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Research article

Properties of λ-pseudo-starlike functions with respect to a boundary point

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Abstract: The purpose of this present paper is to investigate some mapping properties of functions which map the unit disc onto a overlapped leaf-like curve, having real part greater than zero. Also we define a class of λ-pseudo starlike functions related to a leaf-like curve. Integral representation, inequalities for the initial Taylor-Maclaurin coefficients and Fekete-Szegö problem for subclasses of analytic functions related to various conic regions are obtained as our main results.

Keywords: analytic functions; starlike functions; functions with positive real part; subordination; Fekete-Szegö problem; coefficient inequalities; q-calculus
Mathematics Subject Classification: 30C45, 30C80

1. Introduction

We denote by $\mathcal{H}(\Theta)$, the set of all analytic function defined in the open unit disc $\Theta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Also, we will denote by Π, the subclass of $\mathcal{H}(\Theta)$ and has a Taylor series expansion of the form

$$\chi(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n. \tag{1.1}$$

We let $\mathcal{S}$ to denote the class of functions in Π which are *univalent* in Θ. In Π, we classify the collection $\mathcal{F}$ of functions $\vartheta(\zeta) \in \Pi$ with $\vartheta(0) = 1$ and $\Re\, \vartheta(\zeta) > 0$. The class of functions in $\mathcal{F}$ are not univalent. However, if the family of functions in $\mathcal{F}$ are single valued then the set $\mathcal{F}$ is normal and compact [1, p. 136].

For a function $\vartheta(\zeta) \in \mathcal{F}$, it is known that the transformation

$$\Omega(\zeta) = \frac{1-\zeta}{\zeta}\left[(1+\zeta) - (1-\zeta)\vartheta(\zeta)\right] \tag{1.2}$$

is regular in Θ and $Re\left[\Omega(\zeta)\right] \geq 0$ in Θ (see [2, p. 104]). It should be noted that $\vartheta(\zeta) \in \mathcal{F}$ does not imply that $\Omega(\zeta) \in \mathcal{F}$. For example, if we let $\vartheta(\zeta) = \frac{1+\zeta}{1-\zeta}$ in (1.2) which is an extremal function in class $\mathcal{F}$, then $\Omega(\zeta) = 0$. The transformation $\Omega(\zeta)$ is invariant if $\vartheta(\zeta) = \frac{1-\zeta^2}{1-\zeta+\zeta^2}$. Suppose $\vartheta(\zeta) = 1 + \sum_{n=1}^{\infty} \vartheta_n \zeta^n$ is in $\mathcal{F}$, then

$$\Omega(\zeta) = 2 - \vartheta_1 + (2\vartheta_1 - \vartheta_2 - 2)\,\zeta + \sum_{n=2}^{\infty} (2\vartheta_n - \vartheta_{n-1} - \vartheta_{n+1})\,\zeta^n,$$

and for $n \geq 1$ we have (see [3, Theorem 10])

$$\left|\vartheta_{n+1} - \vartheta_n\right| \leq (2n+1)\left|2 - \vartheta_1\right|$$

and

$$\left||\vartheta_{n+1}| - |\vartheta_n|\right| \leq (2n+1)\,(2 - |\vartheta_1|)\,.$$

1.1. Koebe function

It is well-known that *Koebe function* $\vartheta(\zeta) = \zeta/(1-\zeta)^2$ maps unit disc on to entire complex plane except for a horizontal slit from $-1/4$ to $-\infty$ and it acts as an extremal function for most of subclasses in univalent function theory. If we let $\vartheta(\zeta) = \zeta/(1-\zeta)^2$ in (1.2), then $\Omega(\zeta)$ maps open unit disc onto entire complex plane except for a vertical oval slit on the imaginary axis from $-2i$ to $2i$ (see Figure 1). Also $\Omega(\zeta)$ has the following series representation at $\zeta = -1$,

$$\Omega(\zeta) = -1 - 2(\zeta+1) - (\zeta+1)^2 - (\zeta+1)^3 - (\zeta+1)^4 - (\zeta+1)^5 - (\zeta+1)^6 + O[\zeta+1]^7.$$

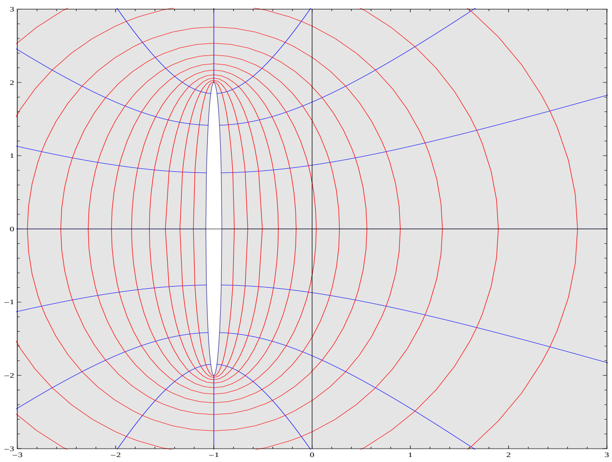
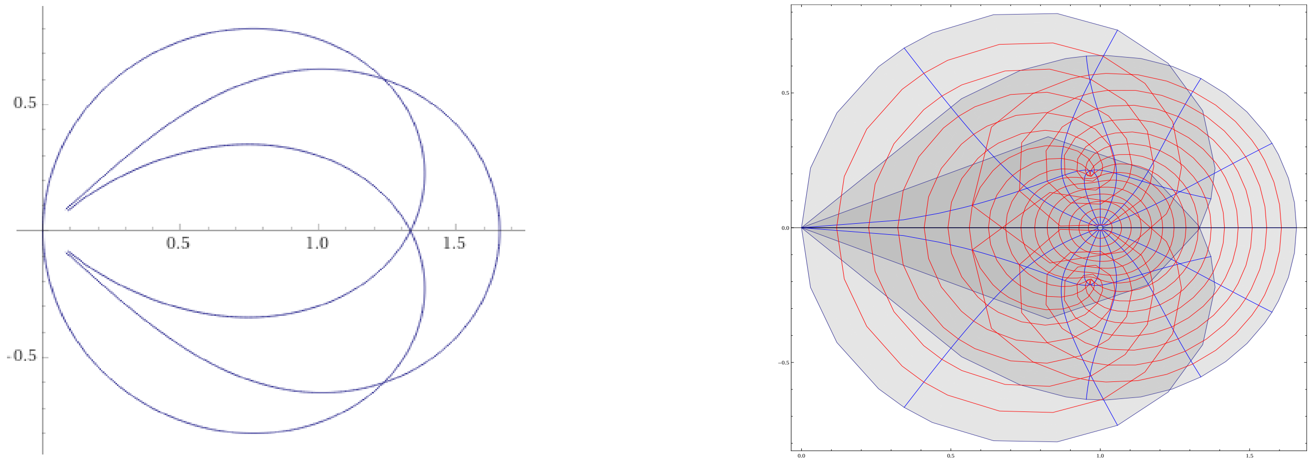


