



---

*Research article*

## Tail risk measures with application for mixtures of elliptical distributions

Pingyun Li and Chuancun Yin\*

School of Statistics and Data Science, Qufu Normal University, Qufu, Shandong 273165, China

\* **Correspondence:** Email: ccyin@qfnu.edu.cn; Tel: 8605374453221.

**Abstract:** In this paper we derive explicit formulas of tail conditional expectation (TCE) and tail variance (TV) for the class of location-scale mixtures of elliptical distributions, which includes the generalized hyper-elliptical (GHE) distribution. We also develop portfolio risk decomposition with TCE for multivariate location-scale mixtures of elliptical distributions. To illustrate our findings, we focus on the generalized hyperbolic (GH) family which is a popular subclass of the GHE for stocks modelling.

**Keywords:** tail conditional expectations; tail variance; portfolio risk decomposition with TCE; location-scale mixtures; elliptical distributions

**Mathematics Subject Classification:** 62E10, 62H05

---

### 1. Introduction and motivation

One of the main challenges faced by financial companies is to evaluate market risks in a set of changes of the basic variables such as stock prices, interest rates or exchange rates. In this regard the Value-at-Risk (VaR) introduced by J. P. Morgan in the mid 1990s has become a standard risk measure of financial market risk. Despite its extensive use, the VaR is not a coherent risk measure because it fails to satisfy subadditivity property (see [1]). The VaR can not determine the expected loss of portfolio in  $q$  worst case, but it defines the minimum loss. Furthermore, the computation of the VaR is based on the assumption that financial data returns follow the normal distribution. However, as shown in the literatures, the underlying distributions of many financial data exhibit skewness, non-symmetric, heavy tails and excess kurtosis (see [10]). They suggest in particular that large losses occur with much higher probability than the normal distribution would suggest.

The tail conditional expectation (TCE) risk measure shares properties that are considered desirable in all cases. For instance, due to the additivity of expectations, TCE allows venture capital to decompose naturally among its various components.

Consider  $X$  to be a loss random variable whose cumulative distribution function (cdf) is denoted by

$F_X(x)$ . The TCE is defined as

$$\text{TCE}_p(X) = E(X|X > x_p), \quad p \in (0, 1),$$

where  $x_p = \inf\{x \in \mathbb{R} : F_X(x) \geq p\} = \text{VaR}_p(X)$ . The TCE has been discussed in many literatures (see e.g., [11, 12, 15, 17]).

The tail conditional expectation risk measure shares properties that are considered desirable in a variety of situations. For instance, due to the additivity of expectations, TCE allows for a natural decomposition of risk capital among its various constituents. The conception of capital allocation principle has long been introduced, in which the capital allocated to each risk unit can be expressed as its contribution to the tail conditional expectation of total risk. Risk allocation can not only help to evaluate and compare the performance of individual risk units, but also help to understand the risk contribution of each unit towards the total risk of the portfolio. Landsman and Valdez [17] derived the portfolio risk decomposition with TCE for the multivariate elliptical distribution. In [18], authors derived the portfolio risk decomposition with TCE for the exponential dispersion model, and Kim [14] for the exponential family class. The allocation for the class of exponential marginal was developed in [20]. The portfolio risk decomposition with TCE was further considered in [19] for the skew-normal distribution. Furman and Landsman [16] for the multivariate Gamma distribution. Cai and Li [2] for the phase-type distribution. Goovaerts et al. [9] and Chiragiev and Landsman [4] have provided the TCE-capital allocation for the multivariate Pareto distribution while Cossette et al. [5] have considered multivariate compound distribution. Ignatieva and Landsman [12] for generalized hyperbolic distribution. Recently, Kim and Kim [15] and Ignatieva and Landsman [12] investigated the TCE allocation for the family of multivariate normal mean-variance mixture distributions and skewed generalized hyperbolic, respectively. The univariate TCE and risk allocation formula for the generalized hyper-elliptical class were available in [13].

Furman and Landsman [8] observed that in many cases the TCE does not provide adequate information about the risks on the right tail. This point can be confirmed by the fact that the TCE does not include the information that the risk deviates from the upper tail expectation. Furman and Landsman [8] introduced the tail variance measure. The tail variance is defined as

$$\text{TV}_p(X) = \text{Var}(X|X > x_p) = E\left((X - \text{TCE}_p(X))^2|X > x_p\right),$$

and it has been discussed in many literatures (see e.g., [8, 15]).

In this paper we consider a class of multivariate location-scale mixtures of elliptical (LSME) distributions which is known to be extremely flexible and contains many special cases as its members. Examples include the generalized hyper-elliptical distribution and generalized hyperbolic distribution.

The rest of the paper is organized as follows. Section 2 reviews the definition and properties of the multivariate LSME class, and introduces the generalized hyper-elliptical distribution as a representative subclass. Section 3 presents a theorem and proves the proposed TCE formula for the LSME and in Section 4, the development is extended to the portfolio risk decomposition with TCE for the multivariate LSME. In Section 5, we develop TV formula for univariate LSME. Section 6 deals with the special case of generalized hyperbolic distribution. Numerical illustration is presented in Section 7. Finally, concluding remarks are presented in Section 8.

## 2. Mixture of elliptical distributions

In this section, we introduce the class of location-scale mixtures of elliptical distributions and some of its properties.

Let  $\Psi_n$  be a class of functions  $\psi(t) : [0, \infty) \rightarrow \mathbb{R}$  such that function  $\psi(\sum_{i=1}^n t_i^2)$  is an  $n$ -dimensional characteristic function.

A random vector  $Y$  is said to have a multivariate elliptical distribution, denoted by  $Y \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ , if its characteristic function can be expressed as

$$\varphi_Y(\mathbf{t}) = \exp(i\mathbf{t}^T \boldsymbol{\mu}) \psi\left(\frac{1}{2} \mathbf{t}^T \boldsymbol{\Sigma} \mathbf{t}\right), \quad (2.1)$$

for column-vector vector  $\boldsymbol{\mu}$ ,  $n \times n$  positive definite scale matrix  $\boldsymbol{\Sigma}$ , and for function  $\psi(t) \in \Psi_n$ , which is called the characteristic generator.

In general, a multivariate elliptical distribution may not have a probability density function (pdf), but if its pdf exists then the form will be

$$f_Y(\mathbf{y}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} g_n\left[\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\mu})\right], \quad (2.2)$$

for function  $g_n(\cdot)$ , which is called the density generator. The condition

$$\int_0^\infty u^{\frac{n}{2}-1} g_n(u) du < \infty \quad (2.3)$$

guarantees  $g_n(u)$  to be the density generator ([7]). In addition, the normalizing constant  $c_n$  is

$$c_n = \frac{\Gamma(\frac{n}{2})}{(2\pi)^{\frac{n}{2}}} \left( \int_0^\infty u^{\frac{n}{2}-1} g_n(u) du \right)^{-1}.$$

Similarly, the elliptical distribution can also be introduced by the density generator and then written  $Y \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ .

From (2.1), it follows that, if  $Y \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$  and  $A$  is  $m \times n$  matrix of rank  $m \leq n$  and  $\mathbf{b}$  is  $m$ -dimensional column-vector, then

$$AY + \mathbf{b} \sim E_m(A\boldsymbol{\mu} + \mathbf{b}, A\boldsymbol{\Sigma}A^T, g_m).$$

The following condition:

$$\int_0^\infty g_1(u) du < \infty$$

guarantees the existence of the mean. If, in addition,  $|\psi'(0)| < \infty$ , the covariance matrix exists and is equal to

$$\text{Cov}(Y) = -\psi'(0)\boldsymbol{\Sigma},$$

(see [3]).

