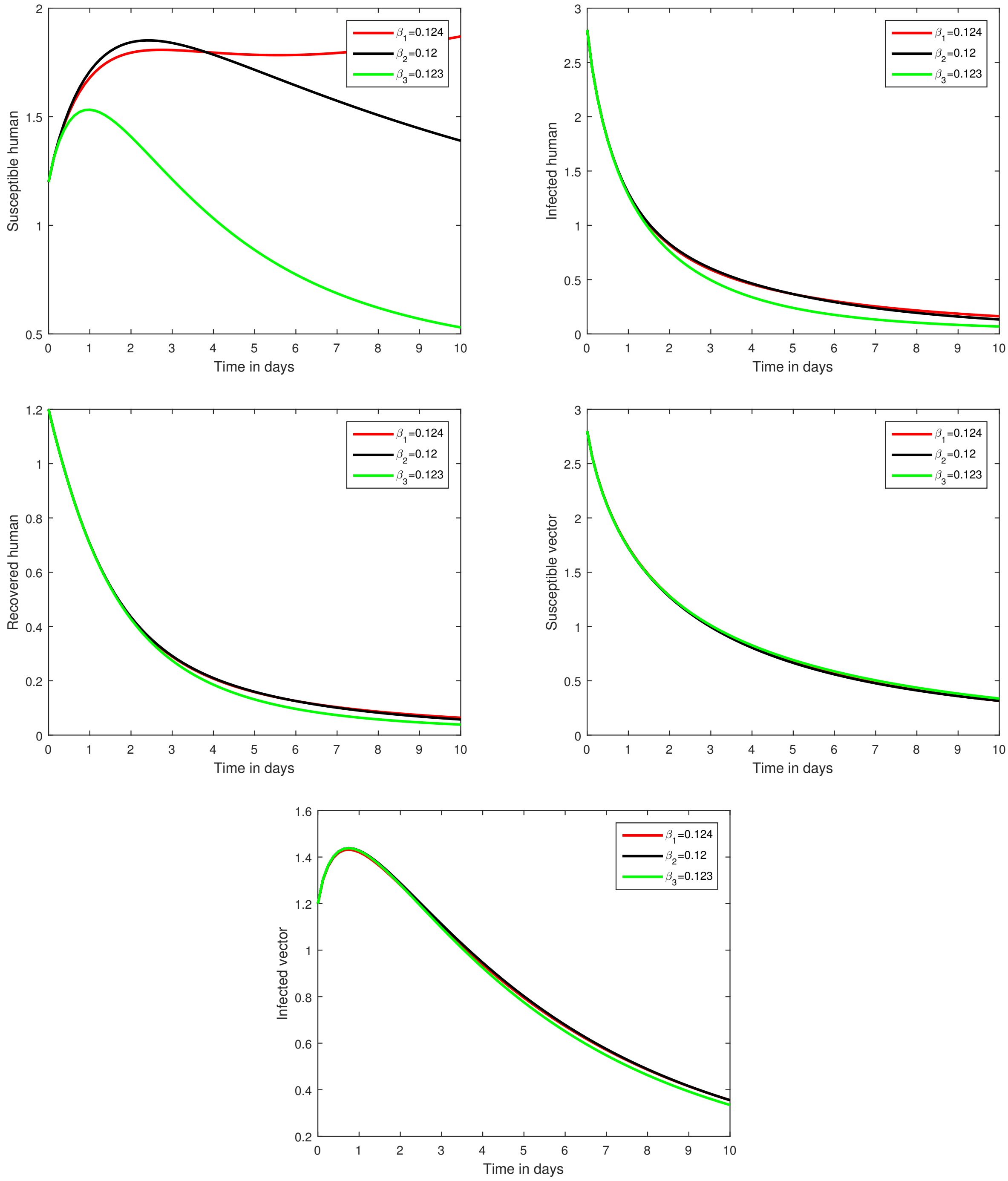

Figure 4. Behavior of the disease for decreasing values of β (disease transmission rate), $\beta_1 = 0.124$, $\beta_2 = 0.12$, $\beta_3 = 0.121$.

7. Sensitivity analysis

Here sensitivity analysis is performed to show that which parameters are useful in reducing the infection spread of the disease. Forward sensitivity structure is taken as a important part of epidemic modelling, albeit its analysis become austere for tough biological dynamics. Epidemiologist and ecologist have given their much attention to the calculation of sensitivity analysis of R_0.

