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Research article

Existence and asymptotic behavior of normalized solutions for the modified Kirchhoff equations in $\mathbb{R}^3$

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Abstract: This paper is concerned with the following modified Kirchhoff type problem

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u - u\Delta(u^2) - \lambda u = |u|^{p-2}u, \quad x\in\mathbb{R}^3,$$

where $a, b > 0$ are constants and $\lambda \in \mathbb{R}$. When $p = \frac{16}{3}$, we prove that the existence of normalized solution with a prescribed L^2-norm for the above equation by applying constrained minimization method. Moreover, when $p \in \left(\frac{10}{3}, \frac{16}{3}\right)$, we prove the existence of mountain pass type normalized solution for the above modified Kirchhoff equation by using the perturbation method. And the asymptotic behavior of normalized solution as $b \to 0$ is analyzed. These conclusions extend some known ones in previous papers.

Keywords: modified Kirchhoff-type equations; L^2-normalized solutions; asymptotic behavior; constrained minimization method; perturbation method
Mathematics Subject Classification: 35B38, 35B40, 35J62

1. Introduction

In this paper, we are dedicated to studying the following modified Kirchhoff type problem

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u - u\Delta(u^2) - \lambda u = |u|^{p-2}u, \quad x\in\mathbb{R}^3, \tag{1.1}$$

where $a, b > 0$ are constants and $\lambda \in \mathbb{R}$. Equation (1.1) appears in a famous physical context. Indeed, if we set $\lambda = 0$, delete the quasilinear term $u\Delta\left(u^2\right)$ and replace $\mathbb{R}^3$ and $|u|^{p-2}u$ by a bounded domain

$\Omega \subset \mathbb{R}^n$ and $f(x,u)$ respectively in (1.1), then we obtain the following Kirchhoff Dirichlet problem:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.2}$$

Equation (1.2) is related to the stationary analogue of the equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u). \tag{1.3}$$

Equation (1.3) was proposed by Kirchhoff in [1] as a generalization of the following well-known D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2\right)\frac{\partial^2 u}{\partial x^2} = f(x,u),$$

which describes free vibrations of elastic strings. Some early classical studies of Kirchhoff equations can be found in Bernstein [2] and Pohožaev [3]. Much attention was received after Lions [4] introducing an abstract functional framework to this problem. For more relevant mathematical and physical background, we refer readers to papers [5–9], and the references therein.

Kirchhoff's model allows for the changes in length of the string produced by transverse vibrations. In (1.2), u denotes the displacement, $f(x,u)$ denotes the external force and b denotes the initial tension while a is related to the intrinsic properties of the string, such as Young's modulus. We point out that such nonlocal problems also appear in other fields as biological systems, where u describes a process which depends on the average of itself, such as population density, see [10, 11] and the references therein.

In mathematics, equation (1.1) is not a pointwise identity because of the appearance of the term $\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u$. Based on such a characteristic, people call it a nonlocal problem. Moreover, problem (1.1) involves the quasilinear term $u\Delta(u^2)$, whose natural energy functional is not well defined in $H^1(\mathbb{R}^3)$ and variational methods cannot be used directly. These cause some mathematical difficulties, and in the meantime make the study of such a problem more interesting.

In the past years, the following quasilinear Schrödinger equation

$$i\partial_t\varphi + \Delta\varphi + \varphi\Delta\left(|\varphi|^2\right) + |\varphi|^{p-2}\varphi = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N \tag{1.4}$$

has attracted considerable attention, where i denotes the imaginary unit and $\varphi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$, $p \in (2, 2\cdot 2^*)$, $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N = 1,2$. Equation (1.4) appears in various physical fields, such as in dissipative quantum mechanics, in plasma physics and in fluid mechanics, see more information in [12, 13]. One usually searches for standing waves solutions of (1.4), i.e. solutions of the form $\varphi(t,x) = e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ is a parameter and $u : \mathbb{R}^N \to \mathbb{R}$ is a function to be founded, then (1.4) is reduced to be the following stationary equation

$$-\Delta u - u\Delta(u^2) - \lambda u = |u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

which has been intensively studied about its existence and multiplicity results by using minimizations, change of variables, Nehari method and perturbation method. See [14–18, 20] and their references therein. Moreover, for the above stationary equation, when cut out the quasilinear term $u\Delta\left(u^2\right)$, substitute $\left(\frac{1}{|x|} * u^2\right)u$ for $|u|^{p-2}u$ and take $\lambda = -1, N = 3$, then we obtain the following Choquard-Pekar equation

$$-\Delta u + u = \left(\frac{1}{|x|} * u^2\right)u, \quad x \in \mathbb{R}^3.$$

The Choquard-Pekar equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity. For the Choquard-type equation and related problems, we refer to [21–24] and references therein.

In the recent years, Feng et al. [25] studied the following modified Kirchhoff type equation

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\right)\Delta u - u\Delta(u^2) + V(x)u = h(x,u), \quad x \in \mathbb{R}^N, \tag{1.5}$$

where $a > 0, b \geq 0, h \in C\left(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}\right)$ and $V \in C\left(\mathbb{R}^N, \mathbb{R}\right)$. Under suitable assumptions on $V(x)$ and $h(x,u)$, some existence results for positive solutions, negative solutions and sequence of high energy solutions were obtained via a perturbation method. Subsequently, in 2015, Wu [26] studied the existence of infinitely many small energy solutions for equation (1.5) by applying Clark's Theorem to a perturbation functional. And in the same year, He [27] proved the existence of infinitely many solutions for equation (1.5) by using the dual method and the non-smooth critical point theory. This year, Wang and Jia [28] has proved the existence of a positive ground state solution for equation (1.5) by relying on a monotonicity trick and a new version of global compactness lemma.

More recently, the physicists are often interested in "normalized solutions", i.e. solutions with prescribed L^2 -norm. Thus it is interesting for us to study whether (1.1) has a normalized solution. When $p = \frac{16}{3}$, equation (1.1) naturally becomes the following form

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u - u\Delta(u^2) - \lambda u = |u|^{\frac{10}{3}}u, \quad x \in \mathbb{R}^3. \tag{1.6}$$

Then, for any fixed $c > 0$, a solution of equation (1.6) with $\left(\int_{\mathbb{R}^3} u^2\right)^{\frac{1}{2}} = c$ can be viewed as a critical point of the following functional

$$I(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + \int_{\mathbb{R}^3} u^2|\nabla u|^2 - \frac{3}{16}\int_{\mathbb{R}^3} |u|^{\frac{16}{3}}$$

constrained on the L^2 -sphere in H :

$$S_c = \{u \in H : \|u\|_2 = c, c > 0\},$$

where $\|u\|_2 := \left(\int_{\mathbb{R}^3} |u|^2\right)^{\frac{1}{2}}$ and $H := \left\{u \in H^1\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} u^2|\nabla u|^2 < +\infty\right\}$. In this case, the parameter λ is not fixed any longer but appears as an associated Lagrange multiplier. We call $(u_c, \lambda_c) \in S_c \times \mathbb{R}$ a couple of solution to equation (1.6) if u_c is a critical point of $I(u)$ constrained on S_c and λ_c is the associated Lagrange parameter.

As a matter of convenience, we set

$$i_c := \inf_{u \in S_c} I(u).$$

Depending on (4.5) in [15], we know that there exists a positive constant C such that

$$\int_{\mathbb{R}^3} |u|^{\frac{16}{3}} \leq C \left(\int_{\mathbb{R}^3} |u|^2 \right)^{\frac{2}{3}} \int_{\mathbb{R}^3} u^2 |\nabla u|^2$$

for any $u \in H$. And then we can set

$$A := \inf_{u \in H \setminus \{0\}} \frac{\left(\int_{\mathbb{R}^3} u^2 \right)^{\frac{2}{3}} \int_{\mathbb{R}^3} u^2 |\nabla u|^2}{\int_{\mathbb{R}^3} |u|^{\frac{16}{3}}} \geq \frac{1}{C} > 0. \tag{1.7}$$

Our first two main results are as follows:

Theorem 1.1. Let $c_* = \left(\frac{16A}{3}\right)^{\frac{3}{4}}$. Then
(1) $i_c = \begin{cases} 0, & 0 < c \leq c_*, \\ -\infty, & c > c_*; \end{cases}$
(2) i_c has no minimizer for all $c > 0$;
(3) $I(u)$ has no critical point on the constraint S_c for all $0 < c \leq c_*$.

Since Liu et al. [16] have been proved that problem (1.6) has at least one nontrivial solution when $p = \frac{16}{3}$, it is reasonable to conjecture that $I(u)$ has at least one critical point constrained on S_c for some $c > c_*$. For this, we give an affirmative answer in this paper. As far as we know, there are very few papers on this respect. To state our main result, we set

$$B_c := \left\{ u \in S_c : \int_{\mathbb{R}^3} u^2 |\nabla u|^2 < \frac{3}{16} \int_{\mathbb{R}^3} |u|^{\frac{16}{3}} \right\}, \tag{1.8}$$

then it follows from Theorem 1.1-(1) that $B_c \neq \emptyset$ for each $c > c_*$. Define

$$D_c := \{ u \in B_c : G(u) = 0 \},$$

where

$$G(u) := a \int_{\mathbb{R}^3} |\nabla u|^2 + b \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + 5 \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 - \frac{15}{16} \int_{\mathbb{R}^3} |u|^{\frac{16}{3}}.$$

Theorem 1.2. Assume that $c > c_*$, where c_* is given in Theorem (1.1). Then there exists a couple of solution $(u_c, \lambda_c) \in D_c \times \mathbb{R}^-$ for equation (1.6) with $I(u_c) = \inf_{u \in D_c} I(u)$.

In order to prove Theorem 1.2, we set

$$m_c = \inf_{u \in D_c} I(u) \tag{1.9}$$

and need to prove that m_c is attained. But there are two difficulties. First, it is not easy to prove that minimizers of m_c are critical points of $I(u)$ constrained on S_c since there may be two Lagrange multipliers. For this problem, we will use the Pohozaev identity and the famous Gagliardo-Nirenberg inequality. Second, it is difficult to show that D_c is weakly closed thanks to lack of compactness for the minimizing sequences. To overcome this difficulty, we construct a Schwartz symmetric minimizing sequence of m_c and prove the strict monotonicity of the function $c \mapsto m_c$ to avoid possible vanishing and dichotomy of the sequence.

In the last part of this paper, we are devoted to studying existence and asymptotic behavior of normalized solution for equation (1.1). For $p \in \left(\frac{10}{3}, \frac{16}{3}\right)$, the corresponding functional of equation (1.1) is defined as

$$J_b(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + \int_{\mathbb{R}^3} u^2|\nabla u|^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p$$

restricted on the constraint

$$E_k = \left\{u \in H\left(\mathbb{R}^3\right) : \|u\|_2^2 = k\right\}, \quad k > 0,$$

where $H\left(\mathbb{R}^3\right)$ is also defined as $H := \left\{u \in H^1\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} u^2|\nabla u|^2 < +\infty\right\}$. Moreover, for convenience, we set

$$m_k = \inf_{u\in E_k} J_b(u).$$

To get critical point of J_b, we need to use minimax and deformation type arguments, which are more or less standard for smooth variational formulations. However, the variational functional J_b here is not well defined in $H^1\left(\mathbb{R}^3\right)$ let alone being smooth. If a small subspace of $H^1\left(\mathbb{R}^3\right)$ is used, then one loses compactness. To overcome this difficulty of lack of differentiability of J_b, we will apply a perturbation method. Thus we should consider the perturbed functional

$$J_{\mu,b}(u) := \frac{\mu}{4}\int_{\mathbb{R}^3} |\nabla u|^4 dx + J_b(u),$$

where $\mu \in (0, 1]$ is a parameter. For any given $c > 0$, we denote

$$T_k = \left\{u \in X : \|u\|_2^2 = k\right\},$$

where $X = W^{1,4}\left(\mathbb{R}^3\right) \cap H^1\left(\mathbb{R}^3\right)$.

One may observe that $J_{\mu,b}(u)$ is well-defined and C^1 in X (see [19]). The idea is to first look for critical points of $J_{\mu,b}$ for $\mu > 0$ small by using minimax and deformation arguments. After that, we consider convergence of critical point as $\mu \to 0$. Here we will employ the techniques developed in [19] to obtain a certain strong convergence to critical point of the original functional J_b.

Before presenting our last two main results, we need to introduce the following known results:

Lemma 1.3. ([15], Theorem 1.5; [29], Theorem 1.5) Assume that $p \in \left(\frac{10}{3}, \frac{16}{3}\right)$. Then there exists a $k(p) > 0$, given by

$$k(p) := \inf\{k > 0 : m_k < 0\},$$

such that

(1) If $k \in (0, k(p))$, $m_k = 0$ and m_k has no minimizer;
(2) If $k = k(p)$, $m_k = 0$ and m_k admits a minimizer;
(3) If $k \in (k(p), \infty)$, $m_k < 0$ and m_k admits a minimizer.

Now we give the last two main results as follows:

Theorem 1.4. Assume that $p \in \left(\frac{10}{3}, \frac{16}{3}\right)$. Then there exist a $k_0 \in (0, k(p))$, a Schwarz symmetric function v_k and a Lagrange multiplier $\lambda_k < 0$ such that for any $k \in (k_0, \infty)$, $(v_k, \lambda_k) \in E_k \times \mathbb{R}^-$ is a couple of weak solution for equation (1.1), and $J_b(v_k) > 0$ is a mountain pass level.