From (2.2) and (2.3),  $g_1(x)$  can be a density generator of univariate elliptical distribution of the random variable  $Y \sim E_1(\mu, \sigma^2, g_1)$  whose pdf can be expressed as

$$f_Y(y) = \frac{c}{\sigma} g_1 \left( \frac{1}{2} \left( \frac{y - \mu}{\sigma} \right)^2 \right),$$

where  $c$  is the normalizing constant. In this paper, we assume

$$\text{Var}(Z) = \sigma_Z^2 < \infty, \quad (2.4)$$

where  $Z = \frac{Y - \mu}{\sigma}$  is the spherical random variable. The cdf of the random variable  $Z$  can be written as the following integration form:

$$F_Z(z) = c \int_{-\infty}^z g_1 \left( \frac{1}{2} u^2 \right) du.$$

We can obtain the mean and variance of  $Z$ :

$$\mu_Z = 0$$

and

$$\sigma_Z^2 = 2c \int_0^{\infty} u^2 g_1 \left( \frac{1}{2} u^2 \right) du = -\psi'(0).$$

Landsman and Valdez [17] showed that

$$f_{Z^*}(z) = \frac{1}{\sigma_Z^2} \bar{G} \left( \frac{1}{2} z^2 \right) \quad (2.5)$$

is the density of another spherical random variable  $Z^*$  associated with  $Z$ , where

$$\bar{G}(z) = \int_z^{\infty} g_1(u) du.$$

The random vector  $X \sim \text{LSME}_n(\mu, \Sigma, \gamma, g_n; \Pi)$  has an  $n$ -dimensional LSME distribution with location parameter  $\mu$ , positive definite scale matrix  $\Sigma$ , if

$$X = m(\Theta) + \Theta^{\frac{1}{2}} \Sigma^{\frac{1}{2}} Y, \quad (2.6)$$

in distribution, where

- (1)  $Y \sim E_n(\mathbf{0}, \mathbf{I}_n, g_n)$ , the  $n$ -dimensional multivariate elliptical variable;
- (2) Non-negative scalar random variable  $\Theta$  is independent of  $Y$ , whose pdf and cdf are  $\pi(\theta)$  and  $\Pi(\theta)$  respectively;
- (3)  $m(\Theta) = \mu + \Theta\gamma$ , where  $\mu = (\mu_1, \dots, \mu_n)^T$  and  $\gamma = (\gamma_1, \dots, \gamma_n)^T$  are constant vectors in  $\mathbb{R}^n$ .

The pdf of the LSME can be written as the following integration form:

$$f_X(\mathbf{x}) = \frac{c_n}{\sqrt{|\Sigma|}} \int_0^{\infty} \frac{1}{\sqrt{\theta}} g_n \left( \frac{1}{2\theta} (\mathbf{x} - \mu - \theta\gamma)^T \Sigma^{-1} (\mathbf{x} - \mu - \theta\gamma) \right) \pi(\theta) d\theta, \quad \mathbf{x} \in \mathbb{R}^n.$$

We find that the conditional distribution of  $X|\theta$  is elliptical, that is

$$X|\Theta = \theta \sim E_n(\mathbf{m}(\theta), \theta\Sigma, g_n). \quad (2.7)$$

We can obtain the mean and covariance of  $X$ :

$$E(X) = E[E(X|\Theta)] = E(\mathbf{m}(\Theta)) = \boldsymbol{\mu} + E(\Theta)\boldsymbol{\gamma}$$

and

$$\begin{aligned} \text{Cov}(X) &= E[\text{Var}(X|\Theta)] + \text{Var}[E(X|\Theta)] = E(\Theta\Sigma) + \text{Var}(\mathbf{m}(\Theta)) \\ &= E(\Theta)\Sigma + \text{Var}(\Theta)\boldsymbol{\gamma}\boldsymbol{\gamma}^T. \end{aligned}$$

The characteristic function of  $X|\Theta = \theta$  exists and equals to

$$\varphi_{X|\Theta = \theta}(\mathbf{t}) = \exp(i\mathbf{t}^T \boldsymbol{\mu}) \exp(i\theta \mathbf{t}^T \boldsymbol{\gamma}) \psi\left(\frac{1}{2}\theta \mathbf{t}^T \Sigma \mathbf{t}\right).$$

Then the characteristic function of the LSME-distributed random vector  $X$  can be written as

$$\begin{aligned} \varphi_X(\mathbf{t}) &= \exp(i\mathbf{t}^T \boldsymbol{\mu}) E\left[\exp(i\theta \mathbf{t}^T \boldsymbol{\gamma}) \psi\left(\frac{1}{2}\theta \mathbf{t}^T \Sigma \mathbf{t}\right)\right] \\ &= \exp(i\mathbf{t}^T \boldsymbol{\mu}) \int_0^\infty \exp(i\theta \mathbf{t}^T \boldsymbol{\gamma}) \psi\left(\frac{1}{2}\theta \mathbf{t}^T \Sigma \mathbf{t}\right) \pi(\theta) d\theta. \end{aligned} \quad (2.8)$$

Under the condition (2.2) and from (2.5), we can conclude  $\bar{G}(z)$  is the density generator of the associated elliptical variable  $Z^*$ , then

$$X^* = \mathbf{m}(\Theta) + \sqrt{\Theta} \sigma Z^* \quad (2.9)$$

is said to have a univariate LSME distributions, denoted by  $X^* \sim \text{LSME}_1(\boldsymbol{\mu}, \sigma^2, \boldsymbol{\gamma}, \Theta, \bar{G}; \Pi)$ .

**Proposition 1.** *If  $X \sim \text{LSME}_n(\boldsymbol{\mu}, \Sigma, \boldsymbol{\gamma}, g_n; \Pi)$  and  $Y = BX + \mathbf{b}$  where  $B$  is  $m \times n$  ( $m \leq n$ ) matrix and  $\mathbf{b}$  is  $m$ -dimensional column-vector, then it holds that  $Y \sim \text{LSME}_m(B\boldsymbol{\mu} + \mathbf{b}, B\Sigma B^T, B\boldsymbol{\gamma}, g_m; \Pi)$ .*

*Proof.* Using the characteristic function (2.8), we write

$$\begin{aligned} \varphi_Y(\mathbf{t}) &= E(e^{i\mathbf{t}^T (BX + \mathbf{b})}) \\ &= \exp(i\mathbf{t}^T \mathbf{b}) \varphi_X(B^T \mathbf{t}) \\ &= \exp(i\mathbf{t}^T \mathbf{b}) \exp(i(B^T \mathbf{t})^T \boldsymbol{\mu}) \int_0^\infty \exp(i\theta (B^T \mathbf{t})^T \boldsymbol{\gamma}) \psi\left(\frac{1}{2}\theta (B^T \mathbf{t})^T \Sigma (B^T \mathbf{t})\right) \pi(\theta) d\theta \\ &= \exp(i\mathbf{t}^T (B\boldsymbol{\mu} + \mathbf{b})) \int_0^\infty \exp(i\theta \mathbf{t}^T B\boldsymbol{\gamma}) \psi\left(\frac{1}{2}\theta \mathbf{t}^T B\Sigma B^T \mathbf{t}\right) \pi(\theta) d\theta, \end{aligned} \quad (2.10)$$

i.e.,  $Y \sim \text{LSME}_m(B\boldsymbol{\mu} + \mathbf{b}, B\Sigma B^T, B\boldsymbol{\gamma}, g_m; \Pi)$ .

**Example 2.1** (Generalized hyper-elliptical distribution). The GHE distribution is constructed by mixing a generalized inverse Gaussian distribution with elliptical distribution. A positive random

variable  $\Theta$  is said to have a generalized inverse Gaussian distribution, denoted by  $\Theta \sim \text{GIG}(\lambda, a, b)$ , if its pdf is given by

$$\pi(\theta; \lambda, a, b) = \frac{a^{-\lambda}(\sqrt{ab})^\lambda}{2K_\lambda(\sqrt{ab})} \theta^{\lambda-1} \exp\left(-\frac{1}{2}(a\theta^{-1} + b\theta)\right), \quad \theta > 0, \quad (2.11)$$

where parameters follow

$$\begin{cases} a \geq 0 & \text{and } b > 0, & \text{if } \lambda > 0, \\ a > 0 & \text{and } b \geq 0, & \text{if } \lambda < 0, \\ a > 0 & \text{and } b > 0, & \text{if } \lambda = 0 \end{cases}$$

and  $K_\lambda(\cdot)$  denotes the modified Bessel function of the second kind with index  $\lambda \in \mathbb{R}$ . A random vector  $\mathbf{X} \sim \text{GHE}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma}, g_n, \lambda, a, b)$  has an  $n$ -dimensional GHE distribution, if there exists a random vector  $\mathbf{Y}$  follows (2.6) such that

$$\mathbf{X} = \mathbf{m}(\Theta) + \Theta^{\frac{1}{2}} \boldsymbol{\Sigma}^{\frac{1}{2}} \mathbf{Y}, \quad (2.12)$$

where  $\Theta \sim \text{GIG}(\lambda, a, b)$ .

