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Research article

Hyers-Ulam-Rassias stability of cubic functional equations in fuzzy normed spaces

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Abstract: In this paper, two cubic functional equations are shown to be equivalent, Hyers-Ulam-Rassias stability of them is proved under some suitable conditions by the fixed point method in fuzzy normed spaces. Moreover, the fuzzy continuity of the solution of the functional equation is discussed.

Keywords: fuzzy normed space; cubic functional equation; Hyers-Ulam-Rassias stability; fuzzy continuity
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1. Introduction

The stability problem of functional equations originated from the stability problem of group homomorphisms proposed by Ulam [18] in 1940. Under what conditions does there exist a group homomorphism near an approximate group homomorphism? If the answer is affirmative, we would say that the equation of homomorphism is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? The first result about the stability problem of functional equations was shown by Hyers [7] in 1941. In 1950, Aoki [1], and in 1978, Rassias [17] proved a generalization of Hyers' theorem for additive and linear mappings, respectively. The result of Rassias has influenced the development of what is now called Hyers-Ulam-Rassias stability theory for functional equations. Several stability results have been recently obtained for various equations, also for mappings with more general domains and ranges (see e.g., [3,5,6,10,11,14,16]).

Jun and Kim [8] firstly proved Hyers-Ulam-Rassias stability of the following functional equation in Banach space:

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is called a cubic functional equation, since $f(x)=cx^3\,(c\in\mathbb{R})$ is its solution. Every solution of cubic functional equation is called cubic mapping. In 2008, Wiwatwanich and Nakmahachchalasint [19] studied Hyers-Ulam-Rassias stability of another cubic functional equation:

$$f(x+3y)-3f(x+y)+3f(x-y)-f(x-3y)=48f(y), \tag{1.2}$$

in Banach space by the direct method.

Kang and Chu [9] investigated the generalized Hyers-Ulam-Rassias stability of an n-dimensional cubic functional equation:

$$f\left(2\sum_{j=1}^{n-1}x_j+x_n\right)+f\left(2\sum_{j=1}^{n-1}x_j-x_n\right)+4\sum_{j=1}^{n-1}f(x_j)$$
$$=16f\left(\sum_{j=1}^{n-1}x_j\right)+2\sum_{j=1}^{n-1}\left[f(x_j+x_n)+f(x_j-x_n)\right], \tag{1.3}$$

in Banach spaces, and proved that Eq (1.1) is equivalent to Eq (1.3).

In 2008, Mirmostafaee and Moslehian [16] introduced three different versions of fuzzy approximate additive function in fuzzy normed space and proved that an approximate additive function can be approximated by additive function under some appropriate conditions. Since then, the stability of functional equation in fuzzy normed space has attracted the attention of scholars. In 2017, Li [12] studied Hyers-Ulam-Rassias stability of the quartic functional equation:

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y), \tag{1.4}$$

in fuzzy normed spaces. In 2021, Wu and Lu [20] establish the stability results concerning the following functional equations:

$$f(ax+by)=rf(x)+sf(y), \tag{1.5}$$

where constants $a,b>0$ and $r,s\in\mathbb{R}$ with $a+b=r+s\neq 1$, and

$$f(x+y+z)=2f\left(\frac{x+y}{2}\right)+f(z), \tag{1.6}$$

in fuzzy normed spaces.

In this paper, we shall prove that Eq (1.1) is equivalent to Eq (1.2), and shall study Hyers-Ulam-Rassias stability of Eq (1.2) in fuzzy normed spaces. For convenience, Hyers-Ulam-Rassias stability is referred to as stability in this paper.

2. Preliminaries

In this section we shall recall some notations and basic results used in this paper.

Definition 2.1 ([2]). Let X be a linear space over a field $\mathbb{R}$. A fuzzy subset N: $X \times \mathbb{R} \to [0, 1]$ is called a fuzzy norm on X if for all $x, y \in X$, and all $t, s \in \mathbb{R}$:

(N1) $\forall t \leq 0$, $N(x,t)=0$;

(N2) $\forall t > 0$, $N(x,t)=1$ if and only if $x=0$;

(N3) $\forall t > 0$, $N(cx,t)=N\left(x,\frac{t}{|c|}\right)$ if $c \neq 0$;

(N4) $N(x+y,s+t) \geq \min\{N(x,s),N(y,t)\}$;

(N5) $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t)=1$;

(N6) $\forall x \neq 0$, $N(x,\cdot)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space.

