Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 8542–8557.
DOI: 10.3934/math.2022476
Received: 09 December 2021
Revised: 06 February 2022
Accepted: 14 February 2022
Published: 28 February 2022

Research article

Geometry of the line space associated to a given dual ruled surface

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Abstract: As a continuation to our results in [1], we study the dual ruled surface defined on the set of dual numbers. The idea of the dual part are defined similar to quaternion space. The dual part of this represents a ruled dual submanifold. The geometric properties of their dual parts are investigated. Some examples are given and plotted.

Keywords: line space; dual space; dual sphere; ruled surfaces
Mathematics Subject Classification: 53A10, 53A25, 53C50

1. Introduction

Dual numbers were introduced in the 19th century by Clifford, and their applications to rigid body kinematics were subsequently generalized by Kafelnikov and study in their principle of transference. The principle of transference, states that if dual numbers replace real ones (see [2,3]); then all relations of vector algebra for intersecting lines are also valid for skew lines. In practice, this means that all rules of vector algebra for the kinematics of a rigid body with a fixed point (spherical kinematics) also hold for motor algebra of a free rigid body (spatial kinematics). As a result, a general rigid body motion can be described by only three dual equations rather than six real ones. For several decades there were attempts to apply dual numbers to rigid body dynamics. Investigators showed that the momentum of a rigid body can be described as a motor that obeys the motor transformation rule, hence, its derivative with respect to time yields the dual force. However, in those investigations, while going from the velocity motor to the momentum motor, there was always a need to expand the equation to six dimensions and to treat the velocity motor as two separated real vectors. This process actually diminishes one of the main advantages of dual numbers-namely, compactness of representation. Screws in the space can be represented by dual vectors at the origin. The components of a dual vector consisting of a line vector at the origin and the perpendicular moment vector for the line vector in the space are equal to Plucker's line coordinates. Furthermore, the space of lines could be represented by points on the unit sphere and points in the tangential planes that affiliated to each point on the sphere (see [1,4]).

1.1. Dual numbers

A dual number $\widehat{x}$ is defined as an ordered pair of real numbers (x, x^*) expressed formally as (see [5]):

$$\widehat{x} = x + \varepsilon x^*, \tag{1.1}$$

where x is referred to the real part and x^* to the dual part. The symbol ε is a multiplier which has the property $\varepsilon^2 = 0$. The algebra of dual numbers results from (1.1). Two dual numbers are equal if and only if their real and dual parts are equal, respectively. As in the case of complex numbers, addition of two dual numbers is defined as (see [4,5])

$$(x + \varepsilon x^*) + (y + \varepsilon y^*) = (x + y) + \varepsilon(x^* + y^*). \tag{1.2}$$

Multiplication of two dual numbers result in (see [6]):

$$(x + \varepsilon x^*)(y + \varepsilon y^*) = xy + \varepsilon(x^* y + xy^*). \tag{1.3}$$

Division of dual numbers $\frac{\hat{x}}{\hat{y}}$ is defined as (see [1]):

$$\frac{\hat{x}}{\hat{y}} = \frac{x}{y} + \varepsilon\left(\frac{x^*}{y} - \frac{xy^*}{y^2}\right), y \neq 0. \tag{1.4}$$

Note that the division is possible and unambiguous only if $y \neq 0$.

1.2. Dual function

Dual function of dual number presents a mapping of a space of dual numbers on itself, namely (see [3,6,7]).

$$\widehat{f(\widehat{x})} = f(x) + \varepsilon f^*(x, x^*) \tag{1.5}$$

where $\widehat{x} = x + \varepsilon x^*$ is a dual variable, f and $f^*$are two, generally different, f^* is a function of two variables. The dual function (1.5) is said to be analytic if it satisfies the following:

$$\widehat{f}(x + \varepsilon x^*) = f + \varepsilon f^* = f(x) + \varepsilon(x^* f'(x) + \widetilde{f}(x)), \quad ' = \tfrac{d}{dx} \tag{1.6}$$

where $\widetilde{f}(x)$ is an arbitrary function of a real part of a dual variable. The analytic condition for dual function is (see [1]):

$$\frac{\partial f^*}{\partial x^*} = \frac{\partial f}{\partial x}. \tag{1.7}$$

The derivative of such a dual function with respect to a dual variable is

$$\frac{d\widehat{f(\widehat{x})}}{d\widehat{x}} = \frac{\partial f^*}{\partial x^*} + \varepsilon \frac{\partial f^*}{\partial x}. \tag{1.8}$$

Taking into account (1.7) we have:

$$\frac{d\widehat{f(x)}}{d\widehat{x}} = \frac{\partial f}{\partial x} + \varepsilon\frac{\partial f^*}{\partial x} = f'(x) + \varepsilon(x^* f''(x) + \widetilde{f}'(x)). \tag{1.9}$$

If a function $f(x)$ has the derivative $f'(x)$, its value for the dual argument $\hat{x} = x + \varepsilon\ x^*$ is denoted by $\hat{f}(\widehat{x})$ or $\hat{f}(x)$. Using the formal Taylor expansion of the function $\hat{f}(x)$ with the property $\varepsilon^2 = 0$; we have (see [6,8]).

$$\hat{f}(x) = f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x). \tag{1.10}$$