### 3. Tail conditional expectation for univariate case

The univariate LSME variable is given by  $n = 1$  in the multivariate definition. That is, the univariate LSME variable  $X \sim \text{LSME}_1(\mu, \sigma^2, \gamma, g_1; \Pi)$  satisfies

$$X = m(\Theta) + \sqrt{\Theta} \sigma Z,$$

where  $Z \sim E_1(0, 1, g_1)$  is the standard elliptical variable, and non-negative scalar random variable  $\Theta$  with pdf  $\pi(\theta)$  is independent of  $Z$ . From (2.7), we have

$$X|\theta \sim E_1(m(\theta), \theta\sigma^2, g_1). \quad (3.1)$$

Assuming that both the conditional distribution and the mixed distribution are continuous, the pdf of  $X$  produced by the mixed distribution can be written as

$$f_X(x) = \int_{\Omega_\theta} f(x|\theta) \pi(\theta) d\theta, \quad (3.2)$$

where  $f(x|\theta)$  is the pdf of  $X|\theta$  and  $\Omega_\theta$  is the support of  $\pi(\theta)$ . Now let  $x_p$  be the quantile of the LSME variable  $X$ . Then the TCE of  $X$  can be expressed as

$$\begin{aligned} E(X|X > x_p) &= \frac{1}{1-p} \int_{x_p}^{\infty} x f_X(x) dx \\ &= \frac{1}{1-p} \int_{\Omega_\theta} \int_{x_p}^{\infty} x f(x|\theta) dx \pi(\theta) d\theta \end{aligned}$$

$$= \frac{1}{1-p} \int_{\Omega_\theta} E_{X|\theta}(X|X > x_p) \bar{F}_{X|\theta}(x_p) \pi(\theta) d\theta. \quad (3.3)$$

The TCE formula for a univariate elliptical distribution is introduced by [17], and equals to

$$\begin{aligned} \text{TCE}_p(X|\theta) &= E_{X|\theta}(X|X > x_p) \\ &= m(\theta) + \frac{\frac{1}{\sqrt{\theta}\sigma} f_{Z^*}(\kappa(x_p; \theta))}{\bar{F}_Z(\kappa(x_p; \theta))} \sigma_Z^2 \theta \sigma^2 \\ &= m(\theta) + \frac{\frac{1}{\sqrt{\theta}\sigma} f_{Z^*}(\kappa(x_p; \theta))}{\bar{F}_Z(\kappa(x_p; \theta))} \text{Var}(X|\theta), \end{aligned} \quad (3.4)$$

where

$$\kappa(x; \theta) = \frac{x - m(\theta)}{\sqrt{\theta}\sigma}.$$

We now give a general TCE formula for the univariate LSME distributions.

**Theorem 1.** Let  $X \sim \text{LSME}_1(\mu, \sigma^2, \gamma, g_1; \Pi)$  and  $\pi'(\theta) = (c')^{-1} \theta \pi(\theta)$  be a mixing pdf with  $c' = E(\Theta) < \infty$ . Then the TCE of  $X$  can be computed by:

$$\begin{aligned} E(X|X > x_p) &= \mu + \frac{\gamma c'}{1-p} \bar{F}_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, g_1; \Pi') + \frac{c' \sigma^2 \sigma_Z^2}{1-p} f_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi'), \end{aligned} \quad (3.5)$$

where  $\Pi'$  is the cdf corresponding to the pdf  $\pi'$ .

*Proof.* From (3.3) and (3.4), we have

$$\begin{aligned} E(X|X > x_p) &= \frac{1}{1-p} \int_0^\infty \left[ m(\theta) + \frac{\frac{1}{\sqrt{\theta}\sigma} f_{Z^*}(\kappa(x_p; \theta))}{\bar{F}_Z(\kappa(x_p; \theta))} \text{Var}(X|\theta) \right] \bar{F}_{X|\theta}(x_p) \pi(\theta) d\theta \\ &= \frac{1}{1-p} \int_{x_p}^\infty \int_0^\infty \frac{m(\theta)}{\sqrt{\theta}\sigma} f_Z(\kappa(x; \theta)) \pi(\theta) d\theta dx \\ &\quad + \frac{\sigma^2 \sigma_Z^2}{1-p} \int_0^\infty \frac{f_{Z^*}(\kappa(x_p; \theta))}{\sqrt{\theta}\sigma} \theta \pi(\theta) d\theta \\ &= \frac{1}{1-p} \int_{x_p}^\infty \mu f(x) dx + \frac{\gamma}{1-p} \int_{x_p}^\infty \int_0^\infty f(x|\theta) \theta \pi(\theta) d\theta dx \\ &\quad + \frac{\sigma^2 \sigma_Z^2}{1-p} \int_0^\infty \frac{f_{Z^*}(\kappa(x_p; \theta))}{\sqrt{\theta}\sigma} \theta \pi(\theta) d\theta. \end{aligned} \quad (3.6)$$

From the definition of LSME distributions and (3.2), we have

$$\int_0^\infty f(x|\theta) \theta \pi(\theta) d\theta = c' \int_0^\infty f(x|\theta) \pi'(\theta) d\theta = c' f_{\text{LSME},1}(x; \mu, \sigma^2, \gamma, g_1; \Pi').$$

As a result, (3.6) can be further simplified

$$\begin{aligned}
 E(X|X > x_p) &= \mu + \frac{\gamma c'}{1-p} \int_{x_p}^{\infty} \int_0^{\infty} f(x|\theta) \pi'(\theta) d\theta dx + \frac{c' \sigma^2 \sigma_Z^2}{1-p} \int_0^{\infty} \frac{f_Z^*(\kappa(x_p; \theta))}{\sqrt{\theta} \sigma} \pi'(\theta) d\theta \\
 &= \mu + \frac{\gamma c'}{1-p} \int_{x_p}^{\infty} f_{\text{LSME},1}(x; \mu, \sigma^2, \gamma, g_1; \Pi') dx + \frac{c' \sigma^2 \sigma_Z^2}{1-p} \int_0^{\infty} \frac{f_Z^*(\kappa(x_p; \theta))}{\sqrt{\theta} \sigma} \pi'(\theta) d\theta \\
 &= \mu + \frac{\gamma c'}{1-p} \bar{F}_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, g_1; \Pi') + \frac{c' \sigma^2 \sigma_Z^2}{1-p} f_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi').
 \end{aligned}$$

**Corollary 1.** Let  $X \sim \text{GHE}_1(\mu, \sigma^2, \gamma, g_1, \lambda, a, b)$ . Assume the conditions in Theorem 1 are satisfied, then the TCE of GHE can be computed by:

$$\begin{aligned}
 \text{TCE}_p(X) &= \mu + \frac{\gamma}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} \bar{F}_{\text{GHE},1}(x_p; \mu, \sigma^2, \gamma, g_1, \lambda + 1, a, b) \\
 &\quad + \frac{\sigma^2 \sigma_Z^2}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} f_{\text{GHE},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}, \lambda + 1, a, b).
 \end{aligned} \tag{3.7}$$

*Proof.* From the GIG density in (2.11), we conclude

$$\theta \pi(\theta; \lambda, a, b) = \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} \pi(\theta; \lambda + 1, a, b),$$

by setting

$$c' = \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})},$$

then

$$\pi'(\theta) = (c')^{-1} \theta \pi(\theta) = \pi(\theta; \lambda + 1, a, b),$$

which also is the GIG density. Using (3.5) we can directly obtain (3.7).

#### 4. Portfolio risk decomposition with TCE for multivariate LSME

Consider a risk vector  $\mathbf{Y} = (Y_1, \dots, Y_n)^T$  and  $S = Y_1 + \dots + Y_n$ . We denote  $s_p$  as the  $p$ -quantile of  $S$ , then

$$E(S|S > s_p) = \sum_{i=1}^n E(Y_i|S > s_p),$$

where  $E(Y_i|S > s_p)$  is the contribution of the  $i$ -th risk to the aggregated risks.



