Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 8491–8509.
DOI: 10.3934/math.2022473
Received: 25 October 2021
Revised: 26 January 2022
Accepted: 07 February 2022
Published: 28 February 2022

Research article

Existence results of nontrivial solutions for a new $p(x)$-biharmonic problem with weight function

Wei Guo, Jinfu Yang and Jiafeng Zhang*

School of Data Science and Information Engineering, Guizhou Minzu University, Guiyang 550025, China

* **Correspondence:** Email: jiafengzhang@163.com.

Abstract: In this paper, we study a class of $p(x)$-biharmonic problems with negative nonlocal terms and weight function. Using the mountain pass theorem and the Ekeland's variational principle, at least three solutions are obtained. We also give some comments on the existence of infinite many solutions for our problem when the nonlinear term is a general function.

Keywords: nontrivial solutions; $p(x)$-biharmonic problem; mountain pass theorem; Ekeland's variational principle; variable exponent
Mathematics Subject Classification: 35D05, 47J20, 35J50

1. Introduction

In this paper, we consider the following $p(x)$-biharmonic problem with weight function

$$\begin{cases} \left(a - b \displaystyle\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\right) \Delta^2_{p(x)} u = \lambda V(x)|u|^{q(x)-2}u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a \geq b > 0$ are constants, $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, p, $q \in C(\overline{\Omega})$ with $1 < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x) < N$, $\lambda > 0$ is a real parameter, and $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is called the $p(x)$-biharmonic operator of fourth order.

Recently, the problems of variable exponent growth have been extensively studied, we can refer to [1, 5, 7, 10, 11, 14, 16, 25]. Literature [41] and [36] are the two closest in time, in literature [41], the authors studied the regularity for minimizers for functionals of double phase with variable exponents, the main purpose of this paper is to provide a regularity theorem for minimizers of a class of integral functionals of the calculus of variations called of double phase type with variable exponents, and authors also proved the existence of a local minimum value of the problem. It is worth mentioning that the existence and multiplicity results of solutions for many problems with variable exponents can be found in [36]. In particular, variable exponent space plays an important role in the studies of electrorheological fluids [42, 43], thermotropic fluids [8], and image processing [2, 12]. The major difference with the constant exponent is that the $p(x)$-biharmonic operator has inhomogeneity, it brings a lot of difficulties, such as many classical theorems, like the Lagrange multiplier theorem, can't be used. The study of variable exponent problems can be traced back to the article published by Orlic in 1931 [40]. However, after this article, the authors did not go further. During this period, many authors carried out a series of research work [38, 39], such as module space [37], etc., and obtained many important properties. Until the 1990s, a lot of work on variable exponent space was carried out on the basis of an article published by two authors, O. Kovacik and J. Rakosnik in 1991, in which the basic theory of variable exponent Lebesgue space and Sobolev space [27] is established. Later, authors such as Fan and Zhao [21] further promoted it. Up to now, more and more people have studied it. In addition, Kirchhoff differential equation, as a typical partial differential equation, has been extensively studied, and its source can be traced back to 1883 [30]. Kirchhoff type differential equation, a special partial differential equation, is a mathematical model proposed by Kirchhoff in 1883 when he studied the free vibration of elastic string. Such model was used to study the variation rule of transverse vibration of telescopic steel wire rope. It is well known that Kirchhoff have created the following model and generalized the d 'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

where ρ, ρ_0, h, E, L are constants, the problem (1.1) is related to the stationary problem of Kirchhoff's model. Its theoretical and practical applications are very broad, such as Non-Newtonian mechanics, medicine, economics, ecology and other fields are involved. The original Kirchhoff's model was utilized to study the problems in one-dimensional case. In 1955, Berger [9] studied the following von Karman plate equation in two-dimensional case

$$\frac{\partial^2 u}{\partial t^2}+\Delta^2 u+\left(Q+\int_\Omega|\nabla u|^2dx\right)\Delta u=f(u,u_t,x),$$

where Δ^2 is called the biharmonic operator. It also plays an important role in engineering, physics and material mechanics. Therefore, in recent decades, researchers have also done a lot of researches on the existence and properties of solutions of biharmonic equations, for example, in [4], the authors used Nehari manifold method and fiber mapping to obtain the existence of two solutions. Furthermore, many authors have extended the p-biharmonic problems to the $p(x)$-biharmonic problems, we refer to [3, 6, 17, 24, 28, 29, 31, 35].

