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Research article

Bi-univalent functions of complex order defined by Hohlov operator associated with legendrae polynomial

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Abstract: In this paper, we introduce and investigate two new subclasses of the function class Σ of bi-univalent functions of complex order defined in the open unit disk, which are associated with the Hohlov operator, satisfying subordinate conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Several (known or new) consequences of the results are also pointed out.

Keywords: analytic functions; univalent functions; bi-univalent functions; bi-starlike and bi-convex functions; Hohlov operator; gaussian hypergeometric function; Dziok-Srivastava operator; legendrae polynomial
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1. Introduction, definitions and preliminaries

Let $\mathfrak{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Further, by $\mathfrak{S}$ we shall denote the class of all functions in $\mathfrak{A}$ which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathfrak{S}$ include (for example) the class $\mathfrak{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathfrak{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$. It is well known that every function $f \in \mathfrak{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathfrak{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). Note that the functions

$$f_1(z) = \frac{z}{1-z}, \qquad f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \qquad f_3(z) = -\log(1-z)$$

with their corresponding inverses

$$f_1^{-1}(w) = \frac{w}{1+w}, \qquad f_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}, \qquad f_3^{-1}(w) = \frac{e^w-1}{e^w}$$

are elements of Σ. This subject has been discussed extensively in the pioneering work by Srivastava et al. [31] who revived the study of analytic and bi-univalent functions in recent years. It was followed by many sequels to Srivastava et al. [31] (see for example, [3–5, 20, 26, 50]).

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ sustaining $f(z) = g(w(z))$. Lately Ma and Minda [23] amalgamated various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this persistence, they considered an analytic function ϕ with positive real part in the unit disk $\mathbb{U}$, $\phi(0) = 1, \phi'(0) > 0$, and ϕ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathfrak{A}$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions of functions $f \in \mathfrak{A}$ satisfying the subordination $1+\frac{zf''(z)}{f'(z)} \prec \phi(z)$.

The convolution or Hadamard product of two functions $f, h \in \mathfrak{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \tag{1.3}$$

where $f(z)$ is given by (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$. In terms of the Hadamard product (or convolution), the Dziok-Srivastava linear operator involving the generalized hypergeometric function, was introduced and studied systematically by Dziok and Srivastava [9, 10] and (subsequently) by many other authors. In our present investigation, we recall a familiar convolution operator $\Im_{a,b,c}$ due to Hohlov [16, 17], which certainly a very specialized case of the widely- (and extensively-) investigated Dziok-Srivastava operator and also much more general convolution operator, known as the Srivastava-Wright operator [32] (also see [33]).

For the complex parameters a, b and c with $c \neq 0, -1, -2, -3, \cdots$, the Gaussian hypergeometric function ${}_2F_1(a, b, c; z)$ is defined as

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

$$= 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}} \frac{z^{n-1}}{(n-1)!} \qquad (z \in \mathbb{U}), \tag{1.4}$$

where $(\alpha)_n$ is the Pochhammer symbol (or the shifted factorial) defined as follows:

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 1 & (n=0) \\ \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1) & (n=1,2,3,\cdots). \end{cases} \tag{1.5}$$

For the positive real values $a,\ b$ and c with $c \neq 0,-1,-2,-3,\cdots$,by using the Gaussian hypergeometric function given by (1.4), Hohlov [16, 17] introduced the familiar convolution operator $\Im_{a,b,c}$ as follows:

$$\begin{aligned} \Im_{a,b;c} f(z) &= z\, {}_2F_1(a,b,c;z) * f(z), \\ &= z + \sum_{n=2}^{\infty} \varphi_n a_n z^n \qquad (z \in \mathbb{U}), \end{aligned} \tag{1.6}$$

where

$$\varphi_n = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}. \tag{1.7}$$

Hohlov [16, 17] discussed some interesting geometrical properties exhibited by the operator $\Im_{a,b;c}$. The three-parameter family of operators $\Im_{a,b;c}$ contains, as its special cases, most of the known linear integral or differential operators. In particular, if $b = 1$ in (1.6), then $\Im_{a,b;c}$ reduces to the Carlson-Shaffer operator.Similarly, it is easily seen that the Hohlov operator $\Im_{a,b;c}$ is also a generalization of the Ruscheweyh derivative operator as well as the Bernardi-Libera-Livingston operator.

