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Research article

Certain new aspects in fuzzy fixed point theory

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Abstract: We will establish several fixed point results in new introduced spaces in this manuscript known as fuzzy rectangular metric-like spaces and rectangular b-metric-like spaces. These new results and spaces will improve the approach of existing ones in the literature. Few non-trivial examples and an application also verify the uniqueness of solution.

Keywords: fuzzy rectangular metric space; fuzzy rectangular b-metric-like space; contraction mappings; fixed point theorems; Fredholm integral equations
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

The symbols in Table 1 are used throughout this study.

Table 1. Abbreviations.

Abbreviations	**Definitions**
FSs	*Fuzzy sets*
CTN	*Continuous triangular norms*
MLSs	*Metric-like spaces*
FMSs	*Fuzzy metric spaces*
FBMLSs	*Fuzzy b-metric-like spaces*

Continued on next page

Abbreviations	Definitions
FRBMSs	*Fuzzy rectangular b-metric spaces*
FRMSs	*Fuzzy rectangular metric spaces*
FRMLSs	*Fuzzy rectangular metric-like spaces*
FRBMLSs	*fuzzy rectangular b-metric-like spaces*
FP	*Fixed point*
E	*[0, 1]*

FSs were introduced by Zadeh [1] as a useful tool for situations where data is ambiguous and FS theory contains the concept of degree of membership. The terms "fuzziness" and "probability" are not interchangeable. The term "probability" refers to the objective uncertainty resulting from a large number of observations. The term "fuzziness" describes a perception of ambiguity. Fuzzy notions are used to describe the degrees of possession of a specific property. The ability of FS theory to tackle issues that fixed point theory finds problematic is what makes it valuable in dealing with control challenges. FSs are used to control ill-defined, complex, or non-linear systems.

Metric FP theory has been extensively investigated due to its vast range of applications in mathematics, science and economics. Harnadi [14] explained MLSs and demonstrated FP results. For an extended multi-valued F-contraction in MLSs, Hammad et al. [13] proposed a modified dynamic process. Alghamdi [10] developed the concept of b-MLSs and provided several couple FP techniques for contraction mappings. Mlaiki et al. [15] introduced the concept of rectangular MLSs and used contraction mappings to demonstrate FP results. Rectangular b-metric spaces were introduced by Georgea et al. [16].

CTNs were proposed by Schweizer and Sklar [8]. FMSs were proposed by Kramosil and Michalek [2], who combined the concepts of FSs with metric spaces. Garbiec [5] gave a fuzzy interpretation of the Banach contraction principle in FMSs, while Kaleva and Seikkala [3] defined a distance between two points in FMSs as a non-negative fuzzy number. Hausdorff topology was defined on FMSs by George and Veermani [4]. In the development of control FMSs, Uddin et al. [7] developed different Banach FP findings. Saleem et al. [17] defined fuzzy double controlled metric spaces and established a number of FP theorems. Uddin et al. [18] used fuzzy contractions of the Suzuki type to solve problems. In fuzzy b-metric spaces, K. Javed et al. [19] showed ordered-theoretic FP findings. In the scenario of orthogonal partial b-metric spaces, K. Javed et al. [20] developed various FP findings. For generalised contractions, Ali et al. [21] demonstrated a number of FP findings. Several FP findings were reported in fuzzy b-metric spaces by Rakic et al. [22]. Rakic et al. [23] proved novel FPs in FMSs for the Ciric type. Debnath et al. [24] demonstrated some incredible FP results.

The concept of fuzzy MLSs was proposed by Shukla and Abbas [11] using the principles of MLSs and FSs. The fuzzy MLSs approach was established by Shukla and Gopal [12], who also demonstrated numerous FP solutions. Javed et al. [6] proposed the concept of FBBMLSs and demonstrated a number of FP results. The concept of FRBMSs was developed by Mehmood et al. [9], and the Banach contraction principle was shown in this context. In this study, we elaborated on the ideas offered in [6,9]. The manuscript's main goals are as follows:

(a) Introduce the concepts of FRMLSs and FRBMLSs;

(b) To establish several FP results;

(c) To enhance existing literature of FMSs and fuzzy FP theory.

In this manuscript, we aim to establish several fixed point results in new introduced spaces in this manuscript known as fuzzy rectangular metric-like spaces and rectangular b-metric-like spaces. Few non-trivial examples and an application also verify the uniqueness of solution.

2. Preliminaries

This section includes some basic definitions that will aid in the comprehension of the main material.

Definition 2.1. [8] A binary operation $*: E \times E \to E$ is known as CTN if

C1. $\kappa * \text{N} = \text{N} * \kappa, (\forall)\ \kappa, \text{N} \in E$;

C2. $*$ is continuous;

C3. $\kappa * 1 = \kappa, (\forall)\ \kappa \in E$;

C4. $(\kappa * \text{N}) * \tilde{u} = \kappa * (\text{N} * \tilde{u}), (\forall)\ \kappa, \text{N}, \tilde{u} \in E$;

C5. If $\kappa \leq \tilde{u}$ and $\text{N} \leq \sigma$, with $\kappa, \text{N}, \tilde{u}, \sigma \in E$, then $\kappa * \text{N} \leq \tilde{u} * \sigma$.

Definition 2.2. [6] Suppose $\mathfrak{R} \neq \emptyset$. A triplet $(\mathfrak{R}, F_{\text{N}}, *)$ is known as FBMLS if $*$ is a CTN, F_{N} is a FS on $\mathfrak{R} \times \mathfrak{R} \times (0, +\infty)$ if for all $\sigma, \varkappa, g \in \mathfrak{R}$ and $\epsilon, s > 0$,

R1. $F_{\text{N}}(\sigma, \varkappa, \epsilon) > 0$;

R2. $F_{\text{N}}(\sigma, \varkappa, \epsilon) = 1$ then $\sigma = \varkappa$;

R3. $F_{\text{N}}(\sigma, \varkappa, \epsilon) = F_{\text{N}}(\varkappa, \sigma, \epsilon)$;

R4. $F_{\text{N}}\big(\sigma, g, \text{N}(\epsilon + s)\big) \geq F_{\text{N}}(\sigma, \varkappa, \epsilon) * F_{\text{N}}(\varkappa, g, s)$;

R5. $F_{\text{N}}(\sigma, \varkappa, .): (0, +\infty) \to E$ is continuous and $\lim_{\epsilon \to +\infty} F_{\text{N}}(\sigma, \varkappa, \epsilon) = 1$.

Definition 2.3. [9] Let $\mathfrak{R} \neq \emptyset$. A triplet $(\mathfrak{R}, \delta_v, *)$ is known as FRMS if $*$ is a CTN, δ_v is a FS on $\mathfrak{R} \times \mathfrak{R} \times [0, +\infty)$ if for all $\sigma, \varkappa, g \in \mathfrak{R}$ and $\epsilon, s, w > 0$,

F1. $\delta_v(\sigma, \varkappa, 0) = 0$;

F2. $\delta_v(\sigma, \varkappa, \epsilon) = 1$ if and only if $\sigma = \varkappa$;

F3. $\delta_v(\sigma, \varkappa, \epsilon) = \delta_v(\varkappa, \sigma, \epsilon)$;

F4. $\delta_v(\sigma, g, \epsilon + s + w) \geq \delta_v(\sigma, \varkappa, \epsilon) * \delta_v(\varkappa, u, s) * \delta_v(u, g, w)$ for all distinct $\varkappa, u \in \mathfrak{R} \backslash \{\sigma, g\}$;

F5. $\delta_v(\sigma, \varkappa, .): (0, +\infty) \to E$ is left continuous and $\lim_{\epsilon \to +\infty} \delta_v(\sigma, \varkappa, \epsilon) = 1$.