1.3. Dual space

The set

$$\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D} = \{\widehat{\underline{x}} : \widehat{\underline{x}} = (x_1 + \varepsilon x_1^*, x_2 + \varepsilon x_2^*, x_3 + \varepsilon x_3^*) \\ = (x_1, x_2, x_3) + \varepsilon(x_1^*, x_2^*, x_3^*) = \underline{x} + \varepsilon\underline{x}^*,\ \underline{x},\ \underline{x}^* \in R^3\},$$

is a module on the ring D and is called the dual space (vector space defined on the field of dual numbers). For any $\widehat{\underline{x}} = \underline{x} + \varepsilon\underline{x}^*, \widehat{\underline{y}} = \underline{y} + \varepsilon\underline{y}^* \in \mathbb{D}^3$, the scalar or inner product and the vector product $< \widehat{\underline{x}}, \widehat{\underline{y}} >$, $\widehat{\underline{x}} \wedge \widehat{\underline{y}}$ of $\widehat{\underline{x}}$ and $\widehat{\underline{y}}$ are defined respectively by (see [9,10]):

$$< \widehat{\underline{x}}, \widehat{\underline{y}} >=< \underline{x}, \underline{y} > +\varepsilon(< \underline{x}, \underline{y}^* > + < \underline{x}^*, \underline{y} >), \tag{1.11}$$

$$\widehat{\underline{x}} \wedge \widehat{\underline{y}} = (\underline{x_2}\ \underline{y_3} - \underline{x_3}\ \underline{y_2}, \underline{x_3}\ \underline{y_1} - \underline{x_1}\ \underline{y_3}, \underline{x_1}\ \underline{y_2} - \underline{x_2}\ \underline{y_1}), \tag{1.12}$$

where $\widehat{x} = (x_i + \varepsilon x_i^*),\ \widehat{y} = (y_i + \varepsilon y_i^*) \in \mathbb{D}^3,\ 1 \leq i \leq 3$.

If $x \neq 0$, the norm$\| \widehat{\underline{x}} \|$ is defined by (see [11,12])

$$\| \widehat{\underline{x}} \| = \sqrt{< \widehat{\underline{x}}, \widehat{\underline{x}} >} = \| \underline{x} \| + \varepsilon\frac{< \underline{x}, \underline{x}^* >}{\| \underline{x} \|}. \tag{1.13}$$

A dual vector $\widehat{\underline{x}} = \underline{x} + \varepsilon\underline{x}^*$ is a dual unit vector which satisfies the following $< \underline{x}, \underline{x}^* >= 0$ and $< \underline{x}, \underline{x} >= 1$. Then, we have that $< \widehat{\underline{x}}, \widehat{\underline{x}} >= 1$, also a dual vector $\widehat{\underline{x}}$ with the unit norm is called a dual unit vector The subset $\mathbb{S}^2 = \{\widehat{\underline{x}} = \underline{x} + \varepsilon\underline{x}^* : \| \widehat{\underline{x}} \| = (1, 0); \underline{x}, \underline{x}^* \in \mathbb{R}^3\} \subset \mathbb{D}^3$ is called the dual unit sphere with the center $\widehat{O}$ in $\mathbb{D}^3$.

1.4. Dual space curve

If every $x_i(u)$ and $x_i^*(u)$, $1 \leq i \leq n$ and $u \in \mathbb{R}^n$, are differentiable dual valued functions, the dual vector field $\widehat{x(u)}$ is defined as the following:

$\widehat{x} : u \subset \mathbb{R}^n \to \mathbb{D}^n$ (see [11]),
$\widehat{x(u)} = x(u) + \varepsilon x^*(u)$.

Dual space curve is a dual vector field of one variable defined as the following:

$$\widehat{x}: I \subset \mathbb{R} \to \mathbb{D}^3,$$

where

$$u \to \widehat{x(u)} = (x_1(u) + \varepsilon x_1^*(u), x_2(u) + \varepsilon x_2^*(u), x_3(u) + \varepsilon x_3^*(u)) \\ = x + \varepsilon x^* \in \mathbb{D}^3,$$

is differentiable. The real part $\underline{x}(u)$ is called the indicatrix of the space curve $\widehat{x}(u)$. The dual arc length of the curve $\widehat{x}(u)$ from u_1 to u is defined as (see [2,12])

$$\int_{u_1}^{u} \| (\underline{\widehat{x}})' \| \, du = \int_{u_1}^{u} \| (\underline{x})' \| \, du + \varepsilon \int_{u_1}^{u} < \underline{T}, (\underline{x}^*(u))' > du = s + \varepsilon s^*$$

where $\underline{T}$ is a unit tangent vector of $\underline{x}(u)$. From now on, we will take the arc length s of $\underline{x}(u)$ as the parameter instead of u.

2. Dual ruled surface

A dual ruled surface results from the motion of line in the dual space $\mathbb{D}^3$, similarly to the way a dual curve represents the motion of a dual point.A dual ruled surface is a surface swept out by a dual straight line L with moving direction $\widehat{w}(u^1)$ along a dual curve $\widehat{\alpha}$. Such a surface always has a parameterization in the ruled form (see [11,13,14])

$$\widehat{\sigma} : \widehat{R}(\widehat{u^1}, \widehat{u^2}) = \widehat{\alpha}(\widehat{u^1}) + \widehat{u^2}\widehat{w}(\widehat{u^1}), \quad \widehat{u^1} \in \widehat{I} \subset \mathbb{D}, \quad \widehat{u^2} \in \mathbb{D} \tag{2.1}$$

where, $\widehat{u}^1 = u^1 + \varepsilon u^{*1}$, $\widehat{u}^2 = u^2 + \varepsilon u^{*2}$, $\widehat{\alpha}(\widehat{u^1}) = \alpha(\widehat{u^1}) + \varepsilon \alpha^*(\widehat{u^1})$, $\widehat{w}(\widehat{u^1}) = w(\widehat{u^1}) + \varepsilon w^*(\widehat{u^1})$.

The dual ruled surface generated by the family $\{\widehat{\alpha}(\widehat{u^1}), \widehat{w}(\widehat{u^1})\}$, where $\widehat{\alpha}(\widehat{u^1})$ is a dual directrix of the dual surface and $\widehat{w}(\widehat{u^1})$ is the unit dual generator.