Recently there has been triggering interest to study bi-univalent function class Σ and obtained non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1.1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1,2\}; \ \ \mathbb{N} := \{1,2,3,\cdots\}$$

is still an open problem(see [3–5, 20, 26, 50]). Many researchers (see [13, 15, 22, 31, 51, 52]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

In 1782, Adrien-Marie Legendre discovered Legendre polynomials, which have plentiful physical applications. The Legendre polynomials $P_n(x)$, intermittently called Legendre functions of the first kind, are the particular solutions to the Legendre differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, n \in \mathbb{N}_0, |x| < 1.$$

Here and in the following, let $\mathbb{C}$ and $\mathbb{N}$ denote the sets of complex numbers and positive integers, respectively, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Legendre polynomials are defined by Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n \quad (n \in \mathbb{N}_0), \tag{1.8}$$

for arbitrary real or complex values of the variable x. The Legendre polynomials Pn(x) are generated by the following function

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n,$$

where the particular branch of $(1-2xt+t^2)^{-\frac{1}{2}}$ is taken to be 1 as $t \to 0$. The first few Legendre polynomials are

$$P_0(x)=1, P_1(x)=x, P_2(x)=\frac{1}{2}(3x^2-1), P_3(x)=\frac{1}{2}(5x^3-3x). \tag{1.9}$$

A general case of the Legendre polynomials and their applications can be found in [18, 24]. The function

$$\phi(z)=\frac{1-z}{\sqrt{1-2z\cos\alpha+z^2}},$$

is in $\mathfrak{P}$ for every real α (see [[14], Page 102], [28]), where $\mathfrak{P}$ is the Caratheodory class defined by

$$\mathfrak{P}=\{p(z)\in\mathfrak{A}: p(0)=1, \Re(p(z))>0, z\in\mathbb{U}\},$$

$p(z)=1+c_1z+c_2z^2+\cdots$. By using (1.8), it is easy to check that

$$\begin{aligned}\phi(z) &= 1+\sum_{n=1}^{\infty}[P_n(\cos\alpha)-P_{n-1}(\cos\alpha)]z^n,\\ &= 1+\sum_{n=1}^{\infty}B_nz^n \qquad (1.10)\end{aligned}$$

where

$$B_n=P_n(\cos\alpha)-P_{n-1}(\cos\alpha).$$

In particular by using (1.9),we get

$$B_1=\cos\alpha-1, B_2=\frac{1}{2}(\cos\alpha-1)(1+3\cos\alpha) \tag{1.11}$$

If we consider

$$\frac{1}{(\phi(z))^2}=\frac{1-2z\cos\alpha+z^2}{(1-z)^2}=1+2(1-\cos\alpha)\frac{z}{(1-z)^2}.$$

From the geometric properties of the Koebe function, the function ϕ maps the unit disc onto the right plane $\Re(w)>0$ minus the slit along the positive real axis from $\frac{1}{|\cos\frac{\alpha}{2}|}$ to ∞. $\phi(\mathbb{U})$ is univalent, symmetric with respect to the real axis and starlike with respect to $\phi(0)=1$.

Motivated by aforementioned study on bi-univalent functions [2, 11, 13, 15, 22, 29–31, 51, 52] and present investigation of bi univalent functions associated with various polynomials as well as by many recent works on the Fekete-Szeg functional and other coefficient estimates (see [1, 7, 25, 34–43, 47–49]) in the present paper we introduce new subclasses of the function class Σ of complex order $\vartheta \in \mathbb{C}\backslash\{0\}$,involving Hohlov operator $\Im_{a,b;c}$ related with legendrae polynomial and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of function class Σ. Several related classes are also considered , and connection to earlier known results are stated.

Definition 1. *A function $f\in\Sigma$ given by (1.1) is said to be in the class $\mathfrak{S}_{\Sigma}^{a,b;c}(\vartheta,\lambda,\phi)$ if the following conditions are satisfied:*

$$1+\frac{1}{\vartheta}\left(\frac{z(\Im_{a,b;c}f(z))'}{(1-\lambda)z+\lambda\Im_{a,b;c}f(z)}-1\right)\prec\phi(z) \qquad (\vartheta\in\mathbb{C}\backslash\{0\};\ 0\leqq\lambda\leqq1;\ z\in\mathbb{U}) \tag{1.12}$$

and

$$1+\frac{1}{\vartheta}\left(\frac{w(\Im_{a,b;c}g(w))'}{(1-\lambda)w+\lambda\Im_{a,b;c}g(w)}-1\right)\prec\phi(w)\qquad(\vartheta\in\mathbb{C}\backslash\{0\};\ 0\leqq\lambda\leqq1;\ w\in\mathbb{U}),\tag{1.13}$$

where the function g is given by(1.2).

On specializing the parameters λ and a, b, c one can state the various new subclasses of Σ as illustrated in the following examples.

Example 2. *For $\lambda = 1$ and $\vartheta \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathfrak{S}_{\Sigma}^{a,b;c}(\vartheta,\phi)$ if the following conditions are satisfied:*

$$1+\frac{1}{\vartheta}\left(\frac{z(\Im_{a,b;c}f(z))'}{\Im_{a,b;c}f(z)}-1\right)\prec\phi(z)\tag{1.14}$$

and

$$1+\frac{1}{\vartheta}\left(\frac{w(\Im_{a,b;c}g(w))'}{\Im_{a,b;c}g(w)}-1\right)\prec\phi(w)\tag{1.15}$$

where $z, w \in \mathbb{U}$ and the function g is given by(1.2).