Definition 7.1. *The normalized forward sensitivity index of the* R_0 *which based on differentiably on a variable* Γ *is given by*

$$S_\Gamma = \frac{\Gamma}{R_0}\frac{\partial R_0}{\partial \Gamma}. \tag{7.1}$$

There are three general techniques which are basically utilized to find the sensitivity indices: (i) By a Latin hypercube sampling technique; (ii) by direct differentiation; (iii) by linearising model and then simulating the obtained set of linear algebraic constraints. Here we adopt the direct differentiation technique because it gives analytical forms for the indices. These indices demonstrates the impact of different supports jointed with the infection spread and also provides us considerable knowledge about the comparative variations between R_0 and various parameters. Definitely, it provides the help in texturing the control techniques.

The parameter values b_1, β_2, β_1, λ_h, δ_h, γ_h, b_2 and β_3 have a positive impact on the reproductive number R_0, which exemplify that the decay or growth of given parameters call by 10% will decrease or increase the reproductive number by 10%, 1.70%, 8.2%, 4.27%, 2.3%, 1.7%, 1.7% and 0.2% respectively. But another side, the parameters index of μ_h, γ_v and δ_v shows that raising their values by 10% will reduce the value of reproductive number R_0 by 9.9%, 2.6% and 1.0% respectively. λ_h has no impact on reproduction number. Graphical representations of the sensitivity index is explored in Figures 5–11 which demonstrate the sensitivity of different parameters.

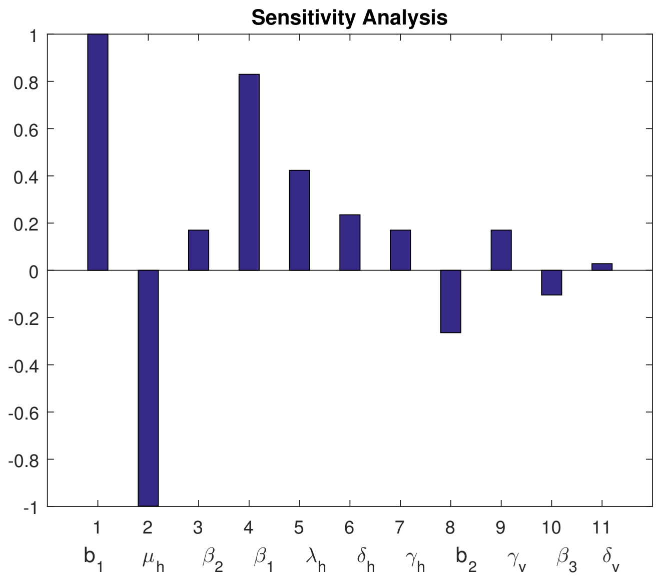

Figure 5. The plot represents the sensitivity indices.

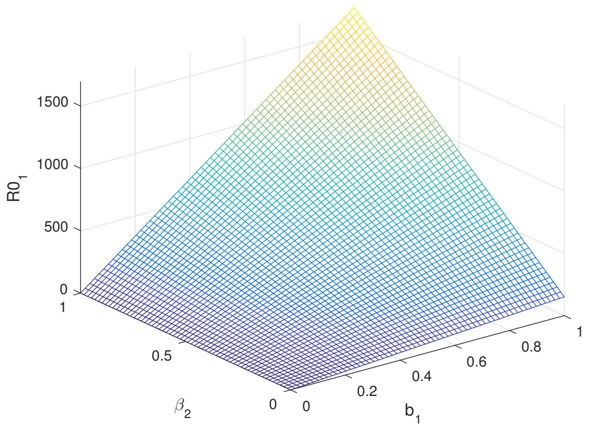

Figure 6. R_0 versus sensitive parameters b_1 and β_2.

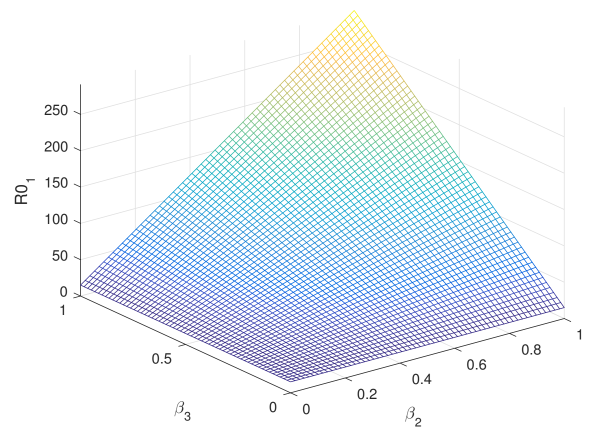

Figure 7. R_0 versus sensitive parameters β_3 and β_2.