Using the formally Taylor expansion and the derivative of a dual function , we can write Eq (2.1) in the dual representation vector as (see [2]),

$$\widehat{R}(\widehat{u^1}, \widehat{u^2}) = R(u^1, u^2) + \varepsilon R^*(u^\eta, u^{*\gamma}) \tag{2.2}$$

where

$$\sigma : R(u^1, u^2) = \alpha(u^1) + u^2 w(u^1) \tag{2.3}$$

$$\sigma^* : R^*(u^\eta, u^{*\gamma}) = \alpha^*(u^1) + u^{*1}\alpha'(u^1) + u^{*2}w(u^1) + u^2(w^*(u^1) + u^{*1}w'(u^1)) \tag{2.4}$$

where $' = \frac{d}{du^1}$ and $R^* = R^*(u^1, u^2, u^{*1}, u^{*2})$ is a regular vector function in four variables u^1, u^2, u^{*1}, u^{*1} This function can be written in the ruled form (see [5]),

$$R^*(u^\eta, u^{*\gamma}) = \alpha^*(u^1) + u^{*1}l(u^1) + u^{*2}w(u^1) + u^2 w^*(u^1) \\ l(u^1) = \alpha'(u^1) + w'(u^1). \tag{2.5}$$

From Eqs (2.3)–(2.5) one can see the following:

Theorem 1. (see [5,12]) For any ruled surface defined in the dual space through the vector field

$\widehat{R}(\widehat{u^{\beta}})$,there are a real ruled surface $R(u^1,u^2)$ and 2-parametric family of ruled surfaces given by $R^*(u^{\eta},u^{*\gamma})$.

The 2-parametric family of ruled surface is defined through the function $R^*(u^{\eta},u^{*\gamma})$ and this function depends on 4 independent parameters $u^{\eta},u^{*\gamma}$, thus we have:

Corollary 1. (see [5]) The vector valued function $R^*(u^{\eta},u^{*\gamma})$ characterizes the space of lines (4-dimensional Grassmann manifold) or line space attached to the dual ruled surface.

Assume that the dual ruled surface is a non cylindrical with $|\widehat{w}(\widehat{u^1})| = 1$, $\widehat{u}^1 \in \widehat{I}$. i.e $<\widehat{w}(\widehat{u}^1),\widehat{w}'(\widehat{u}^1)>=0$ for all $\widehat{u}^1 \in \widehat{I}$.

First we construct a parameterized dual curve $\widehat{\beta}(\widehat{u}^1)$ lies on the trace of $\widehat{R}$; such that $<\widehat{\beta}'(\widehat{u}^1),\widehat{w}'(\widehat{u}^1)>=0$, $\widehat{u}^1 \in I$ that is

$$\widehat{\beta}(\widehat{u}^1) = \widehat{\alpha}(\widehat{u}^1) + \widehat{u^2}(\widehat{u}^1)\widehat{w}(\widehat{u}^1). \tag{2.6}$$

Using the formally Taylor expansion and the derivative of a dual function, Eq (2.6) can be written in the dual representation vector as the following (see [2,5]):

$$\widehat{\beta}(\widehat{u}^1) = \beta(u^1) + \varepsilon\beta^*(u^{\kappa},u^{*1}), \tag{2.7}$$

where

$$\begin{gathered}\beta(u^1) = \alpha(u^1) + u^2(u^1)w(u^1),\\ \beta^*(u^{\kappa},u^{*1}) = \alpha^*(u^1) + u^{*1}\alpha'(u^1) + u^2(u^1)(w^*(u^1) + u^{*1}w'(u^1)) + w(u^1)(u^{2^*}(u^1) + u^{*1}u^{2'}(u^1))\end{gathered} \tag{2.8}$$

for some dual valued function $\widehat{u^2} = \widehat{u^2}(\widehat{u}^1)$.

Assuming the existence of such dual curve $\widehat{\beta}$, one obtains

$$\widehat{\beta}'(\widehat{u^1}) = \widehat{\alpha}' + \widehat{u^2}'\widehat{w} + \widehat{u^2}\widehat{w}'. \tag{2.9}$$

Taylor expansion and the derivative of a dual function gives

$$\widehat{\beta}'(\widehat{u}^1) = \beta'(u^1) + \varepsilon\beta'^*(u^1), \tag{2.10}$$

where

$$\begin{gathered}\beta'(u^1) = \alpha' + u^{2'}(u^1)w + u^2(u^1)w',\\ \beta'^*(u^1,u^{*1}) = \alpha'^* + u^{*1}\alpha'' + u^{2'}(u^1)(w^* + u^{*1}w') + w(u^{2'^*}(u^1) + u^{*1}u^{2''})\\ +u^2(u^1)(w'^* + u^{*1}w'') + w'(u^{2^*}(u^1) + u^{*1}u^{2'}).\end{gathered} \tag{2.11}$$

Since $<\widehat{w}(\widehat{u}^1),\ \widehat{w}'(\widehat{u}^1)>=0$, we have

$$\widehat{u^2} = -\frac{<\widehat{\alpha}',\widehat{w}'>}{<\widehat{w}',\widehat{w}'>}. \tag{2.12}$$

Equation (2.12) can be written in the dual representation vector by using Taylor expansion as the following

$$\widehat{u^2} = u^2(u^1) + \varepsilon u^{2^*}(u^1) \tag{2.13}$$

where

$$u^2(u^1) = -\frac{<\alpha'(u^1),w'(u^1)>}{<w'(u^1),w'(u^1)>},$$
$$u^{2^*}(u^1) = -\frac{<\alpha',w'^*+u^{*1}w''>+<w',\alpha'^*+u^{*1}\alpha''>}{<w',w'>} - \frac{<\alpha',w'><w',w^{*\prime}+u^{*1}w''>}{<w',w'>^2}. \tag{2.14}$$

Thus, if we define $\widehat{\beta}(\widehat{u}^1)$ by Eqs (2.4) and (2.12),we obtain the required dual curve, which is called the *dual line of striction* and its points are called the central points of the dual ruled surface. Thus the striction curve on the real ruled surface σ is given by

$$\beta(u^1) = \alpha(u^1) - \frac{< \alpha'(u^1), w'(u^1) >}{< w'(u^1), w'(u^1) >} w(u^1). \tag{2.15}$$

In this case the line space σ^* degenerate to 1-parametric family of ruled surfaces is given by

$$R^* = R(u^\eta, u^{*\gamma}), \qquad u^{*2} = u^{*2}(u^1)$$

such function are defined explicitly by (2.4) and(2.14) respectively.