Example 3. *For $\lambda = 0$ and $\vartheta \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathfrak{G}_{\Sigma}^{a,b;c}(\vartheta,\phi)$ if the following conditions are satisfied:*

$$1+\frac{1}{\vartheta}\left(\Im_{a,b;c}f(z))'-1\right)\prec\phi(z)\tag{1.16}$$

and

$$1+\frac{1}{\vartheta}\left((\Im_{a,b;c}g(w))'-1\right)\prec\phi(w)\tag{1.17}$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2).

It is of interest to note that for $a = c$ and $b = 1$, the class $\mathfrak{S}_{\Sigma}^{a,b;c}(\vartheta,\lambda,\phi)$ reduces to the following new subclasses

Example 4. *For $\lambda = 1$ and $\vartheta \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathfrak{S}_{\Sigma}^{*}(\vartheta,\phi)$ if the following conditions are satisfied:*

$$1+\frac{1}{\vartheta}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\phi(z)\quad\text{and}\quad1+\frac{1}{\vartheta}\left(\frac{wg'(w)}{g(w)}-1\right)\prec\phi(w),$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2).

Example 5. *For $\lambda = 0$ and $\vartheta \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathfrak{H}_{\Sigma}^{*}(\vartheta,\phi)$ if the following conditions are satisfied:*

$$1+\frac{1}{\vartheta}\left(f'(z)-1\right)\prec\phi(z)\quad\text{and}\quad1+\frac{1}{\vartheta}\left(g'(w)-1\right)\prec\phi(w),$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2)

In the following section we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclasses $\mathfrak{S}_{\Sigma}^{a,b;c}(\vartheta,\lambda,\phi)$ of the function class Σ by employing the techniques used earlier by Deniz in [11].

In order to derive our main results, we shall need the following lemma.

Lemma 6. (see [28]) *If $h \in \mathfrak{P}$, then $|c_k| \leqq 2$ for each k, where $\mathfrak{P}$ is the family of all functions h, analytic in $\mathbb{U}$, for which*

$$\Re\{h(z)\} > 0 \qquad (z \in \mathbb{U}),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

2. Coefficient bounds for the function class $\mathfrak{S}_{\Sigma}^{a,b;c}(\vartheta,\lambda,\phi)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathfrak{S}_{\Sigma}^{a,b;c}(\vartheta,\lambda,\phi)$. Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots$$

or, equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right]$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \cdots\right].$$

Then $p(z)$ and $q(z)$ are analytic in $\mathbb{U}$ with $p(0) = 1 = q(0)$. Since $u, v : \mathbb{U} \to \mathbb{U}$, the functions $p(z)$ and $q(z)$ have a positive real part in $\mathbb{U}$, and $|p_i| \leq 2$ and $|q_i| \leq 2$.

Theorem 7. *Let f be given by* (1.1) *and in the class $\mathfrak{S}_{\Sigma}^{a,b;c}(\vartheta,\lambda,\phi)$. Then*

$$|a_2| \leqq \frac{\sqrt{2}|\vartheta(1-\cos\alpha)|}{\sqrt{\left|[2\vartheta(\lambda^2-2\lambda)(\cos\alpha-1)^2 + (2-\lambda)^2(1-3\cos\alpha)]\varphi_2^2 + 2\vartheta(3-\lambda)(\cos\alpha-1)\varphi_3\right|}} \tag{2.1}$$

and

$$|a_3| \leqq \frac{|\vartheta||\cos\alpha-1|^2}{(2-\lambda)^2\varphi_2^2} + \frac{|\vartheta||\cos\alpha-1|}{(3-\lambda)\varphi_3} \tag{2.2}$$

Proof. It follows from (1.12) and (1.13) that

$$1 + \frac{1}{\vartheta}\left(\frac{z(\mathfrak{I}_{a,b;c}f(z))'}{(1-\lambda)z + \lambda\mathfrak{I}_{a,b;c}f(z)} - 1\right) = \phi(u(z)) \tag{2.3}$$

and

$$1+\frac{1}{\vartheta}\left(\frac{w(\Im_{a,b;c}g(w))'}{(1-\lambda)w+\lambda\Im_{a,b;c}g(w)}-1\right)=\phi(v(w)), \tag{2.4}$$

where $p(z)$ and $q(w)$ in $\mathfrak{P}$ and have the following forms:

$$\phi(u(z))=1+\frac{1}{2}B_1p_1z+\left(\frac{1}{2}B_1(p_2-\frac{p_1^2}{2})+\frac{1}{4}B_2p_1^2\right)z^2+\cdots \tag{2.5}$$

and

$$\phi(v(w))=1+\frac{1}{2}B_1q_1w+\left(\frac{1}{2}B_1(q_2-\frac{q_1^2}{2})+\frac{1}{4}B_2q_1^2\right)w^2+\cdots, \tag{2.6}$$

respectively. Now, equating the coefficients in (2.3) and (2.4), we get

$$\frac{(2-\lambda)}{\vartheta}\varphi_2a_2=\frac{1}{2}B_1p_1, \tag{2.7}$$

$$\frac{(\lambda^2-2\lambda)}{\vartheta}\varphi_2^2a_2^2+\frac{(3-\lambda)}{\vartheta}\varphi_3a_3=\frac{1}{2}B_1(p_2-\frac{p_1^2}{2})+\frac{1}{4}B_2p_1^2, \tag{2.8}$$

$$-\frac{(2-\lambda)}{\vartheta}\varphi_2a_2=\frac{1}{2}B_1q_1 \tag{2.9}$$

and

$$\frac{(\lambda^2-2\lambda)}{\vartheta}\varphi_2^2a_2^2+\frac{(3-\lambda)}{\vartheta}\varphi_3(2a_2^2-a_3)=\frac{1}{2}B_1(q_2-\frac{q_1^2}{2})+\frac{1}{4}B_2q_1^2. \tag{2.10}$$