In [17], the author considered the following problem

$$\begin{cases} M\Big(\int_\Omega\frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big)\Delta(|\Delta u|^{p(x)-2}\Delta u)=f(x,u), & \text{in } \Omega,\\ u=\Delta u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega\in\mathbb{R}(N\ge 2)$ is a bounded domain with a smooth boundary $\partial\Omega$, p is a continuous function on $\overline{\Omega}$ such that $1<p^-:=\inf_{x\in\overline{\Omega}}p(x)\le p^+:=\sup_{x\in\overline{\Omega}}p(x)<+\infty$, the continuous function $M:\mathbb{R}^+\to\mathbb{R}^+$

and the Carathédory function $f : \Omega \times \mathbb{R}$ satisfy some conditions. They established the existence and multiplicity of solutions for problem (1.3) by using variational method and the theory of the variable exponent Sobolev spaces.

Replacing $\left(a - b \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\Delta^2_{p(x)}u$ with $(\Delta(|x|^{p(x)}|\Delta u|^{p(x)-2}\Delta u)$ on the left-hand side of problem (1.1), in [35], existence results for the following perturbed weighted $p(x)$-biharmonic problem with Navier boundary conditions were considered. The author proved the problem has an eigenvalue when there exists $\lambda^* > 0$ such that $\lambda \in (0, \lambda^*)$ and some other hypotheses are satisfied. The other main result of this problem is at least two non-trivial non-negative weak solutions were obtained provided $|V|_{L^\infty(\Omega)} < M$.

In [29], the author considered a class of $p(x)$-biharmonic operators with weights

$$\begin{cases} (\Delta(|\Delta u|^{p_1(x)-2}\Delta u + \Delta(|\Delta u|^{p_2(x)-2}\Delta u) \\ \qquad\qquad = \lambda V_1(x)|u|^{q(x)-2}u - \mu V_2(x)|u|^{\alpha(x)-2}u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where λ, μ are positive real numbers, p_1, p_2, q and α are continuous functions on $\overline{\Omega}$, V_1 and V_2 are weight functions in generalized Lebesgue spaces $L^{s_1(x)}(\Omega)$ and $L^{s_2(x)}(\Omega)$ such that V_1 may change sign in Ω and $V_2 > 0$ on Ω, respectively. The author established the existence results by using variational approaches and Ekeland's variational principle.

In 2020, a class of new problems with nonlocal term were considered in [26]

$$\begin{cases} -\left(a - b \displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) div\left(|\nabla u|^{p(x)-2}\nabla u\right) \\ \qquad\qquad = \lambda|u|^{p(x)-2}u + g(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where the nonlocal term is $a - b \int_\Omega \frac{1}{p(x)}|\nabla u|^2 dx$. The key point in the main results of this literature is to show that the energy functional J has a Mountain Pass energy c. For this type of problems, they have been extensively studied as constant exponent case when $p(x) = 2$, we can refer to [13, 32, 33]. It is worth mentioning that concerning the Kirchhoff type problem, the authors studied the $a + b$ type problem in literature [44, 45]. Precisely, Tang and Cheng in [44] proposed a new approach to recover the compactness for the Palais-Smale sequences; Moreover, Tang and Chen in [45] proposed a new approach to recover the compactness for the minimizing sequences.

Inspired by the above literature [17, 26, 29, 35], in this paper, we study the existence and multiplicity of solutions for a class of $p(x)$-biharmonic problems with weight function and negative nonlocal term.

To prove our first main result, as shown below, we assume the following assumptions:

(M_0) There exists a function $s \in C(\overline{\Omega})$ for all $x \in \overline{\Omega}$, then $1 < q(x) < \alpha(x) < p(x) \le \frac{N}{2} < s(x)$ hold.

(M) $V \in L^{s(x)}(\Omega)$, and $V > 0$ in $\Omega_0 \subset\subset \Omega$, with $|\Omega_0| > 0$.