It is easy to see that (N5) can be implied by (N2) and (N4).

Definition 2.2 ([2]). Let $\{x_n\}$ be a sequence in fuzzy normed linear space (X,N). Then $\{x_n\}$ is said to be convergent if there is $x \in X$ such that $\lim_{n\to\infty} N(x_n - x,t)=1$, $\forall t>0$. In that case x is called the limit of the sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$.

Definition 2.3 ([2]). A sequence $\{x_n\}$ in (X,N) is said to be Cauchy sequence if $\lim_{n\to\infty} N(x_{n+p} - x_n,t)=1$, $\forall t>0$ and $p \in \mathbb{N}$.

If every Cauchy sequence is convergent, then the fuzzy normed space is called a fuzzy Banach space.

The following definition is slightly different from that in [15].

Definition 2.4. Let (Y,N) be a (quasi) fuzzy normed space. A function $f\colon \mathbb{R} \to Y$ is said to be fuzzy continuous at $s_0 \in \mathbb{R}$, if for each $t>0$ and $0<\beta<1$, there is some $\delta>0$ such that $N(f(s)-f(s_0),t) \geq \beta$ for each s with $|s-s_0|<\delta$. f is said to be fuzzy continuous on $\mathbb{R}$, if f is fuzzy continuous at any point of $\mathbb{R}$.

Definition 2.5 ([13]). Let X be a nonempty set. Assume that on the Cartesian product $X \times X$, a distance function $d(x,y)$ $(0 \leq d(x,y) \leq \infty)$ is defined, satisfying the following conditions:

(D1) $d(x,y)=0$ if and only if $x=y$,

(D2) $d(x,y)=d(y,x)$ (symmetry),

(D3) $d(x,y) \leq d(x,z)+d(z,y)$ (triangle inequality),

(D4) every Cauchy sequence in X is convergent.

Then, (X,d) is called a generalized complete metric space.

Theorem 2.6 ([4]). Let (X,d) be a generalized complete metric space and $T:X\to X$ be a strictly contractive mapping with Lipschitz constant L $(L<1)$, that is,

$$d(Tx,Ty) \leq Ld(x,y),\quad \forall x,y \in X.$$

Then for each given $x \in X$, either

$$d\left(T^n x, T^{n+1} x\right) = \infty, \quad \forall n \geq 0;$$

or there exists a natural number n_0 such that

(1) $d\left(T^n x, T^{n+1} x\right) < \infty$ for all $n \geq n_0$;

(2) $\left\{T^n x\right\}$ is convergent to a fixed point y^* of T;

(3) y^* is the unique fixed point of T in the set $\Delta = \left\{y \in X : d\left(T^{n_0} x, y\right) < \infty\right\}$;

(4) $d\left(y, y^*\right) < \frac{1}{1-L} d\left(y, Ty\right)$ for all $y \in \Delta$.

3. Main results

First, we prove that Eq (1.2) is equivalent to Eq (1.1).

Lemma 3.1. Let X and Y be linear spaces and the mapping $f : X \to Y$ satisfy (1.2), then

(1) $f(0)=0$;

(2) f is an odd mapping;

(3) $f(ry)=r^3 f(y)$, $\forall y \in X$, $\forall r \in \mathbb{Q}$.

Proof. (1) Putting $x = y = 0$ in (1.2), we get $f(0)=0$.

(2) Replacing y by $-y$ in (1.2), we get

$$f\left(x-3y\right) - 3f\left(x-y\right) + 3f\left(x+y\right) - f(x+3y) = 48f\left(-y\right). \tag{3.1}$$

Then $f(-y) = -f(y)$, implying that f is an odd mapping.