From (2.7) and (2.9), we find that

$$a_2=\frac{\vartheta B_1p_1}{2(2-\lambda)\varphi_2}=\frac{-\vartheta B_1q_1}{2(2-\lambda)\varphi_2}, \tag{2.11}$$

which implies

$$p_1=-q_1. \tag{2.12}$$

and

$$8(2-\lambda)^2\varphi_2^2a_2^2=\vartheta^2B_1^2(p_1^2+q_1^2). \tag{2.13}$$

Adding (2.8)and (2.10), by using(2.11) and (2.12) ,we obtain

$$4\left([\vartheta(\lambda^2-2\lambda)B_1^2+(2-\lambda)^2(B_1-B_2)]\varphi_2^2+\vartheta(3-\lambda)B_1^2\varphi_3\right)a_2^2=\vartheta^2B_1^3(p_2+q_2). \tag{2.14}$$

Thus, by using (1.11)

$$\begin{aligned}
a_2^2 &= \frac{\vartheta^2B_1^3(p_2+q_2)}{4\left([\vartheta(\lambda^2-2\lambda)B_1^2+(2-\lambda)^2(B_1-B_2)]\varphi_2^2+\vartheta(3-\lambda)B_1^2\varphi_3\right)}\\
&= \frac{\vartheta^2(\cos\alpha-1)^2(p_2+q_2)}{4\left([\vartheta(\lambda^2-2\lambda)(\cos\alpha-1)^2+\frac{1}{2}(2-\lambda)^2(1-3\cos\alpha)]\varphi_2^2+\vartheta(3-\lambda)(\cos\alpha-1)\varphi_3\right)}\\
&= \frac{\vartheta^2(\cos\alpha-1)^2(p_2+q_2)}{2\left([2\vartheta(\lambda^2-2\lambda)(\cos\alpha-1)^2+(2-\lambda)^2(1-3\cos\alpha)]\varphi_2^2+2\vartheta(3-\lambda)(\cos\alpha-1)\varphi_3\right)}.
\end{aligned}$$

(2.15)

Applying Lemma 6 for the coefficients p_2 and q_2, we immediately have

$$|a_2|^2 \leqq \frac{2|\vartheta(1-\cos\alpha)|^2}{\left|[2\vartheta(\lambda^2-2\lambda)(\cos\alpha-1)^2+(2-\lambda)^2(1-3\cos\alpha)]\varphi_2^2+2\vartheta(3-\lambda)(\cos\alpha-1)\varphi_3\right|}. \tag{2.16}$$

Hence,

$$|a_2| \leqq \frac{\sqrt{2}\,|\vartheta(1-\cos\alpha)|}{\sqrt{\left|[2\vartheta(\lambda^2-2\lambda)(\cos\alpha-1)^2+(2-\lambda)^2(1-3\cos\alpha)]\varphi_2^2+2\vartheta(3-\lambda)(\cos\alpha-1)\varphi_3\right|}}.$$

This gives the bound on $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on $|a_3|$, by subtracting (2.10) from (2.8), we get

$$\frac{2(3-\lambda)}{\vartheta}\varphi_3 a_3 - \frac{2(3-\lambda)}{\vartheta}\varphi_3 a_2^2 = \frac{B_1}{2}(p_2-q_2) + \frac{B_2-B_1}{4}(p_1^2-q_1^2). \tag{2.17}$$

It follows from (2.11), (2.12) and (2.17) that

$$\begin{aligned} a_3 &= a_2^2 + \frac{\vartheta B_1(p_2-q_2)}{4(3-\lambda)\varphi_3} \\ &= \frac{\vartheta^2 B_1^2(p_1^2+q_1^2)}{8(2-\lambda)^2\varphi_2^2} + \frac{\vartheta B_1(p_2-q_2)}{4(3-\lambda)\varphi_3} \end{aligned} \tag{2.18}$$

Applying Lemma 6 once again for the coefficients p_2, q_2 and using (1.11), we readily get

$$|a_3| \leqq \frac{|\vartheta(\cos\alpha-1)|^2}{(2-\lambda)^2\varphi_2^2} + \frac{|\vartheta(\cos\alpha-1)|}{(3-\lambda)\varphi_3}.$$

This completes the proof of Theorem 7. □

Fixing $\lambda = 1$ in Theorem 7, we have the following corollary.