Now we take the line of striction $\widehat{\beta} = \widehat{\beta}(\widehat{u}^1)$ as the directrix of the dual ruled surface (2.1) which is given by

$$\widehat{R}(\widehat{u}^1, \widehat{u}^2) = \widehat{\beta}(\widehat{u}^1) + \widehat{u}^2 \widehat{w}(\widehat{u}^1), \tag{2.16}$$

where $\widehat{\beta} = \widehat{\beta}(\widehat{u}^1)$ is given through (2.7), (2.8), (2.13) and (2.14).

Thus, we have proved the following:

Lemma 1. (see [11,12]) For the dual line of striction on the dual ruled surfaces. There exists a real line of striction on the real ruled surface attached to the dual ruled surface.

3. Dual invariants

Here, we try to give dual forms for the invariants attached to the dual ruled surface such as, the parameter of distribution and Gaussian curvature. Using the dual differentiation which is presented in section one, one can obtain the dual tangent vector $\widehat{R}_\alpha$ to the dual ruled surface as in the following.

$$\widehat{R}_1 = R_1(u^1, u^2) + \varepsilon R^*{}_1(u^\kappa, u^{*\gamma}), \quad \widehat{R}_2 = R_2 + \varepsilon R^*{}_2, \quad \widehat{R}_\alpha = \frac{\partial \widehat{R}}{\partial \widehat{u}^\alpha} \tag{3.1}$$

where

$$R_1(u^1, u^2) = \beta' + u^2 w', \quad R^*{}_1(u^\kappa, u^{*\gamma}) = (\beta'^* + u^{*1}\beta'' + u^{*2} w^{*\prime} + u^{*1} w''),$$
$$R_2(u^1, u^{*1}) = w, \qquad R^*{}_2(u^1, u^{*1}) = w^* + u^{*1} w', \tag{3.2}$$

and the dual normal vector field is given as

$$\widehat{R}_1 \wedge \widehat{R}_2 = \widehat{\beta}' \wedge \widehat{w} + \widehat{u}^2 (\widehat{w}' \wedge \widehat{w}). \tag{3.3}$$

The dual representation of (2.18) can be written in the form

$$\widehat{R}_1 \wedge \widehat{R}_2 = \zeta(u^1, u^2) + \varepsilon\zeta^*(u^\eta, u^{*\gamma}), \tag{3.4}$$

where

$$\begin{aligned} \zeta &= \beta' \wedge w + u^2 w' \wedge w, \\ \zeta^* &= (\beta' + u^2 w') \wedge (w^* + w') + (u^2(w^{*\prime} + u^{*1}\beta'') + u^{*2}w' + \beta^* + u^{*1}\beta'') \wedge w. \end{aligned} \tag{3.5}$$

Since $< \widehat{w}', \widehat{w} >= 0$ and $< \widehat{w}', \widehat{\beta}' >= 0$, we conclude that for some dual function $\widehat{\lambda} = \widehat{\lambda}(\widehat{u}^1)$, we have

$$\widehat{\beta}' \wedge \widehat{w} = \widehat{\lambda}\widehat{w}', \tag{3.6}$$

$$|\widehat{R}_1 \wedge \widehat{R}_2|^2 = |\widehat{\lambda}\widehat{w}' + \widehat{u}^2\widehat{w}' \wedge \widehat{w}|^2 = \widehat{\lambda}^2|\widehat{w}'|^2 + \widehat{u}^2|\widehat{w}'|^2 = (\widehat{\lambda}^2 + \widehat{u}^2)|\widehat{w}'|^2. \tag{3.7}$$

The discriminant $\widehat{g}$ of the first fundamental form can be written as

$$\begin{aligned} \widehat{g} &= |\widehat{R}_1 \wedge \widehat{R}_2|^2 = (\widehat{\lambda}^2 + \widehat{u}^2)|\widehat{w}'|^2 \\ &= (\lambda^2 + u^2)|w'|^2 + 2\varepsilon((\lambda^2 + u^2)w'w^{*\prime} + w'^2(\lambda\lambda^* + u^2u^{*2})) \\ &= g + \varepsilon g^*, \end{aligned} \tag{3.8}$$

where

$$g = (\lambda^2 + u^2)|w'|^2, \quad g^* = 2((\lambda^2 + u^2)w'w^{*\prime} + w'^2(\lambda\lambda^* + u^2u^{*2})). \tag{3.9}$$

It follows that the only singular point on the ruled surface (2.15) is along the line of striction $\widehat{u}^2 = 0$, and it will occur if and only if $\widehat{\lambda}(\widehat{u}^1) = 0$. Then from (2.21) one can see that

$$\widehat{\lambda} = \frac{(\widehat{\beta}', \widehat{w}, \widehat{w}')}{|\widehat{w}'|^2}. \tag{3.10}$$

Remark 1. $\widehat{\lambda} = 0 \Rightarrow \lambda = \lambda^* = 0, \widehat{u}^2 = 0 \Rightarrow u^2 = 0, u^{*2} = 0$.

Equation (2.23) can be written in the dual vector representation as

$$\widehat{\lambda}(\widehat{u}^1) = \lambda(u^1) + \varepsilon\lambda^*(u^1, u^{*1}), \tag{3.11}$$

where

$$\begin{aligned} \lambda(u^1) &= \tfrac{(\beta', w, w')}{|w'|^2}, \\ \lambda^*(u^1, u^{*1}) &= \tfrac{2(\beta', w, w') < w', w^{*\prime} + u^{*1}w'' >}{|w'|^4} + \tfrac{(\beta', w'^* + u^{*1}w'', w' + w) + (\beta'^* + u^{*1}\beta'', w, w')}{|w'|^2}. \end{aligned} \tag{3.12}$$

Definition. The dual function $\widehat{\lambda}(\widehat{u}^1)$ is called the dual distribution parameter of the dual ruled surface $\widehat{R}$.