(3) We first prove that

$$f(ny)=n^3 f(y), \quad \forall y \in X, \ \forall n \in \mathbb{N}. \tag{3.2}$$

Setting $x=0$ in (1.2), we get

$$f(3y) = 27f(y). \tag{3.3}$$

Let $x = y$ in (1.2), we get

$$f(4y) = 2f(2y) + 48f(y). \tag{3.4}$$

Let $x = 3y$ in (1.2), we get

$$f(6y) - 3f(4y) + 3f(2y) = 48f(y). \tag{3.5}$$

Substituting (3.3) and (3.4) into (3.5), we get

$$f(2y) = 8f(y). \tag{3.6}$$

Therefore, (3.2) holds for $n = 2$ and $n = 3$.

Now, suppose (3.2) holds whenever $n \leq k$ ($k \geq 3$, $n \in \mathbb{N}$). Next, we shall prove that (3.2) holds when $n = k+1$. In fact, setting $x = (k-2)y$ in (1.2), we get

$$f\left((k+1)y\right)=f\left((k-5)y\right)+3f\left((k-1)y\right)-3f\left((k-3)y\right)+48f(y)$$
$$=\left[(k-5)^3+3(k-1)^3-3(k-3)^3+48\right]f(y)$$
$$=(k+1)^3 f(y).$$

From the principle of induction, we get $f(ny)=n^3 f(y)$, $\forall y\in X,\ \forall n\in\mathbb{N}$.

Last, we shall prove $f(ry)=r^3 f(y),\ \forall y\in X,\ r\in\mathbb{Q}$. Let $r=\frac{m}{n}$, $m,n\in\mathbb{N}$, replacing y by $\frac{y}{n}$, we get

$$\frac{1}{n^3}f(y)=f\left(\frac{y}{n}\right),\ \forall\ y\in X,\ n\in\mathbb{N}.$$

Then,

$$f(ry)=f\left(\frac{m}{n}y\right)=m^3 f\left(\frac{1}{n}y\right)=\left(\frac{m}{n}\right)^3 f(y)=r^3 f(y).$$

The proof ends.

Theorem 3.2. Let X, Y be linear spaces. Then the mapping $f:X\to Y$ satisfies the functional Eq (1.2) if and only if f satisfies the functional Eq (1.1).

Proof. If $f:X\to Y$ satisfies the functional Eq (1.2), from Lemma 3.1, f is an odd mapping and

$$f(3y)=27f(y),\ f(2y)=8f(y).$$

Replacing x by $x-y$ in (1.2), we get

$$f(x+2y)-f(x-4y)-3f(x)+3f(x-2y)=48f(y). \tag{3.7}$$

Replacing x by y and y by x in (3.7), we get

$$f(y+2x)-f(y-4x)-3f(y)+3f(y-2x)=48f(x). \tag{3.8}$$

Setting $y=-y$ in (3.8), we get

$$f(2x-y)+f(4x+y)+3f(y)-3f(2x+y)=48f(x). \tag{3.9}$$

From (3.8) and (3.9), we get

$$4f(2x+y)-4f(2x-y)+f(4x-y)-f(4x+y)-6f(y)=0. \tag{3.10}$$

Replacing x by $x/2$ in (3.10), we get

$$f(2x+y)-f(2x-y)=4f(x+y)-4f(x-y)-6f(y). \tag{3.11}$$

Replacing y by $2y$ in (3.11), we get

$$2f(x+y)-2f(x-y)=f(x+2y)-f(x-2y)-12f(y). \quad (3.12)$$

Exchanging x and y in (3.12), we get the functional Eq (1.1).

Conversely, suppose f satisfies the functional Eq (1.1), by setting $x=y=0$ in (1.1), we get $f(0)=0$. Letting $x=0$ in (1.1), we have $f(-y)=-f(y)$. Setting $y=0$ and $y=x$ in (1.1), respectively, we obtain $f(2x)=8f(x)$ and $f(3x)=27f(x)$. Replacing y by $x+y$ in (1.1), we get

$$f(3x+y)+f(x-y)=2f(2x+y)-2f(y)+12f(x). \quad (3.13)$$

Replacing y by $y-x$ in (1.1), we know

$$f(x+y)+f(3x-y)=2f(y)+2f(2x-y)+12f(x). \quad (3.14)$$

Adding (3.13) and (3.14), and using (1.1), we get the functional Eq (1.2). The proof ends.