Corollary 8. *Let f be given by* (1.1) *and in the class* $\mathfrak{S}_{\Sigma}^{a,b;c}(\vartheta,\phi)$. *Then*

$$|a_2| \leqq \frac{\sqrt{2}\,|\vartheta(1-\cos\alpha)|}{\sqrt{\left|[(1-3\cos\alpha)-2\vartheta(\cos\alpha-1)^2]\varphi_2^2+4\vartheta(\cos\alpha-1)\varphi_3\right|}} \tag{2.19}$$

and

$$|a_3| \leqq \frac{|\vartheta(\cos\alpha-1)|^2}{\varphi_2^2} + \frac{|\vartheta(\cos\alpha-1)|}{2\varphi_3}. \tag{2.20}$$

Taking $a = c$ and $b = 1$, in Corollary 8, we get the following corollary.

Corollary 9. *Let f be given by* (1.1) *and in the class $\mathfrak{S}^{*}_{\Sigma}(\vartheta, \phi)$. Then*

$$|a_2| \leqq \frac{\sqrt{2}\,|\vartheta(1-\cos\alpha)|}{\sqrt{\left|[(1-3\cos\alpha)-2\vartheta(\cos\alpha-1)^2]+4\vartheta(\cos\alpha-1)\right|}} \tag{2.21}$$

and

$$|a_3| \leqq |\vartheta(\cos\alpha-1)|^2 + \frac{|\vartheta(\cos\alpha-1)|}{2}. \tag{2.22}$$

Fixing $\lambda = 0$ in Theorem 7, we have the following corollary.

Corollary 10. *Let f be given by* (1.1) *and in the class $\mathfrak{G}^{a,b;c}_{\Sigma}(\vartheta, \phi)$. Then*

$$|a_2| \leqq \frac{|\vartheta(1-\cos\alpha)|}{\sqrt{\left|2(1-3\cos\alpha)\varphi_2^2+3\vartheta(\cos\alpha-1)\varphi_3\right|}} \tag{2.23}$$

and

$$|a_3| \leqq \frac{|\vartheta(\cos\alpha-1)|^2}{4\varphi_2^2} + \frac{|\vartheta(\cos\alpha-1)|}{3\varphi_3}. \tag{2.24}$$

Taking $a = c$ and $b = 1$, in Corollary 10, we get the following corollary.

Corollary 11. *Let f be assumed by* (1.1) *and in the class $\mathfrak{H}^{*}_{\Sigma}(\vartheta, \phi)$. Then*

$$|a_2| \leqq \frac{|\vartheta(1-\cos\alpha)|}{\sqrt{|2(1-3\cos\alpha)+3\vartheta(\cos\alpha-1)|}} \tag{2.25}$$

and

$$|a_3| \leqq \frac{|\vartheta(\cos\alpha-1)|^2}{4} + \frac{|\vartheta(\cos\alpha-1)|}{3}. \tag{2.26}$$

Due to Zaprawa [53], we prove Fekete-Szegö inequalities [12] for functions $f \in \mathfrak{S}^{a,b;c}_{\Sigma}(\vartheta, \lambda, \phi)$.

Theorem 12. *For $\nu \in \mathbb{R}$, let f be given by* (1.1) *and $f \in \mathfrak{S}^{a,b;c}_{\Sigma}(\vartheta, \lambda, \phi)$, then*

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|\vartheta(\cos\alpha-1)|}{(3-\lambda)\varphi_3} & ; \qquad 0 \leq |h(\nu)| \leq \dfrac{|\vartheta(\cos\alpha-1)|}{4(3-\lambda)\varphi_3} \\ 4\,|h(\nu)| & ; \qquad |h(\nu)| \geq \dfrac{|\vartheta(\cos\alpha-1)|}{4(3-\lambda)\varphi_3} \end{cases}$$

where

$$h(\nu) = \frac{\vartheta^2(1-\nu)(\cos\alpha-1)^2}{2\left([2\vartheta(\lambda^2-2\lambda)(\cos\alpha-1)^2+(2-\lambda)^2(1-3\cos\alpha)]\varphi_2^2+2\vartheta(3-\lambda)(\cos\alpha-1)\varphi_3\right)}. \tag{2.27}$$

Proof. From (2.18), we have

$$a_3 - \nu a_2^2 = \frac{\vartheta B_1(p_2 - q_2)}{4(3-\lambda)\varphi_3} + (1-\nu)\, a_2^2. \tag{2.28}$$

By substituting (2.15) in (2.28), we have

$$\begin{aligned} a_3 - \nu a_2^2 &= \frac{\vartheta(\cos\alpha - 1)(p_2 - q_2)}{4(3-\lambda)\varphi_3} \\ &+ \frac{\vartheta^2(1-\nu)(\cos\alpha - 1)^2(p_2 + q_2)}{2\left([2\vartheta(\lambda^2 - 2\lambda)(\cos\alpha - 1)^2 + (2-\lambda)^2(1 - 3\cos\alpha)]\varphi_2^2 + 2\vartheta(3-\lambda)(\cos\alpha - 1)\varphi_3\right)} \\ &= \left(h(\nu) + \frac{\vartheta(\cos\alpha - 1)}{4(3-\lambda)\varphi_3}\right)c_2 + \left(h(\nu) - \frac{\vartheta(\cos\alpha - 1)}{4(3-\lambda)\varphi_3}\right)d_2, \end{aligned} \tag{2.29}$$

where

$$h(\nu) = \frac{\vartheta^2(1-\nu)(\cos\alpha - 1)^2}{2\left([2\vartheta(\lambda^2 - 2\lambda)(\cos\alpha - 1)^2 + (2-\lambda)^2(1 - 3\cos\alpha)]\varphi_2^2 + 2\vartheta(3-\lambda)(\cos\alpha - 1)\varphi_3\right)}.$$