Definition 2.4. [9] Let $\mathfrak{R} \neq \emptyset$. A triplet $(\mathfrak{R}, \delta_{\text{N}}, *)$ is known as FRBMS if $\text{N} \geq 1$, $*$ is a CTN and δ_{N} is a FS on $\mathfrak{R} \times \mathfrak{R} \times [0, +\infty)$ if for all $\sigma, \varkappa, g \in \mathfrak{R}$ and $\epsilon, s, w > 0$,

L1. $\delta_{\text{N}}(\sigma, \varkappa, 0) = 0$;

L2. $\delta_{\text{N}}(\sigma, \varkappa, \epsilon) = 1$ if and only if $\sigma = \varkappa$;

L3. $\delta_{\text{N}}(\sigma, \varkappa, \epsilon) = \delta_{\text{N}}(\varkappa, \sigma, \epsilon)$;

L4. $\delta_{\text{N}}\big(\sigma, g, \text{N}(\epsilon + s + w)\big) \geq \delta_{\text{N}}(\sigma, \varkappa, \epsilon) * \delta_{\text{N}}(\varkappa, u, s) * \delta_{\text{N}}(u, g, w)$ for all distinct $\varkappa, u \in \mathfrak{R} \backslash \{\sigma, g\}$;

L5. $\delta_{\text{N}}(\sigma, \varkappa, .): (0, +\infty) \to E$ is left continuous and $\lim_{\epsilon \to +\infty} \delta_{\text{N}}(\sigma, \varkappa, \epsilon) = 1$.

3. Main results

In this section, we provide numerous new concepts as generalizations of FRMSs and FRBMSs, as well as several FP results.

Definition 3.1. Suppose $\mathfrak{R} \neq \emptyset$. A triplet $(\mathfrak{R}, L_v, *)$ is known as FRMLS if $*$ is a CTN, L_v is a FS on $\mathfrak{R} \times \mathfrak{R} \times [0, +\infty)$ if for all $\sigma, \varkappa, g \in \mathfrak{R}$ and $\epsilon, s, w > 0$,

S1. $L_v(\sigma, \varkappa, 0) = 0$;

S2. $L_v(\sigma,\varkappa,\epsilon)=1$ implies $\sigma=\varkappa$;
S3. $L_v(\sigma,\varkappa,\epsilon)=L_v(\varkappa,\sigma,\epsilon)$;
S4. $L_v(\sigma,g,\epsilon+s+w)\geq L_v(\sigma,\varkappa,\epsilon)*L_v(\varkappa,u,s)*L_v(u,g,w)$ for all distinct $\varkappa,u\in\Re\backslash\{\sigma,g\}$;
S5. $L_v(\sigma,\varkappa,.):(0,+\infty)\to E$ is left continuous and $\lim_{\epsilon\to+\infty}L_v(\sigma,\varkappa,\epsilon)=1$.

Example 3.1. Suppose $(\Re,d)$ be a rectangular MLS, define $L_v:\Re\times\Re\times[0,+\infty)\to E$ by

$$L_v(\sigma,\varkappa,\epsilon)=\frac{\epsilon}{\epsilon+d(\sigma,\varkappa)},\qquad \text{for all } \sigma,\varkappa\in\Re \text{ and } \epsilon>0,$$

with $*$ be a CTN on $\Re$. Then it is easy to see that $(\Re,L_v,*)$ is a FRMLS.

Example 3.2. Define $L_v:\Re\times\Re\times[0,+\infty)\to E$ by

$$L_v(\sigma,\varkappa,\epsilon)=\frac{\epsilon}{\epsilon+\max\{\sigma,\varkappa\}},\qquad \text{for all } \sigma,\varkappa\in\Re \text{ and } \epsilon>0.$$

CTN is given by $\kappa*\text{N}=\kappa\cdot\text{N}$, then it is obvious that $(\Re,L_v,*)$ is a FRMLS.

Remark 3.1. In the preceding case, the self-distance is not equal to 1, i.e.,

$$L_v(\sigma,\sigma,\epsilon)=\frac{\epsilon}{\epsilon+\max\{\sigma,\sigma\}}=\frac{\epsilon}{\epsilon+\sigma}\neq1.$$

In the case of FRMS, however, the self-distance must be equal to one. As a result, every FRMS is a FRMLS, but the opposite may not be true.

Definition 3.2. Let $\Re\neq\emptyset$ and a triplet $(\Re,\delta,*)$ is known as FRBMLS if $\text{N}\geq1$, $*$ is a CTN and δ is a FS on $\Re\times\Re\times[0,+\infty)$ if for all $\sigma,\varkappa,g\in\Re$ and $\epsilon,s,w>0$,

(a) $\delta(\sigma,\varkappa,0)=0$;
(b) $\delta(\sigma,\varkappa,\epsilon)=1$ implies $\sigma=\varkappa$;
(c) $\delta(\sigma,\varkappa,\epsilon)=\delta(\varkappa,\sigma,\epsilon)$;
(d) $\delta\big(\sigma,g,\text{N}(\epsilon+s+w)\big)\geq\delta(\sigma,\varkappa,\epsilon)*\delta(\varkappa,u,s)*\delta(u,g,w)$ for all distinct $\varkappa,u\in\Re\backslash\{\sigma,g\}$;
(e) $\delta(\sigma,\varkappa,.):(0,+\infty)\to E$ is left continuous and $\lim_{\epsilon\to+\infty}\delta(\sigma,\varkappa,\epsilon)=1$.

Example 3.3. Suppose $(\Re,d)$ be a rectangular b-MLS (RBMLS), define $\delta:\Re\times\Re\times[0,+\infty)\to E$ by

$$\delta(\sigma,\varkappa,\epsilon)=\frac{\epsilon}{\epsilon+d(\sigma,\varkappa)},\qquad \text{for all } \sigma,\varkappa\in\Re \text{ and } \epsilon>0,$$

with CTN $'*'$. Therefore, it is clear that $(\Re,\delta,*)$ is a FRBMLS.

Example 3.4. Define $\delta:\Re\times\Re\times[0,+\infty)\to E$ by

$$\delta(\sigma,\varkappa,\epsilon)=\frac{\epsilon}{\epsilon+\max\{\sigma,\varkappa\}^p},\qquad \text{for all } \sigma,\varkappa\in\Re \text{ and } \epsilon>0.$$

CTN is defined by $\kappa*\text{N}=\kappa\cdot\text{N}$ and $p\geq1$, then it is obvious that $(\Re,\delta,*)$ is a FRBMLS.

Example 3.5. Assume $(\Re,d)$ be a RBMLS, define $\delta:\Re\times\Re\times[0,+\infty)\to E$ by

$$\delta(\sigma,\varkappa,\epsilon)=e^{-\frac{d(\sigma,\varkappa)}{\epsilon}},\qquad \text{for all } \sigma,\varkappa\in\Re \text{ and } \epsilon>0,$$

with CTN $\kappa*\text{N}=\min\{\kappa,\text{N}\}$. Then it is obvious that $(\Re,\delta,*)$ is a FRBMLS.