Lemma 3.3. Let X, (Z,N'), (Y,N) be a linear space, a fuzzy normed space and a fuzzy Banach space, respectively. And let $h: X\to Y$ and $\psi: X\to Z$ be two functions. Set $\Omega=\{g: g: X\to Y, g(0)=0\}$. For any $\eta>0$, define a mapping $d: \Omega\times\Omega\to[0, \infty]$ as

$$d(g,h)=\inf\{\beta\in(0,\infty): N(g(y)-h(y),\beta t)\geq N'(\psi(y),\eta t), \forall y\in X, t>0\},$$

then (Ω,d) is a generalized complete metric space.

Proof. (D1) It is obvious that $d(g,g)=0$. Conversely, suppose $d(g,h)=0$, from the definition of $d(g,h)$, we have

$$N\left(g(y)-h(y),\frac{1}{n}t\right)\geq N'(\psi(y),\eta t),$$

i.e.

$$N(g(y)-h(y),t)\geq N'(\psi(y),n\eta t),$$

for any $n\in\mathbb{N}$, $y\in X$, $t>0$, $\eta>0$. Let $n\to\infty$, we get $N(g(y)-h(y),t)=1$, for any $t>0$, thus $g=h$.

(D2) It is obvious.

(D3) If $d(g,h)=\beta_1$ and $d(h,k)=\beta_2$, $\forall g,h,k\in\Omega$, then for any $\varepsilon_1,\varepsilon_2>0$, we have

$$N(g(y)-h(y),(\beta_1+\varepsilon_1)t)\geq N'(\psi(y),\eta t)$$

and

$$N(h(y)-k(y),(\beta_2+\varepsilon_2)t)\geq N'(\psi(y),\eta t), \quad y\in X, \ t>0, \ \eta>0.$$

Thus

$$N\left(g(y)-k(y),(\beta_1+\varepsilon_1+\beta_2+\varepsilon_2)t\right)$$
$$\geq \min\left\{N\left(g(y)-h(y),(\beta_1+\varepsilon_1)t\right),N\left(h(y)-k(y),(\beta_2+\varepsilon_2)t\right)\right\}$$
$$\geq N'\left(\psi(y),\eta t\right).$$

Hence, $d(g,k)\leq \beta_1+\beta_2+\varepsilon_1+\varepsilon_2$. By the arbitrariness of ε_1 and ε_2, we have

$$d(g,k)\leq d(g,h)+d(h,k).$$

(D4) Let $\{h_n\}$ be a Cauchy sequence in Ω. For the given $y_0\in X$, and any $\lambda\in(0,1)$, since $\lim_{t\to\infty}N'\left(\psi(y),\eta t\right)=1$, there exists $t_0>0$, such that $N'\left(\psi(y_0),\eta t_0\right)>1-\lambda$. For any $\varepsilon>0$, let $0<\beta<\varepsilon/t_0$, since $\{h_n\}$ is a Cauchy sequence, there exists n_0, $d(h_m,h_n)<\beta$ whenever $n,m\geq n_0$. Then we have

$$N\left(h_m(y_0)-h_n(y_0),\varepsilon\right)\geq N\left(h_m(y_0)-h_n(y_0),\beta t_0\right)\geq N'\left(\psi(y_0),\eta t_0\right)>1-\lambda,$$

thus $\{h_n(y)\}$ is a Cauchy sequence. Since (Y,N) is complete, there exists $h:X\to Y$ such that $\{h_n(y)\}$ is convergent to $h(y)$ in Y.