(M_1) Assume that $1 < \alpha < \min\{\frac{n}{p^+}, \frac{np^-}{p^+(n-p^-)}\}$ and $V : \Omega \to [0, \infty)$ belongs to $L^\infty(\Omega)$, denote

$$p^- := \min_{x\in\overline{\Omega}} p(x), \quad \text{and} \quad p^+ = \max_{x\in\overline{\Omega}} p(x).$$

(M_2) For some $x_0 \in \Omega$ and $0 < r < R < \infty$ with $\overline{B_R(x_0)} \subset \Omega$ we have

$$V \text{ vanish in } \overline{B_R(x_0) \setminus B_r(x_0)} \text{ and } V(x) > 0 \text{ for } x \in \Omega \setminus \overline{B_R(x_0) \setminus B_r(x_0)}.$$

(M_3) $q \in C_+(\overline{\Omega})$ is such that $1 < q(x) < p^*(x)\ \ \forall x \in \overline{\Omega}$ and

$$\max_{x \in \overline{B_r(x_0)}} q(x) < p^-\alpha \leq p^+\alpha < \min_{x \in \overline{\Omega \setminus B_R(x_0)}} q(x).$$

The theorem to be proved is as follows:

Theorem 1.1. *Assume that $a \geq b > 0$, p, $q \in C(\overline{\Omega})$, (M) and (M_1)–(M_3) are satisfied and*

$$1 < q^- < q(x) < p^- < p(x) < p^+ < \sqrt{2}p^- < N. \tag{1.4}$$

Then there exist constants λ_*, $\zeta > 0$, such that for any $\lambda \in (0, \lambda_*)$, problem (1.1) has at least three non-trivial solutions provided $|V|_{L^\infty(\Omega)} < \zeta$.

Remark 1.2. Compared with the most of existing work involving $p(x)$-biharmonic operaters, the main difference between them and our problem is that the equation in the present paper contains negative nonlocal term and weight function. In addition, different from other papers [3, 24, 28, 31, 35] dealing with $p(x)$-biharmonic operators, the present paper gives the existence of three nontrivial solutions.

Theorem 1.1 cannot be completed without the proof of the $(PS)_c$ condition, and since problem (1.1) contains a negative nonlocal term, the general method cannot be adopted. Therefore, to complete the proof and solve this difficulty, we use the method in [35]. In addition, a class of problems (4.1), with nonlinear terms and negative nonlocal terms are considered, which is a supplement to problem (1.1).

In Section 2, we introduce the preliminary knowledge that will be involved later, and prove the $(PS)_c$ condition. In Section 3, Theorem 1.1 is proved by proving some lemmas. In Section 4, we supplement the problem (1.1), and an infinite number of solutions are obtained by using the symmetric mountain pass theorem.

2. Preliminaries and functional framework

In order to investigate Eq (1.1) we first review some conclusions about the generalized Lebesgue-Sobolev spaces $W^{1,p(x)}(\Omega)$ and $L^{1,p(x)}(\Omega)$. Let Ω be a bounded domain of $\mathbb{R}^N$, denote $C_+(\overline{\Omega}) = \{p(x) : p(x) \in C(\overline{\Omega}), p(x) > 1,$ for all $x \in \overline{\Omega}\}$ and $p^- = \inf_\Omega p(x) \leq p(x) \leq p^+ = \sup_\Omega p(x) < N$.

For any $p \in C_+(\overline{\Omega})$, we introduce the Lebesgue space of variable exponent endowed with the Luxembourg norm

$$L^{p(\cdot)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\right\}.$$

The Luxembourg norm is as follows

$$\|u\|_{L^{p(x)}(\Omega)} = |u|_{p(\cdot)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\},$$

which is a separable and reflexive Banach space. We can refer to [21] for more relevant knowledge.

No doubt, when $p(x) \equiv p$, the space $L^{p(x)}(\Omega)$ is reduced to the classical Lebesgue space $L^p(\Omega)$ and the norm $|u|_{p(x)}$ reduces to the standard norm in $L^p(\Omega)$

$$\|u\|_{L^p} = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}.$$

For any positive integer k and a given multi-index $\alpha = (\alpha_1, \alpha_1, ..., \alpha_n)$, we define the generalized Sobolev space as

$$W^{k,p(x)}(\Omega) = \{u \in L^p(x)(\Omega) : D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \le k\},$$

where $|\alpha| = \Sigma_{i=1}^n \alpha_i$ and $D^\alpha u = \partial^{|\alpha|} u / \partial^{\alpha_1} x_1 \cdots \partial^{\alpha_N} x_N$.

Endowed with the following norm

$$||u||_{k,p(x)} = \Sigma_{|\alpha|\le k} |D^\alpha u|_{p(x)}.$$

We define the space $W_0^{k,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$, with respect to the norm $\|\cdot\|_{k,p(x)}$. They are separable and reflexive Banach spaces, in addition, we also have similar properties, and the following are some propositions concerning these properties, we refer the reader to [22,27,40].