Example 3.6. Assume $(\Re,d)$ be a RBMLS, define $\delta:\Re\times\Re\times[0,+\infty)\to E$ by

$$\delta(\sigma,\varkappa,\epsilon)=e^{-\frac{\max\{\sigma,\varkappa\}^p}{\epsilon}},\qquad \text{for all } \sigma,\varkappa\in\Re \text{ and } \epsilon>0,$$

with $p\geq1$ and CTN $\kappa*\text{N}=\min\{\kappa,\text{N}\}$. Then it obvious that $(\Re,\delta,*)$ is a FRBMLS.

Remark 3.2. If CTN given by $\kappa * \text{N} = \kappa \cdot \text{N}$, then Example 3.6 is also a FRBMLS.
Remark 3.3. The self distance in FRBMLS may be not equal to 1.

Pick Example 3.6 with $p = 2$, then it yields

$$\delta(\sigma, \sigma, \epsilon) = e^{-\frac{\max\{\sigma,\sigma\}^2}{\epsilon}} = e^{-\frac{\sigma^2}{\epsilon}} \neq 1.$$

Remark 3.4. The preceding statement demonstrates that every FRBMLS is not a FRBMS, because in order to be a FRBMS, self distance must equal 1.
Remark 3.5. FRBMLS may be not continuous.

Example 3.7. Suppose $\mathfrak{R} = [0, +\infty)$, $\delta(\sigma, \varkappa, \epsilon) = \frac{\epsilon}{\epsilon + d(\sigma, \varkappa)}$ for all $\sigma, \varkappa \in \mathfrak{R}, \epsilon > 0$ and

$$d(\sigma, \varkappa) = \begin{cases} 0, & \text{if } \sigma = \varkappa, \\ 2(\sigma + \varkappa)^2, & \text{if } \sigma, \varkappa \in [0,1], \\ \dfrac{1}{2}(\sigma + \varkappa)^2, & \text{otherwise.} \end{cases}$$

If we define CTN by $\kappa * \text{N} = \kappa \cdot \text{N}$, then $(\mathfrak{R}, \delta, *)$ is an FRBMLS. Now, to illustrate continuity, we have

$$\lim_{n \to +\infty} \delta\left(0, 1 - \frac{1}{n}, \epsilon\right) = \lim_{n \to +\infty} \frac{\epsilon}{\epsilon + 2\left(1 - \left(\frac{1}{n}\right)\right)^2} = \frac{\epsilon}{\epsilon + 2} = \delta(0,1,\epsilon).$$

However,

$$\lim_{n \to +\infty} \delta\left(1, 1 - \frac{1}{n}, \epsilon\right) = \lim_{n \to +\infty} \frac{\epsilon}{\epsilon + 2\left(2 - \left(\frac{1}{n}\right)\right)^2} = \frac{\epsilon}{\epsilon + 8} \neq 1 = \delta(1,1,\epsilon).$$

Hence, $(\mathfrak{R}, \delta, *)$ is not continuous.
Definition 3.3. Let $(\mathfrak{R}, \delta, *)$ be a FRBMLS and assume $\{\sigma_n\}$ is a sequence in $\mathfrak{R}$. Then

(a) $\{\sigma_n\}$ is named to be a convergent sequence if there exists $\sigma \in \mathfrak{R}$ such that

$$\lim_{n \to +\infty} \delta(\sigma_n, \sigma, \epsilon) = \delta(\sigma, \sigma, \epsilon), \text{ for all } \epsilon > 0.$$

(b) $\{\sigma_n\}$ is named to be Cauchy sequence if $\lim_{n \to +\infty} \delta(\sigma_n, \sigma_{n+q}, \epsilon)$ is exists and is finite for all $\epsilon > 0$.

(c) If every Cauchy sequence is convergent in $\mathfrak{R}$ then $(\mathfrak{R}, \delta, *)$ is said to be a complete FRBMLS such that

$$\lim_{n \to +\infty} \delta(\sigma_n, \sigma, \epsilon) = \delta(\sigma, \sigma, \epsilon) = \lim_{n \to +\infty} \delta(\sigma_n, \sigma_{n+q}, \epsilon),$$

for all $\epsilon > 0$ and $q \geq 1$.

Remark 3.6. A convergent sequence's limit may not be unique in a FRBMLS.

Consider the FRBMLS in Example 3.4, and describe a sequence as $\sigma_n = 1 - \frac{1}{n}$ for all $n \geq 1$.

If $\sigma \geq 1$, for all $\epsilon > 0$, then

$$\lim_{n\to+\infty}\delta(\sigma_n,\sigma,\epsilon)=\lim_{n\to+\infty}\frac{\epsilon}{\epsilon+\max\{\sigma_n,\sigma\}^p}=\frac{\epsilon}{\epsilon+\sigma^p}=\delta(\sigma,\sigma,\epsilon).$$

That is, the sequence $\{\sigma_n\}$ converges to all $\sigma\geq 1$.

Remark 3.7. It is not necessary for convergent sequence to become Cauchy in a FRBMLS.

Consider the example given in the preceding remark and describe a sequence as $\sigma_n=1+(-1)^n$ for all $n\geq 1$. If $\sigma\geq 2$, for all $\epsilon>0$, then

$$\lim_{n\to+\infty}\delta(\sigma_n,\sigma,\epsilon)=\lim_{n\to+\infty}\frac{\epsilon}{\epsilon+\max\{\sigma_n,\sigma\}^p}=\frac{\epsilon}{\epsilon+\sigma^p}=\delta(\sigma,\sigma,\epsilon).$$

That is, the sequence $\{\sigma_n\}$ converges to all $\sigma\geq 2$ but it is not Cauchy as $\lim_{n\to+\infty}\delta(\sigma_n,\sigma_{n+q},\epsilon)$ does not exist.

Definition 3.4. Let $(\mathfrak{R},\delta,*)$ be a FRBMLS. For $\sigma\in\mathfrak{R}$, $\theta\in(0,1)$, $\epsilon>0$, we define the open ball as $B(\sigma,\theta,\epsilon)=\{\varkappa\in\mathfrak{R}:\delta(\sigma,\varkappa,\epsilon)>1-\theta\}$ (center σ, radius θ with respect ϵo ϵ).

Remark 3.8. FRBMLS may not have to be Hausdorff.

Example 3.8. Let $\mathfrak{R}=\{1,2,3,4\}$. Define $\delta:\mathfrak{R}\times\mathfrak{R}\times[0,+\infty)\to[0,1]$ by

$$\delta(\sigma,\varkappa,\epsilon)=\frac{\epsilon}{\epsilon+\max\{\sigma,\varkappa\}^2},\qquad \text{for all } \sigma,\varkappa\in\mathfrak{R} \text{ and } \epsilon>0.$$

CTN is defined by $\kappa*\text{N}=\kappa\cdot\text{N}$, then $(\mathfrak{R},\delta,*)$ is a FRBMLS.