For any $\beta>0$, since $\{h_n\}$ is a Cauchy sequence, there exists n_0 such that

$$N\left(h_n(y)-h_{n+m}(y),\frac{\beta}{2}t\right)\geq N'\left(\psi(y),\eta t\right)$$

for any $y\in X$, $t>0$, $n>n_0$ and $m\geq 1$. Therefore

$$N\left(h_n(y)-h(y),\beta t\right)\geq \min\left\{N\left(h_n(y)-h_{n+m}(y),\frac{\beta}{2}t\right),N\left(h_{n+m}(y)-h(y),\frac{\beta}{2}t\right)\right\}$$
$$\geq \min\left\{N'\left(\psi(y),\eta t\right),N\left(h_{n+m}(y)-h(y),\frac{\beta}{2}t\right)\right\}.$$

Let $m\to\infty$, we get

$$N\left(h_n(y)-h(y),\beta t\right)\geq \min\left\{N'\left(\psi(y),\eta t\right),1\right\}=N'\left(\psi(y),\eta t\right),\quad \forall y\in X,t>0,\eta>0.$$

Thus $d(h_n h)\leq\beta$ whenever $n\geq n_0$, which implies that $\{h_n\}$ is convergent to h in Ω. Thus, (Ω,d) is a generalized complete metric space. The proof ends.

In the rest of this paper, we focus on the functional Eq (1.2).

For a mapping $f:X\to Y$, for convenience, we define a difference operator Df : $X^2\to Y$ as

$$Df(x,y)=f(x+3y)-3f(x+y)-f(x-3y)+3f(x-y)-48f(y),\quad \forall x,y\in X.$$

Theorem 3.4. Let X, (Z,N'), (Y,N) be a linear space, a fuzzy normed space and a fuzzy Banach space, respectively, and let $0<\alpha<27$. Suppose that the mapping $\varphi:X\times X\to Z$ satisfies

$$N'\left(\varphi\left(0,3y\right),t\right)\geq N'\left(\alpha\varphi\left(0,y\right),t\right) \tag{3.15}$$

and

$$\lim_{n\to\infty} N'\left(\varphi\left(3^n x,3^n y\right),27^n t\right)=1,\ \ \forall x,y\in X,\ \ t>0.$$

If $f:X\to Y$ is φ-approximately cubic in the sense that

$$N\left(Df\left(x,y\right),t\right)\geq N'\left(\varphi\left(x,y\right),t\right),\ \ \forall x,y\in X,\ \ t>0. \tag{3.16}$$

Then

(1) the limitation $c\left(y\right)=\lim_{n\to\infty}\frac{f\left(3^n y\right)}{27^n}$ exists for each $y\in X$, and the mapping $c:X\to Y$ is the unique cubic mapping which satisfies

$$N\left(f\left(y\right)-c\left(y\right),t\right)\geq N'\left(\varphi\left(0,y\right),2\left(27-\alpha\right)t\right),\ \ \forall y\in X,\ \ t>0; \tag{3.17}$$

(2) if the mappings $s\to f\left(sy\right)$ and $s\mapsto\varphi(0,sy)$ are fuzzy continuous for each $y\in X$, then the mapping $s\to c\left(sy\right)$ is also fuzzy continuous, and $c\left(\lambda y\right)=\lambda^3 c\left(y\right)$ holds for all $\lambda\in\mathbb{R}$.

Proof. (1) Consider the set

$$\Omega=\left\{g:g:X\to Y,g\left(0\right)=0\right\},$$

and the mapping,

$$d\left(g,h\right)=\inf\left\{\beta\in\left(0,\infty\right):N\left(g\left(y\right)-h\left(y\right),\beta t\right)\geq N'\left(\varphi\left(0,y\right),54t\right),\forall y\in X,t>0\right\}.$$

Let $\psi\left(y\right)=\varphi\left(0,y\right)$ and $\eta=54$ in Lemma 3.3, then we know that $\left(\Omega,d\right)$ is a generalized complete metric space. Define the mapping $T:\Omega\to\Omega$, $Tg\left(y\right)=\frac{g\left(3y\right)}{27}$, $\forall y\in X$.