Proposition 2.1. *(See [21], Pages 430–431) The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is separable, uniformly convex, reflexive, and its conjugate space is $(L^{p'(x)}(\Omega), |.|_{p'(x)})$, where $p'(x)$ is the conjugate function of $p(x)$, i.e,*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,\ for\ all\ x \in \Omega.$$

For all $u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega uvdx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)} \tag{2.1}$$

holds.

Proposition 2.2. *(See [27], Theorem 2.1) Let $p, q, s : \Omega \to [1, \infty]$ be measurable functions such that*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)},\ for\ a.e.\ x \in \Omega.$$

Let $f \in L^{p(x)}(\Omega), g \in L^{q(x)}(\Omega)$. Then, $fg \in L^{s(x)}(\Omega)$ with

$$|fg|_{s(x)} \le \left(\left(\frac{s}{p}\right)^+ + \left(\frac{s}{q}\right)^+\right)|f|_p|g|_q. \tag{2.2}$$

Proposition 2.3. *(See [20], Proposition 2.5) Let h_1, h_2 and $h_3 : \Omega \to (1, \infty)$ be Lipschitz continuous functions such that $1/h_1(x) + 1/h_2(x) + 1/h_3(x) = 1$, then for any $u \in L^{h_1(x)}(\Omega), v \in L^{h_2(x)}(\Omega)$ and $w \in L^{h_3(x)}(\Omega)$ the following inequality holds*

$$\left|\int_\Omega uvwdx\right| \le \left(\frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-}\right)|u|_{h_1(x)}|v|_{h_2(x)}|w|_{h_3(x)}. \tag{2.3}$$

Proposition 2.4. *(See [21], Theorem 1.11) Let $r, s : \Omega \to [1, \infty)$ be measurable functions such that $r(x) \le s(x)$, then the embedding $L^{s(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$is continuous.*

Proposition 2.5. *(See [21], Theorem 2.3) For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \le p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

Proposition 2.6. *(See [18], Lemma 2.1) Let p and q be measuable functions such that $p \in L^{\infty}(\Omega)$, and $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega), u \neq 0$. Then*

$$\min(|u|^{p^+}_{p(x)q(x)}, |u|^{p^-}_{p(x)q(x)}) \leq \left||u|^{p(x)}\right|_{q(x)} \leq \max(|u|^{p^+}_{p(x)q(x)}, |u|^{p^-}_{p(x)q(x)}).$$

Definition 2.1. *(See [28], Definition 2.3) Assume that spaces E, F are Banach spaces, we define the norm on the space $X := E \bigcap F$ as $\|u\|_X = \|u\|_E + \|u\|_F$, X^* its dual space and $< \cdot, \cdot >$ denote the duality product.*

We also need some properties about the space $X := W_0^{1,p(x)}(\Omega) \bigcap W^{2,p(x)}(\Omega)$. Denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{k,p(x)}(\Omega)$. Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space X endowed with the norm

$$\|u\| = \inf\left\{\mu > 0 : \int_{\Omega}\left|\frac{|\Delta u(x)|}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

According to [23], the norm $\|u\|_{2,p(x)}$ is equivalent to the norm $|\Delta u|_{p(x)}$ in the space X. Consequently, the norm $\|u\|_{2,p(x)}, |\Delta u|_{p(x)}$ and $\|u\|$ are equivalent.

We define the relevant modular on the space $L^{p(x)}(\Omega)$ and give the basic properties needed in this paper as following

$$\rho_{p(\cdot)}(u) := \int_{\Omega} |\Delta u|^{p(x)} dx.$$

Proposition 2.7. *(See [6], Proposition 3.2) Suppose that $u_n, u \in X$ and $p_+ < \infty$. Then the following properties hold:*

(1) $\|u\| < 1 (= 1, > 1) \Leftrightarrow \rho(u) < 1 (= 1. > 1)$;

(2) $\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;

(3) $\|u\| < 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;

(4) $\|u\| \to 0 (\to \infty) \Leftrightarrow \rho(u) \to 0 (\to \infty)$.

Moreover, we define

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N - kp(x)}, & \text{if } kp(x) < N, \\ +\infty, & \text{if } kp(x) \geq N. \end{cases}$$

Let for any $u \in X, L(u) = \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx$, and $A := L' : X \to X^*$, then

$$\langle A(u), v \rangle = \int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \Delta v dx, \text{ for all } u, v \in X.$$

Lemma 2.1. *(See [6], Theorem 3.4)*

(1) $L' : X \to X^*$ is a bounded homeomorphism and strictly monotone operator.