Now, take $\sigma=1$, $\epsilon=20$ and $\varkappa\in\mathfrak{R}$, then

$$\delta(1,2,20)=\frac{20}{20+\max\{1,2\}^2}=\frac{20}{20+4}=\frac{20}{24}=0.8333,$$

$$\delta(1,3,20)=\frac{20}{20+\max\{1,3\}^2}=\frac{20}{20+9}=\frac{20}{29}=0.6896,$$

$$\delta(1,4,20)=\frac{20}{20+\max\{1,4\}^2}=\frac{20}{20+16}=\frac{20}{36}=0.5555.$$

Now, if we take $\theta=0.4$, then

$$B(1,0.4,20)=\{\varkappa\in\mathfrak{R}:\delta(1,\varkappa,20)>0.6\}.$$

Hence, $B(1,0.4,20)=\{2,3\}$ is an open ball. Now, take $\sigma=2$, $\epsilon=10$ and $\varkappa\in\mathfrak{R}$, then

$$\delta(2,1,10)=\frac{10}{10+\max\{2,1\}^2}=\frac{10}{10+4}=\frac{10}{14}=0.7142,$$

$$\delta(2,3,10)=\frac{10}{10+\max\{2,3\}^2}=\frac{10}{10+9}=\frac{10}{19}=0.5263,$$

$$\delta(2,4,10)=\frac{10}{10+\max\{2,4\}^2}=\frac{10}{10+16}=\frac{10}{26}=0.3846.$$

Now, if we take $\theta=0.5$, then

$$B(2,0.5,10)=\{\varkappa\in\mathfrak{R}:\delta(2,\varkappa,10)>0.5\}.$$

Hence, $B(2,0.5,10)=\{1,3\}$ is an open ball. But $B(1,0.4,20)\cap B(2,0.5,10)=\{2,3\}\cap\{1,3\}\neq\emptyset$. This implies that FRBMLS $(\mathfrak{R},\delta_{\text{N}},*)$ is not Hausdorff.

Lemma 3.1. Let $(\Re, \delta, *)$ be a FRBMLS and

$$\delta(\sigma, \varkappa, \varsigma\epsilon) \geq \delta(\sigma, \varkappa, \epsilon), \tag{3.1}$$

for all $\sigma, \varkappa \in \Re$, $0 < \varsigma < 1$ and $\epsilon > 0$, then $\sigma = \varkappa$.
Proof. From (d) of Definition 3.2, it is immediate.
Theorem 3.1. (Banach contraction theorem in fuzzy rectangular b-metric-like spaces)
Suppose $(\Re, \delta, *)$ be a complete FRBMLS with $N \geq 1$ such that

$$\lim_{\epsilon \to +\infty} \delta(\sigma, \varkappa, \epsilon) = 1, \text{ for all } \sigma, \varkappa \in \Re. \tag{3.2}$$

Let $\xi: \Re \to \Re$ be a mapping satisfying

$$\delta(\xi\sigma, \xi\varkappa, \varsigma\epsilon) \geq \delta(\sigma, \varkappa, \epsilon), \tag{3.3}$$

for all $\sigma, \varkappa \in \Re, \varsigma \in \left[0, \frac{1}{N}\right)$. Then ξ has a unique fixed point $u \in \Re$ and $\delta(u, u, \epsilon) = 1$.

Proof. Fix an arbitrary point $\kappa_0 \in \Re$ and for $n = 0,1,2, \dots$, start an iterative process $\kappa_{n+1} = \xi\kappa_n$. Successively applying inequality (3.1), we get for all $n, \epsilon > 0$,

$$\delta(\kappa_n, \kappa_{n+1}, \epsilon) \geq \delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{\varsigma^n}\right). \tag{3.4}$$

Since $(\Re, \delta, *)$ is a FRBMLS. For the sequence $\{\kappa_n\}$, writing $\epsilon = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$ and using the rectangular inequality given in (d) of Definition 3.2 on $\delta(\kappa_n, \kappa_{n+p}, \epsilon)$, we have the following cases.
Case 1. If p is odd, then said $p = 2m + 1$ where $m \in \{1,2,3, \dots\}$, we have

$$\begin{aligned}
&\delta(\kappa_n, \kappa_{n+2m+1}, \epsilon)\\
&\geq \delta\left(\kappa_n, \kappa_{n+1}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+2}, \kappa_{n+2m+1}, \frac{\epsilon}{3N}\right)\\
&\geq \delta\left(\kappa_n, \kappa_{n+1}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+2}, \kappa_{n+3}, \frac{\epsilon}{(3N)^2}\right)\\
&\quad * \delta\left(\kappa_{n+3}, \kappa_{n+4}, \frac{\epsilon}{(3N)^2}\right) * \delta\left(\kappa_{n+4}, \kappa_{n+2m+1}, \frac{\epsilon}{(3N)^2}\right)\\
&\geq \delta\left(\kappa_n, \kappa_{n+1}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+2}, \kappa_{n+3}, \frac{\epsilon}{(3N)^2}\right) * \delta\left(\kappa_{n+3}, \kappa_{n+4}, \frac{\epsilon}{(3N)^2}\right)\\
&\quad * \delta\left(\kappa_{n+4}, \kappa_{n+5}, \frac{\epsilon}{(3N)^3}\right) * \cdots * \delta\left(\kappa_{n+2m}, \kappa_{n+2m+1}, \frac{\epsilon}{(3N)^m}\right).
\end{aligned}$$

Using (d) of Definition 3.2 in the above inequalities, we deduce

$$\begin{aligned}
&\delta(\kappa_n, \kappa_{n+2m+1}, \epsilon)\\
&\geq \delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{3N\varsigma^n}\right) * \delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{3N\varsigma^{n+1}}\right) * \delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^2\varsigma^{n+2}}\right)\\
&\quad * \delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^2\varsigma^{n+3}}\right) * \delta\left(\kappa_{n+4}, \kappa_{n+5}, \frac{\epsilon}{(3N)^3\varsigma^{n+4}}\right) * \cdots * \delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^m\varsigma^{n+m}}\right)\\
&\geq \delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{3N\varsigma^n}\right) * \delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N\varsigma)\varsigma^n}\right) * \delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N\varsigma)^2\varsigma^n}\right) * \delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N\varsigma)^2\varsigma^{n+1}}\right)\\
&\quad * \delta\left(\kappa_{n+4}, \kappa_{n+5}, \frac{\epsilon}{(3N\varsigma)^3\varsigma^{n+1}}\right) * \cdots * \delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N\varsigma)^m\varsigma^{n+m}}\right).
\end{aligned}$$

Case 2. If p is even, then said $p = 2m, m \in \{1,2,3, \cdots\}$, then we have

$$\begin{aligned}\delta(\kappa_n,\kappa_{n+2m},\epsilon) &\geq \delta\left(\kappa_n,\kappa_{n+1},\frac{\epsilon}{3N}\right)*\delta\left(\kappa_{n+1},\kappa_{n+2},\frac{\epsilon}{3N}\right)*\delta\left(\kappa_{n+2},\kappa_{n+2m},\frac{\epsilon}{3N}\right)\\ &\geq \delta\left(\kappa_n,\kappa_{n+1},\frac{\epsilon}{3N}\right)*\delta\left(\kappa_{n+1},\kappa_{n+2},\frac{\epsilon}{3N}\right)*\delta\left(\kappa_{n+2},\kappa_{n+3},\frac{\epsilon}{(3N)^2}\right)\\ &\quad*\delta\left(\kappa_{n+3},\kappa_{n+4},\frac{\epsilon}{(3N)^2}\right)*\delta\left(\kappa_{n+4},\kappa_{n+2m},\frac{\epsilon}{(3N)^2}\right)\\ &\geq \delta\left(\kappa_n,\kappa_{n+1},\frac{\epsilon}{3N}\right)*\delta\left(\kappa_{n+1},\kappa_{n+2},\frac{\epsilon}{3N}\right)*\delta\left(\kappa_{n+2},\kappa_{n+3},\frac{\epsilon}{(3N)^2}\right)\\ &\quad*\delta\left(\kappa_{n+3},\kappa_{n+4},\frac{\epsilon}{(3N)^2}\right)*\delta\left(\kappa_{n+4},\kappa_{n+5},\frac{\epsilon}{(3N)^3}\right)*\cdots\\ &\quad*\delta\left(\kappa_{n+2m-4},\kappa_{n+2m-3},\frac{\epsilon}{(3N)^{m-1}}\right)*\delta\left(\kappa_{n+2m-3},\kappa_{n+2m-2},\frac{\epsilon}{(3N)^{m-1}}\right)\\ &\quad*\delta\left(\kappa_{n+2m-2},\kappa_{n+2m},\frac{\epsilon}{(3N)^{m-1}}\right).\end{aligned}$$