Theorem 1.5. Assume that $p \in \left(\frac{10}{3}, \frac{16}{3}\right)$ and $(v_b, \lambda_b) \in T_k \times \mathbb{R}$ is a couple of critical point of $J_{\mu,b}$. Then for any sequence $\{b_m\}$ satisfying $b_m \to 0^+$ as $m \to +\infty$, there is a subsequence of $\{b_m\}$, still denoted by $\{b_m\}$, such that $v_{b_m} \rightharpoonup v_0$ in $H^1\left(\mathbb{R}^3\right)$, $v_{b_m}\nabla v_{b_m} \rightharpoonup v_0 \nabla v_0$ in $L^2\left(\mathbb{R}^3\right)$ and $\lambda_{b_m} \to \lambda_0$ in $\mathbb{R}$ as $m \to \infty$, where $(v_0, \lambda_0) \in E_k \times \mathbb{R}^-$ is a couple of weak solution to the following equation

$$-a\Delta v - v\Delta\left(v^2\right) - \lambda v = |v|^{p-2}v, \quad \text{in } \mathbb{R}^3.$$

This paper is organized as follows. In Section 2, we give the proof of Theorems 1.1 and 1.2. In Section 3, we first present some preliminary results, and then give the proof of Theorems 1.4 and 1.5.

Notation. Throughout this paper, we let $u^t(x) := t^{\frac{3}{2}}u(tx)$ for $t > 0$. Denote by $C, C_k\,(k = 1, 2, \ldots)$ various positive constants whose exact value is inessential. And denote by $\to$ ($\rightharpoonup$) the strong (weak) convergence. Moreover, for any $1 \le s < \infty, L^s\left(\mathbb{R}^3\right)$ is the usual Lebesgue space endowed with the norm

$$\|u\|_s^s := \int_{\mathbb{R}^3} |u|^s$$

and $W^{1,s}\left(\mathbb{R}^3\right)$ is the usual Sobolev space endowed with the norm

$$\|u\|_{W^{1,s}} := \|\nabla u\|_s + \|u\|_s.$$

For convenience, we denote by X the space $W^{1,4}\left(\mathbb{R}^3\right) \cap H^1\left(\mathbb{R}^3\right)$ equipped with its natural norm $\|\cdot\|_X := \|\cdot\|_{W^{1,4}} + \|\cdot\|_{H^1}$. Recall that a sequence $\{u_n\} \subset H^1\left(\mathbb{R}^3\right)$ is said to be a Palais-Smale sequence for I if $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$. We say I satisfies the Palais-Smale condition if any Palais-Smale sequence contains a convergent subsequence.

2. Proof of Theorems 1.1 and 1.2

In order to prove our theorems, we need to give some preliminary results.

Lemma 2.1. For $c_* = \left(\frac{16A}{3}\right)^{\frac{3}{4}}$, we have

$$i_c = \begin{cases} 0, & 0 < c \le c_*, \\ -\infty, & c > c_*, \end{cases}$$

where $i_c := \inf_{u \in S_c} I(u)$ is defined as in Theorem 1.1.

Proof. (1). By (1.7), one has

$$\int_{\mathbb{R}^3} |u|^{\frac{16}{3}} \le \frac{1}{A}\|u\|_2^{\frac{4}{3}} \int_{\mathbb{R}^3} |u|^2|\nabla u|^2, \quad \forall u \in H.$$

Therefore, for any $c > 0$ and any $u \in S_c$, since $c_* = \left(\frac{16}{3}A\right)^{\frac{3}{4}}$, we get

$$\frac{3}{16}\int_{\mathbb{R}^3} |u|^{\frac{16}{3}} \le \left(\frac{c}{c_*}\right)^{\frac{4}{3}} \int_{\mathbb{R}^3} |u|^2|\nabla u|^2. \tag{2.1}$$

And then, for any $0 < c \leq c_*$ and any $u \in S_c$, it follows from (2.1) that

$$\frac{3}{16}\int_{\mathbb{R}^3}|u|^{\frac{16}{3}} \leq \int_{\mathbb{R}^3}|u|^2|\nabla u|^2,$$

which means

$$I(u) \geq \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 > 0.$$

Thus, we deduce $i_c \geq 0$ by the arbitrary of u.

On the other hand, since $u^t(x) := t^{\frac{3}{2}}u(tx)$ for $t > 0$, to facilitate the estimation of $I(u^t)$, we firstly compute:

$$\int_{\mathbb{R}^3}\left|\nabla u^t\right|^2 = t^2\int_{\mathbb{R}^3}|\nabla u|^2, \quad \int_{\mathbb{R}^3}\left|u^t\right|^2\left|\nabla u^t\right|^2 = t^5\int_{\mathbb{R}^3}u^2\left|\nabla u\right|^2, \quad \int_{\mathbb{R}^3}\left|u^t\right|^{\frac{16}{3}} = t^5\int_{\mathbb{R}^3}|u|^{\frac{16}{3}}.$$

Then we get

$$I(u^t) = \frac{at^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 + t^5\left(\int_{\mathbb{R}^3}|u|^2|\nabla u|^2 - \frac{3}{16}\int_{\mathbb{R}^3}|u|^{\frac{16}{3}}\right) \to 0$$

as $t \to 0^+$. Therefore, $i_c \leq 0$.

So $i_c = 0$ for each $0 < c \leq c_*$.

(2). For any $c > c_* = \left(\frac{16}{3}A\right)^{\frac{3}{4}}$, we obtain $A < \frac{3}{16}c^{\frac{4}{3}}$. By the definition of A there exists $u \in H\backslash\{0\}$ such that

$$\|u\|_2^{\frac{4}{3}}\int_{\mathbb{R}^3}|u|^2|\nabla u|^2 < \frac{3}{16}c^{\frac{4}{3}}\int_{\mathbb{R}^3}|u|^{\frac{16}{3}}.$$

Setting $v := \frac{c}{\|u\|_2}u$, we see that $v \in S_c$ satisfies

$$\begin{aligned}\int_{\mathbb{R}^3}|v|^2|\nabla v|^2 &= \left(\frac{c}{\|u\|_2}\right)^4\int_{\mathbb{R}^3}|u|^2|\nabla u|^2\\ &< \frac{3}{16}\left(\frac{c}{\|u\|_2}\right)^{\frac{16}{3}}\int_{\mathbb{R}^3}|u|^{\frac{16}{3}}\\ &= \frac{3}{16}\int_{\mathbb{R}^3}|v|^{\frac{16}{3}}.\end{aligned}$$

Then for any $t > 0$, one has

$$I(v^t) = \frac{at^2}{2}\int_{\mathbb{R}^3}|\nabla v|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\right)^2 - t^5\left(\frac{3}{16}\int_{\mathbb{R}^3}|v|^{\frac{16}{3}} - \int_{\mathbb{R}^3}|v|^2|\nabla v|^2\right) \to -\infty$$

as $t \to +\infty$, which implies that $i_c = -\infty$ for any $c > c_*$. □

By the proof of Lemma 2.1, it is easy to show the following result.

Corollary 1. For $u \in S_c$, we get

$$\begin{cases} B_c = \emptyset, & 0 < c \le c_*, \\ B_c \neq \emptyset, & c > c_*, \end{cases}$$

where B_c is defined as in (1.8). Moreover, for any $0 < c \le c_*$, we obtain $I(u) > 0$.

Lemma 2.2. i_c has no minimizer for all $c > 0$.

Proof. The proof follows directly from Lemma 2.1 and Corollary 1. □

Lemma 2.3. $I(u)$ has no critical point constrained on S_c for each $c \in (0, c_*]$.

Proof. By contradiction, we suppose that there exist some $c \in (0, c_*]$ and some $u_c \in S_c$ such that $\left(I|_{s_c}\right)'(u_c) = 0$, then there exists a Lagrange multiplier $\lambda_c \in \mathbb{R}$ such that $I'(u_c) - \lambda_c u_c = 0$. Hence by Lemma 5.1 in [30], we know that u_c satisfies the following Pohozaev identity:

$$\frac{a}{2}\int_{\mathbb{R}^3} |\nabla u_c|^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u_c|^2\right)^2 + \int_{\mathbb{R}^3} |u_c|^2 |\nabla u_c|^2 - \frac{3}{2}\lambda_c \int_{\mathbb{R}^3} |u_c|^2 - \frac{9}{16}\int_{\mathbb{R}^3} |u_c|^{\frac{16}{3}} = 0.$$

This together with $I'(u_c) = \lambda_c u_c$ implies that

$$a\int_{\mathbb{R}^3} |\nabla u_c|^2 + b\left(\int_{\mathbb{R}^3} |\nabla u_c|^2\right)^2 + 5\int_{\mathbb{R}^3} |u_c|^2 |\nabla u_c|^2 = \frac{15}{16}\int_{\mathbb{R}^3} |u_c|^{\frac{16}{3}},$$

which implies that $u_c \in B_c$. It is a contradiction with Corollary 1. So the lemma is proved. □

Proof of Theorem 1.1. Theorem 1.1 follows directly from Lemmas 2.1–2.3.

Next, we are absorbed in dealing with the existence of normalized solutions for $I(u)$ restricted to S_c when $c > c_*$. Depending on Corollary 1 and Lemma 2.3, we try to look for normalized solutions constrained on B_c.

Lemma 2.4. For any $u \in B_c$, there exists a unique $\tilde{t} > 0$ such that $I\left(u^{\tilde{t}}\right) = \max_{t>0} I\left(u^t\right)$ and $G\left(u^{\tilde{t}}\right) = 0$, where

$$G(u) = a\int_{\mathbb{R}^3} |\nabla u|^2 + b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + 5\int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 - \frac{15}{16}\int_{\mathbb{R}^3} |u|^{\frac{16}{3}}.$$

Proof. For any $u \in B_c$, we consider the following path $\Phi : (0, +\infty) \to \mathbb{R}$ defined as

$$\Phi(t) = \frac{t^2}{2}a\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{t^4}{4}b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - t^5\left(\frac{3}{16}\int_{\mathbb{R}^3} |u|^{\frac{16}{3}} - \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2\right),$$

namely that $\Phi(t) = I\left(u^t\right)$. By an elementary analysis, it is easy to infer that Φ has a unique positive critical point $\tilde{t}$ corresponding to its maximum, i.e. $\Phi'(\tilde{t}) = 0$ and $\Phi(\tilde{t}) = \max_{t>0} \Phi(t)$. Hence $I\left(u^{\tilde{t}}\right) = \max_{t>0} I\left(u^t\right)$ and

$$a\tilde{t}^2\int_{\mathbb{R}^3} |\nabla u|^2 + b\tilde{t}^4\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + 5\tilde{t}^5\int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 - \frac{15}{16}\tilde{t}^5\int_{\mathbb{R}^3} |u|^{\frac{16}{3}} = 0,$$

which means $G\left(u^{\tilde{t}}\right) = 0$. □

Lemma 2.5. For any $c > c_*$, $I(u)$ is bounded from below and coercive on D_c. Moreover, there exist $C_0, C_1 > 0$ such that $\int_{\mathbb{R}^3} |u|^{\frac{16}{3}} \geq C_0$ and $I(u) \geq C_1$ for all $u \in D_c$.

Proof. For any $c > c_*$, it follows from Lemma 2.4 that $D_c \neq \emptyset$. Then for any $u \in D_c$, we obtain

$$I(u) = I(u) - \frac{1}{5}G(u) = \frac{3}{10}a \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{20}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \geq 0. \tag{2.2}$$

Then I is bounded from below and coercive on D_c.

Moreover, by the Gagliardo-Nirenberg inequality (see [31]), there exists a positive constant C such that

$$\int_{\mathbb{R}^3} |u|^{\frac{16}{3}} \leq C\|\nabla u\|_2^5 \|u\|_2^{\frac{1}{3}}. \tag{2.3}$$

Relying on (2.3), we deduce that there exists $C > 0$ depending only on c such that

$$\left(\frac{1}{C}\int_{\mathbb{R}^3} |u|^{\frac{16}{3}}\right)^{\frac{2}{5}} \leq \|\nabla u\|_2^2 \leq \frac{15}{16}\int_{\mathbb{R}^3} |u|^{\frac{16}{3}} \leq \frac{15C}{16}\|\nabla u\|_2^5,$$

which implies

$$\int_{\mathbb{R}^3} |u|^{\frac{16}{3}} \geq \left(\frac{16}{15C^{\frac{2}{5}}}\right)^{\frac{5}{3}} =: C_0$$

and

$$\|\nabla u\|_2^2 \geq \left(\frac{16}{15C}\right)^{\frac{2}{3}}.$$

The last inequality and (2.2) deduce that

$$I(u) \geq \frac{3}{10}a \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{20}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \geq \frac{3}{10}a\left(\frac{16}{15C}\right)^{\frac{2}{3}} + \frac{b}{20}\left(\frac{16}{15C}\right)^{\frac{4}{3}} =: C_1$$

for all $u \in D_c$. □

Lemma 2.6. The function $c \mapsto m_c$ is strictly decreasing on $(c_*, +\infty)$, where m_c is defined as (1.9).

Proof. By Lemma 2.5, we see that $m_c > 0$. Then for any $c_1, c_2 \in (c_*, +\infty)$ satisfying $c_1 < c_2$, it is enough to prove that $m_{c_2} < m_{c_1}$.