Let  $Y = (Y_1, \dots, Y_n) \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$  and  $S = Y_1 + \dots + Y_n$ , then ([6])

$$\begin{aligned} E(Y_i|S = s) &= \int_{-\infty}^{\infty} y_i f(y_i|s) dy_i \\ &= E(Y_i) + \frac{\text{Cov}(Y_i, S)}{\text{Var}(S)} (s - E(S)). \end{aligned}$$

The contribution of risk  $Y_i$ ,  $1 \leq i \leq n$ , to the total TCE can be expressed as

$$\begin{aligned} E(Y_i|S > s_p) &= \int_{s_p}^{\infty} E(Y_i|S = s) dF_S(s|S > s_p) \\ &= \int_{s_p}^{\infty} E(Y_i|S = s) \frac{f_S(s)}{1 - F_S(s_p)} ds \\ &= \frac{1}{1 - p} \int_{s_p}^{\infty} E(Y_i|S = s) f_S(s) ds. \end{aligned}$$

We now exploit this formulation to the multivariate LSME to obtain its portfolio risk decomposition with TCE.

Let us assume  $\mathbf{X} = (X_1, \dots, X_n)^T \sim \text{LSME}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma}, g_n; \Pi)$ . Denote the  $(i, j)$  element of  $\boldsymbol{\Sigma}$  by  $\sigma_{ij}$ , define

$$S = X_1 + \dots + X_n.$$

Then,  $E(X_i|S = s)$  can be further expanded by conditioning on  $\theta$  as follows:

$$\begin{aligned} E(X_i|S = s) &= \int_{-\infty}^{\infty} x_i f(x_i|s) dx_i = \frac{\int_{-\infty}^{\infty} x_i f(x_i, s) dx_i}{f_S(s)} \\ &= \frac{1}{f_S(s)} \int_{-\infty}^{\infty} x_i \int_0^{\infty} f(x_i, s|\theta) d\pi(\theta) dx_i \\ &= \frac{1}{f_S(s)} \int_{-\infty}^{\infty} x_i \int_0^{\infty} f(x_i|s, \theta) f(s|\theta) \pi(\theta) d\theta dx_i \\ &= \frac{1}{f_S(s)} \int_0^{\infty} \left[ \int_{-\infty}^{\infty} x_i f(x_i|s, \theta) dx_i \right] f(s|\theta) \pi(\theta) d\theta. \end{aligned} \quad (4.1)$$

To deal with the inner integral, we define a matrix  $B_i$  of size  $2 \times n$ :

$$B_i = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & 1 \end{bmatrix}. \quad (4.2)$$

The first row vector has 1 in the  $i$ th position. If we keep the general form

$$\mathbf{m}(\theta) = (m_1(\theta), \dots, m_n(\theta))^T,$$

we have

$$B_i \mathbf{X} | \theta = (X_i, S | \theta)^T = B_i \mathbf{m}(\theta) + \theta^{\frac{1}{2}} B_i \boldsymbol{\Sigma}^{\frac{1}{2}} \mathbf{Y},$$

here  $(X_i, S|\theta)^T$  stands for a random column vector of size  $2 \times 1$ , with each element being  $X_i|\theta$  and  $S|\theta$ , respectively. Thus, the joint distribution of  $(X_i, S)$  under the condition of  $\Theta = \theta$  is a bivariate elliptical distribution

$$(X_i, S|\theta)^T \sim E_2(B_i \mathbf{m}(\theta), \theta B_i \Sigma B_i^T, g_2),$$

where the mean vector and covariance matrix of  $(X_i, S|\theta)$  are given by

$$E(B_i X|\theta) = B_i \mathbf{m}(\theta) = (E(X_i|\theta), E(S|\theta))^T = \begin{bmatrix} m_i(\theta) \\ \sum_{j=1}^n m_j(\theta) \end{bmatrix},$$

$$\text{Cov}(B_i X|\theta) = -\psi'(0) \theta B_i \Sigma B_i^T$$

$$= -\psi'(0) \begin{bmatrix} \theta \sigma_{ii} & \theta \sum_{j=1}^n \sigma_{ij} \\ \theta \sum_{j=1}^n \sigma_{ij} & \theta \sigma_S^2 \end{bmatrix},$$

where

$$\sigma_S^2 = \mathbf{1}^T \Sigma \mathbf{1} = \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij}.$$

Therefore, if we impose another condition on  $S$ , we see that  $f(x_i|s, \theta)$  is an elliptical density. In particular

$$\begin{aligned} \int_{-\infty}^{\infty} x_i f(x_i|s, \theta) dx_i &= E(X_i|S = s, \Theta = \theta) \\ &= E(X_i|\theta) + \frac{\text{Cov}(X_i, S|\theta)}{\text{Var}(S|\theta)} (s - E(S|\theta)) \\ &= m_i(\theta) + \frac{-\psi'(0) \sum_{j=1}^n \sigma_{ij}}{-\psi'(0) \sigma_S^2} \left( s - \sum_{j=1}^n m_j(\theta) \right) \\ &= m_i(\theta) + \frac{\sum_{j=1}^n \sigma_{ij}}{\sigma_S^2} \left( s - \sum_{j=1}^n m_j(\theta) \right). \end{aligned}$$

Consequently

$$\begin{aligned} E(X_i|S = s) &= \frac{1}{f_S(s)} \int_0^{\infty} \left[ \int_{-\infty}^{\infty} x_i f(x_i|s, \theta) dx_i \right] f(s|\theta) \pi(\theta) d\theta \\ &= \frac{1}{f_S(s)} \int_0^{\infty} \left[ m_i(\theta) + \frac{\sum_{j=1}^n \sigma_{ij}}{\sigma_S^2} \left( s - \sum_{j=1}^n m_j(\theta) \right) \right] \\ &\quad \times \frac{1}{\sqrt{\theta} \sigma_S} f_Z \left( \frac{s - \sum_{j=1}^n m_j(\theta)}{\sqrt{\theta} \sigma_S} \right) \pi(\theta) d\theta. \end{aligned} \tag{4.3}$$

Eventually

$$E(X_i|S > s_p) = \frac{1}{1-p} \int_{s_p}^{\infty} E(X_i|S = s) f_S(s) ds$$

$$\begin{aligned}
&= \frac{1}{1-p} \int_{s_p}^{\infty} \int_0^{\infty} \left[ m_i(\theta) + \frac{\sum_{j=1}^n \sigma_{ij}}{\sigma_S^2} \left( s - \sum_{j=1}^n m_j(\theta) \right) \right] \\
&\quad \times \frac{1}{\sqrt{\theta} \sigma_S} f_Z \left( \frac{s - \sum_{j=1}^n m_j(\theta)}{\sqrt{\theta} \sigma_S} \right) \pi(\theta) d\theta ds. \tag{4.4}
\end{aligned}$$

This expression, though complex, can produce a closed-form quantity to properly select  $\pi(\theta)$  and  $m_j(\theta)$ .

The portfolio risk decomposition with TCE is additive, that is, the sum of all portfolio risk decomposition must amount to the TCE for  $S$ . We can verify this

$$\begin{aligned}
&\sum_{i=1}^n \mathbb{E}(X_i | S > s_p) \\
&= \frac{1}{1-p} \sum_{i=1}^n \int_{s_p}^{\infty} \int_0^{\infty} \left[ m_i(\theta) + \frac{\sum_{j=1}^n \sigma_{ij}}{\sigma_S^2} \left( s - \sum_{j=1}^n m_j(\theta) \right) \right] \frac{1}{\sqrt{\theta} \sigma_S} f_Z \left( \frac{s - \sum_{j=1}^n m_j(\theta)}{\sqrt{\theta} \sigma_S} \right) \pi(\theta) d\theta ds \\
&= \frac{1}{1-p} \int_{s_p}^{\infty} \int_0^{\infty} \left[ \sum_{i=1}^n m_i(\theta) + \frac{\sum_{i=1}^n \sum_{j=1}^n \sigma_{ij}}{\sigma_S^2} \left( s - \sum_{j=1}^n m_j(\theta) \right) \right] \frac{1}{\sqrt{\theta} \sigma_S} f_Z \left( \frac{s - \sum_{j=1}^n m_j(\theta)}{\sqrt{\theta} \sigma_S} \right) \pi(\theta) d\theta ds \\
&= \frac{1}{1-p} \int_{s_p}^{\infty} \int_0^{\infty} \left[ \sum_{i=1}^n m_i(\theta) + \left( s - \sum_{j=1}^n m_j(\theta) \right) \right] \frac{1}{\sqrt{\theta} \sigma_S} f_Z \left( \frac{s - \sum_{j=1}^n m_j(\theta)}{\sqrt{\theta} \sigma_S} \right) \pi(\theta) d\theta ds \\
&= \frac{1}{1-p} \int_{s_p}^{\infty} \int_0^{\infty} s \frac{1}{\sqrt{\theta} \sigma_S} f_Z \left( \frac{s - \sum_{j=1}^n m_j(\theta)}{\sqrt{\theta} \sigma_S} \right) \pi(\theta) d\theta ds \\
&= \mathbb{E}(S | S > s_p),
\end{aligned}$$

as required. Now the general portfolio risk decomposition with TCE formula for the multivariate LSME distributions class in presented is a more concrete and compact manner when  $\mathbf{m}(\theta)$  is linear in  $\theta$ .