Now we prove T is strictly contractive with Lipschitz constant $\frac{\alpha}{27}$. For the given $g,h\in\Omega$, set $d\left(g,h\right)=\gamma$. If $\gamma=\infty$, it is obvious that

$$d\left(Tg,Th\right)\leq\frac{\alpha}{27}d\left(g,h\right).$$

If $\gamma\in\left[0,\infty\right)$, then for any $t>0$, $\varepsilon>0$, we have the following inequality by (N5):

$$N\left(g\left(y\right)-h\left(y\right),\left(\gamma+\varepsilon\right)t\right)\geq N'\left(\varphi\left(0,y\right),54t\right),\ \ \forall y\in X.$$

Therefore, using (N3) and (3.15), we get

$$N\left(Tg(y)-Th(y),\frac{\alpha(\gamma+\varepsilon)t}{27}\right)=N\left(\frac{g(3y)}{27}-\frac{h(3y)}{27},\frac{\alpha(\gamma+\varepsilon)t}{27}\right)$$

$$=N\left(g(3y)-h(3y),\alpha(\gamma+\varepsilon)t\right)\geq N'\left(\varphi(0,3y),54\alpha t\right)\geq N'\left(\alpha\varphi(0,y),54\alpha t\right)$$

$$=N'\left(\varphi(0,y),54t\right),\quad \forall y\in X,\ t>0.$$

Hence $d(Tg,Th)\leq\frac{\alpha(\gamma+\varepsilon)}{27}$. Since ε is arbitrary, then $d(Tg,Th)\leq\frac{\alpha}{27}d(g,h)$.

Next, setting $x=0$ in (3.16), we can obtain

$$N\left(\frac{f(3y)}{27}-f(y),t\right)\geq N'\left(\varphi(0,y),54t\right).$$

Then we get that $d(Tf,f)\leq 1<\infty$. From Theorem 2.6, we have the followings:

(a) $\{T^n f\}$ is convergent to a fixed point c of T, that is $\lim_{n\to\infty}d\left(T^n f,c\right)=0$,

(b) c is the unique fixed point of T, that is, $c(3y)=27c(y),\ \forall y\in X$,

(c) $d(f,c)\leq\frac{1}{1-\alpha/27}d(Tf,f)\leq\frac{1}{1-\alpha/27}=\frac{27}{27-\alpha}$,

which implies that

$$c(y)=\lim_{n\to\infty}\frac{f\left(3^n y\right)}{27^n}$$

and

$$N\left(f(y)-c(y),t\right)\geq N'\left(\varphi(0,y),2(27-\alpha)t\right),\quad \forall y\in X,\ t>0.$$

Replacing x, y by $3^n x$, $3^n y$, respectively in (3.16), we get

$$N\left(\frac{Df\left(3^n x,3^n y\right)}{27^n},t\right)=N\left(Df\left(3^n x,3^n y\right),27^n t\right)\geq N'\left(\varphi\left(3^n x,3^n y\right),27^n t\right).$$

Since

$$\lim_{n\to\infty}N'\left(\varphi\left(3^n x,3^n y\right),27^n t\right)=1$$

and

$$c(y)=\lim_{n\to\infty}\frac{f\left(3^n y\right)}{27^n},$$

we have $N\left(D(c(y)),t\right)=1,\ \forall t>0$, thus $D\left(c(y)\right)=0,\ \forall y\in X$. That is, c is the cubic mapping which satisfies the Eq (1.2).

To prove the uniqueness of c, let us assume that there exists a cubic mapping $q:X\to Y$ which satisfies (1.2) and (3.17). From Lemma 3.1 we have

$$c\left(3^n y\right)=27^n c(y),\ \ q\left(3^n y\right)=27^n q(y),\ \forall n\in\mathbb{N}.$$

Then

$$N\left(c(y)-q(y),t\right)=N\left(\frac{c\left(3^n y\right)}{27^n}-\frac{q\left(3^n y\right)}{27^n},t\right)$$

$$\geq \min\left\{N\left(\frac{c\left(3^n y\right)}{27^n}-\frac{f\left(3^n y\right)}{27^n},\frac{t}{2}\right),N\left(\frac{f\left(3^n y\right)}{27^n}-\frac{q\left(3^n y\right)}{27^n},\frac{t}{2}\right)\right\}$$

$$\geq N'\left(\varphi\left(0,3^n y\right),27^n\left(27-\alpha\right)t\right)$$

$$\geq N'\left(\varphi(0,y),\frac{27^n\left(27-\alpha\right)t}{\alpha^n}\right),\ \ \forall t>0.$$

Since

$$\lim_{n\to\infty}\frac{27^n\left(27-\alpha\right)t}{\alpha^n}=\infty,\ \ \lim_{n\to\infty}N'\left(\varphi(0,y),\frac{27^n\left(27-\alpha\right)t}{\alpha^n}\right)=1,\ \ \forall t>0.$$

Then $N\left(c(y)-q(y),t\right)=1$, $\forall t>0$. Thus, $c(y)=q(y)$.