Using (3.4) in the above inequalities, we deduce

$$\begin{aligned}&\delta(\kappa_n,\kappa_{n+2m},\epsilon)\\ &\geq \delta\left(\kappa_0,\kappa_1,\frac{\epsilon}{3N\varsigma^n}\right)*\delta\left(\kappa_0,\kappa_1,\frac{\epsilon}{3N\varsigma^{n+1}}\right)*\delta\left(\kappa_0,\kappa_1,\frac{\epsilon}{(3N)^2\varsigma^{n+2}}\right)\\ &\quad*\delta\left(\kappa_0,\kappa_1,\frac{\epsilon}{(3N)^2\varsigma^{n+3}}\right)*\delta\left(\kappa_0,\kappa_{n+5},\frac{\epsilon}{(3N)^3\varsigma^{n+4}}\right)*\cdots*\delta\left(\kappa_0,\kappa_1,\frac{\epsilon}{(3N)^{m-1}\varsigma^{n+2m-2}}\right)\\ &\geq \delta\left(\kappa_0,\kappa_1,\frac{\epsilon}{3N\varsigma^n}\right)*\delta\left(\kappa_0,\kappa_1,\frac{\epsilon}{(3N\varsigma)\varsigma^n}\right)*\delta\left(\kappa_0,\kappa_1,\frac{\epsilon}{(3N\varsigma)^2\varsigma^n}\right)*\delta\left(\kappa_0,\kappa_1,\frac{\epsilon}{(3N\varsigma)^2\varsigma^{n+1}}\right)\\ &\quad*\delta\left(\kappa_{n+4},\kappa_{n+5},\frac{\epsilon}{(3N\varsigma)^3\varsigma^{n+1}}\right)*\cdots*\delta\left(\kappa_0,\kappa_1,\frac{\epsilon}{(3N\varsigma)^{m-1}\varsigma^{n+m-1}}\right).\end{aligned}$$

Therefore, from $\lim_{\epsilon\to+\infty}\delta(\sigma,\varkappa,\epsilon)=1$, Cases 1, 2 and (3.2) conclude that for all $p\in\{1,2,3,\cdots\}$, we have

$$\lim_{n\to+\infty}\delta\big(\kappa_n,\kappa_{n+p},\epsilon\big)=1.$$

Hence, $\{\kappa_n\}$ is a Cauchy sequence. Since $(\Re,\delta,*)$ is a complete FRBMLS, so there exists $u\in\Re$ such that

$$\lim_{n\to+\infty}\delta(\sigma_n,u,\epsilon)=\delta(u,u,\epsilon)=\lim_{n\to+\infty}\delta\big(\sigma_n,\sigma_{n+q},\epsilon\big)=1,\qquad \text{for all } \epsilon>0 \text{ and } q\geq 1.$$

Now, we examine that u is a fixed point of ξ.

$$\begin{aligned}\delta(u,\xi u,\epsilon) &\geq \delta\left(u,\kappa_n,\frac{\epsilon}{3N}\right)*\delta\left(\kappa_n,\kappa_{n+1},\frac{\epsilon}{3N}\right)*\delta\left(\kappa_{n+1},\xi u,\frac{\epsilon}{3N}\right)\\ &\geq \delta\left(u,\kappa_n,\frac{\epsilon}{3N}\right)*\delta\left(\xi\kappa_{n-1},\xi\kappa_n,\frac{\epsilon}{3N}\right)*\delta\left(\xi\kappa_n,\xi u,\frac{\epsilon}{3N}\right)\\ &\geq \delta\left(u,\kappa_n,\frac{\epsilon}{3N}\right)*\delta\left(\kappa_{n-1},\kappa_n,\frac{\epsilon}{3N\varsigma}\right)*\delta\left(\kappa_n,u,\frac{\epsilon}{3N\varsigma}\right)\\ &\to 1*1*1=1 \text{ as } n\to+\infty.\end{aligned}$$

Hence, u is a fixed point of ξ.

Uniqueness: Let v is another fixed point of ξ for some $v\in\Re$, then

$$\delta(v,u,\epsilon)=\delta(\xi v,\xi u,\epsilon)\geq\delta\left(v,u,\frac{\epsilon}{\varsigma}\right)=\delta\left(\xi v,\xi u,\frac{\epsilon}{\varsigma}\right)$$
$$\geq\delta\left(v,u,\frac{\epsilon}{\varsigma^2}\right)\geq\cdots\geq\delta\left(v,u,\frac{\epsilon}{\varsigma^n}\right)\to 1 \text{ as } n\to+\infty,$$

and by using the fact $\lim_{\epsilon\to+\infty}\delta(\sigma,\varkappa,\epsilon)=1$. Thus, $u=v$. Hence, the fixed point is unique.

Theorem 3.2. (Banach contraction theorem in fuzzy rectangular metric-like spaces)

Suppose $(\Re,\delta,*)$ be a FRMLS such that

$$\lim_{\epsilon\to+\infty}\delta(\sigma,\varkappa,\epsilon)=1,\qquad \text{for all } \sigma,\varkappa\in\Re.$$

Let $\xi:\Re\to\Re$ be a mapping satisfying

$$\delta(\xi\sigma,\xi\varkappa,\varsigma\epsilon)\geq\delta(\sigma,\varkappa,\epsilon),$$

for all $\sigma,\varkappa\in\Re,\varsigma\in[0,1)$. Then ξ has a unique fixed point $u\in\Re$ and $\delta(u,u,\epsilon)=1$.

Proof. It is immediate if we take $\text{N}=1$ in the above theorem.

Example 3.10. Let $\Re=[0,1]$, define $\delta:\Re\times\Re\times[0,+\infty)\to[0,1]$ by

$$\delta(\sigma,\varkappa,\epsilon)=\frac{\epsilon}{\epsilon+\max(\sigma+\varkappa)^2},$$

for all $\sigma,\varkappa\in\Re$ and $\epsilon>0$, with CTN $\kappa*\text{N}=\kappa.\text{N}$. Then it is obvious that $(\Re,\delta,*)$ is a complete FRBMLS.