Figure 1. Image of $\Omega(\zeta)$ if $\vartheta(\zeta) = \zeta/(1-\zeta)^2$.

1.2. Overlapped clover leaf-like domain

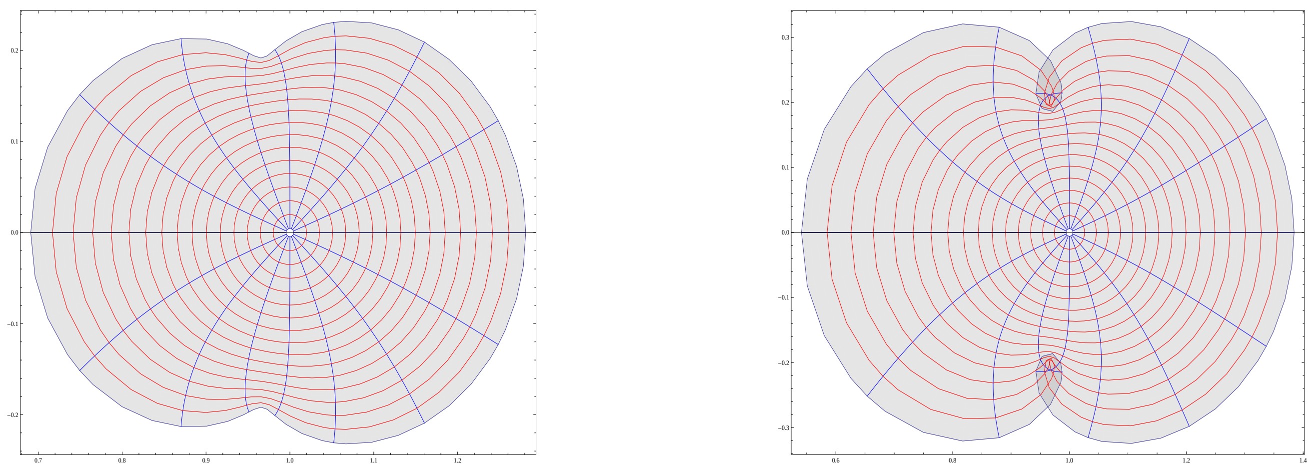
It is well-known that function $\zeta + \sqrt{1+\zeta^2}$ maps open unit disc onto a shell-shaped region. Now if we let $\vartheta(\zeta) = \zeta + \sqrt{1+\zeta^2}$ (the branch of the square root is chosen to be the principal one) in (1.2), then $\Omega(\zeta)$ maps Θ onto the right half plane, having real part greater than 0 and symmetric with respect to real axis. Also we observe that the image of a unit circle under the function $\Omega(\zeta)$ is translated into a overlapped clover leaf like curve (see Figure 2). The image of $\Omega(\zeta)$ becomes circular when the radius of the disc is smaller (see Figure 3). Clearly, the function $\Omega(\zeta)$ is not univalent in Θ and has series expansion of the form

$$\Omega(\zeta) = 1 - \frac{\zeta}{2} - \frac{3\zeta^3}{8} - \frac{\zeta^4}{4} + \frac{\zeta^5}{16} + \frac{\zeta^6}{8} - \frac{3\zeta^7}{128} - \frac{5\zeta^8}{64} + \frac{3\zeta^9}{256} + \frac{7\zeta^{10}}{128} + O[\zeta]^{10}. \tag{1.3}$$



(a) (b)

Figure 2. (a) Mapping of $|\zeta| = 1$ under $\Omega(\zeta)$ if $\vartheta(\zeta) = \zeta + \sqrt{1+\zeta^2}$; (b) Mapping of $|\zeta| < 1$ under $\Omega(\zeta)$ if $\vartheta(\zeta) = \zeta + \sqrt{1+\zeta^2}$.



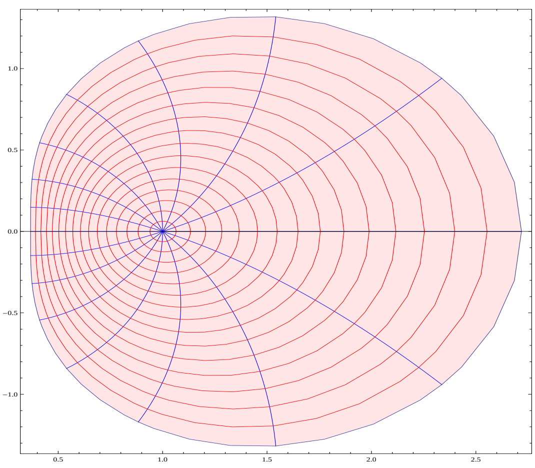
(a) (b)

Figure 3. (a) Image of $|\zeta| < 0.5$ under $\Omega(\zeta)$ if $\vartheta(\zeta) = \zeta + \sqrt{1+\zeta^2}$; (b) Image of $|\zeta| < 0.65$ under $\Omega(\zeta)$ if $\vartheta(\zeta) = \zeta + \sqrt{1+\zeta^2}$.