By the definition of m_{c_1} and Lemma 2.4, there exists $u_n \in D_{c_1}$ such that $I(u_n) \leq m_{c_1} + \frac{1}{n}$ and $I(u_n) = \max_{t>0} I(u_n^t)$. Setting $v_n(x) := \left(\frac{c_1}{c_2}\right)^{\frac{1}{2}} u_n\left(\frac{c_1}{c_2}x\right)$, then $\|v_n\|_2 = c_2$ and $\|\nabla v_n\|_2 = \|\nabla u_n\|_2$. And we have

$$\int_{\mathbb{R}^3} |v_n|^2 |\nabla v_n|^2 = \left(\frac{c_1}{c_2}\right)\int_{\mathbb{R}^3} |u_n|^2 |\nabla u_n|^2 \leq \int_{\mathbb{R}^3} |u_n|^2 |\nabla u_n|^2 \tag{2.4}$$

and

$$\int_{\mathbb{R}^3} |v_n|^{\frac{16}{3}} = \left(\frac{c_2}{c_1}\right)^{\frac{1}{3}} \int_{\mathbb{R}^3} |u_n|^{\frac{16}{3}} > \int_{\mathbb{R}^3} |u_n|^{\frac{16}{3}},$$

i.e. $v_n \in B_{c_2}$ since $u_n \in D_{c_1}$. Then by Lemma 2.4 there exists a sequence $\{t_n\} \subset \mathbb{R}^+$ such that $v_n^{t_n} \in D_{c_2}$ and $I\left(v_n^{t_n}\right) = \max_{t>0} I\left(v_n^t\right)$.

Furthermore, there exists $C > 0$ independent of n such that $t_n \geq C$ for all $n \in \mathbb{N}^+$. Suppose not, we assume that $t_n \to 0$ as $n \to +\infty$. By the definition of $\{v_n\}$, we see that $\{v_n\}$ is uniformly bounded in H. Then we conclude from Lemma 2.5 that $0 < m_{c_2} \leq \lim_{n\to+\infty} I\left(v_n^{t_n}\right) \to 0$, which is impossible. Therefore, by Lemma 2.5 and (2.4) one has

$$\begin{aligned} m_{c_2} \leq I\left(v_n^{t_n}\right) &\leq I\left(u_n^{t_n}\right) - \left[\left(\frac{c_2}{c_1}\right)^{\frac{1}{3}} - 1\right] t_n^5 \int_{\mathbb{R}^3} |u_n|^{\frac{16}{3}} \\ &< \max_{t>0} I\left(u_n^t\right) - \left[\left(\frac{c_2}{c_1}\right)^{\frac{1}{3}} - 1\right] t_n^5 C_0 \\ &\leq I(u_n) - \left[\left(\frac{c_2}{c_1}\right)^{\frac{1}{3}} - 1\right] t_n^5 C_0 \\ &\leq m_{c_1} + \frac{1}{n} - \left[\left(\frac{c_2}{c_1}\right)^{\frac{1}{3}} - 1\right] C^5 C_0, \end{aligned}$$

where C_0 is a positive constant independent of n given in Lemma 2.5. Thus we get $m_{c_2} < m_{c_1}$, which implies that the proof is completed. □

Lemma 2.7. For any $c > c_*$, each minimizer of m_c is a critical point of $I(u)$ constrained on S_c.

Proof. We define $\widetilde{D}(c) := \{u \in S_c \mid G(u) = 0\}$, and then we easily know that $\widetilde{D}(c) = D(c)$. Suppose that $u \in D_c$ is a minimizer of m_c, then u is also a minimizer of $\widetilde{m}_c$. Hence by standard arguments, there exist $\lambda, \theta \in \mathbb{R}$ such that $I'(u) - \lambda u - \theta G'(u) = 0$, namely that

$$\begin{aligned} -(1-2\theta)a\Delta u - (1-4\theta)b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u - (1-5\theta)\, u\Delta\left(u^2\right) \\ - (1-5\theta)\, |u|^{\frac{10}{3}} u = \lambda u. \end{aligned} \tag{2.5}$$

Thus we need to prove that $\theta = 0$.

By contradiction, we suppose that $\theta \neq 0$. By (2.5), we know that u satisfies the following Pohozaev identity

$$\begin{aligned} P(u) := \frac{1-2\theta}{2} a \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1-4\theta}{2} b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + (1-5\theta)\int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 \\ - \frac{3}{2}\lambda \int_{\mathbb{R}^3} u^2 - \frac{9}{16}(1-5\theta)\int_{\mathbb{R}^3} |u|^{\frac{16}{3}} = 0. \end{aligned}$$

This together with (2.5) implies that

$$\begin{aligned} (1-2\theta)a\int_{\mathbb{R}^3} |\nabla u|^2 + (1-4\theta)\, b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + 5(1-5\theta)\int_{\mathbb{R}^3} |u|^2|\nabla u|^2 \\ - \frac{15}{16}(1-5\theta)\int_{\mathbb{R}^3} |u|^{\frac{16}{3}} = 0, \end{aligned}$$

which means

$$G(u)-\theta\left[2a\int_{\mathbb{R}^3}|\nabla u|^2+4b\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2+25\left(\int_{\mathbb{R}^3}|u|^2|\nabla u|^2-\frac{3}{16}\int_{\mathbb{R}^3}|u|^{\frac{16}{3}}\right)\right]=0.$$

This equality combined with $G(u)=0$ and $\theta\neq 0$ implies that $\int_{\mathbb{R}^3}|\nabla u|^2=0$, which contradicts Lemma 2.5. So $\theta=0$, and then u is a critical point of $I(u)$ constrained on S_c . □

Lemma 2.8. Suppose that $\{u_n\}\subset H$ is a bounded sequence of Schwartz Symmetric functions satisfying $u_n\rightharpoonup u$ in H, then

$$\int_{\mathbb{R}^3}|\nabla u|^2+5\int_{\mathbb{R}^3}|u|^2|\nabla u|^2\leq\liminf_{n\to+\infty}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2+5\int_{\mathbb{R}^3}|u_n|^2\,|\nabla u_n|^2\right).$$

Proof. The proof is similar to Lemma 4.3 in [15], so we omit it here. □

Proof of Theorem 1.2. Assume that $\{u_n\}\subset D_c$ is a minimizing sequence of m_c, it follows from Lemma 2.5 that $\{u_n\}$ is uniformly bounded in H. To obtain a minimizer of m_c, let $\{v_n\}$ be the sequence of Schwartz symmetric functions for $\{u_n\}$. Then by the Pólya-Szegö inequality ([15], Lemma 4.3), one has

$$\begin{aligned}&\int_{\mathbb{R}^3}|\nabla v_n|^2\leq\int_{\mathbb{R}^3}|\nabla u_n|^2,\quad\int_{\mathbb{R}^3}|v_n|^2=\int_{\mathbb{R}^3}|u_n|^2,\quad\int_{\mathbb{R}^3}|v_n|^{\frac{16}{3}}=\int_{\mathbb{R}^3}|u_n|^{\frac{16}{3}},\\&\int_{\mathbb{R}^3}|\nabla v_n|^2+5\int_{\mathbb{R}^3}|v_n|^2\,|\nabla v_n|^2\leq\int_{\mathbb{R}^3}|\nabla u_n|^2+5\int_{\mathbb{R}^3}|u_n|^2\,|\nabla u_n|^2.\end{aligned}\tag{2.6}$$

By (2.6), we infer that the sequence $\{v_n\}$ is also uniformly bounded in H. And we obtain

$$G(v_n)\leq G(u_n)=0.\tag{2.7}$$

Since $\{v_n\}$ is uniformly bounded, up to a subsequence, there exists $v\in H$ such that

$$\begin{cases}v_n\rightharpoonup v, & \text{in } H,\\ v_n\to v, & \text{in } L^p\left(\mathbb{R}^3\right),\quad\forall p\in(2,2\cdot 2^*).\end{cases}\tag{2.8}$$

Moreover, by (2.6) and Lemma 2.5, one obtains

$$\int_{\mathbb{R}^3}|v|^{\frac{16}{3}}=\lim_{n\to+\infty}\int_{\mathbb{R}^3}|v_n|^{\frac{16}{3}}=\lim_{n\to+\infty}\int_{\mathbb{R}^3}|u_n|^{\frac{16}{3}}\geq C_0>0,\tag{2.9}$$

where $C_0>0$ is a constant given in Lemma 2.5. (2.9) implies that $v\neq 0$. Setting $\beta:=\|v\|_2$, then $\beta\in(0,c]$. By Lemma 2.8 and (2.7)–(2.9), we deduce

$$G(v)\leq\liminf_{n\to+\infty}G(v_n)\leq 0,$$

i.e. $v \in B_\beta$. So it follows from Corollary 1 that $\beta \in (c_*, c]$. By Lemma 2.4, there exists a unique $t \in (0,1]$ such that $v^t \in D_\beta$. Then by Lemma 2.6 we have

$$\begin{aligned} m_\beta \leq I(v^t) = I(v^t) - \frac{1}{5}G(v^t) &= \frac{3a}{10}t^2\int_{\mathbb{R}^3}|\nabla v|^2 + \frac{b}{20}t^4\left(\int_{\mathbb{R}^3}|\nabla v|^2\right)^2 \\ &\leq \liminf_{n\to+\infty}\left[\frac{3}{10}a\int_{\mathbb{R}^3}|\nabla v_n|^2 + \frac{b}{20}\left(\int_{\mathbb{R}^3}|\nabla v_n|^2\right)^2\right] \\ &\leq \liminf_{n\to+\infty}\left[\frac{3}{10}a\int_{\mathbb{R}^3}|\nabla u_n|^2 + \frac{b}{20}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2\right] \\ &= \liminf_{n\to+\infty}\left(I(u_n) - \frac{1}{5}G(u_n)\right) \\ &= m_c \leq m_\beta, \end{aligned}$$

where the equality holds only for $\beta = c$ and $t = 1$. Therefore, $\beta = c$ and $I(v) = m_c$. And then we get a minimizer $v \in D_c$ of m_c. By Lemma 2.7, we know that v is a critical point of $I(v)$ constrained on S_c. Thus, there exists $\lambda_c \in \mathbb{R}$ such that $I'(v) - \lambda_c v = 0$, namely that

$$\begin{aligned} \lambda_c c^2 &= \langle I'(v), v\rangle \\ &= a\int_{\mathbb{R}^3}|\nabla v|^2 + b\left(\int_{\mathbb{R}^3}|\nabla v|^2\right)^2 + 4\int_{\mathbb{R}^3}|v|^2|\nabla v|^2 - \int_{\mathbb{R}^3}|v|^{\frac{16}{3}}. \end{aligned}$$

This together with $G(v) = 0$ deduces that

$$\lambda_c c^2 \leq \frac{15}{16}\int_{\mathbb{R}^3}|v|^{\frac{16}{3}} - \int_{\mathbb{R}^3}|v|^{\frac{16}{3}} < 0,$$

i.e. $\lambda_c < 0$. So $(v, \lambda_c) \in S_c \times \mathbb{R}^-$ is a couple of solution to problem (1.6).

3. Proof of Theorems 1.4 and 1.5

In this section, we need some notations and useful preliminary results. For the sake of convenience, we set

$$\begin{aligned} &\mathcal{A}(u) = \int_{\mathbb{R}^3}|\nabla u|^4, \ \ \mathcal{B}(u) = \int_{\mathbb{R}^3}|\nabla u|^2, \ \ C(u) = \int_{\mathbb{R}^3}|u|^2|\nabla u|^2, \\ &\mathcal{D}(u) = \int_{\mathbb{R}^3}|u|^2, \ \ \mathcal{E}(u) = \int_{\mathbb{R}^3}|u|^p. \end{aligned}$$

Correspondingly, setting $u^t(x) = t^{\frac{3}{2}}u(tx)$ for $t > 0$, by calculation we can easily get

$$\begin{aligned} &\mathcal{A}(u^t) = t^7\mathcal{A}(u), \ \ \mathcal{B}(u^t) = t^2\mathcal{B}(u), \ \ C(u^t) = t^5C(u), \\ &\mathcal{D}(u^t) = \mathcal{D}(u), \ \ \mathcal{E}(u^t) = t^{\frac{3}{2}(p-2)}\mathcal{E}(u) \end{aligned}$$

and

$$J_{\mu,b}(u^t) = \frac{\mu}{4}t^7\mathcal{A}(u) + \frac{a}{2}t^2\mathcal{B}(u) + \frac{b}{4}t^4\left(\mathcal{B}(u)\right)^2 + t^5C(u) - \frac{t^{\frac{3}{2}(p-2)}}{p}\mathcal{E}(u).$$

Lemma 3.1. Assume that $p \in \left(\frac{10}{3}, 6\right)$. Then for $l > 0$, setting

$$C_l := \left\{ u \in T_k : \int_{\mathbb{R}^3} \left(1 + |u|^2\right) |\nabla u|^2 = l \right\},$$

there exists a $l_0 > 0$ sufficiently small such that for all $l \in (0, l_0]$ and all $\mu > 0$,

$$J_{\mu,b}(u) \geq \frac{1}{2} \min \left\{ \frac{a}{2}, 1 \right\} l > 0 \quad \text{and} \quad Q_{\mu,b}(u) \geq \frac{1}{2} \min \{a, 5\} l > 0$$

for all $u \in C_l$, where

$$\begin{aligned} Q_{\mu,b}(u) &:= \frac{d}{dt} J_{\mu,b}\left(u^t\right)\Big|_{t=1} \\ &= \frac{7}{4}\mu\mathcal{A}(u) + a\mathcal{B}(u) + b\left(\mathcal{B}(u)\right)^2 + 5\mathcal{C}(u) - \frac{3(p-2)}{2p}\mathcal{E}(u). \end{aligned}$$