(2) L' is a mapping of type S_+, namely, if $u_n \rightharpoonup u$ and $\limsup \langle L(u_n) - L(u), u_n - u \rangle \leq 0$, imply $u_n \to u$ (strongly) in X.

Remark 2.1. *(See [28], Remark 3.2)* We denote by $s'(x)$ the conjugate of the functions $s(x)$, $r(x) := \frac{s(x)q(x)}{s(x)-q(x)}$. Then the following embedding properties hold.

Under assumption (M_0) and (M), we have $\max(r(x), s'(x)q(x)) < p^*(x)$, for all $x \in \overline{\Omega}$. It follows that the embeddings $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ and $X \hookrightarrow L^{r(x)}(\Omega)$ are compact and continuous.

Under Remark 2.1, we have for all $u \in X$

$$\int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx \leq \frac{1}{q^-}|V|_{s(x)}||u|^{q(x)}|_{s'(x)} \leq \begin{cases} \frac{1}{q^-}|V|_{s(x)}|u|^{q^-}_{s'(x)q(x)}, & \text{if } |u|_{s'(x)q(x)} \leq 1, \\ \frac{1}{q^-}|V|_{s(x)}|u|^{q^+}_{s'(x)q(x)}, & \text{if } |u|_{s'(x)q(x)} > 1. \end{cases}$$

and

$$|u|_{s'(x)q(x)} \leq C\|u\|, \quad \text{for all } u \in X.$$

A function $u \in X$ is a weak solution of (1.1), if

$$\left(a - b\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta\varphi dx = \lambda\int_\Omega V(x)|u|^{q(x)-2}u\varphi dx,$$

where $\varphi \in X$.

Next, we will prove Theorem 1.1 using the variational method. Define the energy functional $J : X \to \mathbb{R}$ associated with problem (1.1) by

$$J(u) = a\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)^2 - \lambda\int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx, \tag{2.4}$$

for all $u \in X$ is well defined and of C^1 class on X. Also, we have

$$\begin{aligned}\langle J'(u), v\rangle = &\left(a - b\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v dx \\ &- \lambda\int_\Omega V(x)|u|^{q(x)-2}uv dx,\end{aligned} \tag{2.5}$$

for all $u, v \in X$. Therefore, we can find the weak solutions of problem (1.1) as the critical points of functional J. To simplify the notation, we are going to represent the norm X by $\|.\|$ instead of $\|.\|_X$.

Related definitions of $(PS)_c$ have been given in Definition 3.1 of literature [26], where if any sequence $u_n \in X$ satisfying:

$$J(u_n) \to c \text{ and } J'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty, \tag{2.6}$$

has a convergent subsequence, it is said to satisfy the $(PS)_c$ condition.

Next, we prove that the functional J satisfies the compactness condition.

Lemma 2.2. *Assume that $a \geq b > 0, p, q \in C(\overline{\Omega})$, then the functional $J(u)$ satisfies the $(PS)_c$ condition at the level $c < \frac{a^2}{2b}$.*

Proof. The proof we're going to do is divided into two steps.

Step 1. We prove that $\{u_n\}$ is bounded in X. Let $\{u_n\} \subset X$ be a $(PS)_c$ sequence for J such that $c < \frac{a^2}{2b}$.

Our main tool in this step is based on proving by contradiction, we assume that, passing eventually to a subsequence, still denote by $\{u_n\}$, we have $\|u_n\| \to \infty$ as $n \to \infty$. By (2.4), for n large enough, we can get