Define $\xi:\Re\to\Re$ by $\xi(\sigma)=\frac{1-2^{-\sigma}}{3}$. Then

$$\begin{aligned}\delta(\xi\sigma,\xi\varkappa,\varsigma\epsilon)&=\delta\left(\frac{1-2^{-\sigma}}{3},\frac{1-2^{-\varkappa}}{3},\varsigma\epsilon\right)\\&=\frac{\varsigma\epsilon}{\varsigma\epsilon+\left(\frac{1-2^{-\sigma}}{3}+\frac{1-2^{-\varkappa}}{3}\right)^2}\\&=\frac{9\varsigma\epsilon}{9\varsigma\epsilon+\left(2-(2^{-\sigma}+2^{-\varkappa})\right)^2}\\&\geq\frac{9\varsigma\epsilon}{9\varsigma\epsilon+(\sigma+\varkappa)^2}\geq\frac{\epsilon}{\epsilon+(\sigma+\varkappa)^2}\\&=\delta(\sigma,\varkappa,\epsilon),\end{aligned}$$

for all $\sigma,\varkappa\in\Re$, where $\varsigma\in\left[\frac{1}{2},1\right)$. Thus, all the conditions of Theorem 3.1 satisfied and hence, 0 is the unique fixed point of ξ.

Theorem 3.3. Let $(\Re,\delta,*)$ be a complete FRBMLS with $\text{N}\geq 1$ such that

$$\lim_{\epsilon\to+\infty}\delta(\sigma,\varkappa,\epsilon)=1,\ \text{for all } \sigma,\varkappa\in\Re. \tag{3.5}$$

Let $\xi:\Re\to\Re$ be a mapping satisfying

$$\frac{1}{\delta(\xi\sigma,\xi\varkappa,\epsilon)}-1\leq\varsigma\left[\frac{1}{\delta(\sigma,\varkappa,\epsilon)}-1\right], \tag{3.6}$$

for all $\sigma, \varkappa \in \Re, \varsigma \in \left[0, \frac{1}{N}\right)$. Then ξ has a unique fixed point $u \in \Re$ and $\delta(u, u, \epsilon) = 1$.

Proof. Let $(\Re, \delta, *)$ be a complete FRBMLS. For arbitrary $\kappa_0 \in \Re$, define a sequence $\{\kappa_n\}$ in $\Re$ by

$$\kappa_1 = \xi\kappa_0,\ \kappa_2 = \xi^2\kappa_0 = \xi\kappa_1, \dots, \kappa_n = \xi^n\kappa_0 = \xi\kappa_{n-1} \text{ for all } n \in \mathbb{N}.$$

If $\kappa_n = \kappa_{n-1}$ for some $n \in \mathbb{N}$, then κ_n is a fixed point of ξ. We assume that $\kappa_n \neq \kappa_{n-1}$ for all $n \in \mathbb{N}$. For $\epsilon > 0$ and $n \in \mathbb{N}$, we get from (3.6) that

$$\frac{1}{\delta(\kappa_n, \kappa_{n+1}, \epsilon)} - 1 = \frac{1}{\delta(\xi\kappa_{n-1}, \xi\kappa_n, \epsilon)} - 1 \leq \varsigma\left[\frac{1}{\delta(\kappa_{n-1}, \kappa_n, \epsilon)} - 1\right].$$

We have

$$\begin{aligned}\frac{1}{\delta(\kappa_n, \kappa_{n+1}, \epsilon)} &\leq \frac{\varsigma}{\delta(\kappa_{n-1}, \kappa_n, \epsilon)} + (1-\varsigma) = \frac{\varsigma}{\delta(\xi\kappa_{n-2}, \xi\kappa_{n-1}, \epsilon)} + (1-\varsigma) \\ &\leq \frac{\varsigma^2}{\delta(\kappa_{n-2}, \kappa_{n-1}, \epsilon)} + \varsigma(1-\varsigma) + (1-\varsigma), \forall\epsilon > 0.\end{aligned}$$

Continuing this way, we get

$$\begin{aligned}\frac{1}{\delta(\kappa_n, \kappa_{n+1}, \epsilon)} &\leq \frac{\varsigma^n}{\delta(\kappa_0, \kappa_1, \epsilon)} + \varsigma^{n-1}(1-\varsigma) + \varsigma^{n-2}(1-\varsigma) + \cdots + \varsigma(1-\varsigma) + (1-\varsigma) \\ &\leq \frac{\varsigma^n}{\delta(\kappa_0, \kappa_1, \epsilon)} + (\varsigma^{n-1} + \varsigma^{n-2} + \cdots + 1)(1-\varsigma) \leq \frac{\varsigma^n}{\delta(\kappa_0, \kappa_1, \epsilon)} + (1-\varsigma^n).\end{aligned}$$

We have

$$\frac{1}{\frac{\varsigma^n}{\delta(\kappa_0, \kappa_1, \epsilon)} + (1-\varsigma^n)} \leq \delta(\kappa_n, \kappa_{n+1}, \epsilon),\ \forall\epsilon > 0,\ n \in \mathbb{N}. \tag{3.7}$$

Since $(\Re, \delta, *)$ be a FRBMLS for the sequence $\{\kappa_n\}$, writing $\epsilon = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$ and using the rectangular inequality given in (d) of Definition 3.2 on $\delta(\kappa_n, \kappa_{n+p}, \epsilon)$, in the following cases.

Case 1. If p is odd, then said $p = 2m + 1$ where $m \in \{1,2,3,\dots\}$, we have

$$\begin{aligned}&\delta(\kappa_n, \kappa_{n+2m+1}, \epsilon) \\ &\geq \delta\left(\kappa_n, \kappa_{n+1}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+2}, \kappa_{n+2m+1}, \frac{\epsilon}{3N}\right) \\ &\geq \delta\left(\kappa_n, \kappa_{n+1}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+2}, \kappa_{n+3}, \frac{\epsilon}{(3N)^2}\right) \\ &\quad * \delta\left(\kappa_{n+3}, \kappa_{n+4}, \frac{\epsilon}{(3N)^2}\right) * \delta\left(\kappa_{n+4}, \kappa_{n+2m+1}, \frac{\epsilon}{(3N)^2}\right) \\ &\geq \delta\left(\kappa_n, \kappa_{n+1}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+2}, \kappa_{n+3}, \frac{\epsilon}{(3N)^2}\right) * \delta\left(\kappa_{n+3}, \kappa_{n+4}, \frac{\epsilon}{(3N)^2}\right) \\ &\quad * \delta\left(\kappa_{n+4}, \kappa_{n+5}, \frac{\epsilon}{(3N)^3}\right) * \cdots * \delta\left(\kappa_{n+2m}, \kappa_{n+2m+1}, \frac{\epsilon}{(3N)^m}\right).\end{aligned}$$

By using (3.7) in the above inequality, we have

$$
\begin{aligned}
&\delta(\kappa_n, \kappa_{n+2m+1}, \epsilon) \\
&\geq \frac{1}{\frac{\varsigma^n}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{3N}\right)} + (1-\varsigma^n)} * \frac{1}{\frac{\varsigma^{n+1}}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{3N}\right)} + (1-\varsigma^{n+1})} * \frac{1}{\frac{\varsigma^{n+2}}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^2}\right)} + (1-\varsigma^{n+2})} \\
&\qquad * \frac{1}{\frac{\varsigma^{n+3}}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^2}\right)} + (1-\varsigma^{n+3})} * \cdots * \frac{1}{\frac{\varsigma^{n+2m}}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^m}\right)} + (1-\varsigma^{n+2m})} \\
&\geq \frac{1}{\frac{\varsigma^n}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{3N}\right)} + (1-\varsigma^n)} * \frac{1}{\frac{\varsigma(\varsigma^{n})}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{3N}\right)} + (1-\varsigma(\varsigma^{n}))} * \frac{1}{\frac{\varsigma^2(\varsigma^{n})}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^2}\right)} + (1-\varsigma^2(\varsigma^{n}))} \\
&\qquad * \frac{1}{\frac{\varsigma^2(\varsigma^{n+1})}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^2}\right)} + (1-\varsigma^2(\varsigma^{n+1}))} * \cdots * \frac{1}{\frac{\varsigma^m(\varsigma^{n+m})}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^m}\right)} + (1-\varsigma^m(\varsigma^{n+m}))}.
\end{aligned}
$$