Thus by taking modulus of (2.29), we conclude that

$$\left|a_3 - \nu a_2^2\right| \le \begin{cases} \dfrac{|\vartheta(\cos\alpha - 1)|}{(3-\lambda)\varphi_3} & ; \quad 0 \le |h(\nu)| \le \dfrac{|\vartheta(\cos\alpha - 1)|}{4(3-\lambda)\varphi_3} \\ 4\,|h(\nu)| & ; \quad |h(\nu)| \ge \dfrac{\vartheta(\cos\alpha - 1)}{4(3-\lambda)\varphi_3} \end{cases} \tag{2.30}$$

where $h(\nu)$ is given by (2.27). □

By taking $\nu = 1$ in above Theorem one can easily state the following:

Remark 13. *Let the function f be assumed by* (1.1) *and $f \in \mathfrak{S}_{\Sigma}^{a,b;c}(\vartheta, \lambda, \phi)$. Then*

$$\left|a_3 - a_2^2\right| \le \frac{|\vartheta(\cos\alpha - 1)|}{(3-\lambda)\varphi_3}.$$

3. Subclass of bi-univalent function $\mathfrak{M}_{\Sigma}^{a,b,c}(\tau, \phi)$

In [27], Obradovic et.al gave some criteria for univalence expressing by $\Re(f'(z)) > 0$, for the linear combinations

$$\tau\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\tau)\frac{1}{f'(z)} > 0, \qquad (\tau \ge 1, z \in \mathbb{U}).$$

Based on the above definitions recently, Lashin in [19] introduced and studied the new subclasses of bi-univalent function.

Definition 14. *A function $f \in \Sigma$ given by* (1.1) *is said to be in the class $\mathfrak{M}_{\Sigma}^{a,b,c}(\tau, \phi)$ if it satisfies the following conditions :*

$$\tau\left(1 + \frac{z(\Im_{a,b;c}f(z))''}{(\Im_{a,b;c}f(z))'}\right) + (1-\tau)\frac{1}{(\Im_{a,b;c}f(z))'} \prec \phi(z) \tag{3.1}$$

and

$$\tau\left(1 + \frac{w(\Im_{a,b;c}g(w))''}{(\Im_{a,b;c}g(w))'}\right) + (1-\tau)\frac{1}{(\Im_{a,b;c}g(w))'} \prec \phi(w) \tag{3.2}$$

where $\tau \geq 1, z, w \in \mathbb{U}$ and the function g is given by (1.2).

Remark 15. *For a function $f \in \Sigma$ given by* (1.1)*, is said to be in the class $\mathfrak{M}_{\Sigma}^{a,b,c}(1, \phi) \equiv \mathfrak{K}_{\Sigma}^{a,b,c}(\phi)$ if it satisfies the following conditions :*

$$\left(1 + \frac{z(\Im_{a,b;c}f(z))''}{(\Im_{a,b;c}f(z))'}\right) \prec \phi(z) \text{ and } \left(1 + \frac{w(\Im_{a,b;c}g(w))''}{(\Im_{a,b;c}g(w))'}\right) \prec \phi(w)$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2).

Theorem 16. *Let f be given by* (1.1)*and $f \in \mathfrak{M}_{\Sigma}^{a,b,c}(\tau, \phi)$, $\tau \geq 1$. Then*

$$|a_2| \leq \min\begin{cases} \frac{|\cos\alpha - 1|}{2(2\tau-1)\varphi_2}, \\ \frac{\sqrt{2}(1-\cos\alpha)}{\sqrt{\left|(2(1+\tau)(1-\cos\alpha)+4(2\tau-1)^2(3\cos\alpha-1))\varphi_2^2\right|}} \end{cases} \tag{3.3}$$

and

$$|a_3| \leq \min\begin{cases} \frac{|1-\cos\alpha|}{3(3\tau-1)\varphi_3} + \frac{(1-\cos\alpha)^2}{4(2\tau-1)^2\varphi_2^2}, \\ \frac{|1-\cos\alpha|}{3(3\tau-1)\varphi_3} + \frac{2(1-\cos\alpha)^2}{\left|(2(1+\tau)(1-\cos\alpha)+8(2\tau-1)^2(3\cos\alpha-1))\varphi_2^2\right|}. \end{cases} \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that

$$\tau\left(1 + \frac{z(\Im_{a,b;c}f(z))''}{(\Im_{a,b;c}f(z))'}\right) + (1-\tau)\frac{1}{(\Im_{a,b;c}f(z))'} = \phi(u(z)) \tag{3.5}$$

and

$$\tau\left(1 + \frac{w(\Im_{a,b;c}g(w))''}{(\Im_{a,b;c}g(w))'}\right) + (1-\tau)\frac{1}{(\Im_{a,b;c}g(w))'} = \phi(v(w)). \tag{3.6}$$