$$
\begin{aligned}
&C + \|u_n\| \\
&\quad \geq \frac{2p^{-2} - (p^+)^2}{2p^+} J(u_n) - < J'(u_n), u_n > \\
&\quad \geq a\left(\frac{2p^{-2} - (p^+)^2}{2(p^+)^2} - 1\right)\|u_n\|^{p^-} + b\left(\frac{-\frac{2p^{-2}-(p^+)^2}{2p^+}}{2(p^-)^2} + \frac{1}{p^+}\right)\|u_n\|^{2p^-} \\
&\quad - \lambda\left(\frac{2p^{-2} - (p^+)^2}{2p^+q^-} - 1\right)\int_\Omega V(x)|u_n|^{q(x)}dx \\
&\quad \geq a\left(\frac{2p^{-2} - (p^+)^2}{2(p^+)^2} - 1\right)\|u_n\|^{p^-} + b\left(\frac{-\frac{2p^{-2}-(p^+)^2}{2p^+}}{2(p^-)^2} + \frac{1}{p^+}\right)\|u_n\|^{2p^-} \\
&\quad - \lambda\left(\frac{2p^{-2} - (p^+)^2}{2p^+q^-} - 1\right)|V|_{s(x)}\|u_n|^{q(x)}|_{s'(x)} \\
&\quad \geq a\left(\frac{2p^{-2} - (p^+)^2}{2(p^+)^2} - 1\right)\|u_n\|^{p^-} + b\left(\frac{-\frac{2p^{-2}-(p^+)^2}{2p^+}}{2(p^-)^2} + \frac{1}{p^+}\right)\|u_n\|^{2p^-} \\
&\quad - \lambda\left(\frac{2p^{-2} - (p^+)^2}{2p^+q^-} - 1\right)|V|_{s(x)} \min(|u_n|^{q^+}_{s'(x)q(x)}, |u_n|^{q^-}_{s'(x)q(x)}) \\
&\quad \geq a\left(\frac{2p^{-2} - (p^+)^2}{2(p^+)^2} - 1\right)\|u_n\|^{p^-} + b\left(\frac{-\frac{2p^{-2}-(p^+)^2}{2p^+}}{2(p^-)^2} + \frac{1}{p^+}\right)\|u_n\|^{2p^-} \\
&\quad - \lambda\left(\frac{2p^{-2} - (p^+)^2}{2p^+q^-} - 1\right)|V|_{s(x)} \min(C^{q^+}\|u_n\|^{q^+}, C^{q^-}\|u_n\|^{q^-}).
\end{aligned}
$$

By the above last inequality, we have

$$
\begin{aligned}
C + \|u_n\| + \lambda\left(\frac{2p^{-2} - (p^+)^2}{2p^+q^-} - 1\right)|V|_{s(x)} \min\left(C^{q^+}\|u_n\|^{q^+}, C^{q^-}\|u_n\|^{q^-}\right) \\
\geq a\left(\frac{2p^{-2} - (p^+)^2}{2(p^+)^2} - 1\right)\|u_n\|^{p^-} + b\left(\frac{-\frac{2p^{-2}-(p^+)^2}{2p^+}}{2(p^-)^2} + \frac{1}{p^+}\right)\|u_n\|^{2p^-}.
\end{aligned}
$$

Dividing the above inequality by $\|u_n\|^{2p^-}$, and since (1.4) holds, passing to the limit as $n \to \infty$, we get a contradiction. Thus, $\{u_n\}$ is bounded in X.

Step 2. In the second part, we will show that $\{u_n\}$ has a convergent subsequence in X. According to Proposition 2.5, the embedding

$$X \hookrightarrow L^{h(x)}(\Omega)$$

is compact, where $1 \leq h(x) < p^*(x)$. Passing, if necessary, to a subsequence, there exists $u \in X$ such that

$$u_n \rightharpoonup u \text{ in } X, u_n \to u \text{ in } L^{h(x)}(\Omega), u_n(x) \to u(x) \text{ a.e. in } \Omega. \tag{2.7}$$

From Hölder's inequality and (2.7), we can inferred that

$$\begin{aligned}\left|\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n-u)dx\right| &\le |V|_{s(x)}||u_n|^{q(x)-2}u_n(u_n-u)|_{s'(x)}\\ &\le \left|V|_{s(x)}||u_n|^{q(x)-2}u_n\right|_{\frac{q(x)}{q(x)-1}}|u_n-u|_{\frac{s(x)q(x)}{s(x)-q(x)}}\\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n-u)dx=0. \tag{2.8}$$

Due to (2.6), we have

$$\langle J'(u_n),u_n-u\rangle\to 0.$$

Therefore

$$\begin{aligned}\langle J'(u),u_n-u\rangle=&\left(a-b\int_\Omega\frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\right)\int_\Omega|\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n-\Delta u)dx\\ &-\lambda\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n-u)dx\to 0.\end{aligned}$$

Hence, we can infer from (2.8) that

$$\left(a-b\int_\Omega\frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\right)\int_\Omega|\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n-\Delta u)dx\to 0. \tag{2.9}$$

Since $\{u_n\}$ is bounded in X, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega\frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\to t_0\ge 0 \text{ as } n\to\infty.$$

If $t_0=0$, then $\{u_n\}$ strongly converges to $u=0$ in X and the proof is finished. If $t_0>0$, we will discuss the following two cases respectively:

Case 1. If $t_0\neq\frac{a}{b}$ then $a-b\int_\Omega\frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\to 0$ is not true and no subsequence of $\{a-b\int_\Omega\frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\}$ converges to zero. Therefore, there exists $\delta>0$ such that $\left|a-b\int_\Omega\frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\right|>\delta>0$ when n is large enough. Obviously it is concluded that

$$\left\{a-b\int_\Omega\frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\right\}\text{ is bounded.} \tag{2.10}$$

Case 2. If $t_0=\frac{a}{b}$ then $a-b\int_\Omega\frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\to 0$.