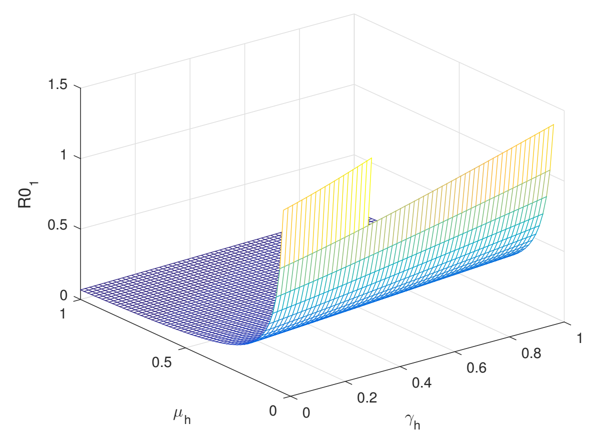

Figure 8. R_0 versus sensitive parameters μ_h and γ_h.

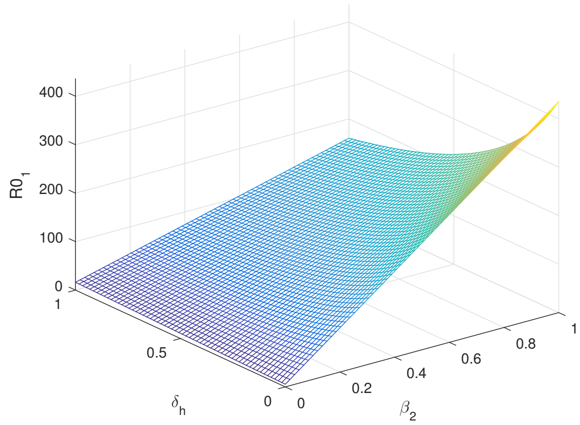

Figure 9. R_0 versus sensitive parameters δ_h and β_2.

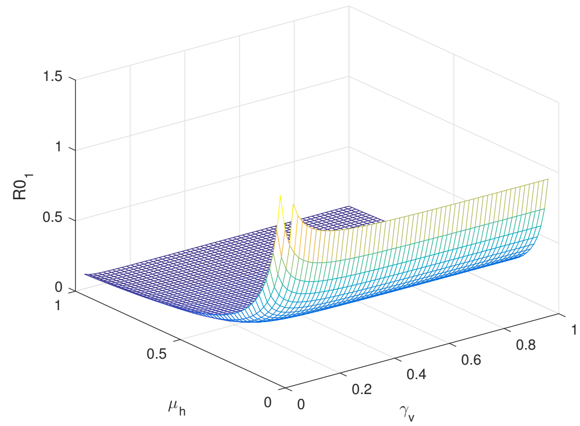

Figure 10. R_0 versus sensitive parameters μ_h and γ_v.

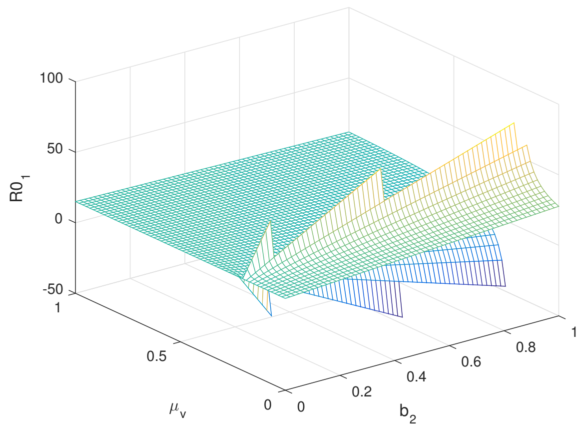

Figure 11. R_0 versus sensitive parameters μ_v and β_2.

8. Conclusions

In this research, we have simulated a leptospirosis model dynamics by using a non-classical type derivative with the application of Banach contraction theorem and Picard successive approximation algorithm. Here the fractional order derivative called Caputo-Febrizo of order α has been used which exists a non-singular exponentially decay type kernel. The given analysis have been justified by applying well-known Sumudu transform alongwith proposed iterative scheme. Using the Banach theorem, the existence of unique equilibrium solution have been exemplified. With the help of Matlab, we have also presented numerical simulations to the approximate solutions which show the effectiveness of the theoretical results. From the given simulations, we concluded that the mentioned non-classical type model gives more emphatic structure as compare to the integer-order dynamics. Hence we can say that given variable order structure approaches to time responses with super-slow evolutions and super-fast transients directed to the steady-state, which impacts can not be smoothly textured by the ordinary order systems. In the future, the above model texture can be further rebuilt by more effective and advance models of dynamical systems. One may use Atangana Baleanu fractional derivative or Hilfer fractional derivative to compare the results with our model. One can also employ the fractional optimal control strategy to decrease the infected people and increased the susceptible people by choosing suitable control variables. The stochastic version of this model will also be useful to show the effect of white noise on the model.

Acknowledgments

This research was supported by King Mongkut's University of Technology Thonburi's Postdotoral Fellowship.

Conflict of interest

The authors declare no conflicts of interest.

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