**Theorem 2.** Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)^T \sim \text{LSME}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma}, g_n; \Pi)$  and denote the pdf of  $S = \mathbf{1}^T \mathbf{X}$  by  $f_S(s)$ . Let  $\pi'(\theta) = (c')^{-1} \theta \pi(\theta)$  be a mixing pdf with  $c' = \mathbb{E}(\Theta) < \infty$ . Then the portfolio risk decomposition with TCE for the  $i$ -th marginal variable is given by

$$\begin{aligned}
&\mathbb{E}(X_i | S > s_p) \\
&= b_{0,i} + b_{1,i} \mathbb{E}(S | S > s_p) + \frac{b_{2,i}}{1-p} c' \bar{F}_{\text{LSME},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, g_1; \Pi'), \tag{4.5}
\end{aligned}$$

where  $\Pi'$  is the cdf corresponding to the pdf  $\pi'$ , the coefficients  $b_{0,i}$ ,  $b_{1,i}$ , and  $b_{2,i}$  are defined as

$$b_{0,i} = \mu_i - b_{1,i} \sum_{j=1}^n \mu_j; \quad b_{1,i} = \frac{\sum_{j=1}^n \sigma_{ij}}{\sigma_S^2}; \quad b_{2,i} = \gamma_i - b_{1,i} \sum_{j=1}^n \gamma_j,$$

and  $s_p$  is the  $p$ -quantile of  $S$ .

*Proof.* Let  $m_i(\theta) = \mu_i + \theta \gamma_i$ , and from (4.3) we have

$$\mathbb{E}(X_i | S = s)$$

$$\begin{aligned}
&= \frac{1}{f_S(s)} \int_0^\infty \left[ \mu_i + \frac{\sum_{j=1}^n \sigma_{ij}}{\sigma_S^2} (s - \sum_{j=1}^n \mu_j) + \left( \gamma_i - \sum_{j=1}^n \gamma_j \frac{\sum_{j=1}^n \sigma_{ij}}{\sigma_S^2} \right) \theta \right] \\
&\quad \times \frac{1}{\sqrt{\theta} \sigma_S} f_Z \left( \frac{s - \sum_{j=1}^n \mu_j - \theta \sum_{j=1}^n \gamma_j}{\sqrt{\theta} \sigma_S} \right) \pi(\theta) d\theta \\
&= \frac{1}{f_S(s)} \int_0^\infty [b_{0,i} + b_{1,i}s + b_{2,i}\theta] \frac{1}{\sqrt{\theta} \sigma_S} f_Z \left( \frac{s - \sum_{j=1}^n \mu_j - \theta \sum_{j=1}^n \gamma_j}{\sqrt{\theta} \sigma_S} \right) \pi(\theta) d\theta \\
&= \frac{1}{f_S(s)} \left[ b_{0,i} f_S(s) + b_{1,i} s f_S(s) + b_{2,i} c' f_{\text{LSME},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, g_1; \Pi') \right] \\
&= b_{0,i} + b_{1,i} s + b_{2,i} c' \frac{f_{\text{LSME},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, g_1; \Pi')}{f_S(s)}.
\end{aligned}$$

By inserting this into the portfolio risk decomposition with TCE formulation (4.4), we complete the proof as

$$\begin{aligned}
\mathbb{E}(X_i | S > s_p) &= \int_{s_p}^\infty \mathbb{E}(X_i | S = s) f(s | S > s_p) ds \\
&= \int_{s_p}^\infty \mathbb{E}(X_i | S = s) \frac{f_S(s)}{1-p} ds \\
&= \frac{1}{1-p} \int_{s_p}^\infty (b_{0,i} + b_{1,i}s) f_S(s) ds + \frac{1}{1-p} b_{2,i} c' \int_{s_p}^\infty f_{\text{LSME},1}(s; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, g_1; \Pi') ds \\
&= b_{0,i} + b_{1,i} \mathbb{E}(S | S > s_p) + \frac{b_{2,i}}{1-p} c' \bar{F}_{\text{LSME},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, g_1; \Pi'). \tag{4.6}
\end{aligned}$$

Notice that  $\sum_{i=1}^n b_{0,i} = 0$ ,  $\sum_{i=1}^n b_{1,i} = 1$ , and  $\sum_{i=1}^n b_{2,i} = 0$ , which can be used to verify that the sum of these portfolio risk decomposition amounts to  $\mathbb{E}(S | S > s_p)$ .

**Corollary 2.** Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)^T \sim \text{GHE}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma}, g_n, \lambda, a, b)$ . The portfolio risk decomposition with TCE for the  $i$ -th marginal variable is given by

$$\begin{aligned}
&\mathbb{E}(X_i | S > s_p) \\
&= \mu_i + \frac{\sigma_Z^2 \sum_{j=1}^n \sigma_{ij}}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} f_{\text{GHE},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, \bar{G}; \lambda + 1, a, b) \\
&\quad + \frac{\gamma_i}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} \bar{F}_{\text{GHE},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, g_1, \lambda + 1, a, b). \tag{4.7}
\end{aligned}$$

*Proof.* We can know  $S \sim \text{GHE}_1(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, g_1, \lambda + 1, a, b)$  by using Proposition 1. Using (4.5), we see that TCE of  $S$  is given by

$$\begin{aligned}
&\mathbb{E}(S | S > s_p) \\
&= \mu_i + \frac{\sigma_Z^2 \sum_{j=1}^n \sigma_{ij}}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} f_{\text{GHE},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, \bar{G}, \lambda + 1, a, b) \\
&\quad + \frac{\gamma_i}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} \bar{F}_{\text{GHE},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, g_1, \lambda + 1, a, b).
\end{aligned}$$

Therefore

$$\begin{aligned}
 & E(X_i | S > s_p) \\
 &= \mu_i + b_{1,i} \frac{\sigma_S^2 \sigma_Z^2}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} f_{\text{GHE},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, \bar{G}, \lambda+1, a, b) \\
 &+ \frac{\gamma_i}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} \bar{F}_{\text{GHE},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, g_1, \lambda+1, a, b) \\
 &= \mu_i + \frac{\sigma_Z^2 \sum_{j=1}^n \sigma_{ij}}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} f_{\text{GHE},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, \bar{G}, \lambda+1, a, b) \\
 &+ \frac{\gamma_i}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} \bar{F}_{\text{GHE},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, g_1, \lambda+1, a, b).
 \end{aligned}$$

## 5. Tail variance for univariate case

The TV of the univariate elliptical distribution is introduced by [8]. From (3.1), We can write the TV for  $X|\theta$  as

$$\text{TV}_p(X|\theta) = \text{Var}(X|\theta) \left[ r(\kappa(x_p; \theta)) + h_{Z,Z^*}(\kappa(x_p; \theta)) (\kappa(x_p; \theta) - h_{Z,Z^*}(\kappa(x_p; \theta)) \sigma_Z^2) \right],$$

where  $\kappa(x; \theta)$  is the same as in (3.4),

$$r(z) = \frac{\bar{F}_{Z^*}(z)}{\bar{F}_Z(z)}$$

is the distorted ratio function, and

$$h_{Z,Z^*}(z) = \frac{f_{Z^*}(z)}{F_Z(z)}$$

is the distorted hazard function.

TV can be rewritten as:

$$\begin{aligned}
 \text{TV}_p(X) &= \text{Var}(X|X > x_p) = E \left[ (X - \text{TCE}_p(X))^2 | X > x_p \right] \\
 &= E(X^2 | X > x_p) - [\text{TCE}_p(X)]^2.
 \end{aligned} \tag{5.1}$$

Consequently, we need to derive the second order conditional tail moment  $E(X^2 | X > x_p)$ . We now provide its analytic expression in the following result.