(2) For any $t>0$, since $\lim_{n\to\infty}\frac{2\cdot 27^n\left(27-\alpha\right)t}{6\alpha^n}=\infty$, we have

$$\lim_{n\to\infty}N'\left(\varphi(0,y),\frac{2\cdot 27^n\left(27-\alpha\right)t}{6\alpha^n}\right)=1. \tag{3.18}$$

For the given $y\in X$, $s_0\in\mathbb{R}$, $t>0$ and $0<\beta<1$, it follows from (3.18) that there exists sufficiently large $n_0\in\mathbb{N}$ such that

$$N'\left(\varphi(0,s_0 y),\frac{2\cdot 27^{n_0}\left(27-\alpha\right)t}{6\alpha^{n_0}}\right)>\beta. \tag{3.19}$$

Using (3.17), we get

$$N\left(c(s_0 y)-\frac{f\left(3^{n_0}s_0 y\right)}{27^{n_0}},\frac{t}{3}\right)=N\left(\frac{c\left(3^{n_0}s_0 y\right)}{27^{n_0}}-\frac{f\left(3^{n_0}s_0 y\right)}{27^{n_0}},\frac{t}{3}\right)$$

$$\geq N'\left(\varphi(0,s_0 y),\frac{2\cdot 27^{n_0}\left(27-\alpha\right)t}{3\alpha^{n_0}}\right)\geq N'\left(\varphi(0,s_0 y),\frac{2\cdot 27^{n_0}\left(27-\alpha\right)t}{6\alpha^{n_0}}\right)>\beta. \tag{3.20}$$

Since mappings $s\mapsto f(2^{n_0}sy)$ and $s\mapsto\varphi(0,sy)$ are fuzzy continuous at s_0, we know that there exists $0<\delta<1$ such that

$$N\left(f\left(3^{n_0}sy\right)-f\left(3^{n_0}s_0y\right),\frac{27^{n_0}t}{3}\right)>\beta \tag{3.21}$$

and

$$N'\left(\varphi(0,sy)-\varphi(0,s_0y),\frac{2\cdot 27^{n_0}(27-\alpha)t}{6\alpha^{n_0}}\right)>\beta \tag{3.22}$$

whenever $0<|s-s_0|<\delta$. Then, by (3.17), (3.19) and (3.22), we get

$$N\left(c(sy)-\frac{f\left(3^{n_0}sy\right)}{27^{n_0}},\frac{t}{3}\right)=N\left(\frac{c\left(3^{n_0}sy\right)}{27^{n_0}}-\frac{f\left(3^{n_0}sy\right)}{27^{n_0}},\frac{t}{3}\right)$$

$$\geq N'\left(\varphi(0,sy),\frac{2\cdot 27^{n_0}(27-\alpha)t}{3\alpha^{n_0}}\right)$$

$$\geq \min\left\{N'\left(\varphi(0,sy)-\varphi(0,s_0y),\frac{2\cdot 27^{n_0}(27-\alpha)t}{6\alpha^{n_0}}\right),N'\left(\varphi(0,s_0y),\frac{2\cdot 27^{n_0}(27-\alpha)t}{6\alpha^{n_0}}\right)\right\}$$

$$>\beta. \tag{3.23}$$

Therefore, by (3.20), (3.21) and, (3.23), we have

$$N\left(c(sy)-c(s_0y),t\right)$$

$$\geq \min\left\{N\left(c(sy)-\frac{f\left(3^{n_0}sy\right)}{27^{n_0}},\frac{t}{3}\right),N\left(\frac{f\left(3^{n_0}sy\right)}{27^{n_0}}-\frac{f\left(3^{n_0}s_0y\right)}{27^{n_0}},\frac{t}{3}\right),N\left(c(s_0y)-\frac{f\left(3^{n_0}s_0x\right)}{27^{n_0}},\frac{t}{3}\right)\right\}$$

$$>\beta.$$

This means that $s\to c(sy)$ is fuzzy continuous.