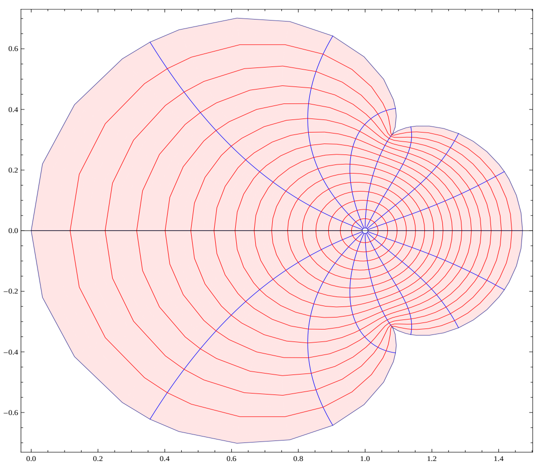
1.3. Starlike domain

Taking $\vartheta(\zeta) = e^{\zeta}$ in (1.2), we find that $\Omega(\zeta)$ maps open unit disc onto a domain which is starlike with respect to 1 (see Figure 4) and has series representation of the form

$$\Omega(\zeta) = 1 - \frac{\zeta}{2} - \frac{\zeta^2}{6} - \frac{5\zeta^3}{24} - \frac{11\zeta^4}{120} - \frac{19\zeta^5}{720} - \frac{29\zeta^6}{5040} - \frac{41\zeta^7}{40320} - \frac{11\zeta^8}{72576} - \frac{71\zeta^9}{3628800} + O[\zeta]^{10}.$$



(a)



(b)

Figure 4. (a) Mapping of $|\zeta| < 1$ under the transformation $\vartheta(\zeta) = e^{\zeta}$; (b) Mapping of $|\zeta| < 1$ under the transformation of $\Omega(\zeta)$ if $\vartheta(\zeta) = e^{\zeta}$.

For $\chi \in \Pi$ given by (1.1) and $0 < q < 1$, the *Jackson's q-derivative operator* or *q-difference operator* for a function $\chi \in \Pi$ is defined by (see [4, 5])

$$\mathfrak{D}_q\chi(\zeta) := \begin{cases} \chi'(0), & \text{if } \zeta = 0, \\ \dfrac{\chi(\zeta) - \chi(q\zeta)}{(1-q)\zeta}, & \text{if } \zeta \neq 0. \end{cases} \tag{1.4}$$

From (1.4), if χ has the power series expansion (1.1) we can easily see that $\mathfrak{D}_q\chi(\zeta) = 1 + \sum_{n=2}^{\infty} [n]_q a_n \zeta^{n-1}$, for $\zeta \neq 0$, where the *q-integer number* $[n]_q$ is defined by

$$[n]_q := \frac{1-q^n}{1-q},$$

and note that $\lim_{q\to 1^-} \mathfrak{D}_q\chi(\zeta) = \chi'(\zeta)$. Throughout this paper, we let denote

$$([n]_q)_k := [n]_q[n+1]_q[n+2]_q \ldots [n+k-1]_q.$$

The *q-Jackson integral* is defined by (see [6])

$$I_q[\chi(\zeta)] := \int_0^{\zeta} \chi(t) d_q t = \zeta(1-q) \sum_{n=0}^{\infty} q^n \chi(\zeta q^n), \tag{1.5}$$

provided the q-series converges. Further observe that

$$\mathfrak{D}_q I_q \chi(\zeta) = \chi(\zeta) \qquad \text{and} \qquad I_q \mathfrak{D}_q \chi(\zeta) = \chi(\zeta) - \chi(0),$$

where the second equality holds if χ is continuous at $\zeta = 0$. The significance of the q-derivative operator $\mathfrak{D}_q$ is quite evident by its applications in the study of several subclasses of analytic functions. The firm base of the usage of the q-calculus in the context of *geometric function theory* was efficiently established, and the use of the generalized basic (or q-) hypergeometric functions in *geometric function theory* was made by Srivastava (see, for details, [7, 8]). Recently lots of results have been established in this duality theory (see [9–18]).

Let $\chi(\zeta)$ and $g(\zeta)$ be analytic in Θ. Then we say that the function $\chi(\zeta)$ is *subordinate* to $g(\zeta)$ in Θ, if there exists an *Schwarz function* $w(\zeta)$ in Θ such that $|w(\zeta)| < |\zeta|$ and $\chi(\zeta) = g(w(\zeta))$, denoted by $\chi(\zeta) \prec g(\zeta)$. If $g(\zeta)$ is univalent in Θ, then the subordination is equivalent to $\chi(0) = g(0)$ and $\chi(\Theta) \subset g(\Theta)$.

For $0 \leq \eta < 1$, let $\mathcal{S}^*(\eta)$ and $\mathcal{C}(\eta)$ denote the classes of *starlike functions of order* η and *convex functions of order* η, respectively. Using the concept of subordination for holomorphic functions, Ma and Minda [19] introduced the classes

$$\mathcal{S}^*(\psi) = \left\{\chi \in \Pi : \frac{\zeta\chi'(\zeta)}{\chi(\zeta)} \prec \psi(\zeta)\right\} \quad \text{and} \quad \mathcal{C}(\psi) = \left\{\chi \in \Pi : 1 + \frac{\zeta\chi''(\zeta)}{\chi'(\zeta)} \prec \psi(\zeta)\right\},$$

where $\psi \in \mathcal{F}$ with $\psi'(0) > 0$ maps Θ onto a region starlike with respect to 1 and symmetric with respect to real axis. By choosing ϕ to map unit disc onto some specific regions like parabolas, cardioid, lemniscate of Bernoulli, booth lemniscate in the right-half of the complex plane, various interesting subclasses of starlike and convex functions can be obtained.