From (3.5) and (3.6), we have

$$\begin{aligned} 1 + 2(2\tau-1)\varphi_2 a_2 z + \left[3(3\tau-1)\varphi_3 a_3 + 4(1-2\tau)\varphi_2^2 a_2^2\right] z^2 + \cdots \\ = 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2\right) z^2 + \cdots, \end{aligned}$$

and

$$1 - 2(2\tau-1)\varphi_2 a_2 w + \left(2(5\tau-1)\varphi_2^2 a_2^2 - 3(3\tau-1)\varphi_3 a_3\right) w^2 - \cdots$$

$$= 1 + \frac{1}{2}B_1 q_1 w + \left(\frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2\right) w^2 + \cdots .$$

Now, equating the coefficients, we get

$$2(2\tau - 1)\varphi_2 a_2 = \frac{1}{2}B_1 p_1, \tag{3.7}$$

$$3(3\tau - 1)\varphi_3 a_3 + 4(1 - 2\tau)\varphi_2^2 a_2^2 = \frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2, \tag{3.8}$$

$$-2(2\tau - 1)\varphi_2 a_2 = \frac{1}{2}B_1 q_1, \tag{3.9}$$

and

$$2(5\tau - 1)\varphi_2^2 a_2^2 - 3(3\tau - 1)\varphi_3 a_3 = \frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1 = -q_1 \tag{3.11}$$

From (3.7) by using (1.11),

$$|a_2| \leq \frac{B_1}{2(2\tau - 1)\varphi_2} \tag{3.12}$$

$$\leq \frac{|\cos\alpha - 1|}{2(2\tau - 1)\varphi_2}. \tag{3.13}$$

Also

$$\begin{aligned} 32(2\tau - 1)^2\varphi_2^2 a_2^2 &= B_1^2(p_1^2 + q_1^2) \\ a_2^2 &= \frac{B_1^2(p_1^2 + q_1^2)}{32(2\tau - 1)^2\varphi_2^2} \end{aligned} \tag{3.14}$$

Thus by (1.11), we get

$$|a_2| \leq \frac{B_1}{2(2\tau - 1)\varphi_2} = \frac{|\cos\alpha - 1|}{2(2\tau - 1)\varphi_2} \tag{3.15}$$

Now from (3.8), (3.10) and using (3.14), we obtain

$$\left(2(1 + \tau)B_1^2 - 8(2\tau - 1)^2(B_2 - B_1)\right)\varphi_2^2 a_2^2 = \frac{B_1^3(p_2 + q_2)}{2}. \tag{3.16}$$

Thus, by (3.16) we obtain

$$\begin{aligned} a_2^2 &= \frac{B_1^3(p_2 + q_2)}{2\left(2(1 + \tau)B_1^2 - 8(2\tau - 1)^2(B_2 - B_1)\right)\varphi_2^2} \\ |a_2|^2 &= \frac{2|B_1|^3}{\left|(2(1 + \tau)B_1^2 + 8(2\tau - 1)^2(B_1 - B_2))\varphi_2^2\right|} \\ &= \frac{2|1 - \cos\alpha|^2}{\left|(2(1 + \tau)(1 - \cos\alpha) + 4(2\tau - 1)^2(3\cos\alpha - 1))\varphi_2^2\right|} \end{aligned}$$

$$|a_2| \leq \frac{\sqrt{2}|1-\cos\alpha|}{\sqrt{\left|(2(1+\tau)(1-\cos\alpha)+4(2\tau-1)^2(3\cos\alpha-1))\varphi_2^2\right|}}$$

From (3.8) from (3.10) and using(3.11), we get

$$a_3 = \frac{B_1(p_2-q_2)}{12(3\tau-1)\varphi_3} + a_2^2. \tag{3.17}$$

Then taking modulus, we get

$$|a_3| \leq \frac{B_1}{3(3\tau-1)\varphi_3} + |a_2^2|. \tag{3.18}$$

Using (3.12) and (3.15), we get

$$\begin{aligned} |a_3| &\leq \frac{B_1}{3(3\tau-1)\varphi_3} + \frac{B_1^2}{4(2\tau-1)^2\varphi_2^2} \\ &= \frac{|1-\cos\alpha|}{3(3\tau-1)\varphi_3} + \frac{(1-\cos\alpha)^2}{4(2\tau-1)^2\varphi_2^2}. \end{aligned}$$

Now by using (3.16) in (3.18) ,

$$\begin{aligned} |a_3| &\leq \frac{|B_1|}{3(3\tau-1)\varphi_3} + |a_2^2| \\ &= \frac{|1-\cos\alpha|}{3(3\tau-1)\varphi_3} + \frac{2(1-\cos\alpha)^2}{\left|(2(1+\tau)(1-\cos\alpha)+8(2\tau-1)^2(3\cos\alpha-1))\varphi_2^2\right|}. \end{aligned}$$