Proof. When $p \in \left(\frac{10}{3}, 6\right)$, by the Gagliardo-Nirenberg inequality, there exists a $L_1 = L_1(p) > 0$, such that for all $u \in X$

$$\begin{aligned} \int_{\mathbb{R}^3} |u|^p &\leq L_1 \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{3(p-2)}{4}} \left(\int_{\mathbb{R}^3} |u|^2 \right)^{\frac{6-p}{4}} \\ &\leq L_1 \left(\int_{\mathbb{R}^3} \left(1 + |u|^2\right) |\nabla u|^2 \right)^{\frac{3(p-2)}{4}} \left(\int_{\mathbb{R}^3} |u|^2 \right)^{\frac{6-p}{4}}. \end{aligned}$$

Thus, when $k \in (0, k(p))$, for a constant $L_2 = L_2(p) > 0$ independent of $k > 0$ and for all $u \in T_k$, we obtain

$$J_{\mu,b}(u) \geq \min \left\{ \frac{a}{2}, 1 \right\} \int_{\mathbb{R}^3} \left(1 + |u|^2\right) |\nabla u|^2 - L_2 \left(\int_{\mathbb{R}^3} \left(1 + |u|^2\right) |\nabla u|^2 \right)^{\frac{3(p-2)}{4}}. \tag{3.1}$$

Note that $\frac{3(p-2)}{4} > 1$ as $p \in \left(\frac{10}{3}, 6\right)$. Therefore, by (3.1), we infer that there exists $l_0 > 0$ small enough such that for all $l \in (0, l_0]$

$$J_{\mu,b}(u) \geq \frac{1}{2} \min \left\{ \frac{a}{2}, 1 \right\} l > 0, \quad \text{for all } u \in C_l.$$

Next, by the similar way, we get there exists a constant $L_3 = L_3(p) > 0$ such that for all $u \in T_k$

$$Q_{\mu,b}(u) \geq \min \{a, 5\} \int_{\mathbb{R}^3} \left(1 + |u|^2\right) |\nabla u|^2 - L_3 \left(\int_{\mathbb{R}^3} \left(1 + |u|^2\right) |\nabla u|^2 \right)^{\frac{3(p-2)}{4}},$$

which means there exists $l_0 > 0$ small enough such that for all $l \in (0, l_0]$

$$Q_{\mu,b}(u) \geq \frac{1}{2} \min \{a, 5\} l > 0, \quad \text{for all } u \in C_l.$$

This proof is completed. □

Lemma 3.2. Assume that $p \in \left(\frac{10}{3}, \frac{16}{3}\right)$. Then there exist $k_0 \in (0, k(p))$ and $\mu_0 > 0$ small enough such that for any fixed $k \in [k_0, \infty)$ and $\mu \in (0, \mu_0)$ the functional $J_{\mu,b}$ has a mountain pass geometry on the constraint T_k, that is, there exist $(u_0, u_1) \in T_k \times T_k$ both Schwarz symmetric, such that

$$\eta_{\mu,b}(k) = \inf_{g\in\Gamma_k} \max_{t\in[0,1]} J_{\mu,b}(g(t)) > \max\left\{J_{\mu,b}(u_0), J_{\mu,b}(u_1)\right\},$$

where

$$\Gamma_k = \{g \in C([0,1], T_k) : g(0) = u_0, g(1) = u_1\}.$$

Proof. Firstly, to choose $u_0 \in T_k$, we consider, for any arbitrary Schwartz symmetric function, the scaling

$$v^\theta(x) := \theta^{3/2} v(\theta x), \forall \theta > 0,$$

which deduces that

$$v^\theta \in T_k, \forall\theta > 0, \quad \lim_{\theta\to 0} J_{\mu,b}\left(v^\theta\right) = 0 \quad \text{and} \quad \lim_{\theta\to 0} \int_{\mathbb{R}^3} \left(1 + \left|v^\theta\right|^2\right)\left|\nabla v^\theta\right|^2 = 0.$$

Hence setting $u_0 = v^{\theta_0}$ for a fixed $\theta_0 > 0$ sufficiently small, we get

$$J_{\mu,b}(u_0) \le \frac{1}{4}\min\left\{\frac{a}{2}, 1\right\} l_0 \quad \text{and} \quad \int_{\mathbb{R}^3} \left(1 + |u_0|^2\right)|\nabla u_0|^2 < l_0.$$

Secondly, to choose u_1 we distinguish the cases $k > k(p)$ and $k \le k(p)$.

Case 1: When $k > k(p)$, we follow from Lemma 1.3-(3) that J_b has a global minimizer $u_k \in H^1\left(\mathbb{R}^3\right)$ satisfying $J_b(u_k) = m_k < 0$. Without restriction we can assume that u_k is Schwarz symmetric. And then combining Lemma 4.6 in [15] and Lemma 5.10 in [17], we see that $u_k \in X$ has an exponential decrease at infinity. Thus setting $u_1 = u_k$ and taking $l_0 > 0$ smaller if necessary, we infer that

$$J_b(u_1) < 0 \quad \text{and} \quad \int_{\mathbb{R}^3} \left(1 + |u_1|^2\right)|\nabla u_1|^2 > l_0,$$

where the value $l_0 > 0$ is defined in Lemma 3.1. Now taking $\mu_0 > 0$ small enough, by continuity we get $J_{\mu,b}(u_1) < 0$ for all $\mu \in (0, \mu_0)$.

Case 2: When $k \le k(p)$, we know that for $k = k(p)$ J_b has a global minimizer $u_{k(p)}$, which is Schwarz symmetric and satisfies $J_b\left(u_{k(p)}\right) = 0$. We set

$$l_1 = \int_{\mathbb{R}^3} \left(1 + \left|u_{k(p)}\right|^2\right)\left|\nabla u_{k(p)}\right|^2.$$

Restricting $l_0 > 0$ in Lemma 3.1 if necessary we can assume that $2l_0 < l_1$. By continuity there exists a $t_0 < 1$ such that for any $t \in (t_0, 1)$

$$J_b\left(\sqrt{t}u_{k(p)}\right) < \frac{1}{2}\min\left\{\frac{a}{2}, 1\right\} l_0$$

and

$$\int_{\mathbb{R}^N} \left(1 + \left|\sqrt{t}u_{k(p)}\right|^2\right)\left|\nabla\left(\sqrt{t}u_{k(p)}\right)\right|^2 \ge \frac{3}{2} l_0,$$

where $\sqrt{t}u_k(p) \in T_k$ with $k = tk(p)$. Now we set $k_0 = t_0 k(p)$, and then for each $k \in (k_0, k(p))$ and $t \in (t_0, 1)$, we can choose $u_1 = \sqrt{t}u_{k(p)}$ such that $u_1 \in T_k$. Finally taking $\mu_0 > 0$ small enough we have by continuity that $J_{\mu,b}(u_1) < \frac{1}{2}\min\left\{\frac{a}{2}, 1\right\} l_0$ for all $\mu \in (0, \mu_0)$. □

Lemma 3.3. Assume that $p \in \left(\frac{10}{3}, \frac{16}{3}\right)$. For any fixed $k, \mu > 0$, if there exists a sequence $\{u_n\} \subset T_k$ such that $\left\{J_{\mu,b}(u_n)\right\} \subset \mathbb{R}$ is bounded, then $\{u_n\}$ is bounded in X.

Proof. By the Gagliardo-Nirenberg-Sobolev inequality, for any $u \in X$ there holds

$$\int_{\mathbb{R}^3} |u|^s \le C\|u\|_2^{\frac{12-s}{5}} \left(\int_{\mathbb{R}^3} |u|^2|\nabla u|^2\right)^{\frac{3(s-2)}{10}}, \tag{3.2}$$

where $s \in (2, 12)$ and $C > 0$ is a constant independent of u. Thus we have

$$\begin{aligned} J_{\mu,b}(u_n) \ge& \frac{\mu}{4}\|\nabla u_n\|_4^4 + \min\{\frac{a}{2}, 1\} \int_{\mathbb{R}^3} \left(1 + |u_n|^2\right)|\nabla u_n|^2 \\ &- C\left(\int_{\mathbb{R}^3} \left(1 + |u_n|^2\right)|\nabla u_n|^2\right)^{\frac{3(p-2)}{10}}. \end{aligned} \tag{3.3}$$

When $p \in \left(\frac{10}{3}, \frac{16}{3}\right)$, we get $\frac{3(p-2)}{10} < 1$. Then by (3.3) and the boundedness of $\left\{J_{\mu,b}(u_n)\right\}$, we infer that $\left\{\int_{\mathbb{R}^3} \left(1 + |u_n|^2\right)|\nabla u_n|^2\right\}$ and $\left\{\|\nabla u_n\|_4^4\right\}$ are bounded for fixed $\mu > 0$. Moreover, due to (3.2), we obtain $\left\{\|u_n\|_4^4\right\}$ is also bounded. Thus $\{u_n\}$ is bounded in X. □

Next, we shall prove that $\eta_{\mu,b}(k)$ is indeed a critical value for $J_{\mu,b}$ restricted on T_k. To this end, we first show that there exists a bounded Palais-Smale sequence at the mountain pass level $\eta_{\mu,b}(k)$. To find such a Palais-Smale sequence, we adopt the approach developed by Jeanjean [32], already applied in [33]. Set

$$\widetilde{\eta}_{\mu,b}(k) := \inf_{\widetilde{g}\in\widetilde{\Gamma}_k} \max_{0\le t\le 1} \widetilde{J}_{\mu,b}(\widetilde{g}(t)),$$

where

$$\widetilde{\Gamma}_k := \{\widetilde{g} \in C([0, 1], T_k \times \mathbb{R}) : \widetilde{g}(0) = (u_0, 0), \widetilde{g}(1) = (u_1, 0)\}$$

and

$$\widetilde{J}_{\mu,b} : T_k \times \mathbb{R} \to \mathbb{R},\ (u, \theta) \to J_{\mu,b}(\kappa(u, \theta))$$

for $\kappa(u, \theta) := e^{\frac{3}{2}\theta}u\left(e^\theta x\right)$. Clearly, for any $g \in \Gamma_k$, we have $\widetilde{g} := (g, 0) \in \widetilde{\Gamma}_k$. Based on this fact, we get that the maps

$$\varphi : \Gamma_k \to \widetilde{\Gamma}_k,\ \ g \mapsto \varphi(g) := (g, 0) \text{ and } \psi : \widetilde{\Gamma}_k \to \Gamma_k,\ \ \widetilde{g} \mapsto \psi(\widetilde{g}) := \kappa \circ \widetilde{g}$$

satisfy

$$\widetilde{J}_{\mu,b}(\varphi(g)) = J_{\mu,b}(g) \quad \text{and} \quad J_{\mu,b}(\psi(\widetilde{g})) = \widetilde{J}_{\mu,b}(\widetilde{g}),$$

which means $\widetilde{\eta}_{\mu,b}(k) = \eta_{\mu,b}(k)$.

The lemma below has been established by the Ekeland variational principle ([32], Lemma 2.3). Hereinafter we denote by $\mathcal{W}$ the set $X \times \mathbb{R}$ equipped with the norm $\|\cdot\|_{\mathcal{W}}^2 = \|\cdot\|_X^2 + |\cdot|_{\mathbb{R}}^2$ and denote by $\mathcal{W}^*$ its dual space.

Lemma 3.4. Let $\varepsilon > 0$. Suppose that $\widetilde{g}_0 \in \widetilde{\Gamma}_k$ satisfies

$$\max_{0\le t\le 1} \widetilde{J}_{\mu,b}(\widetilde{g}_0(t)) \le \widetilde{\eta}_{\mu,b}(k) + \varepsilon.$$

Then there exists a pair of $(u_0, \theta_0) \in T_k \times \mathbb{R}$ such that:
(1) $\widetilde{J}_{\mu,b}(u_0, \theta_0) \in \left[\widetilde{\eta}_{\mu,b}(k) - \varepsilon, \widetilde{\eta}_{\mu,b}(k) + \varepsilon\right]$;
(2) $\min\limits_{0\le t\le 1} \|(u_0, \theta_0) - \widetilde{g}_0(t)\|_{\mathcal{W}} \le \sqrt{\varepsilon}$;
(3) $\left\| \left(\widetilde{J}_{\mu,b}\big|_{T_k\times\mathbb{R}} \right)' (u_0, \theta_0) \right\|_{\mathcal{W}^*} \le 2\sqrt{\varepsilon}$, namely that

$$\left| \left\langle \widetilde{J}'_{\mu,b}(u_0, \theta_0), z \right\rangle_{\mathcal{W}^*\times\mathcal{W}} \right| \le 2\sqrt{\varepsilon}\|z\|_{\mathcal{W}}$$

holds for all $z \in \widetilde{T}_{(u_0,\theta_0)} := \{(z_1, z_2) \in \mathcal{W}, \langle u_0, z_1\rangle_{L^2} = 0\}$.

Lemma 3.5. Assume that $p \in \left(\frac{10}{3}, \frac{16}{3}\right)$. Then for any fixed $k \in [k_0, \infty)$, where k_0 is given in Lemma 3.2, there exist a sequence $\{v_n\} \subset T_k$ and a sequence $\{\xi_n\} \subset X$ of Schwarz symmetric functions satisfying

$$\begin{cases} J_{\mu,b}(v_n) \to \eta_{\mu,b}(k) > 0, \\ \|J'_{\mu,b}\big|_{T_k}(v_n)\|_{X^*} \to 0, \\ Q_{\mu,b}(v_n) \to 0, \\ \|v_n - \xi_n\|_X \to 0, \end{cases} \tag{3.4}$$

as $n \to \infty$, where X^* denotes the dual space of X.