The dual unit normal vector field on the dual surface (2.15) is

$$\widehat{N} = \frac{\widehat{R}_1 \wedge \widehat{R}_2}{|\widehat{R}_1 \wedge \widehat{R}_2|} = \frac{\widehat{R}_1 \wedge \widehat{R}_2}{\sqrt{\widehat{g}}}, \tag{3.13}$$

then from Eqs (3.4) and (3.8) we have

$$\begin{aligned}\widehat{N} &= \tfrac{\zeta(u^1,u^2)+\varepsilon\zeta^*(u^\eta,u^{*\gamma})}{\sqrt{g+\varepsilon g^*}}\\ &= \tfrac{\zeta}{g^2} + \varepsilon\,\tfrac{\zeta^* g^2 - g^*\zeta}{g^3} = N + \varepsilon N^*.\end{aligned} \tag{3.14}$$

Thus we have the following interesting results

Lemma 2. The unit dual normal vector field of two parts, one is the real unit normal vector field and the other is the vector field depends on the line space attached to the dual ruled surface.

The same lemma can be reformulated for the dual parameter of distribution and the line of striction as shown in the Eqs (2.4), (2.16), (3.11), respectively.

The coefficients of the first fundamental form are given in the following form

$$\begin{aligned}\widehat{g}_{11} &=< \widehat{\beta}',\widehat{\beta}' > +(u^2)^2 < \widehat{w}',\widehat{w}' >= (< \beta',\beta' > +(u^2)^2 < w', w' >)\\ &+\varepsilon(2 < \beta',\beta^{*\prime} > +2(u^2)^2 < \widehat{w}',\widehat{w^*}' >) = g_{11} + \varepsilon {g^*}_{11},\end{aligned} \tag{3.15}$$

$$\widehat{g}_{12} =< \widehat{\beta}',\widehat{w} >=< \beta', w > +\varepsilon(< \beta', w^* > + < \beta^{*\prime}, w >) = g_{12} + \varepsilon {g^*}_{12},\ \widehat{g}_{22} = 1.$$

Using the definition of the Gauss curvature $\widehat{K}$ and routine calculations one obtains

$$\widehat{K} = -\frac{\widehat{\lambda}^2|\widehat{w}'|^4}{(\widehat{\lambda}^2 + (\widehat{u}^2)^2)^2|\widehat{w}'|^4} = -\frac{\widehat{\lambda}^2}{(\widehat{\lambda}^2 + (\widehat{u}^2)^2)^2}. \tag{3.16}$$

From (2.24) we have

$$\widehat{K}(\widehat{u}^1,\widehat{u}^2) = K(u^1,u^2) + \varepsilon K^*(u^\kappa, u^{*\gamma}), \tag{3.17}$$

where

$$K(u^1,u^2) = -\tfrac{\lambda^2}{\overline{\lambda}^2},\quad \overline{\lambda} = \lambda^2 + (u^2)^2,\quad K^*(u^\kappa,u^{*\gamma}) = \tfrac{(2\lambda\lambda^* + 2u^{*1}\lambda\lambda')((u^2)^2-\lambda^2)+2\lambda^2u^2u^{*2}}{\overline{\lambda}^3}. \tag{3.18}$$

Remark 2. The function K is the Gauss curvature of the real ruled surface σ.

4. Special cases

The vector valued function (2.5) defines a 2-parametric family of ruled surfaces σ^*. Here we give a classification to the ruled surfaces belonging to this family, which are called partially dual ruled surfaces.

(I) Consider $u^{*1} = 0$ and $\widehat{u}^2 = u^2 + \varepsilon u^{*2}$ and using Eqs (2.3)–(2.5), we have a partially dual ruled surface σ_I given by

$$\widehat{\sigma}_I : \widehat{R}(u^1,\widehat{u}^2) = R(u^1,u^2) + \varepsilon R^*(u^1,u^2,u^{*2}), \tag{4.1}$$

where

$$\sigma_I : R(u^1,u^2) = \alpha(u^1) + u^2 w(u^1), \tag{4.2}$$

$$\sigma^*{}_I : R^*(u^\kappa, u^{*2}) = \alpha^*(u^1) + u^{*2}w(u^1) + u^2 w^*(u^1). \tag{4.3}$$

It is easy to see that the function R^* defines 3-parametric family of lines $\sigma^*{}_{I_i}, i = 1, 2, 3$. Now, we take the line of striction as the directrix of the dual ruled surface and using (2.16), (3.1)–(3.10) we have the following:

Lemma 3. The dual parameter of distribution $\widehat{\lambda_1}$ is given as

$$\widehat{\lambda_1}(\widehat{u}^1) = \lambda_1(u^1) + \varepsilon\lambda_1{}^*(u^1), \tag{4.4}$$

where

$$\lambda_1(u^1) = \frac{(\beta', w, w')}{|w'|^2}, \quad \lambda_1{}^*(u^1) = \frac{2(\beta', w, w') < w', w^{*\prime} >}{|w'|^4} + \frac{(\beta', w'^*, w' + w) + (\beta'^*, w, w')}{|w'|^2}. \tag{4.5}$$

Lemma 4. The dual Gaussian curvature $\widehat{K_2}$ is given as

$$\widehat{K_1}(\widehat{u}^1, \widehat{u}^2) = K_1(u^1, u^2) + \varepsilon K_1{}^*(u^\eta, u^{*2}), \tag{4.6}$$

$$K_1(u^1, u^2) = -\frac{\lambda_1{}^2}{\overline{\lambda_1}}, \quad \overline{\lambda} = \lambda_1{}^2 + (u^2)^2, \quad K_1{}^*(u^\eta, u^{*2}) = \frac{2\lambda_1\lambda_1{}^*((u^2)^2 - \lambda_1{}^2) + 2\lambda_1{}^2 u^2 u^{*2}}{\overline{\lambda_1}^3}. \tag{4.7}$$

Remark 3. The functions λ_1, K_1 is defined for the real ruled surface σ_I.