Case 2. If p is even, then said $p = 2m, m \in \{1,2,3,\cdots\}$, then we have

$$
\begin{aligned}
&\delta(\kappa_n, \kappa_{n+2m}, \epsilon) \\
&\geq \delta\left(\kappa_n, \kappa_{n+1}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+2}, \kappa_{n+2m}, \frac{\epsilon}{3N}\right) \\
&\geq \delta\left(\kappa_n, \kappa_{n+1}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+2}, \kappa_{n+3}, \frac{\epsilon}{(3N)^2}\right) * \delta\left(\kappa_{n+3}, \kappa_{n+4}, \frac{\epsilon}{(3N)^2}\right) \\
&\qquad * \delta\left(\kappa_{n+4}, \kappa_{n+2m}, \frac{\epsilon}{(3N)^2}\right) \\
&\geq \delta\left(\kappa_n, \kappa_{n+1}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+1}, \kappa_{n+2}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+2}, \kappa_{n+3}, \frac{\epsilon}{(3N)^2}\right) * \delta\left(\kappa_{n+3}, \kappa_{n+4}, \frac{\epsilon}{(3N)^2}\right) \\
&\qquad * \delta\left(\kappa_{n+4}, \kappa_{n+5}, \frac{\epsilon}{(3N)^3}\right) * \cdots * \delta\left(\kappa_{n+2m-4}, \kappa_{n+2m-3}, \frac{\epsilon}{(3N)^{m-1}}\right) \\
&\qquad * \delta\left(\kappa_{n+2m-3}, \kappa_{n+2m-2}, \frac{\epsilon}{(3N)^{m-1}}\right) * \delta\left(\kappa_{n+2m-2}, \kappa_{n+2m}, \frac{\epsilon}{(3N)^{m-1}}\right) \\
&\geq \frac{1}{\frac{\varsigma^n}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{3N}\right)} + (1-\varsigma^n)} * \frac{1}{\frac{\varsigma^{n+1}}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{3N}\right)} + (1-\varsigma^{n+1})} * \frac{1}{\frac{\varsigma^{n+2}}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^2}\right)} + (1-\varsigma^{n+2})} \\
&\qquad * \frac{1}{\frac{\varsigma^{n+3}}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^2}\right)} + (1-\varsigma^{n+3})} * \cdots * \frac{1}{\frac{\varsigma^{n+2m-2}}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^{m-1}}\right)} + (1-\varsigma^{n+2m-2})} \\
&\geq \frac{1}{\frac{\varsigma^n}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{3N}\right)} + (1-\varsigma^n)} * \frac{1}{\frac{\varsigma(\varsigma^{n})}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{3N}\right)} + (1-\varsigma(\varsigma^{n}))} * \frac{1}{\frac{\varsigma^2(\varsigma^{n})}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^2}\right)} + (1-\varsigma^2(\varsigma^{n}))} \\
&\qquad * \frac{1}{\frac{\varsigma^2(\varsigma^{n+1})}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^2}\right)} + (1-\varsigma^2(\varsigma^{n+1}))} * \cdots * \frac{1}{\frac{\varsigma^{m-1}(\varsigma^{n+m-1})}{\delta\left(\kappa_0, \kappa_1, \frac{\epsilon}{(3N)^m}\right)} + (1-\varsigma^{m-1}(\varsigma^{n+m-1}))}.
\end{aligned}
$$

We deduce from the cases 1 and 2 that

$$\lim_{n\to+\infty} \delta\left(\kappa_n, \kappa_{n+p}, \epsilon\right) = 1 \text{ for all } \epsilon > 0,\ p \geq 1.$$

Therefore, $\{\kappa_n\}$ is a Cauchy sequence in $(\Re, \delta, *)$. By the completeness of $(\Re, \delta, *)$, there exists $u \in \Re$ such that

$$\lim_{n\to+\infty} \delta(\kappa_n, u, \epsilon) = \lim_{n\to+\infty} \delta\left(\kappa_n, \kappa_{n+p}, \epsilon\right) = \lim_{n\to+\infty} \delta(u, u, \epsilon) = 1, \forall\, \epsilon > 0, p \geq 1. \tag{3.6}$$

Now we prove that u is a fixed point for ξ. For this we obtain from (3.6) that

$$\frac{1}{\delta(\xi\kappa_n, \xi u, \epsilon)} - 1 \leq \varsigma\left[\frac{1}{\delta(\kappa_n, u, \epsilon)} - 1\right] = \frac{\varsigma}{\delta(\kappa_n, u, \epsilon)} - \varsigma,$$

$$\frac{1}{\frac{\varsigma}{\delta(\kappa_n, u, \epsilon)} + 1 - \varsigma} \leq \delta(\xi\kappa_n, \xi u, \epsilon),$$

and

$$\frac{1}{\delta(\xi\kappa_{n-1}, \xi\kappa_n, \epsilon)} - 1 \leq \varsigma\left[\frac{1}{\delta(\kappa_{n-1}, \kappa_n, \epsilon)} - 1\right] = \frac{\varsigma}{\delta(\kappa_{n-1}, \kappa_n, \epsilon)} - \varsigma,$$

$$\frac{1}{\frac{\varsigma}{\delta(\kappa_{n-1}, \kappa_n, \epsilon)} + 1 - \varsigma} \leq \delta(\xi\kappa_{n-1}, \xi\kappa_n, \epsilon).$$

Using the above inequalities, we deduce

$$\begin{aligned}\delta(u, \xi u, \epsilon) &\geq \delta\left(u, \kappa_n, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_n, \kappa_{n+1}, \frac{\epsilon}{3N}\right) * \delta\left(\kappa_{n+1}, \xi u, \frac{\epsilon}{3N}\right)\\ &\geq \delta\left(u, \kappa_n, \frac{\epsilon}{3N}\right) * \delta\left(\xi\kappa_{n-1}, \xi\kappa_n, \frac{\epsilon}{3N}\right) * \delta\left(\xi\kappa_n, \xi u, \frac{\epsilon}{3N}\right)\\ &\geq \delta\left(u, \kappa_n, \frac{\epsilon}{3N}\right) * \frac{1}{\frac{\varsigma}{\delta\left(\kappa_{n-1}, \kappa_n, \frac{\epsilon}{3N}\right)} + 1 - \varsigma} * \frac{1}{\frac{\varsigma}{\delta\left(\kappa_n, u, \frac{\epsilon}{3N}\right)} + 1 - \varsigma}.\end{aligned}$$

Taking limit as $n \to +\infty$ and utilizing (3.8) in the preceding inequality, we examine that $\delta(u, \xi u, \epsilon) = 1$, that is, $\xi u = u$. Hence, u is a fixed point of ξ and $\delta(u, u, \epsilon) = 1$ for all $\epsilon > 0$.
Now we prove the uniqueness of u of ξ. Let v be another fixed point of ξ, such that $\delta(u, v, \text{t}) < 1$ for some $\epsilon > 0$, and follows from (3.6) that

$$\frac{1}{\delta(u, v, \epsilon)} - 1 = \frac{1}{\delta(\xi u, \xi v, \epsilon)} - 1 \leq \varsigma\left[\frac{1}{\delta(u, v, \epsilon)} - 1\right] < \frac{1}{\delta(u, v, \epsilon)} - 1$$

a contradiction. Therefore, we must have $\delta(u, v, \epsilon) = 1$, for all $\epsilon > 0$, and hence $u = v$.