Proof. By the definition of $\eta_{\mu,b}(k)$, we see that for each $n \in \mathbb{N}^+$, there exists $g_n \in \Gamma_k$ such that

$$\max_{0\le t\le 1} J_{\mu,b}(g_n(t)) \le \eta_{\mu,b}(k) + \frac{1}{n}.$$

Denote by g_n^* the Schwarz symmetrization of $g_n \in \Gamma_k$. Then by the Pólya-Szegö inequality $\|\nabla u^*\|_q^q \le \|\nabla u\|_q^q, \forall q \in [1, \infty)$, and using Lemma 4.3 in [15], one has

$$\max_{0\le t\le 1} J_{\mu,b}(g_n^*(t)) \le \max_{0\le t\le 1} J_{\mu,b}(g_n(t)).$$

Since $\widetilde{\eta}_{\mu,b}(k) = \eta_{\mu,b}(k)$ and $\widetilde{g}_n = (g_n^*, 0) \in \widetilde{\Gamma}_k$, we deduce

$$\max_{0\le t\le 1} \widetilde{J}_{\mu,b}(\widetilde{g}_n(t)) \le \widetilde{\eta}_{\mu,b}(k) + \frac{1}{n}.$$

Thus relying on Lemma 3.4, we obtain a sequence $\{(u_n, \theta_n)\} \subset T_k \times \mathbb{R}$ such that
(i) $\widetilde{J}_{\mu,b}(u_n, \theta_n) \in \left[\eta_{\mu,b}(k) - \frac{1}{n}, \eta_{\mu,b}(k) + \frac{1}{n}\right]$;
(ii) $\min\limits_{0\le t\le 1} \left\|(u_n, \theta_n) - (g_n^*(t), 0)\right\|_{\mathcal{W}} \le \sqrt{\frac{1}{n}}$;
(iii) $\left\| \left(\widetilde{J}_{\mu,b}\big|_{T_k\times\mathbb{R}} \right)' (u_n, \theta_n) \right\|_{\mathcal{W}^*} \le 2\sqrt{\frac{1}{n}}$, i.e.

$$\left| \left\langle \widetilde{J}'_{\mu,b}(u_n, \theta_n), z \right\rangle_{\mathcal{W}^*\times\mathcal{W}} \right| \le 2\sqrt{\frac{1}{n}}\|z\|_{\mathcal{W}}$$

holds for all $z \in \widetilde{T}_{(u_n,\theta_n)} := \{(z_1, z_2) \in \mathcal{W}, \langle u_n, z_1\rangle_{L^2} = 0\}$.

We claim that for each $n \in \mathbb{N}^+$, there exists $t_n \in [0, 1]$ such that $v_n = \kappa(u_n, \theta_n)$ and $\xi_n := g_n^*(t_n)$ satisfy (3.4). Firstly, depending on (i), we deduce that $J_{\mu,b}(v_n) \to \eta_{\mu,b}(k)$ as $n \to \infty$, since $J_{\mu,b}(v_n) =$

$J_{\mu,b}(\kappa(u_n,\theta_n)) = \widetilde{J}_{\mu,b}(u_n,\theta_n)$. Furthermore, by Lemma 3.1, we get $\eta_{\mu,b}(k) > 0$. Secondly, for any $(\phi, r) \in \mathcal{W}$, notice that

$$\begin{aligned}
&\left\langle \widetilde{J}'_{\mu,b}(u,\theta),(\phi,r)\right\rangle_{\mathcal{W}^*\times\mathcal{W}}\\
=&\frac{7}{4}\mu re^{7\theta}\int_{\mathbb{R}^3}|\nabla u|^4+\mu e^{7\theta}\int_{\mathbb{R}^3}|\nabla u|^2\nabla u\nabla\phi+are^{2\theta}\int_{\mathbb{R}^3}|\nabla u|^2+ae^{2\theta}\int_{\mathbb{R}^3}\nabla u\nabla\phi\\
&+bre^{4\theta}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2+be^{4\theta}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)\int_{\mathbb{R}^3}\nabla u\nabla\phi+5re^{5\theta}\int_{\mathbb{R}^3}|u|^2|\nabla u|^2\\
&+2e^{5\theta}\int_{\mathbb{R}^3}|\nabla u|^2u\phi+2e^{5\theta}\int_{\mathbb{R}^3}|u|^2\nabla u\nabla\phi-\frac{3(p-2)}{2p}re^{\frac{3(p-2)}{2}\theta}\int_{\mathbb{R}^3}|u|^p\\
&-e^{\frac{3(p-2)}{2}\theta}\int_{\mathbb{R}^3}|u|^{p-2}u\phi,
\end{aligned}$$

then we deduce

$$\begin{aligned}
Q_{\mu,b}(v_n)&=\frac{7}{4}\mu\mathcal{A}(v_n)+a\mathcal{B}(v_n)+b(\mathcal{B}(v_n))^2+5\mathcal{C}(v_n)-\frac{3(p-2)}{2p}\mathcal{E}(v_n)\\
&=\frac{7}{4}\mu e^{7\theta_n}\int_{\mathbb{R}^3}|\nabla u_n|^4+ae^{2\theta_n}\int_{\mathbb{R}^3}|\nabla u_n|^2+be^{4\theta_n}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2\\
&\quad+5e^{5\theta_n}\int_{\mathbb{R}^3}|u_n|^2|\nabla u_n|^2-\frac{3(p-2)}{2p}e^{\frac{3(p-2)}{2}\theta_n}\int_{\mathbb{R}^3}|u_n|^p\\
&=\left\langle \widetilde{J}'_{\mu,b}(u_n,\theta_n),(0,1)\right\rangle_{\mathcal{W}^*\times\mathcal{W}}.
\end{aligned}$$

Hence, by (iii), we easily obtain $Q_{\mu,b}(v_n) \to 0$ as $n \to \infty$ for $(0,1) \in \widetilde{T}_{(u_n,\theta_n)}$.

Thirdly, we will prove that

$$\left\| J'_{\mu,b}\big|_{T_k}(v_n)\right\|_{X^*} \to 0$$

as $n \to \infty$. We claim that for $n \in \mathbb{N}$ sufficiently large,

$$\left|\left\langle J'_{\mu,b}(v_n),\omega\right\rangle_{X^*\times X}\right| \le \frac{8}{\sqrt{n}}\|\omega\|_X^2,\quad \forall\omega\in T_{v_n},$$

where $T_{v_n} = \{\omega \in X, \langle v_n, \omega\rangle_{L^2} = 0\}$. To this end, for $\omega \in T_{v_n}$, setting $\widetilde{\omega} = \kappa(\omega, -\theta_n)$, by simple calculation we get

$$\left\langle J'_{\mu,b}(v_n),\omega\right\rangle_{X^*\times X}=\left\langle \widetilde{J}'_{\mu,b}(u_n,\theta_n),(\widetilde{\omega},0)\right\rangle_{\mathcal{W}^*\times\mathcal{W}}.$$

Since $\int_{\mathbb{R}^3} u_n\widetilde{\omega} = \int_{\mathbb{R}^3} v_n\omega$, we get $(\widetilde{\omega}, 0) \in \widetilde{T}_{(u_n,\theta_n)} \Leftrightarrow \omega \in T_{v_n}$. Moreover, by (ii), one has

$$|\theta_n|=|\theta_n-0|\le\min_{0\le t\le1}\left\|(u_n,\theta_n)-(g_n^*(t),0)\right\|_{\mathcal{W}}\le\frac{1}{\sqrt{n}}. \tag{3.5}$$

And by simple calculations, we can easily get

$$\|(\widetilde{\omega},0)\|_{\mathcal{W}}^2=\|\widetilde{\omega}\|_X^2\le4\|\omega\|_X^2,$$

which together with (iii) implies that

$$\left\langle J'_{\mu,b}\left(v_n\right),\omega\right\rangle_{X^*\times X}=\left\langle \widetilde{J}'_{\mu,b}\left(u_n,\theta_n\right),\left(\widetilde{\omega},0\right)\right\rangle_{\mathcal{W}^*\times\mathcal{W}}\le\frac{2}{\sqrt{n}}\|(\widetilde{\omega},0)\|^2_{\mathcal{W}}$$
$$\le\frac{8}{\sqrt{n}}\|\omega\|^2_X. \tag{3.6}$$

Thus by (3.6) we deduce

$$\left\|J'_{\mu,b}\Big|_{T_k}\left(v_n\right)\right\|_{X^*}=\sup_{\omega\in T_{v_n},\|\omega\|_X\le1}\left|\left\langle J'_{\mu,b}\left(v_n\right),\omega\right\rangle_{X^*\times X}\right|\le\frac{8}{\sqrt{n}}\to0$$

as $n\to\infty$.

In the end, for each $n\in\mathbb{N}^+$, it follows from (ii) that there exists $t_n\in[0,1]$ such that $\left\|(u_n,\theta_n)-(g_n^*(t_n),0)\right\|_{\mathcal{W}}\to0$. This implies that

$$\left\|u_n-g_n^*(t_n)\right\|_X\to0.$$

Thus from (3.5) and

$$\|v_n-\xi_n\|_X=\left\|\kappa\left(u_n,\theta_n\right)-g_n^*(t_n)\right\|_X\le\|\kappa\left(u_n,\theta_n\right)-u_n\|_X+\left\|u_n-g_n^*(t_n)\right\|_X,$$

we conclude that $\|v_n-\xi_n\|_X\to0$ as $n\to\infty$. This proof is completed. □

Next, we will show the compactness of the Palais-Smale sequence $\{v_n\}$ obtained in Lemma 3.5. To this aim, we need to first give two useful lemmas.

Lemma 3.6. Let $p\in\left(\frac{10}{3},\frac{16}{3}\right)$, $\lambda\in\mathbb{R}$. If $v\in H\left(\mathbb{R}^3\right)$ is a weak solution of equation (1.1), then $Q_b(v)=0$, where $Q_b(v):=Q_{\mu,b}(v)\Big|_{\mu=0}$. Moreover, if $\lambda\ge0$, one has $v=0$.

Proof. If $v\in H\left(\mathbb{R}^3\right)$ is a weak solution of problem (1.1), then v satisfies the following Pohozaev identity

$$\frac{a}{2}\int_{\mathbb{R}^3}|\nabla v|^2+\frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla v|^2\right)^2+\int_{\mathbb{R}^3}|v|^2|\nabla v|^2-\frac{3}{p}\int_{\mathbb{R}^3}|v|^p=\frac{3}{2}\lambda\int_{\mathbb{R}^3}|v|^2.$$

Multiplying (1.1) by v and integrating on $\mathbb{R}^3$, we derive the following identity

$$a\int_{\mathbb{R}^3}|\nabla v|^2+b\left(\int_{\mathbb{R}^3}|\nabla v|^2\right)^2+4\int_{\mathbb{R}^3}|v|^2|\nabla v|^2-\int_{\mathbb{R}^3}|v|^p=\lambda\int_{\mathbb{R}^3}|v|^2.$$

Hence one has immediately

$$Q_b(v)=a\int_{\mathbb{R}^3}|\nabla v|^2+b\left(\int_{\mathbb{R}^3}|\nabla v|^2\right)^2+5\int_{\mathbb{R}^3}|v|^2|\nabla v|^2-\frac{3(p-2)}{2p}\int_{\mathbb{R}^3}|v|^p=0.$$

Also with simple calculations, we get

$$0\le a\int_{\mathbb{R}^3}|\nabla v|^2+b\left(\int_{\mathbb{R}^3}|\nabla v|^2\right)^2+4\int_{\mathbb{R}^3}|v|^2|\nabla v|^2\le\frac{3(p-2)}{p-6}\lambda\int_{\mathbb{R}^3}|v|^2.$$

Based on the above facts, we can easily get the following two conclusions:

(1) If $\lambda>0$, we get $v=0$ immediately;

(2) If $\lambda=0$, we have $\mathcal{B}(v)=0$ and $C(v)=0$. Then by $Q_b(v)=0$, we infer $v=0$. □

Lemma 3.7. ([34], Lemma 3) Let F be a C^1 functional on X. Then if $\{x_n\} \subset T_k$ is bounded in X, we obtain

$$F'|_{T_k}(x_n) \to 0 \text{ in } X^*\left(\mathbb{R}^3\right) \Longleftrightarrow F'(x_n) - \langle F'(x_n), x_n\rangle x_n \to 0 \text{ in } X^*\left(\mathbb{R}^3\right)$$

as $n \to \infty$.

Proposition 1. Assume that $p \in \left(\frac{10}{3}, \frac{16}{3}\right)$. Let $\{v_n\} \subset T_k$ be the Palais-Smale sequence as obtained in Lemma 3.5. Then there exist $v_\mu \in X\backslash\{0\}$ and $\lambda_\mu \in \mathbb{R}$ such that, passing to a subsequence,
1) $v_n \rightharpoonup v_\mu > 0$, in X;
2) $J'_{\mu,b}(v_n) - \lambda_\mu v_n \to 0$, in X^*;
3) $J'_{\mu,b}\left(v_\mu\right) - \lambda_\mu v_\mu = 0$, in X^*.
Moreover, if $\lambda_\mu < 0$, we get

$$\lim_{n\to\infty} \left\|v_n - v_\mu\right\|_X = 0. \tag{3.7}$$

Proof. By Lemma 3.3 we see that $\{v_n\}$ is bounded in X. This implies the boundedness of the Schwarz symmetric sequences $\{\xi_n\}$ obtained in Lemma 3.5. Thus by Proposition 1.7.1 in [35], we conclude that passing to a subsequence, there exists $v_\mu \in X$, which is non-negative and Schwarz symmetric, such that

$$\begin{aligned} &\xi_n \rightharpoonup v_\mu \geq 0, \text{ in } X; \\ &\xi_n \to v_\mu, \text{ in } L^q\left(\mathbb{R}^3\right), \ \forall q \in (2, 2^*). \end{aligned}$$

By interpolation, we obtain

$$\xi_n \to v_\mu, \text{ in } L^q\left(\mathbb{R}^3\right), \ \forall q \in (2, 2\cdot 2^*).$$

Since $\left\|v_n - v_\mu\right\|_q \leq \|v_n - \xi_n\|_q + \left\|\xi_n - v_\mu\right\|_q$, one has

$$v_n \to v_\mu, \text{ in } L^q\left(\mathbb{R}^3\right), \ \forall q \in (2, 2\cdot 2^*). \tag{3.8}$$

At this moment we firstly show that $v_\mu \neq 0$. By contradiction, assume that $v_\mu = 0$. Then by (3.8) we get $\|v_n\|_p \to 0$, and using $Q_{\mu,b}(v_n) \to 0$ we deduce that

$$\mathcal{A}(v_n) \to 0, \ \ \mathcal{B}(v_n) \to 0 \text{ and } C(v_n) \to 0.$$

This leads to $J_{\mu,b}(v_n) \to 0$, which contradicts the fact that $J_{\mu,b}(v_n) \to \eta_{\mu,b}(k) > 0$. Thus Point 1) is established.