Geometric interpretation

For the vector valued function $R^* = R^*(u^1, u^2, u^{*2})$ given by Eq (3.3), we consider the following cases:

(1) $u^2 = const = c_1$, characterizes a dual ruled surface $\sigma^*{}_{I_1}$ defined as

$$\sigma^*{}_{I_1} : \widehat{R}(u^1, u^{*2}) = \widetilde{r}(u^1) + u^{*2}\overline{r}(u^1), \tag{4.8}$$

where

$$\widetilde{r}(u^1) = \alpha^*(u^1) + c_1 w^*(u^1), \quad \overline{r}(u^1) = w(u^1). \tag{4.9}$$

(2) $u^{*2} = const = c_2$, define a dual ruled surface $\sigma^*{}_{I_2}$ as

$$\widehat{R}(u^1, u^2) = \widetilde{r}(u^1) + u^2\overline{r}(u^1), \tag{4.10}$$

where

$$\widetilde{r}(u^1) = \alpha^*(u^1) + c_2 w(u^1), \quad \overline{r}(u^1) = w^*(u^1). \tag{4.11}$$

(3) $u^1 = const = c_1$, describes a degenerate ruled surface $\sigma^*{}_{I_3}$ (plane) passing through the point $\alpha(u^1)$ and contains the fixed directions $w^*(u^1), w$.

(II) Consider $u^{*2} = 0$ and $\widehat{u}^1 = u^1 + \varepsilon u^{*1}$ and using Eqs (2.3)–(2.5) we have a partially dual ruled surface σ_{II} given as

$$\widehat{\sigma}_{II} : \widehat{R}(\widehat{u^1}, u^2) = R(u^1, u^2) + \varepsilon R^*(u^\eta, u^{*1}), \tag{4.12}$$

where

$$\sigma_{\text{II}} : R(u^1, u^2) = \alpha(u^1) + u^2 w(u^1), \tag{4.13}$$

$$\sigma^*{}_{\text{II}} : R^*(u^\eta, u^{*1}) = \alpha^*(u^1) + u^{*1}\alpha'(u^1) + u^2(w^*(u^1) + u^{*1}w'(u^1)). \tag{4.14}$$

Similarly as in case one, it is easy to see that the function R^* defines a 3-parametric family of lines $\sigma^*{}_{\text{II}_i}, i = 1, 2, 3$ as in case I. Thus we have the following:
Lemma 5. The dual parameter of distribution $\widehat{\lambda_2}$ is given as

$$\widehat{\lambda_2}(\widehat{u}^1) = \lambda_2(u^1) + \varepsilon\lambda_2{}^*(u^1, u^{*1}), \tag{4.15}$$

where

$$\begin{aligned} \lambda_2(u^1) &= \tfrac{(\beta', w, w')}{|w'|^2}, \\ \lambda_2{}^*(u^1, u^{*1}) &= \tfrac{2(\beta', w, w') < w', w^{*\prime} + u^{*1} w'' >}{|w'|^4} + \tfrac{(\beta', w'^{*} + u^{*1} w'', w' + w) + (\beta'^{*} + u^{*1}\beta'', w, w')}{|w'|^2}. \end{aligned} \tag{4.16}$$

Lemma 6. The dual Gaussian curvature $\widehat{K_2}$ is given as

$$\widehat{K_2}(\widehat{u}^1, u^2) = K_2(u^1, u^2) + \varepsilon K_2^*(u^\eta, u^{*1}), \tag{4.17}$$

where

$$K_2(u^1, u^2) = -\tfrac{\lambda_2{}^2}{\overline{\lambda_2}^2}, \quad \overline{\lambda_2} = \lambda_2{}^2 + (u^2)^2, \; K_2{}^*(u^\eta, u^{*1}) = \tfrac{2\lambda_2\lambda_2{}^* + 2u^{*1}\lambda_2\lambda_2{}'((u^2)^2 - \lambda_2{}^2)}{\overline{\lambda_2}^3}. \tag{4.18}$$

Remark 4. The vector function R represents the real part of the dual ruled surface which coincident with the well known construction of the ruled surface, while the function $R^*(u^\gamma, u^{*\gamma})$ represents the dual part of the dual ruled surface.

Remark 5. The dual part $R^*(u^\eta, u^{*\gamma})$ represents 2-parametric family of ruled surfaces (line space) attached to a given a dual ruled surface

Remark 6. From (3.30), (3.32) one can see that the real parts of the parameter of distribution and the Gauss curvature have the same values for the partially ruled surface, $K_1 = K_2, \lambda_1 = \lambda_2, \overline{\lambda_1} = \overline{\lambda_2}$ (defined on areal ruled surface).

Remark 7. The two partially ruled surfaces $\widehat{\sigma}_I, \widehat{\sigma}_{\text{II}}$ have the same geometric interpretation, i.e., consists of two ruled surface and a plane.

5. Application

As an application to the construction of the dual ruled surfaces we consider the dual helicoid given by
Example 1. The dual vector function:

$$\widehat{\sum}_1 : \widehat{R}(\widehat{u}^1, \widehat{u}^2) = (\widehat{u}^2 \cos\widehat{u}^1, \widehat{u}^2 \sin\widehat{u}^1, \widehat{u}^1), \quad \widehat{u}^2 \neq 0. \tag{5.1}$$

This vector function can be written in the dual form as

$$\sigma_I : \widehat{R}(\widehat{u}^1, \widehat{u}^2) = R(u^1, u^2) + \varepsilon R^*(u^\eta, u^{*\gamma}), \tag{5.2}$$

where

$$\sigma_I^* : R(u^1, u^2) = (u^2 \cos u^1, u^2 \sin u^1, u^1), \tag{5.3}$$

$$R^*(u^\eta, u^{*\gamma}) = (u^{*2} \cos u^1 - u^2 u^{*1} \sin u^1, u^{*2} \sin u^1 + u^2 u^{*1} \cos u^1, u^{*1}). \tag{5.4}$$

Also, we can write this equation in the dual ruled surface formula as in the following

$$\widehat{R}(\widehat{u}^1, \widehat{u}^2) = \widehat{\beta}(\widehat{u}^1) + \widehat{u}^2 \widehat{w}(\widehat{u}^1), \tag{5.5}$$

where

$$\widehat{\beta}(\widehat{u}^1) = (0, 0, \widehat{u}^1), \quad \widehat{w}(\widehat{u}^1) = (\cos \widehat{u}^1, \sin \widehat{u}^1, 0). \tag{5.6}$$