Set

$$\varphi(u)=\lambda\int_\Omega\frac{V(x)}{q(x)}|u|^{q(x)}dx,\ \text{for all } u\in X.$$

Then

$$\langle\varphi'(u),v\rangle=\lambda\int_\Omega V(x)|u|^{q(x)-2}v dx,\ \text{for all } u,v\in X.$$

From the above equation, we have

$$\langle \varphi'(u_n) - \varphi'(u), v\rangle = \lambda \int_\Omega V(x)(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u)v dx.$$

We complete the proof with such a lemma:

Lemma 2.3. *Let $u_n, u \in X$ such that (2.7) holds. Then, passing to a subsequence, if necessary, the following properties hold:*
(i) $\int_\Omega V(x)(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u)v dx \to 0 \ (n \to \infty)$;
(ii)$\langle \varphi'(u_n) - \varphi'(u), v\rangle \to 0$, $(n \to \infty)$ $v \in X$.

Proof. Due to (2.7), we have $u_n \to u$ in $L^{q(x)}(\Omega)$. Then we get

$$|u_n|^{q(x)-2}u_n \to |u|^{q(x)-2}u \text{ in } L^{\frac{q(x)}{q(x)-1}}(\Omega). \tag{2.11}$$

So from the Hölder inequality, we have

$$\begin{aligned} \left|\int_\Omega V(x)(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u)v dx\right| &\le |V|_{s(x)} \left||u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right|_{\frac{q(x)}{q(x)-1}} |v|_{\frac{s(x)a(x)}{s(x)-q(x)}} \\ &\le C|V|_{s(x)} \left||u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right|_{\frac{q(x)}{q(x)-1}} \|v\| \\ &\to 0. \end{aligned} \tag{2.12}$$

The proof of *(ii)* can also be obtained by slightly modifying the proof above. So we will not prove it in detail here. So we end up with both $\|\varphi'(u_n) - \varphi'(u)\|_{X^*} \to 0$ and $\varphi'(u_n) \to \varphi'(u)$.

Next, we will make the proof for Case 2:
According to Lemma 2.3 and since

$$\langle J'(u), v\rangle = (a - b\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx) \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx - \langle \varphi'(u), v\rangle,$$

$$\langle J'(u_n), v\rangle \to 0$$

and

$$a - b\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx \to 0,$$

so there are $\varphi'(u_n) \to 0$ $(n \to \infty)$, i.e.,

$$\langle \varphi'(u), v\rangle = \lambda \int_\Omega V(x)|u|^{q(x)-2}uv dx, \text{ for all } v \in X,$$

and so

$$\lambda V(x)|u(x)|^{q(x)-2}u(x) = 0, \text{ for a.e. } x \in \Omega,$$

by the fundamental lemma of the variational method (see [47]). It follows that $u = 0$. So

$$\varphi(u_n) = \lambda \int_\Omega \frac{V(x)}{q(x)}|u_n|^{q(x)}dx \to \lambda \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx = 0.$$

So when $t_0 = \frac{a}{b}$, we have

$$
\begin{aligned}
J(u_n) = & a \int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} dx - \frac{b}{2} \left(\int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} dx \right)^2 \\
& - \lambda \int_\Omega \frac{V(x)}{q(x)} |u_n|^{q(x)} dx \\
& \to \frac{a^2}{2b}.
\end{aligned}
\tag{2.13}
$$

This is a contradiction since $J(u_n) \to c < \frac{a^2}{2b}$, then $a - b \int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} dx \to 0$ is not true. And it's similar to Case 1, we have that

$$
\left\{ a - b \int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} dx \right\} \text{ is bounded.} \tag{2.14}
$$

From what has been discussed above, it can be inferred that

$$
\int_\Omega |\Delta u_n|^{p(x)-2} \Delta u_n (\Delta u_n - \Delta u) dx \to 0.
$$

Thus, from Lemma 2.1, we can deduce that $u_n \to u$ strongly in X as $n \to \infty$, which implied that J satisfies the $(PS)_c$ condition.