Since $\{v_n\} \subset X$ is bounded, by Lemma 3.7 we obtain

$$J'_{\mu,b}\Big|_{T_k}(v_n) \to 0 \text{ in } X^* \Longleftrightarrow J'_{\mu,b}(v_n) - \left\langle J'_{\mu,b}(v_n), v_n\right\rangle v_n \to 0 \text{ in } X^*.$$

Thus for any $\omega \in X$,

$$\begin{aligned} &\left\langle J'_{\mu,b}(v_n) - \left\langle J'_{\mu,b}(v_n), v_n\right\rangle v_n, \omega\right\rangle \\ &= \mu\int_{\mathbb{R}^3} |\nabla v_n|^2 \nabla v_n \nabla\omega + \left(a + b\int_{\mathbb{R}^3} |\nabla v_n|^2\right)\int_{\mathbb{R}^3} \nabla v_n \nabla\omega \\ &\quad + 2\int_{\mathbb{R}^3}\left(v_n\omega\,|\nabla v_n|^2 + |v_n|^2 \nabla v_n \nabla\omega\right) - \int_{\mathbb{R}^3} |v_n|^{p-2} v_n\omega \\ &\quad - \lambda_n \int_{\mathbb{R}^3} v_n\omega \to 0, \end{aligned} \tag{3.9}$$

where

$$\lambda_n = \frac{1}{\|v_n\|_2^2}\left[\mu\mathcal{A}(v_n) + a\mathcal{B}(v_n) + b(\mathcal{B}(v_n))^2 + 4C(v_n) - \mathcal{E}(v_n)\right].$$

So $\left\langle J'_\mu(v_n), v_n\right\rangle - \lambda_n \|v_n\|_2^2 \to 0$. Furthermore, we easily obtain $\{\lambda_n\}$ is bounded since $\left\langle J'_\mu(v_n), v_n\right\rangle$ is bounded. Thus there exists $\lambda_\mu \in \mathbb{R}$, such that up to a subsequence, $\lambda_n \to \lambda_\mu$. This and (3.9) imply Point 2).

To prove Point 3), we follow from Point 2) that it is enough to show that for any $\omega \in X$,

$$\left\langle J'_{\mu,b}(v_n) - \lambda_\mu v_n, \omega\right\rangle \to \left\langle J'_{\mu,b}\left(v_\mu\right) - \lambda_\mu v_\mu, \omega\right\rangle. \tag{3.10}$$

Since $v_n \rightharpoonup v_\mu$ in X, then we obtain

$$\begin{gathered}\int_{\mathbb{R}^3} \nabla v_n \nabla\omega \to \int_{\mathbb{R}^N} \nabla v_\mu \nabla\omega,\\ \int_{\mathbb{R}^3} |v_n|^{p-2} v_n\omega \to \int_{\mathbb{R}^3} \left|v_\mu\right|^{p-2} v_\mu\omega,\\ \int_{\mathbb{R}^3} v_n\omega \to \int_{\mathbb{R}^3} v_\mu\omega.\end{gathered}$$

Notice that

$$\begin{aligned}&\left\langle J'_{\mu,b}(v_n) - \lambda_\mu v_n, \omega\right\rangle\\ &= \mu\int_{\mathbb{R}^3} |\nabla v_n|^2 \nabla v_n\nabla\omega + \left(a + b\int_{\mathbb{R}^3} |\nabla v_n|^2\right)\int_{\mathbb{R}^3} \nabla v_n\nabla\omega - \lambda_\mu\int_{\mathbb{R}^3} v_n\omega\\ &\quad + 2\int_{\mathbb{R}^3}\left(v_n\omega|\nabla v_n|^2 + |v_n|^2\nabla v_n\nabla\omega\right) - \int_{\mathbb{R}^3} |v_n|^{p-2} v_n\omega,\end{aligned}$$

so we only need to prove that

$$\int_{\mathbb{R}^3} |\nabla v_n|^2 \nabla v_n\nabla\omega \to \int_{\mathbb{R}^3} \left|\nabla v_\mu\right|^2 \nabla v_\mu\nabla\omega; \tag{3.11}$$

$$\int_{\mathbb{R}^3}\left(v_n\omega|\nabla v_n|^2 + |v_n|^2\nabla v_n\nabla\omega\right) \to \int_{\mathbb{R}^3}\left(v_\mu\omega\left|\nabla v_\mu\right|^2 + \left|v_\mu\right|^2\nabla v_\mu\nabla\omega\right). \tag{3.12}$$

We easily obtain $\left\{|\nabla v_n|^2\nabla v_n\right\}$ is bounded in $L^{4/3}\left(\mathbb{R}^3\right)$ since $\{\nabla v_n\}$ is bounded in $L^4\left(\mathbb{R}^3\right)$. Thus $|\nabla v_n|^2\nabla v_n \rightharpoonup \left|\nabla v_\mu\right|^2\nabla v_\mu$ in $L^{4/3}\left(\mathbb{R}^3\right)$, and then we get (3.11) by weak convergence for any $\nabla\omega \in L^4\left(\mathbb{R}^3\right)$. Similarly, by the Young inequality, one has

$$\begin{aligned}\left(|v_n||\nabla v_n|^2\right)^{4/3} &\le \tfrac{1}{3}|v_n|^4 + \tfrac{2}{3}|\nabla v_n|^4,\\ \left(|v_n|^2|\nabla v_n|\right)^{4/3} &\le \tfrac{2}{3}|v_n|^4 + \tfrac{1}{3}|\nabla v_n|^4.\end{aligned}$$

These yield that both $\left\{|v_n||\nabla v_n|^2\right\}$ and $\left\{|v_n|^2|\nabla v_n|\right\}$ are bounded in $L^{4/3}\left(\mathbb{R}^3\right)$, since $\{v_n\}$ is bounded in X. Thus (3.12) holds by a similar argument. At this point, (3.10) holds and we have proved Point 3).

Finally, we follow from Points 2) and 3) that

$$\left\langle J'_{\mu,b}(v_n) - \lambda_\mu v_n, v_n\right\rangle \to \left\langle J'_{\mu,b}\left(v_\mu\right) - \lambda_\mu v_\mu, v_\mu\right\rangle = 0.$$

Using (3.8) we obtain that

$$\begin{aligned}&\mu\|\nabla v_n\|_4^4+a\|\nabla v_n\|_2^2+b\|\nabla v_n\|_2^4-\lambda_\mu\|v_n\|_2^2+4\int_{\mathbb{R}^3}|v_n|^2|\nabla v_n|^2\\ \to&\mu\left\|\nabla v_\mu\right\|_4^4+a\left\|\nabla v_\mu\right\|_2^2+b\left\|\nabla v_\mu\right\|_2^4-\lambda_\mu\left\|v_\mu\right\|_2^2+4\int_{\mathbb{R}^3}\left|v_\mu\right|^2\left|\nabla v_\mu\right|^2.\end{aligned}$$

If $\lambda_\mu<0$, this together with $v_n\rightharpoonup v_\mu$ in X implies that (3.7) holds. This proof is completed. □

Corollary 2. Assume that $p\in\left(\frac{10}{3},\frac{16}{3}\right)$. Then for any $k\in[k_0,\infty)$, where k_0 is given in Lemma 3.2, there exists $\mu_0>0$ such that for each $\mu\in(0,\mu_0)$, the functional $J_{\mu,b}$ has a critical point v_μ, which is Schwarz symmetric and satisfies $J_{\mu,b}\left(v_\mu\right)\le\eta_{\mu,b}(k)$ on $T_{\bar{k}}$ with $0<\bar{k}\le k$, i.e., there exists $\lambda_\mu\in\mathbb{R}$ such that $J'_{\mu,b}\left(v_\mu\right)-\lambda_\mu v_\mu=0$. Moreover, if $\lambda_\mu<0$, then one has $J_{\mu,b}\left(v_\mu\right)=\eta_{\mu,b}(k)$.

Proof. It follows directly from Lemma 3.5 and Proposition 1. □

Finally, the proof of Theorems 1.4 and 1.5 is based on the following convergence result for the perturbation functional $J_{\mu,b}$.

Proposition 2. For any fixed $k>0$, let $\mu_m\to0$ as $m\to\infty$. Assume that $\{w_m\}\subset T_{k_m}$ is a sequence of Schwarz symmetric functions and $\{\lambda_m\}\subset\mathbb{R}$, and they satisfy

$$0<\delta_0\le k_m\le k,\quad\left|J_{\mu_m,b}(w_m)\right|\le C\text{ and }J'_{\mu_m,b}(w_m)-\lambda_m w_m=0,$$

where $\delta_0>0$, $C>0$ are independent of $m\in\mathbb{N}$. Then there exist $w_k\in H^1\left(\mathbb{R}^3\right)\cap L^\infty\left(\mathbb{R}^3\right)\backslash\{0\}$ and $\lambda_k\in\mathbb{R}$ such that passing to a subsequence, we get

$$\begin{aligned}&\lambda_m\to\lambda_k,\ \text{in }\mathbb{R},\\&J'_b(w_k)-\lambda_k w_k=0,\end{aligned}\tag{3.13}$$

as $m\to\infty$. Moreover, if $\lambda_k<0$, then

$$\begin{aligned}&w_m\to w_k,\ \text{in }H^1\left(\mathbb{R}^3\right),\\&w_m\nabla w_m\to w_k\nabla w_k,\ \text{in }L^2\left(\mathbb{R}^3\right),\\&\mu_m\|\nabla w_m\|_4^4\to0,\end{aligned}\tag{3.14}$$

as $m\to\infty$. Thus w_k is a critical point of J_b on $E_{k'}$ for $k'=\lim\limits_{m\to\infty}k_m$.

Proof. First, since $0<\delta_0\le k_m\le k$, $\left|J_{\mu_m,b}(w_m)\right|\le C$ and $J'_{\mu_m,b}(w_m)-\lambda_m w_m=0$, it follows from the proof of Lemma 3.3 and Proposition 1 that $\left\{\int_{\mathbb{R}^3}|\nabla w_m|^2\right\}$, $\left\{\int_{\mathbb{R}^3}|w_m|^2|\nabla w_m|^2\right\}$ and $\{\lambda_m\}$ are all bounded. Therefore, passing to a subsequence, $\lambda_m\to\lambda_k\in\mathbb{R}$, and noting that $\{w_m\}\subset T_k$ is Schwarz symmetric, by reusing Proposition 1.7.1 in [35], we deduce, up to a subsequence, that

$$\begin{aligned}&w_m\rightharpoonup w_k,\ \text{in }H^1\left(\mathbb{R}^3\right),\\&w_m\to w_k,\ \text{in }L^q\left(\mathbb{R}^3\right),\forall q\in(2,2\cdot2^*),\\&w_m\nabla w_m\rightharpoonup w_k\nabla w_k,\ \text{in }L^2\left(\mathbb{R}^3\right),\\&w_m\to w_k,\ \text{a.e. in }\mathbb{R}^3,\end{aligned}\tag{3.15}$$

for some $w_k \in X$. Since $\{w_m\}$ satisfies $J'_{\mu_m,b}(w_m) - \lambda_m w_m = 0$, one has

$$\begin{aligned}
&\mu_m \int_{\mathbb{R}^3} |\nabla w_m|^2 \nabla w_m \nabla\phi + a \int_{\mathbb{R}^3} \nabla w_m \nabla\phi + b\left(\int_{\mathbb{R}^3} |\nabla w_m|^2\right) \int_{\mathbb{R}^3} \nabla w_m \nabla\phi \\
&+ 2 \int_{\mathbb{R}^3} \left(w_m\phi |\nabla w_m|^2 + |w_m|^2 \nabla w_m \nabla\phi\right) - \lambda_m \int_{\mathbb{R}^3} w_m\phi = \int_{\mathbb{R}^3} |w_m|^{p-2} w_m\phi
\end{aligned} \tag{3.16}$$

for any $\phi \in X$. Then by the Sobolev inequality and the Moser iteration, referring to Theorem 3.1 in [25], we can get

$$\|w_m\|_{L^\infty(\mathbb{R}^3)} \le C \text{ and } \|w_k\|_{L^\infty(\mathbb{R}^3)} \le C.$$

Now we prove that w_k satisfies that

$$\langle J'_b(w_k) - \lambda_k w_k, \phi\rangle = 0,\ \forall\phi \in H^1\left(\mathbb{R}^3\right) \cap L^\infty\left(\mathbb{R}^3\right).$$