Robertson [3] introduced the following class of analytic functions which satisfy the condition

$$\Re\left\{\frac{2\zeta G'(\zeta)}{G(\zeta)} + \frac{1+\zeta}{1-\zeta}\right\} > 0, \quad (G(0) = 1;\ \zeta \in \Theta).$$

Starlike functions with respect to a boundary point didn't receive much attention of the researchers, for developments pertaining to starlike functions with respect to a boundary point refer to the recent study by Lecko, Murugusundaramoorthy and Sivasubramanian in [20, 21] and references provided therein.

Throughout this study, $w(\zeta) = \sum_{n=1}^{\infty} w_n \zeta^n$, $\zeta \in \Theta$ will denote the Schwarz function such that $w(0) = 0$ and $|w(\zeta)| < 1$, $\zeta \in \Theta$.

We now begin with the following difinition.

Definition 1.1. For $0 \leq \alpha \leq 1$, we say that the function $\chi \in \Pi$ belongs to the class $\mathcal{KL}^{\alpha}(\psi)$ if it satisfies the subordination condition

$$1 + \frac{1-\zeta}{\zeta}\left[(1+\zeta) - (1-\zeta)\left(\frac{\alpha\zeta^2\chi''(\zeta) + \zeta\chi'(\zeta)}{(1-\alpha)\chi(\zeta) + \alpha\zeta\chi'(\zeta)}\right)\right] \prec \psi(\zeta), \tag{1.6}$$

where $\psi \in \mathcal{F}$ and has a series expansion of the form

$$\psi(\zeta) = \delta + A_1\zeta + A_2\zeta^2 + A_3\zeta^3 + \cdots, \qquad (A_1 \neq 0;\ \delta \in \mathbb{C} \setminus \{0\};\ \zeta \in \Theta). \tag{1.7}$$

Remark 1.1. We note that the function

$$P(\zeta) := 1 + \frac{1-\zeta}{\zeta}\left[(1+\zeta) - (1-\zeta)\left(\frac{\alpha\zeta^2\chi''(\zeta) + \zeta\chi'(\zeta)}{(1-\alpha)\chi(\zeta) + \alpha\zeta\chi'(\zeta)}\right)\right]$$

is well-defined and analytic in Θ. Further, $\Re[P(\zeta)] > 0$ if $\Re\left(\frac{\alpha\zeta^2\chi''(\zeta)+\zeta\chi'(\zeta)}{(1-\alpha)\chi(\zeta)+\alpha\zeta\chi'(\zeta)}\right) > 0$ and is chosen so that this function does not vanish in Θ (see [3, Lemma 1]).

Just swapping the geometry in the definition of $\mathcal{KL}^{\alpha}(\psi)$ from left hand side to right hand side, we now define the following.

Definition 1.2. For a function $\vartheta(\zeta) \in \mathcal{F}$, $0 \leq \alpha \leq 1$, $\delta \in \mathbb{C} \setminus \{0\}$ and $\lambda \geq 1$ is real , we say that the function $\chi \in \Pi$ belongs to the class $\mathcal{CL}^{\alpha}_{\lambda}(\Omega)$ if it satisfies the subordination condition

$$\delta + \frac{\zeta\left[\alpha\zeta\chi''(\zeta) + \chi'(\zeta)\right]^{\lambda}}{(1-\alpha)\chi(\zeta) + \alpha\zeta\chi'(\zeta)} \prec 1 + \Omega(\zeta), \tag{1.8}$$

where $\Omega(\zeta)$ is defined as in (1.2) with $\vartheta(0) = 1$.

Remark 1.2. The left hand side of (1.8) was motivated by the so-called λ-pseudo starlike functions introduced and studied by Babalola in [22].

For completeness, we will now define q-analogue of the $\mathcal{KL}^{\alpha}(\psi)$ and $\mathcal{CL}^{\alpha}_{\lambda}(\Omega)$ as follows.

Definition 1.3. For $0 \leq \alpha \leq 1$, we say that the function $\chi \in \Pi$ belongs to the class $\mathcal{KQ}^{\alpha}(\psi)$ if it satisfies the subordination condition

$$1 + \frac{1-\zeta}{\zeta}\left[(1+\zeta) - \zeta(1-\zeta)\left(\frac{\alpha q\zeta^2\mathfrak{D}_q(\mathfrak{D}_q([\chi(\zeta)]) + \zeta\mathfrak{D}_q\chi(\zeta)}{(1-\alpha)\chi(\zeta) + \alpha\zeta\mathfrak{D}_q\chi(\zeta)}\right)\right] \prec \psi(\zeta), \tag{1.9}$$

where $\psi \in \mathcal{F}$ is given by (1.7).

Definition 1.4. For $0 \leq \alpha \leq 1$ and $\lambda \geq 1$ is real, we say that the function $\chi \in \Pi$ belongs to the class $\mathcal{CQ}^{\alpha}_{\lambda}(\Omega)$ if it satisfies the subordination condition

$$\delta + \left(\frac{\zeta[\alpha q\zeta\mathfrak{D}_q(\mathfrak{D}_q(\chi(\zeta)) + \mathfrak{D}_q\chi(\zeta)]^{\lambda}}{(1-\alpha)\chi(\zeta) + \alpha\zeta\mathfrak{D}_q\chi(\zeta)}\right) \prec 1 + \Omega(\zeta), \tag{1.10}$$

where $\Omega(\zeta)$ is defined as in (1.2).