By Lemma 3.1, we have $c(ry)=r^3c(y)$, $\forall\ y\in X$, $r\in\mathbb{Q}$. Then for any $\lambda\in\mathbb{R}$, there exists rational number sequence r_n such that $r_n\to\lambda$. Since $c(sy)$ is fuzzy continuous with respect to s, we have

$$c(\lambda y)=c\left(\lim_{n\to\infty}r_ny\right)=\lim_{n\to\infty}c(r_ny)=\lim_{n\to\infty}r_n^{\ 3}c(y)=\lambda^3c(y).$$

The proof ends.

In the case $\alpha>27$, corresponding to Theorem 3.4, we can get the following conclusion.

Theorem 3.5. Let X, (Z,N'), (Y,N) be a linear space, a fuzzy normed space and a fuzzy Banach space, respectively, and let $\alpha>27$, suppose that the mapping $\varphi: X\times X\to Z$ satisfies:

$$N'\left(\varphi\left(0,\frac{y}{3}\right),t\right)\geq N'\left(\varphi(0,y),\alpha t\right) \tag{3.24}$$

and

$$\lim_{n\to\infty} N'\left(\varphi\left(\frac{x}{3^n},\frac{y}{3^n}\right),\frac{t}{27^n}\right)=1,\ \ \forall x,y\in X,\ \ t>0.$$

If $f:X\to Y$ is φ-approximately cubic in the sense that

$$N\left(Df\left(x,y\right),t\right)\geq N'\left(\varphi\left(x,y\right),t\right),\ \ \forall x,y\in X,\ \ t>0. \tag{3.25}$$

Then

(1) the limitation $c\left(y\right)=\lim_{n\to\infty}27^n f\left(\frac{y}{3^n}\right)$ exists for each $y\in X$, and the mapping $c:X\to Y$ is the unique cubic mapping which satisfies

$$N\left(f\left(y\right)-c\left(y\right),t\right)\geq N'\left(\varphi(0,y),\frac{2(\alpha-27)}{\alpha}t\right),\ \ \forall y\in X,\ \ t>0; \tag{3.26}$$

(2) if the mappings $s\to f\left(sy\right)$ and $s\mapsto\varphi(0,sy)$ are fuzzy continuous for each $y\in X$, then the mapping $s\mapsto c(sy)$is also fuzzy continuous, and $c\left(\lambda y\right)=\lambda^3 c\left(y\right)$ holds for all $\lambda\in\mathbb{R}$.

Proof. The proof is similar to Theorem 3.4, we only give a framework of the proof for the existence. Let

$$\Omega=\left\{g:g:X\to Y,g\left(0\right)=0\right\},$$

and let

$$d\left(g,h\right)=\inf\left\{\beta\in\left(0,\infty\right):N\left(g\left(y\right)-h\left(y\right),\beta t\right)\geq N'\left(\varphi\left(0,y\right),2t\right),\forall y\in X,t>0\right\}.$$

We can prove that $\left(\Omega,d\right)$ is a generalized complete metric space. Define the mapping $T:\Omega\to\Omega$, $Tg\left(y\right)=27g\left(\frac{y}{3}\right)$, $\forall y\in X$. Then, T is strictly contractive with Lipschitz constant $\frac{27}{\alpha}$. From Theorem 2.6, $\left\{T^n f\right\}$ is convergent to a fixed point c of T, and $c:X\to Y$ is the unique cubic mapping which satisfies (3.26). The proof ends.

4. Conclusions

In this paper, the equivalence of the two equations is proved and we establish Hyers-Ulam-Rassias stability of a cubic functional equation in fuzzy normed spaces by using fixed point alternative theorem.

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Conflict of interest

The authors declare that there is no conflict of interest in this paper.

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