□

Due to Zaprawa [53], we prove Fekete-Szegö inequalities [12] for functions $f \in \mathfrak{M}_{\Sigma}^{a,b,c}(\tau,\phi)$

Theorem 17. *For $\nu \in \mathbb{R}$, let f be given by* (1.1) *and $f \in \mathfrak{M}_{\Sigma}^{a,b,c}(\tau,\phi)$, then*

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|\cos\alpha - 1|}{3(3\tau-1)\varphi_3} & ; 0 \leq |h(\nu)| \leq \dfrac{|\cos\alpha-1|}{12(3\tau-1)\varphi_3} \\ 4\,|h(\nu)| & ; |h(\nu)| \geq \dfrac{|\cos\alpha-1|}{12(3\tau-1)\varphi_3} \end{cases}$$

where

$$h(\nu) = \frac{(1-\nu)(1-\cos\alpha)^2}{2\left|(2(1+\tau)(1-\cos\alpha)+4(2\tau-1)^2(3\cos\alpha-1))\varphi_2^2\right|}.$$

Proof. From (3.17), we have

$$a_3 - \nu a_2^2 = \frac{B_1(p_2-q_2)}{12(3\tau-1)\varphi_3} + (1-\nu)a_2^2. \tag{3.19}$$

By substituting (3.16) in (3.19), we have

$$\begin{aligned} a_3 - \nu a_2^2 &= \frac{B_1(p_2 - q_2)}{12(3\tau - 1)\varphi_3} + \frac{B_1^3(p_2 + q_2)(1 - \nu)}{2\left(2(1 + \tau)B_1^2 - 8(2\tau - 1)^2(B_2 - B_1)\right)\varphi_2^2} \\ &= \left(h(\nu) + \frac{(\cos\alpha - 1)}{12(3\tau - 1)\varphi_3}\right)p_2 + \left(h(\nu) - \frac{(\cos\alpha - 1)}{12(3\tau - 1)\varphi_3}\right)q_2, \end{aligned} \tag{3.20}$$

where

$$h(\nu) = \frac{(1 - \nu)(1 - \cos\alpha)^2}{2\left|(2(1 + \tau)(1 - \cos\alpha) + 4(2\tau - 1)^2(3\cos\alpha - 1))\varphi_2^2\right|} \tag{3.21}$$

Thus by taking modulus of (3.20), we get

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|\cos\alpha - 1|}{3(3\tau - 1)\varphi_3} & ; 0 \leq |h(\nu)| \leq \dfrac{|\cos\alpha - 1|}{12(3\tau - 1)\varphi_3} \\ 4\,|h(\nu)| & ; |h(\nu)| \geq \dfrac{|\cos\alpha - 1|}{12(3\tau - 1)\varphi_3} \end{cases} \tag{3.22}$$

where $h(\nu)$ is given by (3.21). □

By taking $\nu = 1$ in above Theorem one can easily state the following:

Remark 18. *Let f be given by* (1.1) *and $f \in \mathfrak{M}_{\Sigma,}^{a,b,c}(\tau, \phi)$. Then*

$$\left|a_3 - a_2^2\right| \leq \frac{|\cos\alpha - 1|}{3(3\tau - 1)\varphi_3}.$$

4. Concluding remarks

If $a = 1$, $b = 1 + \delta$, $c = 2 + \delta$ with $\Re(\delta) > -1$, then the operator $\Im_{a,b,c}f$ turns into familiar Bernardi operator [6]

$$B_f(z) = [\Im_{a,b,c}(f)](z) = \frac{1 + \delta}{z^\delta} \int_0^1 t^{\delta - 1} f(t)dt.$$

$\Im_{1,1,2}f$ and $\Im_{1,2,3}f$ are the famous Alexander [14] and Libera [21] operators, respectively. Further if $b = 1$ in (1.6), then $\Im_{a,b;c}$ immediately yields the Carlson-Shaffer operator $L(a, c)(f) := \Im_{a,1,c}f$ [8]. So, numerous other interesting corollaries and consequences of our main results (which are asserted by Theorem 7-Theorem 17) can be derived similarly. Further by fixing $\alpha = \pi$ one can state the referents new results for the function classes defined in this paper. The facts involved may be left as an exercise for the interested reader. Also, motivating further researches on the subject-matter of this,we have chosen to draw the attention of the interested readers toward a considerably large number of related recent publications(see,for example, [38,44–46]). and developments in the area of mathematical analysis. In conclusion,we choose to reiterate an important observation,which was presented in the recently-published review-cum-expository review article by Srivastava ([38], p. 340),who pointed out the fact that the results for the above-mentioned or new $q-$ analogues can easily(and possibly trivially)be translated into the corresponding results for the so-called $(p; q)-$analogues(with $0 < |q| < p \leq 1$)by applying some obvious parametric and argument variations with the additional parameter p being redundant.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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