Remark 1.3. Even though the classes that were defined are closely related to the various subclasses of analytic functions, but unfortunately we couldn't find any special cases which could be readily obtained by giving some fixed values to the parameters involved.

2. Preliminaries

In this section we state the results that would be used to establish our main results which can be found in the standard text in *geometric function theory*.

Lemma 2.1. *[23, p. 41] If* $\vartheta(\zeta) = 1 + \sum_{n=1}^{\infty} \vartheta_n\zeta^n \in \mathcal{F}$, *then* $|\vartheta_n| \leq 2$ *for all* $n \geq 1$, *and the inequality is sharp for* $\vartheta_{\lambda}(\zeta) = \dfrac{1+\lambda\zeta}{1-\lambda\zeta}$, $|\lambda| \leq 1$.

Lemma 2.2. *[19] If* $\vartheta(\zeta) = 1 + \sum_{n=1}^{\infty} \vartheta_n \zeta^n \in \mathcal{F}$*, and* v *is complex number, then*

$$\left|\vartheta_2 - v\vartheta_1^2\right| \leq 2\max\{1; |2v-1|\},$$

and the result is sharp for the functions

$$\vartheta_1(\zeta) = \frac{1+\zeta}{1-\zeta} \quad \text{and} \quad \vartheta_2(\zeta) = \frac{1+\zeta^2}{1-\zeta^2}.$$

3. Integral representation

Theorem 3.1. *If* $\chi \in \mathcal{KL}^{\alpha}(\psi)$*, then*

$$\chi(\zeta) = \begin{cases} \zeta \exp\left\{\int_0^{\zeta} \frac{1}{t}\left[\frac{1+t}{1-t} + \frac{t}{(1-t)^2}\{1-\psi[w(t)]\} - 1\right] dt\right\}, & \text{if } \alpha = 0 \\ \int_0^{\zeta} \exp\left\{\int_0^{\eta} \frac{1}{t}\left[\frac{1+t}{1-t} + \frac{t}{(1-t)^2}\{1-\psi[w(t)]\} - 1\right] dt\right\} d\eta, & \text{if } \alpha = 1. \end{cases} \tag{3.1}$$

Proof. By the definition of $\mathcal{KL}^{\alpha}(\psi)$, we have

$$1 + \frac{1-\zeta}{\zeta}\left[(1+\zeta) - (1-\zeta)\left(\frac{\alpha\zeta^2\chi''(\zeta) + \zeta\chi'(\zeta)}{(1-\alpha)\chi(\zeta) + \alpha\zeta\chi'(\zeta)}\right)\right] = \psi[w(\zeta)]. \tag{3.2}$$

Suppose that $F_\alpha(\zeta) = (1-\alpha)\chi(\zeta) + \alpha\zeta\chi'(\zeta)$, then the condition (3.2) can be rewritten as

$$\frac{F'_\alpha(\zeta)}{F_\alpha(\zeta)} - \frac{1}{\zeta} = \frac{1}{\zeta}\left[\frac{1+\zeta}{1-\zeta} + \frac{\zeta}{(1-\zeta)^2}\{1-\psi[w(\zeta)]\} - 1\right].$$

Integrating the above expression, we have (integrating ζ_0 to ζ with $\zeta_0 \neq 0$ and then let $\zeta_0 \to 0$),

$$\log\left\{\frac{F_\alpha(\zeta)}{\zeta}\right\} = \int_0^{\zeta} \frac{1}{t}\left[\frac{1+t}{1-t} + \frac{t}{(1-t)^2}\{1-\psi[w(t)]\} - 1\right] dt.$$

Or equivalently,

$$F_\alpha(\zeta) = \zeta \exp\left\{\int_0^{\zeta} \frac{1}{t}\left[\frac{1+t}{1-t} + \frac{t}{(1-t)^2}\{1-\psi[w(t)]\} - 1\right] dt\right\}.$$

Thus, if $\chi \in \mathcal{KL}^{\alpha}(\psi)$, then we have

$$\chi(\zeta) = \zeta \exp\left\{\int_0^{\zeta} \frac{1}{t}\left[\frac{1+t}{1-t} + \frac{t}{(1-t)^2}\{1-\psi[w(t)]\} - 1\right] dt\right\}, \quad \text{if } \alpha = 0, \tag{3.3}$$

and

$$\zeta\chi'(\zeta) = \zeta \exp\left\{\int_0^{\zeta} \frac{1}{t}\left[\frac{1+t}{1-t} + \frac{t}{(1-t)^2}\{1-\psi[w(t)]\} - 1\right] dt\right\}, \quad \text{if } \alpha = 1. \tag{3.4}$$

On simplifying and integrating (3.4), we can establish the assertion of Theorem 3.1. □

To obtain the integral representation of the form (3.1) for the q-analogue class, we need the following result obtained by Agrawal and Sahoo [24]. For $\chi \in \mathcal{H}(\Theta)$ and $0 < q < 1$, we have

$$I_q \frac{\mathfrak{D}_q \chi(\zeta)}{\chi(\zeta)} = \frac{q-1}{\ln q} \log \chi(\zeta), \tag{3.5}$$

where $I_q \chi$ is the Jackson q-integral, defined as in (1.5).