Using Eqs (2.25) and (2.29), we have

$$\lambda_I = 1, \quad \lambda_I^* = 1. \tag{5.7}$$

From (5.7), (5.8) we have the well known results for the parameter of distibution, and Gaussian curvature for the helicoid given by

$$K_I = \frac{1}{(1+(u^2)^2)^2}, \quad K^*{}_I = 2\frac{2(u^2)^2 + u^2 u^{*2} - 1}{(1+(u^2)^2)^3}. \tag{5.8}$$

Example 2. Consider the dual ruled surface:

$$\widehat{\Sigma}_{\text{II}} : \widehat{R}(\widehat{u}^1, \widehat{u}^2) = (\widehat{u}^2 \cos \widehat{u}^1, \widehat{u}^2 \sin \widehat{u}^1, \cosh \widehat{u}^1), \quad \widehat{u}^2 \neq 0, \tag{5.9}$$

or in the dual form:

$$\widehat{R}(\widehat{u}^1, \widehat{u}^2) = R(u^1, u^2) + \varepsilon R^*(u^\eta, u^{*\gamma}), \tag{5.10}$$

where

$$\sigma_{\text{II}} : R(u^1, u^2) = (u^2 \cos u^1, u^2 \sin u^1, \cosh u^1), \tag{5.11}$$

and

$$\sigma^*{}_{\text{II}} : R^*(u^\eta, u^{*\gamma}) = (u^{*2} \cos u^1 - u^2 u^{*1} \sin u^1, u^{*2} \sin u^1 + u^2 u^{*1} \cos u^1, u^{*1} \sinh u^1). \tag{5.12}$$

As a similar way to example one, we have

$$\lambda_{\text{II}} = \sinh u^1, \quad \lambda^*{}_{\text{II}} = u^{*1} \cosh u^1, \tag{5.13}$$

$$K_{\text{II}} = \frac{\sinh u^1}{(\sinh u^1 + (u^2)^2)^2}, \quad K^*{}_{\text{II}} = \frac{2u^{*1} \sinh u^1((u^2)^2 - \sinh u^1) + 2u^{*2} u^2 \sinh^2}{(\sinh u^1 + (u^2)^2)^2}. \tag{5.14}$$

6. Conclusions

The vector valued function $R^*(u^\kappa, u^{*\gamma})$, which is defined in Eq (3.4) depends on 4 independent parametric $u^\kappa, u^{*\gamma}$, thus it characterizes the space of lines (4-dimensional Grassmann manifold). The space of lines and their subfamilies of ruled surface are constructed and plotted in Figures 1–11, for the two dual ruled surfaces (4.1) and (4.9) respectively.

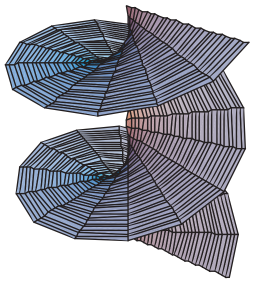

Figure 1. Partially dual (pure real) σ_I.

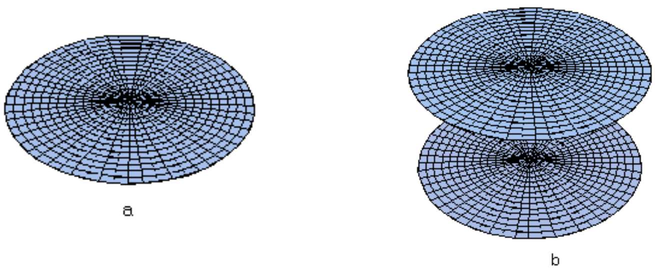

Figure 2. Partially dual $\sigma^*{}_{I_1}$ (a) $u^{*1} = 1$, (b) $u^{*1} = 0$.

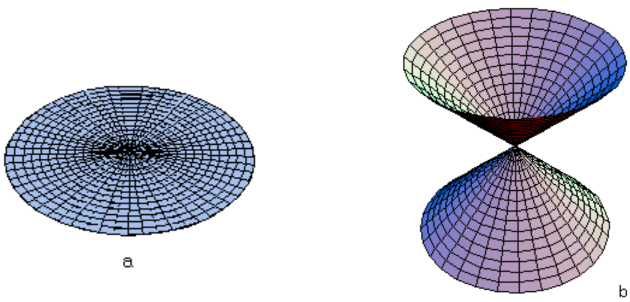

Figure 3. Partially dual $\sigma^*{}_{I_2}$ (a) $u^2 = const$, (b) $u^{*1} = const$.

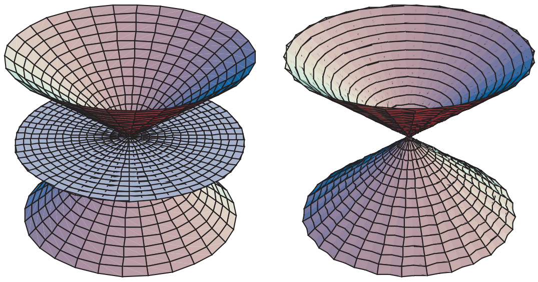

Figure 4. Partially dual ${\sigma^*}_{I_3}$ (a) $u^2 = const$, (b) $u^2 = 0$, $(u^{*1}, u^{*2}) \neq (0, 0)$.

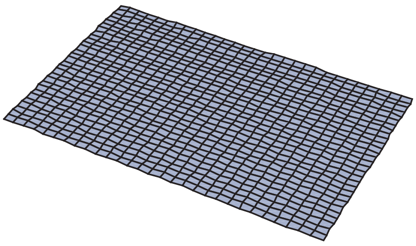

Figure 5. Partially dual $u^2 = const$.

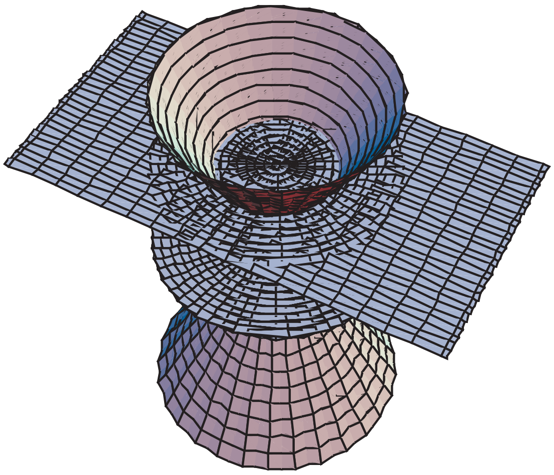

Figure 6. Line space ${\sigma^*}_I$ for dual helicoid.