In (3.16), choosing $\phi = \psi \exp(-Lw_m)$ with $\psi \in C_0^\infty\left(\mathbb{R}^3\right)$, $\psi \ge 0$ and $L > 0$. Then one has

$$\begin{aligned}
0 =&\mu_m \int_{\mathbb{R}^3} |\nabla w_m|^2 \nabla w_m (\nabla\psi \exp(-Lw_m) - L\psi \exp(-Lw_m) \nabla w_m) \\
&+ \left(a + b\int_{\mathbb{R}^3} |\nabla w_m|^2\right) \int_{\mathbb{R}^3} \nabla w_m (\nabla\psi \exp(-Lw_m) - L\psi \exp(-Lw_m) \nabla w_m) \\
&+ 2 \int_{\mathbb{R}^3} |w_m|^2 \nabla w_m (\nabla\psi \exp(-Lw_m) - L\psi \exp(-Lw_m) \nabla w_m) \\
&+ 2 \int_{\mathbb{R}^3} w_m\psi \exp(-Lw_m) |\nabla w_m|^2 - \lambda_m \int_{\mathbb{R}^3} w_m\psi \exp(-Lw_m) \\
&- \int_{\mathbb{R}^3} |w_m|^{p-2} w_m\psi \exp(-Lw_m).
\end{aligned}$$

This together with $\mu_m L \int_{\mathbb{R}^3} \psi \exp(-Lw_m) |\nabla w_m|^4 \ge 0$ implies that

$$\begin{aligned}
0 \le&\mu_m \int_{\mathbb{R}^3} |\nabla w_m|^2 \nabla w_m \nabla\psi \exp(-Lw_m) \\
&+ \left(a + b\int_{\mathbb{R}^3} |\nabla w_m|^2\right) \int_{\mathbb{R}^3} \nabla w_m \nabla\psi \exp(-Lw_m) \\
&+ 2 \int_{\mathbb{R}^3} |w_m|^2 \nabla w_m \nabla\psi \exp(-Lw_m) \\
&- \int_{\mathbb{R}^3} |\nabla w_m|^2 \psi \exp(-Lw_m) \left[L\left(a + b\int_{\mathbb{R}^3} |\nabla w_m|^2 + 2w_m^2\right) - 2w_m\right] \\
&- \lambda_m \int_{\mathbb{R}^3} w_m\psi \exp(-Lw_m) - \int_{\mathbb{R}^3} |w_m|^{p-2} w_m\psi \exp(-Lw_m).
\end{aligned} \tag{3.17}$$

Taking $L > 1$ such that $La > 1$, we get

$$\int_{\mathbb{R}^3} |\nabla w_m - \nabla w_k|^2 \psi \exp(-Lw_m) \left[L\left(a + b\int_{\mathbb{R}^3} |\nabla w_m|^2 + 2w_m^2\right) - 2w_m\right] \ge 0,$$

which means

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla w_m|^2 \psi \exp(-Lw_m)\left[L\left(a+b\int_{\mathbb{R}^3}|\nabla w_m|^2+2w_m^2\right)-2w_m\right]\\
&\geq \int_{\mathbb{R}^3}\left(2\nabla w_m\nabla w_k-|\nabla w_k|^2\right)\psi\exp(-Lw_m)\left[L\left(a+b\int_{\mathbb{R}^3}|\nabla w_m|^2+2w_m^2\right)-2w_m\right]\\
&\to \int_{\mathbb{R}^3}|\nabla w_k|^2\psi\exp(-Lw_k)\left[L\left(a+b\int_{\mathbb{R}^3}|\nabla w_k|^2+2w_k^2\right)-2w_k\right].
\end{aligned}$$

Because $\mu_m \to 0$ and $\|w_m\|_{L^\infty(\mathbb{R}^3)} \leq C$, (3.15) implies

$$\mu_m \int_{\mathbb{R}^3} |\nabla w_m|^2 \nabla w_m \nabla\psi \exp(-Lw_m) \to 0$$

as $m \to \infty$. By the weak convergence of w_m, Hölder inequality and Lebesgue's dominated convergence theorem, we deduce

$$\begin{gathered}
a\int_{\mathbb{R}^3}\nabla w_m\nabla\psi\exp(-Lw_m)\to a\int_{\mathbb{R}^3}\nabla w_k\nabla\psi\exp(-Lw_k),\\
b\int_{\mathbb{R}^3}|\nabla w_m|^2\int_{\mathbb{R}^3}\nabla w_m\nabla\psi\exp(-Lw_m)\to b\int_{\mathbb{R}^3}|\nabla w_k|^2\int_{\mathbb{R}^3}\nabla w_k\nabla\psi\exp(-Lw_k),\\
\int_{\mathbb{R}^3}|w_m|^2\nabla w_m\nabla\psi\exp(-Lw_m)\to\int_{\mathbb{R}^3}|w_k|^2\nabla w_k\nabla\psi\exp(-Lw_k),\\
\int_{\mathbb{R}^3}w_m\psi\exp(-Lw_m)\to\int_{\mathbb{R}^3}w_k\psi\exp(-Lw_k)
\end{gathered}$$

and

$$\int_{\mathbb{R}^3}|w_m|^{p-2}w_m\psi\exp(-Lw_m)\to\int_{\mathbb{R}^3}|w_k|^{p-2}w_k\psi\exp(-Lw_k).$$

Hence, by (3.17), we get

$$\begin{aligned}
&\left(a+b\int_{\mathbb{R}^3}|\nabla w_k|^2\right)\int_{\mathbb{R}^3}\nabla w_k\nabla\left(\psi\exp(-Lw_k)\right)\\
&+2\int_{\mathbb{R}^3}|w_k|^2\nabla w_k\nabla\left(\psi\exp(-Lw_k)\right)+2\int_{\mathbb{R}^3}w_k\left(\psi\exp(-Lw_k)\right)|\nabla w_k|^2\\
&-\lambda_k\int_{\mathbb{R}^3}w_k\left(\psi\exp(-Lw_k)\right)-\int_{\mathbb{R}^3}|w_k|^{p-2}w_k\left(\psi\exp(-Lw_k)\right)\geq 0.
\end{aligned}\tag{3.18}$$

Let $\varphi \in C_0^\infty\left(\mathbb{R}^3\right)$ satisfy $\varphi \geq 0$. Choose a sequence of non-negative functions $\psi_m \in C_0^\infty\left(\mathbb{R}^3\right)$ such that $\psi_m \to \varphi\exp(Lw_k)$ in $H^1\left(\mathbb{R}^3\right)$, $\psi_m \to \varphi\exp(Lw_k)$ a.e. in $\mathbb{R}^3$, and ψ_m is uniformly bounded in $L^\infty\left(\mathbb{R}^3\right)$. Taking $\psi = \psi_m$ in (3.18) and letting $m \to \infty$, we get

$$\begin{aligned}
&\left(a+b\int_{\mathbb{R}^3}|\nabla w_k|^2\right)\int_{\mathbb{R}^3}\nabla w_k\nabla\varphi+2\int_{\mathbb{R}^3}|w_k|^2\nabla w_k\nabla\varphi\\
&\quad+2\int_{\mathbb{R}^3}w_k\varphi|\nabla w_k|^2-\lambda_k\int_{\mathbb{R}^3}w_k\varphi-\int_{\mathbb{R}^3}|w_k|^{p-2}w_k\varphi\geq 0.
\end{aligned}$$

The opposite inequality can be obtained in a similar way. Therefore, for any $\varphi \in C_0^\infty\left(\mathbb{R}^3\right)$, we have

$$\begin{aligned}\left(a + b\int_{\mathbb{R}^3}|\nabla w_k|^2\right)\int_{\mathbb{R}^3}\nabla w_k\nabla\varphi + 2\int_{\mathbb{R}^3}|w_k|^2\nabla w_k\nabla\varphi \\ + 2\int_{\mathbb{R}^3}w_k\varphi\,|\nabla w_k|^2 - \lambda_k\int_{\mathbb{R}^3}w_k\varphi - \int_{\mathbb{R}^3}|w_k|^{p-2}w_k\varphi = 0.\end{aligned} \tag{3.19}$$

This proves (3.13).

Now by approximation again, we follow from (3.19) that

$$\begin{aligned}a\int_{\mathbb{R}^3}|\nabla w_k|^2 + b\left(\int_{\mathbb{R}^3}|\nabla w_k|^2\right)^2 + 4\int_{\mathbb{R}^3}|w_k|^2|\nabla w_k|^2 - \lambda_k\int_{\mathbb{R}^3}|w_k|^2 \\ - \int_{\mathbb{R}^3}|w_k|^p = 0.\end{aligned} \tag{3.20}$$

Taking $\phi = w_m$ in (3.16), then we have

$$\begin{aligned}\mu_m\int_{\mathbb{R}^3}|\nabla w_m|^4 + a\int_{\mathbb{R}^3}|\nabla w_m|^2 + b\left(\int_{\mathbb{R}^3}|\nabla w_m|^2\right)^2 \\ + 4\int_{\mathbb{R}^3}|w_m|^2|\nabla w_m|^2 - \lambda_m\int_{\mathbb{R}^3}|w_m|^2 = \int_{\mathbb{R}^3}|w_m|^p,\end{aligned}$$

and then

$$\begin{aligned}\mu_m\int_{\mathbb{R}^3}|\nabla w_m|^4 + a\int_{\mathbb{R}^3}|\nabla w_m|^2 + b\left(\int_{\mathbb{R}^3}|\nabla w_m|^2\right)^2 \\ + 4\int_{\mathbb{R}^3}|w_m|^2|\nabla w_m|^2 - \lambda_k\int_{\mathbb{R}^3}|w_m|^2 = \int_{\mathbb{R}^3}|w_m|^p + o(1),\end{aligned} \tag{3.21}$$

since $\lambda_m \to \lambda_k$ and $\lim_{m\to\infty}\int_{\mathbb{R}^3}|w_m|^2 = k' \geq \delta_0 > 0$. Thus, if $\lambda_k < 0$, depending on $\int_{\mathbb{R}^3}|w_m|^p \to \int_{\mathbb{R}^3}|w_k|^p$ in (3.15), we conclude from (3.15), (3.20) and (3.21) that

$$\begin{aligned}\mu_n\int_{\mathbb{R}^3}|\nabla w_m|^4 \to 0, \int_{\mathbb{R}^3}|\nabla w_m|^2 \to \int_{\mathbb{R}^3}|\nabla w_k|^2 \\ \int_{\mathbb{R}^3}|w_m|^2|\nabla w_m|^2 \to \int_{\mathbb{R}^3}|w_k|^2|\nabla w_k|^2, \int_{\mathbb{R}^3}|w_m|^2 \to \int_{\mathbb{R}^3}|w_k|^2\end{aligned}$$

as $m \to \infty$. Since $\|w_k\|_{L^\infty(\mathbb{R}^3)} \leq C$, by (3.13) we get that $w_k \in H^1\left(\mathbb{R}^3\right) \cap L^\infty\left(\mathbb{R}^3\right)\backslash\{0\}$ is a critical point of J_b on $E_{k'}$. The proof is completed. □

At this point, we can prove our last two main results.

Proof of Theorem 1.4. First, we need to show that $J_{\mu,b}\left(v_\mu\right) \leq C$. By the definition of $\eta_{\mu,b}(v_\mu)$ and Corollary 2, we get

$$0 < J_{\mu,b}\left(v_\mu\right) \leq \eta_{\mu,b}(k) \leq \eta_{1,b}(k), \tag{3.22}$$

where $\eta_{1,b}(k)$ is independent of $\mu > 0$. Next, fix $k > 0$ and take $\mu_m \to 0$. By Corollary 2 there exist a sequence of Schwarz symmetric functions w_m on T_{k_m} and $\lambda_m \in \mathbb{R}$ such that $0 < k_m \leq k$, $J_{\mu_m,b}\left(w_m\right) \leq$

$\eta_{\mu_m,b}(k)$ and $J'_{\mu_m,b}(w_m) - \lambda_m w_m = 0$. Then we get

$$\begin{aligned} &\mu_m \int_{\mathbb{R}^3} |\nabla w_m|^4 + a \int_{\mathbb{R}^3} |\nabla w_m|^2 + b \left(\int_{\mathbb{R}^3} |\nabla w_m|^2 \right)^2 \\ &+ 4 \int_{\mathbb{R}^3} |w_m|^2 |\nabla w_m|^2 - \int_{\mathbb{R}^3} |w_m|^p = \lambda_m \int_{\mathbb{R}^3} |w_m|^2 . \end{aligned} \tag{3.23}$$

And $\{(w_m, \lambda_m)\}$ satisfies the following Pohozaev identity

$$\begin{aligned} &\frac{-\mu_m}{12} \int_{\mathbb{R}^3} |\nabla w_m|^4 + \frac{a}{6} \int_{\mathbb{R}^3} |\nabla w_m|^2 + \frac{b}{6} \left(\int_{\mathbb{R}^3} |\nabla w_m|^2 \right)^2 \\ &+ \frac{1}{3} \int_{\mathbb{R}^3} |w_m|^2 |\nabla w_m|^2 - \frac{1}{p} \int_{\mathbb{R}^3} |w_m|^p = \frac{\lambda_m}{2} \int_{\mathbb{R}^3} |w_m|^2 . \end{aligned} \tag{3.24}$$

Thus, combing (3.23) and (3.24), we get $Q_{\mu_m,b}(w_m) = 0$.

We claim that $k_m \geq \delta_0$ for some $\delta_0 > 0$. In fact if $k_m \to 0$, then by (3.2) we infer $w_m \to 0$ in $L^p\left(\mathbb{R}^3\right)$. This fact together with $Q_{\mu_m,b}(w_m) = 0$ means that $\int_{\mathbb{R}^3} \left(1 + w_m^2\right) |\nabla w_m|^2 \to 0$. Then by Lemma 3.1, we easily obtain a contradiction, namely that the claim is proved.