**Proposition 2.** Assume a random variable  $X \sim \text{LSME}_1(\mu, \sigma^2, \gamma, g_1; \Pi)$ . Let  $\pi'(\theta) = (c')^{-1} \theta \pi(\theta)$  and  $\pi''(\theta) = (c'')^{-1} \theta^2 \pi(\theta)$  be two different mixing pdfs with  $c' = E(\Theta) < \infty$  and  $c'' = E(\Theta^2) < \infty$  respectively. Then

$$\begin{aligned}
 & E(X^2 | X > x_p) \\
 &= \frac{\sigma^2 \sigma_Z^2}{1-p} \left[ (x_p + \mu) c' f_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi') + \gamma c'' f_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi'') \right]
 \end{aligned}$$

$$\begin{aligned}
& + c' \bar{F}_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi') \Big] + \frac{\gamma}{1-p} \left[ 2\mu c' \bar{F}_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, g_1; \Pi') \right. \\
& \left. + \gamma c'' \bar{F}_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, g_1; \Pi'') \right] + \mu^2, \tag{5.2}
\end{aligned}$$

where  $\Pi'$  and  $\Pi''$  are two cdfs corresponding to the two different pdfs  $\pi'$  and  $\pi''$ , respectively.

*Proof.* From observing

$$\begin{aligned}
\mathbb{E}(X^2|X > x_p) &= \frac{1}{1-p} \int_{x_p}^{\infty} x^2 f_X(x) dx \\
&= \frac{1}{1-p} \int_0^{\infty} \int_{x_p}^{\infty} x^2 f_{X|\theta}(x) \pi(\theta) dx d\theta \\
&= \frac{1}{1-p} \int_0^{\infty} \mathbb{E}_{X|\theta}(X^2|X > x_p) \bar{F}_{X|\theta}(x_p) \pi(\theta) d\theta.
\end{aligned}$$

To deal with the second order conditional tail moment in the integration, we write it as

$$\mathbb{E}_{X|\theta}(X^2|X > x_p) = \text{TV}_p(X|\theta) + [\text{TCE}_p(X|\theta)]^2. \tag{5.3}$$

From [17], we know

$$\text{TCE}_p(X|\theta) = m(\theta) + h_{Z,Z^*}(\kappa(x_p; \theta)) \frac{\text{Var}(X|\theta)}{\sqrt{\theta}\sigma},$$

taking  $\text{Var}(X|\theta) = \theta\sigma^2\sigma_Z^2$  into consideration, then (5.3) becomes

$$\begin{aligned}
\mathbb{E}_{X|\theta}(X^2|X > x_p) &= \text{Var}(X|\theta) \left[ r(\kappa(x_p; \theta)) + h_{Z,Z^*}(\kappa(x_p; \theta)) (\kappa(x_p; \theta) - h_{Z,Z^*}(\kappa(x_p; \theta)) \sigma_Z^2) \right] \\
&\quad + \left( m(\theta) + h_{Z,Z^*}(\kappa(x_p; \theta)) \frac{\text{Var}(X|\theta)}{\sqrt{\theta}\sigma} \right)^2 \\
&= \text{Var}(X|\theta) r(\kappa(x_p; \theta)) + \text{Var}(X|\theta) h_{Z,Z^*}(\kappa(x_p; \theta)) \frac{x_p - m(\theta)}{\sqrt{\theta}\sigma} \\
&\quad - \text{Var}(X|\theta) (h_{Z,Z^*}(\kappa(x_p; \theta)))^2 \sigma_Z^2 + m^2(\theta) + 2m(\theta) h_{Z,Z^*}(\kappa(x_p; \theta)) \frac{\text{Var}(X|\theta)}{\sqrt{\theta}\sigma} \\
&\quad + (h_{Z,Z^*}(\kappa(x_p; \theta)))^2 \frac{\theta\sigma^2\sigma_Z^2 \text{Var}(X|\theta)}{\theta\sigma^2} \\
&= m^2(\theta) + \text{Var}(X|\theta) \left( r(\kappa(x_p; \theta)) + h_{Z,Z^*}(\kappa(x_p; \theta)) \frac{x_p + m(\theta)}{\sqrt{\theta}\sigma} \right).
\end{aligned}$$

As a result

$$\begin{aligned}
\mathbb{E}(X^2|X > x_p) &= \frac{1}{1-p} \int_0^{\infty} \mathbb{E}_{X|\theta}(X^2|X > x_p) \bar{F}_{X|\theta}(x_p) \pi(\theta) d\theta \\
&= \frac{1}{1-p} \int_0^{\infty} \text{Var}(X|\theta) \left[ \frac{x_p + m(\theta)}{\sqrt{\theta}\sigma} h_{Z,Z^*}(\kappa(x_p; \theta)) + r(\kappa(x_p; \theta)) \right]
\end{aligned}$$

$$\begin{aligned}
& \times \bar{F}_{X|\theta}(x_p)\pi(\theta)d\theta + \frac{1}{1-p} \int_0^\infty m^2(\theta)\bar{F}_{X|\theta}(x_p)\pi(\theta)d\theta \\
& = \frac{1}{1-p} \int_0^\infty \text{Var}(X|\theta) \left[ \frac{x_p + m(\theta)}{\sqrt{\theta}\sigma} f_{Z^*}(\kappa(x_p; \theta)) + \bar{F}_{Z^*}(\kappa(x_p; \theta)) \right] \\
& \quad \times \pi(\theta)d\theta + \frac{1}{1-p} \int_0^\infty \bar{F}_{X|\theta}(x_p)\pi(\theta) (\mu^2 + 2\mu\theta\gamma + \theta^2\gamma^2) d\theta \\
& = \frac{\sigma^2\sigma_Z^2}{1-p} \left[ (x_p + \mu)c' f_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi') + \gamma c'' f_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi'') \right. \\
& \quad \left. + c' \bar{F}_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi') \right] + \frac{\gamma}{1-p} \left[ 2\mu c' \bar{F}_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, g_1; \Pi') \right. \\
& \quad \left. + \gamma c'' \bar{F}_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, g_1; \Pi'') \right] + \mu^2.
\end{aligned}$$

**Theorem 3.** Under assumptions of Proposition 2, the TV of  $X$  is given by

$$\begin{aligned}
& \text{TV}_p(X) \\
& = \frac{\sigma^2\sigma_Z^2}{1-p} \left[ (x_p - \mu)c' f_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi') + \gamma c'' f_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi'') \right. \\
& \quad \left. + c' \bar{F}_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi') \right] + \frac{\gamma^2}{1-p} c'' \bar{F}_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, g_1; \Pi'') \\
& \quad - \left( \frac{c'}{1-p} \right)^2 \left[ \gamma \bar{F}_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, g_1; \Pi') + \sigma^2\sigma_Z^2 f_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi') \right]^2, \quad (5.4)
\end{aligned}$$

where  $\Pi'$  and  $\Pi''$  are two cdfs corresponding to the two different pdfs  $\pi'$  and  $\pi''$ , respectively.

*Proof.* From the Theorem 1, the TCE formula is

$$\begin{aligned}
& E(X|X > x_p) \\
& = \mu + \frac{\gamma c'}{1-p} \bar{F}_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, g_1; \Pi') + \frac{c' \sigma^2 \sigma_Z^2}{1-p} f_{\text{LSME},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \Pi').
\end{aligned}$$

Hence, the result can be derived by using Proposition 2 and (5.1).