Theorem 3.2. *If $\chi \in \mathcal{KQ}^{\alpha}(\psi)$, then*

$$\chi(\zeta) = \begin{cases} \zeta \exp\left(\frac{\ln q}{q-1} \int_0^\zeta \frac{1}{t}\left[\frac{1+t}{1-t} + \frac{t}{(1-t)^2}\{1-\psi[w(t)]\} - 1\right] d_q t\right), & \text{if } \alpha = 0 \\ \int_0^\zeta \exp\left(\frac{\ln q}{q-1} \int_0^\eta \frac{1}{t}\left[\frac{1+t}{1-t} + \frac{t}{(1-t)^2}\{1-\psi[w(t)]\} - 1\right] d_q t\right) d_q \eta, & \text{if } \alpha = 1. \end{cases} \tag{3.6}$$

Proof. Suppose that $\mathcal{G}_\alpha(\zeta) = (1-\alpha)\chi(\zeta) + \alpha\zeta\mathfrak{D}_q\chi(\zeta)$, then the condition (1.9) can be rewritten as

$$\frac{\mathfrak{D}_q \mathcal{G}_\alpha(\zeta)}{\mathcal{G}_\alpha(\zeta)} - \frac{1}{\zeta} = \frac{1}{\zeta}\left[\frac{1+\zeta}{1-\zeta} + \frac{\zeta}{(1-\zeta)^2}\{1-\psi[w(\zeta)]\} - 1\right].$$

Applying the q-integration to the above expression (see (3.5)), we have,

$$\frac{q-1}{\ln q} \log\left\{\frac{\mathcal{G}_\alpha(\zeta)}{\zeta}\right\} = \int_0^\zeta \frac{1}{t}\left[\frac{1+t}{1-t} + \frac{t}{(1-t)^2}\{1-\psi[w(t)]\} - 1\right] d_q t.$$

Or equivalently,

$$\mathcal{G}_\alpha(\zeta) = \zeta \exp\left\{\frac{\ln q}{q-1} \int_0^\zeta \frac{1}{t}\left[\frac{1+t}{1-t} + \frac{t}{(1-t)^2}\{1-\psi[w(t)]\} - 1\right] d_q t\right\}.$$

Thus, if $\chi \in \mathcal{KQ}^{\alpha}(\psi)$, then we have

$$\chi(\zeta) = \zeta \exp\left\{\frac{\ln q}{q-1} \int_0^\zeta \frac{1}{t}\left[\frac{1+t}{1-t} + \frac{t}{(1-t)^2}\{1-\psi[w(t)]\} - 1\right] d_q t\right\}, \quad \text{if } \alpha = 0,$$

and

$$\chi(\zeta) = \int_0^\zeta \exp\left\{\frac{\ln q}{q-1} \int_0^\eta \frac{1}{t}\left[\frac{1+t}{1-t} + \frac{t}{(1-t)^2}\{1-\psi[w(t)]\} - 1\right] d_q t\right\} d_q \eta, \quad \text{if } \alpha = 1.$$

Hence the proof of Theorem 3.2 is completed. □

Theorem 3.3. *If $\chi \in C\mathcal{L}_\lambda^{\alpha}(\Omega)$, then*

$$\chi(\zeta) = \left\{\frac{\lambda-1}{\lambda} \int_0^\zeta \left[\frac{1-\delta+\Omega(w(t))}{t}\right]^{1/\lambda} dt\right\}^{\frac{\lambda}{\lambda-1}}, \quad (\text{if } \alpha = 0) \tag{3.7}$$

and

$$\chi(\zeta) = \int_0^\zeta \frac{1}{\eta}\left\{\frac{\lambda-1}{\lambda} \int_0^\eta \left[\frac{1-\delta+\Omega(w(t))}{t}\right]^{1/\lambda} dt\right\}^{\frac{\lambda}{\lambda-1}} d\eta, \quad (\text{if } \alpha = 1). \tag{3.8}$$

Proof. If we let $F_\alpha(\zeta) = (1-\alpha)\chi(\zeta) + \alpha\zeta\chi'(\zeta)$, then Eq (1.8) can be rewritten as

$$\frac{\zeta[F'_\alpha(\zeta)]^\lambda}{F_\alpha(\zeta)} \prec 1-\delta+\Omega(\zeta). \tag{3.9}$$

The condition (3.9) is equivalent to

$$\left[(F_\alpha(\zeta))^{1-1/\lambda}\right]' = (1-1/\lambda)\,\frac{\{1-\delta+\Omega[w(\zeta)]\}^{1/\lambda}}{\zeta^{1/\lambda}}.$$

Now following the steps as in Theorem 3.1, we can establish the assertion of Theorem 3.3. □

Theorem 3.4. *If $\chi \in \mathcal{CQ}^\alpha_\lambda(\Omega)$, then*

$$\chi(\zeta) = \left\{\frac{\lambda-1}{\lambda}\int_0^\eta \left[\frac{1-\delta+\Omega(w(t))}{t}\right]^{1/\lambda} d_qt\right\}^{\frac{\lambda}{\lambda-1}}, \quad \text{if } \alpha = 0 \tag{3.10}$$

and

$$\chi(\zeta) = \int_0^\zeta \frac{1}{\eta}\left\{\frac{\lambda-1}{\lambda}\int_0^\eta \left[\frac{1-\delta+\Omega(w(t))}{t}\right]^{1/\lambda} d_qt\right\}^{\frac{\lambda}{\lambda-1}} d_q\eta, \quad \text{if } \alpha = 1. \tag{3.11}$$

Proof. By the definition $C\mathcal{L}^\alpha_\lambda(\Omega)$, we can rewrite (1.10) as

$$\left(\frac{\zeta[\alpha q\zeta\mathfrak{D}_q(\mathfrak{D}_q(\chi(\zeta)) + \mathfrak{D}_q\chi(\zeta)]^\lambda}{(1-\alpha)\chi(\zeta)+\alpha\zeta\mathfrak{D}_q\chi(\zeta)}\right) = 1-\delta+\Omega[w(\zeta)]. \tag{3.12}$$

Taking $\mathcal{G}_\alpha(\zeta) = (1-\alpha)\chi(\zeta) + \alpha\zeta\mathfrak{D}_q[\chi(\zeta)]$, we can rewrite (3.12) by

$$\frac{\mathfrak{D}_q\mathcal{G}_\alpha(\zeta)}{[\mathcal{G}_\alpha(\zeta)]^{1/\lambda}} = \left[\frac{1-\delta+\Omega[w(\zeta)]}{\zeta}\right]^{1/\lambda}. \tag{3.13}$$