Now applying Proposition 2 to $\{w_m\}$, we conclude that there exist $\lambda_k \in \mathbb{R}$ and $w_k \neq 0$ such that $w_m \to w_k$ in $L^p\left(\mathbb{R}^3\right)$, $\liminf_{m\to\infty} \|w_m\|_2^2 \geq \|w_k\|_2^2$ and $J'_b(w_k) - \lambda_k w_k = 0$. Furthermore, by Lemma 3.6, we infer that $\lambda_k < 0$. Going back we may say that $\lambda_k < 0$ for m large (or μ_m small). Then by Corollary 2 we get $k_m = k$, $w_m \in E_k$ and $J_{\mu_m,b}(w_m) = \eta_{\mu_m,b}(k)$ for all m large. Using Proposition 2 again we get $w_k \in E_k$ is a critical point of J_b on E_k. The proof of Theorem 1.4 is completed.

Proof of Theorem 1.5. For any $b > 0$, we follow from Proposition 1 that $J_{\mu,b}$ has a couple of critical point $(v_b, \lambda_b) \in T_k \times \mathbb{R}$. Furthermore, similar to the proof of Lemma 3.6, we get $\lambda_b < 0$.

We claim that for any sequence $\{b_m\}$ satisfying $b_m \to 0^+$ as $m \to +\infty$, $\{v_{b_m}\}$ is bounded in T_k. For $b > 0$ small, one has

$$\begin{aligned} \eta_{\mu,b_m}(k) :&= \inf_{g\in\Gamma_k} \max_{0\leq t\leq 1} J_{\mu,b_m}(g(t)) \\ &\leq \inf_{g\in\Gamma_k} \max_{0\leq t\leq 1} J_{\mu,1}(g(t)) \\ &< +\infty. \end{aligned}$$

Notice that $\{(v_{b_m}, \lambda_{b_m})\} \subseteq T_k \times \mathbb{R}^-$ is a sequence of critical point of $J_{\mu,b}$ with $b = b_m$, then similar to (3.23) and (3.24), we get $Q_{\mu,b_m}(v_{b_m}) = 0$. This fact implies that

$$\begin{aligned} &J_{\mu,b_m}(v_{b_m}) - \frac{2}{3(p-2)} Q_{\mu,b_m}(v_{b_m}) \\ &= \frac{(3p-20)\mu}{12(p-2)} \mathcal{A}(v_{b_m}) + \frac{(3p-10)a}{6(p-2)} \mathcal{B}(v_{b_m}) \\ &\quad + \frac{(3p-14)b_m}{12(p-2)} \left(\mathcal{B}(v_{b_m})\right)^2 + \frac{3p-16}{3(p-2)} C(v_{b_m}) \\ &= \eta_{\mu,b_m}(k). \end{aligned}$$

So $\{\mathcal{A}(v_{b_m})\}, \{\mathcal{B}(v_{b_m})\}$ and $\{C(v_{b_m})\}$ are all bounded in $\mathbb{R}$, namely that $\{v_{b_m}\}$ is bounded in X.

By the boundedness of $\{v_{b_m}\}$ in X, we can easily deduce that $\{\lambda_{b_m}\}$ is bounded in $\mathbb{R}$. Then there exist a subsequence of $\{b_m\}$, still denoted by $\{b_m\}$, and $\lambda_0 \le 0$ such that $\lambda_{b_m} \to \lambda_0$ and

$$\begin{cases} v_{b_m} \rightharpoonup v_0 & \text{in } H^1\left(\mathbb{R}^3\right), \\ v_{b_m} \to v_0 & \text{in } L^p\left(\mathbb{R}^3\right), \forall q \in (2, 2 \cdot 2^*), \\ v_{b_m} \to v_0 & \text{a.e. in } \mathbb{R}^3, \end{cases}$$

as $m \to +\infty$. This together with (3.11) and (3.12) implies that (v_0, λ_0) is a couple of critical point of $J_{\mu,0}$, namely that

$$\mu \int_{\mathbb{R}^3} |\nabla v_0|^4 + a \int_{\mathbb{R}^3} |\nabla v_0|^2 + 4 \int_{\mathbb{R}^3} |v_0|^2 |\nabla v_0|^2 - \int_{\mathbb{R}^3} |v_0|^p = \lambda_0 \int_{\mathbb{R}^3} |v_0|^2 .$$

We claim that $\lambda_0 < 0$. By contradiction, if $\lambda_0 = 0$, referring to Lemma 3.6, we get $v_0 = 0$, which means $v_{b_m} \to 0$ in $L^p\left(\mathbb{R}^3\right)$. And depending on $Q_{\mu,b_m}(v_{b_m}) = 0$, we infer that $\int_{\mathbb{R}^3}\left(1 + v_{b_m}^2\right)\left|\nabla v_{b_m}\right|^2 \to 0$, which means that we get a contradiction due to Lemma 3.1. Therefore, we know that $(v_0, \lambda_0) \in T_k \times \mathbb{R}^-$ is a couple of critical point of $J_{\mu,0}$.

Next, similar to (3.22), we can get $0 < J_{\mu,0}(v_0) \le \eta_{\mu,0}(k) \le \eta_{1,0}(k)$. Thus, referring to the proof of Theorem 1.4, we get $(v_0, \lambda_0) \in E_k \times \mathbb{R}^-$ is a couple of weak solution to the following equation

$$-a\Delta v - v\Delta\left(v^2\right) - \lambda v = |v|^{p-2}v, \quad \text{in } \mathbb{R}^3.$$

This proof is ended.

4. Conclusions

In this work, we have achieved two main results. On the one hand, when $p = \frac{16}{3}$, we prove that problem (1.6) has at least one normalized solution on D_c by making use of constrained minimization method. On the other hand, when $p \in \left(\frac{10}{3}, \frac{16}{3}\right)$, we prove the existence and asymptotic behavior of normalized solutions for equation (1.1) by using the perturbation method. Therefore, to some extent, we have improved and extended the results of the existing literature.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

1. G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.
2. S. Bernstein, Sur une classe d'équations fonctionelles aux dérivées partielles, *Bull. Acad. Sci. URSS. Sér. Math. (Izvestia Akad. Nauk SSSR)*, **4** (1940), 17–26.
3. S. I. Pohožaev, A certain class of quasilinear hyperbolic equations, *Mat. Sb. (N.S.)*, **96** (1975), 152–166. https://doi.org/10.1136/vr.96.8.166

4. J. L. Lions, On some questions in boundary value problems of mathematical physics, in *Contemporary Developments in Continuum Mechanics and Partial Differential Equations (Proceedings of International Symposium, Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977)*, In: *North-Holland Math. Stud.*, vol. 30, North-Holland, Amsterdam, (1978), 284–346. https://doi.org/10.1016/S0304-0208(08)70870-3
5. A. Arosio, S. Panizzi, On the well-posedness of the Kirchhoff string, *Trans. Amer. Math. Soc.*, **348** (1996), 305–330. https://doi.org/10.1090/S0002-9947-96-01532-2
6. G. B. Li, C. L. Xiang, Nondegeneracy of positive solutions to a Kirchhoff problem with critical Sobolev growth, *Appl. Math. Lett.*, **86** (2018), 270–275. https://doi.org/10.1016/j.aml.2018.07.010
7. S. S. Lu, An autonomous Kirchhoff-type equation with general nonlinearity in $\mathbb{R}^N$, *Nonlinear Anal. RWA*, **34** (2017), 233–249. https://doi.org/10.1016/j.nonrwa.2016.09.003
8. Q. L. Xie, Singular perturbed Kirchhoff type problem with critical exponent, *J. Math. Anal. Appl.*, **454** (2017), 144–180. https://doi.org/10.1016/j.jmaa.2017.04.048
9. C. Vetro, Variable exponent p(x)-Kirchhoff type problem with convection, *J. Math. Anal. Appl.*, **506** (2022). https://doi.org/10.1016/j.jmaa.2021.125721
10. A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Anal.*, **94** (2014), 156–170. https://doi.org/10.1016/j.na.2013.08.011
11. M. Q. Xiang, Vicenţiu D. Rădulescu, B. L. Zhang, Nonlocal Kirchhoff problems with singular exponential nonlinearity, *Appl. Math. Optim.*, **84** (2021), 915–954. https://doi.org/10.1007/s00245-020-09666-3
12. M. V. Goldman, M. Porkolab, Upper hybrid solitons ans oscillating two-stream instabilities, *Phys. Fluids*, **19** (1976), 872–881. https://doi.org/10.1063/1.861553
13. R. W. Hasse, A general method for the solution of nonlinear soliton and kink Schrödinger equations, *Z. Phys. B*, **37** (1980), 83–87. https://doi.org/10.1159/000225502
14. M. Colin, L. Jeanjean, Solutions for quasilinear Schrödinger equation: a dual approach, *Nonlinear Anal.*, **56** (2004), 213–226. https://doi.org/10.1016/j.na.2003.09.008
15. M. Colin, L. Jeanjean, M. Squassina, Stability and instability results for standing waves of quasilinear Schrödinger equations, *Nonlinearity*, **23** (2010), 1353–1385. https://doi.org/10.1088/0951-7715/23/6/006
16. J. Q. Liu, Y. Q. Wang, Z. Q. Wang, Soliton solutions for quasilinear Schrödinger equations II, *J. Differential Equations*, **187** (2003), 473–493. https://doi.org/10.1016/S0022-0396(02)00064-5
17. J. Q. Liu, Y. Q. Wang, Z. Q. Wang, Solutions for quasilinear Schrödinger equations via the Nehari method, *Comm. Partial Differential Equations*, **29** (2004), 879–901. https://doi.org/10.1081/PDE-120037335
18. J. Q. Liu, Y. Q. Wang, Z. Q. Wang, Multiple sign-changing solutions for quasilinear elliptic equations via perturbation method, *Comm. Partial Differential Equations*, **39** (2014), 2216–2239. https://doi.org/10.1080/03605302.2014.942738
19. J. Q. Liu, Y. Q. Wang, Z. Q. Wang, Quasilinear elliptic equations via perturbation method, *Proc. Amer. Math. Soc.*, **141** (2013), 253–263. https://doi.org/10.1090/S0002-9939-2012-11293-6
20. C. Huang, G. Jia, Existence of positive solutions for supercritical quasilinear Schrödinger ellitic equations, *J. Math. Anal. Appl.*, **472** (2019), 705–727.

21. V. Moroz, J. van Schaftingen, A guide to the Choquard equation, *J. Fixed Point Theory Appl.*, **19** (2017), 773–813. https://doi.org/10.1007/s11784-016-0373-1
22. J. K. Xia, Z. Q. Wang, Saddle solutions for the Choquard equation, *Calc. Var. Partial Differ. Equ.*, **58** (2019), 85. https://doi.org/10.1007/s00526-019-1546-8
23. D. D. Qin, Vicenţiu D. Rădulescu, X. H. Tang, Ground states and geometrically distinct solutions for periodic Choquard-Pekar equations, *J. Differential Equations*, **275** (2021), 652–683. https://doi.org/10.1016/j.jde.2020.11.021
24. B. B. V. Maia, On a class of p(x)-Choquard equations with sign-changing potential and upper critical growth, *Rend. Circ. Mat. Palermo (2) Suppl.*, **70** (2021), 1175–1199. https://doi.org/10.1007/s12215-020-00553-y
25. Z. H. Feng, X. Wu, H. X. Li, Multiple solutions for a modified Kirchhoff-type equation in $\mathbb{R}^N$, *Math. Methods Appl. Sci.*, **38** (2015), 708–725. https://doi.org/10.1002/mma.3102
26. K. Wu, X. Wu, Infinitely many small energy solutions for a modified Kirchhoff-type equation in $\mathbb{R}^N$, *Comput. Math. Appl.*, **70** (2015), 592–602. https://doi.org/10.1016/j.camwa.2015.05.014
27. X. M. He, Multiplicity of Solutions for a Modified Schrödinger-Kirchhoff-Type Equation in $\mathbb{R}^N$, *Discrete Dyn. Nat. Soc.*, **2015** (2015).
28. Z. X. Wang, G. Jia, Existence of solutions for modified Kirchhoff-type equation without the Ambrosetti-Rabinowitz condition, *AIMS Math.*, **6** (2021), 4614–4637. https://doi.org/10.3934/math.2021272
29. L. Jeanjean, T. J. Luo, Sharp non-existence results of prescribed L^2-norm solutions for some class of Schrödinger-Poisson and quasi-linear equations, *Z. Angew. Math. Phys.*, **64** (2013), 937–954. https://doi.org/10.1007/s00033-012-0272-2
30. C. Huang, G. Jia, Infinitely many sign-changing solutions for modified Kirchhoff-type equations in $\mathbb{R}^3$, *Complex Var Elliptic Equ.*, **66** (2021), 2090–2116. https://doi.org/10.1080/17476933.2020.1807964
31. M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Commun. Math. Phys.*, **87** (1983), 567–576. https://doi.org/10.1007/BF01208265
32. L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.*, **28** (1997), 1633–1659. https://doi.org/10.1016/S0362-546X(96)00021-1
33. T. Bartsch, S. De Valeriola, Normalized solutions of nonlinear Schrödinger equations, *Arch. Math.*, **100** (2013), 75–83. https://doi.org/10.1007/s00013-012-0468-x
34. H. Berestycki, P. L. Lions, Nonlinear scalar field equations, II existence of infinitely many solutions, *Arch. Ration. Mech. Anal.*, **82** (1983), 347–375. https://doi.org/10.1007/BF00250556
35. T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University, New York, 2003.

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