**Corollary 3.** Let  $X \sim \text{GHE}_1(\mu, \sigma^2, \gamma, g_1, \lambda, a, b)$ . Assume the conditions in Theorem 3 are satisfied, then the TV for GHE can be computed by:

$$\begin{aligned}
\text{TV}_p(X) & = \frac{\sigma^2\sigma_Z^2}{1-p} \left[ (x_p - \mu) \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} f_{\text{GHE},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \lambda + 1, a, b) \right. \\
& \quad \left. + \gamma \frac{a}{b} \frac{K_{\lambda+2}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} f_{\text{GHE},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \lambda + 2, a, b) \right. \\
& \quad \left. + \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} \bar{F}_{\text{GHE},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \lambda + 1, a, b) \right] \\
& \quad + \frac{\gamma^2}{1-p} \frac{a}{b} \frac{K_{\lambda+2}(\sqrt{ab})}{K_\lambda(\sqrt{ab})} \bar{F}_{\text{GHE},1}(x_p; \mu, \sigma^2, \gamma, g_1, \lambda + 2, a, b)
\end{aligned}$$

$$\begin{aligned}
& - \left( \frac{1}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} \right)^2 \left[ \gamma \bar{F}_{\text{GHE},1}(x_p; \mu, \sigma^2, \gamma, g_1, \lambda+1, a, b) \right. \\
& \left. + \sigma^2 \sigma_Z^2 f_{\text{GHE},1}(x_p; \mu, \sigma^2, \gamma, \bar{G}; \lambda+1, a, b) \right]^2. \tag{5.5}
\end{aligned}$$

*Proof.* From the GIG density in (2.11), we conclude

$$\theta \pi(\theta; \lambda, a, b) = \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} \pi(\theta; \lambda+1, a, b)$$

and

$$\theta^2 \pi(\theta; \lambda, a, b) = \frac{a}{b} \frac{K_{\lambda+2}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} \pi(\theta; \lambda+2, a, b).$$

By setting

$$c' = \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})}, \quad c'' = \frac{a}{b} \frac{K_{\lambda+2}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})},$$

the two pdfs can be presented as

$$\pi'(\theta) = (c')^{-1} \theta \pi(\theta) = \pi(\theta; \lambda+1, a, b)$$

and

$$\pi''(\theta) = (c'')^{-1} \theta^2 \pi(\theta) = \pi(\theta; \lambda+2, a, b),$$

which also are two GIG pdfs. Using (5.4) we can directly obtain (5.5).

## 6. Examples

**Example 6.1** (Generalized hyperbolic distribution). If  $\boldsymbol{\mu} = \mathbf{0}$ ,  $\boldsymbol{\Sigma} = \mathbf{I}_n$  and density generator  $g(u) = e^{-u}$  in (2.2), then the elliptical vector  $\mathbf{Y}$  is said to have a multivariate normal distribution, denoted by  $\mathbf{Y} \sim N_n(\mathbf{0}, \mathbf{I}_n)$ . Letting  $\mathbf{Y} \sim N_n(\mathbf{0}, \mathbf{I}_n)$  in (2.6), then the random vector  $\mathbf{X} \sim \text{GH}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma}, \lambda, a, b)$  is an  $n$ -dimensional generalized hyperbolic (GH) distribution. Therefore, the pdf of the GH distribution is (see [15])

$$\begin{aligned}
& f_{\text{GH}_n}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\gamma}, \lambda, a, b) \\
& = c \frac{K_{\lambda-(n/2)} \left( \sqrt{(a + (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})) (b + \boldsymbol{\gamma}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\gamma})} \right) e^{(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\gamma}}}{\left( a + (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right) (b + \boldsymbol{\gamma}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\gamma})^{\frac{n}{4} - \frac{\lambda}{2}}},
\end{aligned}$$

where the normalizing constant is

$$c = \frac{(ab)^{-\frac{\lambda}{2}} b^{\lambda} (b + \boldsymbol{\gamma}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\gamma})^{(n/2) - \lambda}}{(2\pi)^{n/2} |\boldsymbol{\Sigma}|^{1/2} K_{\lambda} \sqrt{ab}}.$$



From Corollary 1, TCE of GH distribution is given by

$$\begin{aligned} \text{TCE}_p(X) = & \mu + \frac{\gamma}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} \bar{F}_{\text{GH},1}(x_p; \mu, \sigma^2, \gamma, \lambda+1, a, b) \\ & + \frac{\sigma^2}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} f_{\text{GH},1}(x_p; \mu, \sigma^2, \gamma, \lambda+1, a, b). \end{aligned}$$

From Corollary 2, portfolio risk decomposition with TCE for the  $i$ -th marginal of GH distribution is given by

$$\begin{aligned} & E(X_i | S > s_p) \\ = & \mu_i + \frac{\sum_{j=1}^n \sigma_{ij}}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} f_{\text{GH},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, \lambda+1, a, b) \\ & + \frac{\gamma_i}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} \bar{F}_{\text{GH},1}(s_p; \mathbf{1}^T \boldsymbol{\mu}, \mathbf{1}^T \boldsymbol{\Sigma} \mathbf{1}, \mathbf{1}^T \boldsymbol{\gamma}, \lambda+1, a, b). \end{aligned}$$

From Corollary 3, TV of GH distribution is given by

$$\begin{aligned} \text{TV}_p(X) = & \frac{\sigma^2}{1-p} \left[ (x_p - \mu) \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} f_{\text{GH},1}(x_p; \mu, \sigma^2, \gamma, \lambda+1, a, b) \right. \\ & + \gamma \frac{a}{b} \frac{K_{\lambda+2}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} f_{\text{GH},1}(x_p; \mu, \sigma^2, \gamma, \lambda+2, a, b) \\ & \left. + \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} \bar{F}_{\text{GH},1}(x_p; \mu, \sigma^2, \gamma, \lambda+1, a, b) \right] \\ & + \frac{\gamma^2}{1-p} \frac{a}{b} \frac{K_{\lambda+2}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} \bar{F}_{\text{GH},1}(x_p; \mu, \sigma^2, \gamma, \lambda+2, a, b) \\ & - \left( \frac{1}{1-p} \sqrt{\frac{a}{b}} \frac{K_{\lambda+1}(\sqrt{ab})}{K_{\lambda}(\sqrt{ab})} \right)^2 \left[ \gamma \bar{F}_{\text{GH},1}(x_p; \mu, \sigma^2, \gamma, \lambda+1, a, b) \right. \\ & \left. + \sigma^2 f_{\text{GH},1}(x_p; \mu, \sigma^2, \gamma, \lambda+1, a, b) \right]^2. \end{aligned}$$

## 7. Numerical illustration

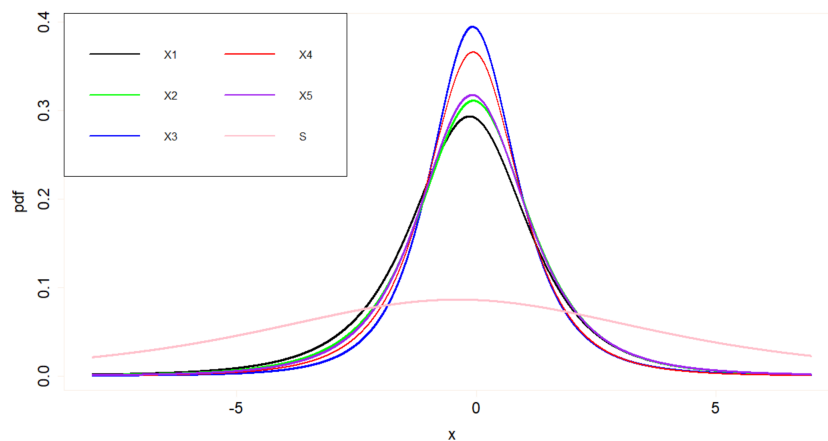
In this section we discuss the TV of five stocks (Amazon, Goldman Sachs, IBM, Google, and Apple) and aggregate portfolio covering a time frame from the 1st of January 2015 to the 1st of January 2017. Ignatieva and Landsman [12] fitted a GH model to five stocks and aggregate portfolio, and obtained the following parameter set based on the maximum likelihood technique.

$$\lambda = -1.18336, \quad a = 1.272016, \quad \psi = 0.348483,$$

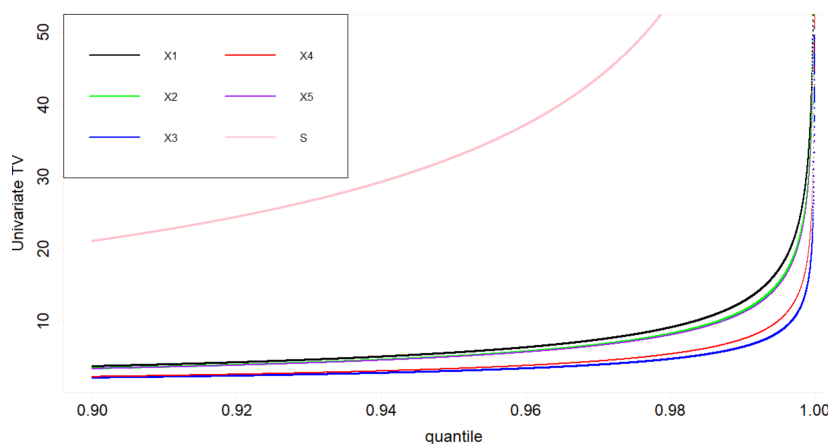
$$\boldsymbol{\mu} = \begin{pmatrix} -0.09977 \\ -0.04555 \\ -0.09355 \\ -0.03669 \\ -0.10367 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} -0.08626 \\ -0.00803 \\ 0.07928 \\ -0.05230 \\ 0.08534 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} 3.387 & 1.407 & 1.103 & 1.828 & 1.354 \\ 1.407 & 3.014 & 1.288 & 1.209 & 1.434 \\ 1.103 & 1.288 & 1.870 & 1.061 & 1.155 \\ 1.828 & 1.209 & 1.061 & 2.171 & 1.220 \\ 1.354 & 1.434 & 1.155 & 1.220 & 2.891 \end{pmatrix}.$$