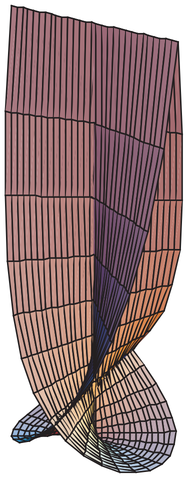

Figure 7. Partially dual (pure real) σ_{II}.

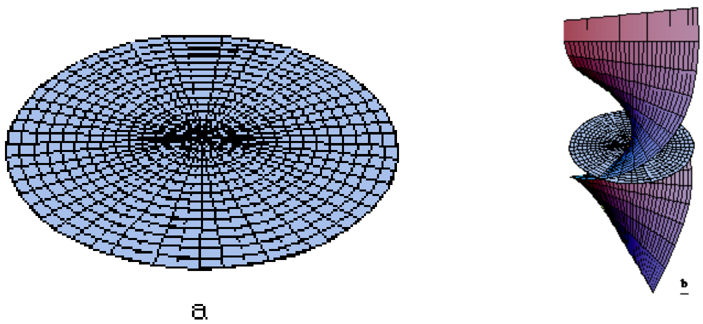

Figure 8. Partially dual $\sigma^{*}{}_{\text{II}_1}$ (a) $u^{*1} = 1$, (b) $u^{*1} = 0$.

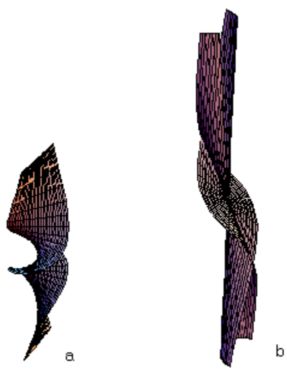

Figure 9. Partially dual $\sigma^{*}{}_{\text{II}_2}$ (a) $u^2 = const$, (b) $u^{*1} = const$.

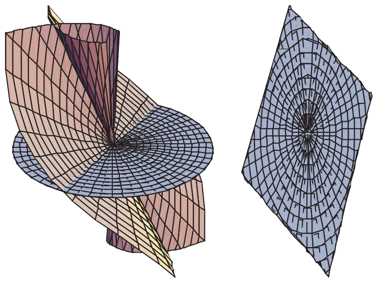

Figure 10. Partially dual ${\sigma^*}_{\Pi_3}$ (a) $u^2 = const$, (b) $u^2 = 0$, $(u^{*1}, u^{*2}) \neq (0, 0)$.

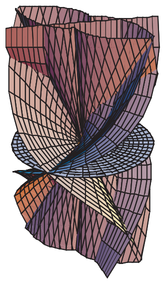

Figure 11. Line space ${\sigma^*}_{\Pi}$ for the dual ruled surface $\widehat{\Sigma}_{\Pi}$.

Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. N. H. Abdel-All, M. Soliman, R. A. Huesien, A. A. Ali, Dual construction of developable ruled surface, *J. Am. Sci.*, **7** (2011), 789–793. https://doi.org/10.7537/marsjas070411.109
2. L. M. Hsia, A. T. Yang, On the principle of transference in three-dimensional kinematics, *J. Mech. Des.*, **103** (1981), 652–656. https://doi.org/10.1115/1.3254966
3. J. M. Selig, Note on the principle of transference, *Am. Soc. Mech. Eng.*, 1986.
4. V. Brodsky, M. Shoham, Dual numbers representation of rigid body dynamics, *Mech. Mach. Theory*, **34** (2012), 693–718. https://doi.org/10.1016/S0094-114X(98)00049-4
5. R. Ding, Y. Zhang, Dual space drawing methods for ruled surfaces with particular shapes, *Int. J. Comput. Sci. Net.*, **6** (2006), 1–12.
6. M. K. Karacan, B. Bukcu, N. Yuksel, On the dual Bishop Darboux rotation axis of the dual space curve, *APPS. Appl. Sci.*, **10** (2008), 115–120.
7. A. Yücesan, A. C. Cöken, N. Ayyildiz, On the dual Darboux rotation axis of the timelike dual space curve, *Balk. J. Geom. Appl.*, **7** (2002), 137–142.
8. A. Yücesan, N. Ayyildiz, A. C. Cöken, On rectifying dual space curves, *Rev. Mat. Complut.*, **20** (2007), 497–506.

9. H. Pottmanna, M. Peternella, B. Ravanib, An introduction to line geometry with applications, *Comput.-Aided Design.*, **31** (1999), 3–16. https://doi.org/10.1016/S0010-4485(98)00076-1
10. J. Mahovsky, B. Wyvill, Fast ray-axis aligned bounding box overlap tests with Plucker coordinates, *J. Graphics Tools*, **9** (2004), 35–46. https://doi.org/10.1080/10867651.2004.10487597
11. Y. Li, Y. Zhu, Q. Y. Sun, Singularities and dualities of pedal curves in pseudo-hyperbolic and de Sitter space, *Int. J. Geom. Methods Mod. Phys.*, **18** (2021), 1–31. https://doi.org/10.1142/S0219887821500080
12. G. R. Veldkamp, On the use of dual numbers, vectors and matrices in instantaneous spatial kinematics, *Mech. Mach. Theory*, **11** (1976), 141–156. https://doi.org/10.1016/0094-114X(76)90006-9
13. F. Messelmi, Analysis of dual functions, *Ann. Rev. Chaos Theory, Bifurcations Dyn. Syst.*, **4** (2013), 37–54. https://doi.org/10.13140/2.1.1006.4006
14. Y. Li, Z. Wang, T. Zhao, Geometric algebra of singular ruled surfaces, *Adv. Appl. Clifford Algebras*, **31** (2021), 19. https://doi.org/10.1007/s00006-020-01097-1