In quantum calculus, it is impossible to have a general chain rule. However we know that

$$\mathfrak{D}_q[\log\mathcal{G}_\alpha(\zeta)] = \frac{\ln q}{q-1}\frac{\mathfrak{D}_q\mathcal{G}_\alpha(\zeta)}{\mathcal{G}_\alpha(\zeta)}.$$

Hence we can easily prove that

$$\mathfrak{D}_q\left[\{\mathcal{G}_\alpha(\zeta)\}^{\frac{\lambda-1}{\lambda}}\right] = \frac{\lambda-1}{\lambda}\mathfrak{D}_q[\mathcal{G}_\alpha(\zeta)]\,\{\mathcal{G}_\alpha(\zeta)\}^{-1/\lambda}.$$

Using this equality, the condition (3.13) can be equivalently written as

$$\mathfrak{D}_q\left[(\mathcal{G}_\alpha(\zeta))^{\frac{\lambda-1}{\lambda}}\right] = \frac{\lambda-1}{\lambda}\left[\frac{\{1-\delta+\Omega[w(\zeta)]\}}{\zeta}\right]^{1/\lambda}.$$

Following the steps as in Theorem 3.2, we can establish the assertion of Theorem 3.4. □

4. Bounds for the initial coefficients and Fekete-Szegö inequalities

Theorem 4.1 and Theorem 4.2 establish the estimates for the initial coefficients and Fekete-Szegö inequalities for the class $\mathcal{KL}^{\alpha}(\psi)$ and $C\mathcal{L}_{\lambda}^{\alpha}(\Omega)$, respectively.

Theorem 4.1. *Let $\chi \in \mathcal{KL}^{\alpha}(\psi)$ have a Taylor series expansion of the form (1.1) and $\psi(\zeta) = \delta + A_1\zeta + A_2\zeta^2 + A_3\zeta^3 + \cdots$ ($A_1 > 0$; $\delta \in \mathbb{C} \setminus \{0\}$; $\zeta \in \Theta$), then*

$$|a_2| \leq \frac{|\delta - 3|}{(1+\alpha)}, \tag{4.1}$$

$$|a_3| \leq \frac{(|A_1| + |\delta - 4|^2 + 1)}{2(1+2\alpha)}, \tag{4.2}$$

and $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{(|A_1| + |\delta - 4|^2 + 1)}{2(1+2\alpha)} + \frac{|\mu(\delta - 3)^2|}{(1+\alpha)^2}. \tag{4.3}$$

Proof. Let $\chi \in \mathcal{KL}^{\alpha}(\psi)$. Then by the definition of subordination, there is a function $w(\zeta)$ such that

$$1 + \frac{1-\zeta}{\zeta}\left[(1+\zeta) - (1-\zeta)\left(\frac{\alpha\zeta^2\chi''(\zeta) + \zeta\chi'(\zeta)}{(1-\alpha)\chi(\zeta) + \alpha\zeta\chi'(\zeta)}\right)\right] = \psi\left[w(\zeta)\right].$$

Define the function $\vartheta(\zeta)$ by

$$\vartheta(\zeta) = 1 + l_1\zeta + l_2\zeta^2 + \cdots = \frac{1 + w(\zeta)}{1 - w(\zeta)} \prec \frac{1+\zeta}{1-\zeta}, \quad (\zeta \in \Theta). \tag{4.4}$$

We note that $\vartheta(0) = 1$ and $\vartheta \in \mathcal{F}$ (see Lemma 2.1). Using (4.4), it is easy to see that

$$w(\zeta) = \frac{\vartheta(\zeta) - 1}{\vartheta(\zeta) + 1} = \frac{1}{2}\left[l_1\zeta + \left(l_2 - \frac{l_1^2}{2}\right)\zeta^2 + \left(l_3 - l_1 l_2 + \frac{l_1^3}{4}\right)\zeta^3 + \cdots\right],$$

so we have

$$\begin{aligned}&(1-\zeta^2) - (1-\zeta)^2\frac{\alpha\zeta^2\chi''(\zeta) + \zeta\chi'(\zeta)}{(1-\alpha)\chi(\zeta) + \alpha\zeta\chi'(\zeta)}\\ &= (\delta - 1)\zeta + \frac{A_1\ell_1}{2}\zeta^2 + \frac{A_1}{2}\left\{\ell_2 - \frac{\ell_1^2}{2}\left(1 - \frac{A_2}{A_1}\right)\right\}\zeta^3 + \cdots.\end{aligned} \tag{4.5}$$

The left hand side of (4.5) will be

$$\begin{aligned}(1-\zeta^2) - (1-\zeta)^2\frac{\alpha\zeta^2\chi''(\zeta) + \zeta\chi'(\zeta)}{(1-\alpha)\chi(\zeta) + \alpha\zeta\chi'(\zeta)} &= \left[2 - (1+\alpha)\,a_2\right]\zeta\\ -\left\{2\,(1+2\alpha)\,a_3 - (1+\alpha)^2 a_2^2 - 2(1+\alpha)a_2\right\}\zeta^2 + \cdots.\end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we have

$$a_2 = -\frac{(\delta - 3)}{(1+\alpha)}, \tag{4.7}$$

and

$$a_3 = -\frac{1}{2(1+2\alpha)}\left[\frac{A_1\ell_1}{2} - (\delta-4)^2 + 1\right]. \tag{4.8}$$

By applying Lemma 2.1 in (4.8), we obtain (4.2). Using (4.7) and (4.8) together with Lemma 2.2 we can establish (4.3). □