For the risk analysis, we denote five stocks as  $X_1, \dots, X_5$ . We also consider aggregate portfolio  $S$  where each stock has equal weight for simplicity, so that the aggregate portfolio is defined as  $S = X_1 + \dots + X_5$ . Figure 1 shows the densities for five stocks  $X_i, i = 1, \dots, 5$  and aggregate portfolio  $S$ . The pdf of  $S$  has the largest variance, and Amazon has the largest dispersion among five stocks. IBM has the smallest dispersion. Figure 2 presents the TV for five stocks  $X_i, i = 1, \dots, 5$  and aggregate portfolio  $S$ . All the risk measures increase over the quantile with the TV. Also Figure 2 shows the differences in the TV measure along with five stocks and aggregate portfolio. For the same quantile, the TV of Apple is the largest one and the TV of IBM is the smallest one among the five stocks.



**Figure 1.** Densities for  $X_i, i = 1, \dots, 5$  and  $S$  for GH.



**Figure 2.** TV for  $X_i, i = 1, \dots, 5$  and  $S$  for GH.

## 8. Conclusions

In this paper we generalize the tail risk measure and portfolio risk decomposition with TCE formula derived by [15] for the class of multivariate normal mean-variance mixture distributions to the larger class of multivariate elliptical location-scale mixtures distributions. A prominent member in the normal mean-variance mixture class is the generalized hyperbolic (GH) distribution, which itself can construct a Lévy process. The GH is a special case of normal mean-variance mixture random variable with  $\mathbf{X} \sim N_n(\mathbf{0}, \mathbf{I}_n)$  and the distribution of  $\Theta$  given by a generalized inverse Gaussian (GIG) distribution with three parameters (see [12, 15] for details). Prominent member in the elliptical location-scale mixtures class is the generalized hyper-elliptical (GHE) distribution. The GHE distribution provides excellent fit to univariate and multivariate data, allowing to capture a long right tail in the distribution of losses even more effectively than the GH distribution considered in [12]. And GHE is a special case of elliptical location-scale mixtures random variable with  $\mathbf{X} \sim N_n(\mathbf{0}, \mathbf{I}_n)$  and the distribution of  $\Theta$  given by a generalized inverse Gaussian (GIG) distribution with three parameters. Although the univariate TCE and portfolio risk decomposition with TCE formula for the GHE class was available in [13], it can be derived more efficiently and seen as a special case of TCE for the unified location-scale mixtures of elliptical distributions and risk allocation formula in Theorems 1 and 2, respectively. And the univariate TV formula for the GHE class can be derived efficiently and seen as a special case of TV for the unified location-scale mixtures of elliptical distributions in Theorem 3.

## Acknowledgments

The research was supported by the National Natural Science Foundation of China (No. 12071251).

## Conflict of interest

The authors declare no conflict of interest.

## References

1. P. Artzner, F. Delbaen, J. M. Eber, D. Heath, Coherent measures of risk, *Math. Financ.*, **9** (1999), 203–228. <https://doi.org/10.1111/1467-9965.00068>
2. J. Cai, H. Li, Conditional tail expectations for multivariate phase-type distributions, *J. Appl. Probab.*, **42** (2005), 810–825. <https://doi.org/10.1239/jap/1127322029>
3. S. Cambanis, S. Huang, G. Simons, On the theory of elliptically contoured distributions, *J. Multivariate Anal.*, **11** (1981), 368–385. [https://doi.org/10.1016/0047-259X\(81\)90082-8](https://doi.org/10.1016/0047-259X(81)90082-8)
4. A. Chiragiev, Z. Landsman, Multivariate pareto portfolios: TCE-based capital allocation and divided differences, *Scand. Actuar. J.*, **2007** (2007), 261–280. <https://doi.org/10.1080/03461230701554007>
5. H. Cossette, M. Mailhot, E. Marceau, TVaR-based capital allocation for multivariate compound distributions with positive continuous claim amounts, *Insur. Math. Econ.*, **50** (2012), 247–256. <https://doi.org/10.1016/j.insmatheco.2011.11.006>

6. J. Dhaene, L. Henrard, Z. Landsman, A. Vandendorpe, S. Vanduffel, Some results on the CTE-based capital allocation rule, *Insur. Math. Econ.*, **42** (2008), 855–863. <https://doi.org/10.2139/ssrn.905211>
7. K. T. Fang, S. Kotz, K. W. Ng, *Symmetric multivariate and related distributions*, New York: CRC Press, 1990. <https://doi.org/10.1201/9781351077040>
8. E. Furman, Z. Landsman, Tail variance premium with applications for elliptical portfolio of risks, *Astin Bull.*, **36** (2006), 433–462. <https://doi.org/10.2143/AST.36.2.2017929>
9. M. J. Goovaerts, V. D. B. Eddy, R. J. A. Laeven, Managing economic and virtual economic capital within financial conglomerates, *N. Am. Actuar. J.*, **9** (2005), 77–89. <https://doi.org/10.1080/10920277.2005.10596212>
10. C. W. J. Granger, Time series concepts for conditional distributions, *Oxford B. Econ. Stat.*, **65** (2003), 689–701. <https://doi.org/10.1046/j.0305-9049.2003.00094.x>
11. K. Ignatieva, Z. Landsman, Estimating the tails of loss severity via conditional risk measures for the family of symmetric generalised hyperbolic distributions, *Insur. Math. Econ.*, **65** (2015), 172–186. <https://doi.org/10.1016/j.insmatheco.2015.09.007>
12. K. Ignatieva, Z. Landsman, Conditional tail risk measures for the skewed generalised hyperbolic family, *Insur. Math. Econ.*, **86** (2019), 98–114. <https://doi.org/10.1016/j.insmatheco.2019.02.008>
13. K. Ignatieva, Z. Landsman, A class of generalised hyper-elliptical distributions and their applications in computing conditional tail risk measures, *Insur. Math. Econ.*, **101** (2021), 437–465. <https://doi.org/10.1016/j.insmatheco.2021.08.011>
14. J. H. T. Kim, Conditional tail moments of the exponential family and its related distributions, *N. Am. Actuar. J.*, **14** (2010), 198–216. <https://doi.org/10.1080/10920277.2010.10597585>
15. J. H. T. Kim, S. Y. Kim, Tail risk measures and risk allocation for the class of multivariate normal mean-variance mixture distributions, *Insur. Math. Econ.*, **86** (2019), 145–157. <https://doi.org/10.1016/j.insmatheco.2019.02.010>
16. Z. Landsman, E. Furman, Risk capital decomposition for a multivariate dependent gamma portfolio, *Insur. Math. Econ.*, **37** (2005), 635–649. <https://doi.org/10.1016/j.insmatheco.2005.06.006>
17. Z. Landsman, E. Valdez, Tail conditional expectations for elliptical distributions, *N. Am. Actuar. J.*, **7** (2003), 55–71. <https://doi.org/10.1080/10920277.2003.10596118>
18. Z. Landsman, E. Valdez, Tail conditional expectations for exponential dispersion models, *Astin Bull.*, **35** (2005), 189–209. <https://doi.org/10.1017/S0515036100014124>
19. R. Vernic, Multivariate skew-normal distributions with applications in insurance, *Insur. Math. Econ.*, **38** (2006), 413–426. <https://doi.org/10.1016/j.insmatheco.2005.11.001>
20. R. Vernic, Capital allocation for Sarmanov’s class of distributions, *Methodol. Comput. Appl.*, **19** (2017), 311–330. <https://doi.org/10.1007/s11009-016-9483-x>



AIMS Press

©2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)