Corollary 3.1. Let $(\Re, \delta, *)$ be a complete FRBMLS and a mapping $\xi: \Re \to \Re$ satisfying

$$\frac{1}{\delta(\xi^n\sigma, \xi^n\varkappa, \epsilon)} - 1 \leq \varsigma\left[\frac{1}{\delta(\sigma, \varkappa, \epsilon)} - 1\right].$$

For some $n \in \mathbb{N}$, $\forall \sigma, \varkappa \in \Re$, $\epsilon > 0$, where $\varsigma \in \left[0, \frac{1}{N}\right)$. Then ξ has a unique fixed point $u \in \Re$ and $\delta(u, u, \epsilon) = 1$, $\forall \epsilon > 0$.

Proof. $u \in \Re$ is a unique fixed point of ξ^n by using Theorem 3.3, and $\delta(u, u, \epsilon) = 1$, $\forall \epsilon > 0$. ξu

is also a fixed point of ξ^n as $\xi^n(\xi u) = \xi u$ and from Theorem 3.3, $\xi u = u$, u is a unique fixed point, since the unique fixed point of ξ is also a unique fixed point of ξ^n.

Example 3.10. Let $\mathfrak{R} = [0,1]$, define $\delta: \mathfrak{R} \times \mathfrak{R} \times [0, +\infty) \to [0,1]$ by

$$\delta(\sigma, \varkappa, \epsilon) = e^{-\frac{\max(\sigma+\varkappa)^2}{\epsilon}},$$

for all $\sigma, \varkappa \in \mathfrak{R}$ and $\epsilon > 0$, with CTN $\kappa * \text{N} = \kappa . \text{N}$. Then it is obvious that $(\mathfrak{R}, \delta, *)$ is a complete FRBMLS. Define $\xi: \mathfrak{R} \to \mathfrak{R}$ by

$$\xi(\sigma) = \begin{cases} 0, & \text{if } \sigma = 1, \\ \dfrac{\sigma}{10}, & \text{otherwise.} \end{cases}$$

Then ξ verifies the contractive form in Theorem 3.3, where $\varsigma \in \left[\frac{1}{2}, 1\right)$, with unique fixed point 0 and $\delta(0,0,\epsilon) = 1$ for all $\epsilon > 0$. Hence, all conditions of Theorem 3.3 are satisfied.

4. Application

An application of Theorem 3.1's integral equation is presented in this section. We show that an integral equation of the type

$$\sigma(j) = g(j) + \int_0^j F\big(j, r, \sigma(r)\big)dr, \tag{4.1}$$

for all $j \in [0, l]$ where $l > 0$, has a solution. Let $C([0, l], \mathbb{R})$ be the space of all continuous functions defined on $[0, l]$ with CTN $\kappa * \text{N} = \kappa . \text{N}$ for all $\kappa, \text{N} \in [0,1]$ and define a complete FRBMLS by

$$\delta(\sigma, \varkappa, \epsilon) = \sup_{j \in [0,l]} \frac{\epsilon}{\epsilon + \big(\sigma(j) + \varkappa(j)\big)^p} \text{ for all } \sigma, \varkappa \in C([0, l], \mathbb{R}), p \geq 1 \text{ and } \epsilon > 0.$$

Theorem 4.1. Let $\xi: C([0, l], \mathbb{R}) \to C([0, l], \mathbb{R})$ be the integral operator given by

$$\xi\big(\sigma(j)\big) = g(j) + \int_0^j F\big(j, r, \sigma(r)\big)dr, \qquad g \in C([0, l], \mathbb{R}),$$

where $F \in C([0, l] \times [0, l] \times \mathbb{R}, \mathbb{R})$ satisfies the following conditions:

There exists $f: [0, l] \times [0, l] \to [0, + + \infty]$ such that for all $r, j \in [0, l]$, $f(j, r) \in L^1([0, l], \mathbb{R})$ and for all $\sigma, \varkappa \in C([0, l], \mathbb{R})$, we have

$$\Big(F\big(j, r, \sigma(r)\big) + F\big(j, r, \varkappa(r)\big)\Big)^p \leq f^p(j, r)\big(\sigma(r) + \varkappa(r)\big)^p$$

and

$$\sup_{j \in [0,l]} \int_0^j f^p(j, r)dr \leq \varsigma < 1.$$

Then the integral equation has the solution $\sigma_* \in C([0, l], \mathbb{R})$.

Proof. For all $\sigma, \varkappa \in C([0, l], \mathbb{R})$, we have

$$\begin{aligned}
&\delta(\xi(\sigma(j)),\xi(\varkappa(j)),\varsigma\epsilon) \\
&= \sup_{j\in[0,l]} \frac{\varsigma\epsilon}{\varsigma\epsilon + \left(\xi(\sigma(j)) + \xi(\varkappa(j))\right)^p} \\
&\geq \sup_{j\in[0,l]} \frac{\varsigma\epsilon}{\varsigma\epsilon + \int_0^j \left(F(j,r,\sigma(r)) + F(j,r,\varkappa(r))\right)^p dr} \\
&\geq \sup_{j\in[0,l]} \frac{\varsigma\epsilon}{\varsigma\epsilon + \int_0^j f^p(j,r)\left(\sigma(r)+\varkappa(r)\right)^p dr} \\
&\geq \frac{\varsigma\epsilon}{\varsigma\epsilon + \left(\sigma(r)+\varkappa(r)\right)^p \sup_{j\in[0,l]} \int_0^j f^p(j,r)dr} \\
&\geq \frac{\varsigma\epsilon}{\varsigma\epsilon + \left(\sigma(r)+\varkappa(r)\right)^p} \geq \frac{\epsilon}{\epsilon + \left(\sigma(r)+\varkappa(r)\right)^p} \\
&= \delta(\sigma,\varkappa,\epsilon).
\end{aligned}$$

Hence, σ_* is a fixed point of ξ, which is the solution of integral equation (4.1).

Remark 4.1. In the above theorem, if we take $p = 1$, then application holds for FRMLS.

5. Conclusions

In this manuscript, we established several fixed point results in new introduced spaces in this manuscript known as fuzzy rectangular metric-like spaces and rectangular b-metric-like spaces. Few non-trivial examples and an application also verify the uniqueness of solution. Fixed point theory receives a lot of attention since it has so many applications in mathematics, science, and economics. Using the ideas presented in the paper, several types of fixed point solutions for single and multi-valued mappings can be established. Intuitionistic fuzzy rectangular metric-like spaces, intuitionistic fuzzy rectangular b-metric-like spaces, Fuzzy controlled rectangular metric-like spaces, and other mathematical structures can be used to further extend the principles provided.

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Conflict of interest

The authors declare that they have no competing interests.

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