Theorem 4.2. *Let $\chi \in C\mathcal{L}_\lambda^\alpha(\Omega)$ have a Taylor series expansion of the form (1.1) and $\Omega(\zeta)$ be defined as in* (1.2)*, then*

$$|a_2| \leq \frac{16}{(2\lambda-1)(1+\alpha)}, \tag{4.9}$$

$$|a_3| \leq \frac{16}{(3\lambda-1)(1+2\alpha)}\left[3 + \max\left\{1, \left|\frac{2(2\lambda^2-4\lambda+1)(2\vartheta_1-\vartheta_2-2)}{(2\lambda-1)^2} - 1\right|\right\}\right], \tag{4.10}$$

and $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{16}{(3\lambda-1)(1+2\alpha)}\left[\max\left\{1, \left|\frac{2(2\vartheta_1-\vartheta_2-2)}{(2\lambda-1)^2}\left(\frac{2\mu(3\lambda-1)(1+2\alpha)}{(1+\alpha)^2} + \left(2\lambda^2-4\lambda+1\right)\right) - 1\right|\right\} + 3\right]. \tag{4.11}$$

Proof. Let $\chi \in C\mathcal{L}_\lambda^\alpha(\Omega)$. Then by the definition of subordination, there is a function $w(\zeta)$ such that

$$\delta + \frac{\zeta\left[\alpha\zeta\chi''(\zeta) + \chi'(\zeta)\right]^\lambda}{(1-\alpha)\chi(\zeta) + \alpha\zeta\chi'(\zeta)} = 1 + \Omega[w(\zeta)]. \tag{4.12}$$

Equivalently, the Eq (4.12) can be rewritten in the form

$$1 + (2\lambda-1)(1+\alpha)a_2\zeta + \left[2(1+2\alpha)a_3 + (2\lambda^2-4\lambda+1)(1+\alpha)^2a_2^2\right]\zeta^2 + \cdots = (3-\vartheta_1) + (2\vartheta_1-\vartheta_2-2)\,l_1\zeta + \left[(2\vartheta_1-\vartheta_2-2)\,l_2 + l_1^2\left(\frac{5\vartheta_2}{2} - \vartheta_3 - 2\vartheta_1 + 1\right)\right]\zeta^2 + \cdots. \tag{4.13}$$

From (4.13), we have

$$a_2 = \frac{(2\vartheta_1-\vartheta_2-2)\,l_1}{(2\lambda-1)(1+\alpha)}, \tag{4.14}$$

$$a_3 = \frac{1}{(3\lambda-1)(1+2\alpha)}\left[(2\vartheta_1-\vartheta_2-2)\,l_2 + l_1^2\left(\frac{5\vartheta_2}{2} - \vartheta_3 - 2\vartheta_1 + 1\right) - \frac{(2\lambda^2-4\lambda+1)(2\vartheta_1-\vartheta_2-2)^2\,l_1^2}{(2\lambda-1)^2}\right]. \tag{4.15}$$

By applying Lemma 2.1 in (4.15), we obtain (4.10). Using (4.14) and (4.15) together with Lemma 2.2, we can get (4.11) □

We just state the following result which can be obtained by retracing the steps as in Theorem 4.1 and Theorem 4.2, respectively.

Theorem 4.3. *Let $\chi \in \mathcal{KQ}^{\alpha}(\psi)$ have a Taylor series expansion of the form (1.1) and $\psi(\zeta) = 1 + A_1\zeta + A_2\zeta^2 + A_3\zeta^3 + \cdots$, $(A_1 > 0;\ \zeta \in \Theta)$, then*

$$|a_2| \leq \frac{|\delta - 3|}{q(1+\alpha q)},$$

$$|a_3| \leq \frac{[q|A_1| + |\delta - 4| + 1]}{q^2(q+1)[1+\alpha q(q+1)]},$$

and $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{[q|A_1| + |\delta - 4| + 1]}{q^2(q+1)[1+\alpha q(q+1)]} + \frac{4|\mu||\delta - 3|^2}{q^2(1+\alpha q)^2}.$$

Theorem 4.4. *Let $\chi \in \mathcal{CQ}_{\lambda}^{\alpha}(\Omega)$ have a Taylor series expansion of the form (1.1) and $\Omega(\zeta)$ is defined as in (1.2), then*

$$|a_2| \leq \frac{16}{[(1+q)\lambda - 1](1+\alpha q)}$$

$$|a_3| \leq \frac{16}{[1+\alpha q(1+q)][\lambda(1+q+q^2) - 1]}$$
$$\left[3 + \max\left\{1, \left|\frac{\left[\lambda(\lambda-1)(1+q)^2 - 2(1+q)\lambda + 2\right](2\vartheta_1 - \vartheta_2 - 2)}{[\lambda(1+q) - 1]^2} - 1\right|\right\}\right],$$

and $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{16}{[1+\alpha q(1+q)][\lambda(1+q+q^2) - 1]}\left[\max\left\{1, \left|\frac{(2\vartheta_1 - \vartheta_2 - 2)}{[(1+q)\lambda - 1]^2}\right.\right.\right.$$
$$\left.\left.\left.\left(\frac{2\mu[1+\alpha q(1+q)][\lambda(1+q+q^2) - 1])}{(1+\alpha q)^2} + \left[\lambda(\lambda-1)(1+q)^2 - 2(1+q)\lambda + 2\right]\right) - 1\right|\right\} + 3\right].$$

5. Conclusions

Inspired by the class of starlike functions with respect to boundary point, we familiarised a subclass of analytic functions by exposing certain analytic description subordinate to a common function. To add more versatility to our study, we defined a new family including the study of pseudo starlike functions. Integral representation and the solution to the Fekete-Szegö problem of the function class introduced here have been investigated. Further, by replacing the ordinary differentiation with the quantum differentiation we have attempted at the discretization of the results.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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