3. Proof of Theorem 1.1

We prove the conditions for satisfying the Mountain Pass theorem (see [47]) by proving the following lemma.

Lemma 3.1. *The following assertions hold:*
(i) There exists $\zeta > 0$ such that for any $|V|_{L^\infty(\Omega)} < \zeta$, there exist $\rho, \gamma > 0$ such that $J(u) \geq \gamma, \forall u \in X$ with $\|u\| = \rho$;
(ii) There exists $\psi \in X, \psi > 0$ such that $\lim\limits_{t \to \infty} J(t\psi) = -\infty$;
(iii) There exists $\omega \in X, \omega > 0$ such that $J(t\omega) < 0$ for all $t > 0$ small enough.

Proof. (i) Define the function $q_1 : \overline{B_r(x_0)} \to (1, +\infty)$ by $q_1(x) = q(x)\ \forall x \in \overline{B_r(x_0)}$ and the function $q_2 : \overline{\Omega \setminus B_R(x_0)} \to (1, \infty)$ by $q_2(x) = q(x)\ \forall x \in \overline{\Omega \setminus B_R(x_0)}$.

Set $q_1^- = \min\limits_{x \in \overline{B_r(x_0)}} q_1(x)$, $q_1^+ = \max\limits_{x \in \overline{B_r(x_0)}} q_1(x)$, $q_2^- = \min\limits_{x \in \overline{\Omega \setminus B_R(x_0)}} q_2(x)$ and $q_2^+ = \min\limits_{x \in \overline{\Omega \setminus B_R(x_0)}} q_2(x)$. Due to the condition (M), we have

$$
1 < q_1^- \leq q_1^+ < p^- \alpha \leq p^+ \alpha < q_2^- \leq q_2^+ < p^*(x) \ \ \forall x \in \overline{\Omega}.
$$

Thus, X is continuously embedded in $L^{q_i^\pm}$ $(i = 1, 2)$.

Then, there exist two positive constants C_1, C_2 such that

$$
\int_{B_r(x_0)} |u|^{q_1(x)} dx \leq C_1 (\|u\|^{q_1^-} + \|u\|^{q_1^+}), \ \forall u \in X; \tag{3.1}
$$

and

$$\int_{\Omega\setminus B_R(x_0)} |u|^{q_2(x)}dx \le C_2(\|u\|^{q_2^-} + \|u\|^{q_2^+}),\ \forall u \in X. \tag{3.2}$$

The inequalities (3.1) and (3.2) have been proved in [15].

Applying the inequalities and Propositions described above, we obtain

$$\begin{aligned}
J(u) = {} & a\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)^2 \\
& - \lambda \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx \\
\ge {} & \frac{a}{p^+}\|u\|^{\alpha p^-} - \frac{b}{2(p^-)^2}\|u\|^{2\alpha p^+} \\
& - \lambda\left[\int_{B_r(x_0)} \frac{V(x)}{q(x)}|u|^{q(x)}dx + \int_{\Omega\setminus B_R(x_0)} \frac{V(x)}{q(x)}|u|^{q(x)}dx\right] \\
\ge {} & \frac{a}{p^+}\|u\|^{\alpha p^-} - \frac{b}{2(p^-)^2}\|u\|^{2\alpha p^+} \\
& - \lambda\frac{C_3|V|_{L^\infty(\Omega)}}{q^-}\left(\|u\|^{q_1^-} + \|u\|^{q_1^+} + \|u\|^{q_2^-} + \|u\|^{q_2^+}\right) \\
\ge {} & \frac{a}{p^+}\rho^{\alpha p^-} - \frac{b}{2(p^-)^2}\rho^{2\alpha p^+} \\
& - \lambda\frac{C_3|V|_{L^\infty(\Omega)}}{q^-}\left(\rho^{q_1^-} + \rho^{q_1^+} + \rho^{q_2^-} + \rho^{q_2^+}\right).
\end{aligned}$$

for all $u \in X$ with $\|u\| < 1$. Therefore, we infer that any ζ satisfies

$$\zeta \ge \frac{\left(\frac{a\rho^{\alpha p^-}}{p^+} - \frac{b\rho^{2\alpha p^+}}{2(p^-)^2} - \gamma\right)\cdot q^-}{\lambda C_3(\rho^{q_1^-} + \rho^{q_1^+} + \rho^{q_2^-} + \rho^{q_2^+})}.$$

(ii) Let $\psi \in C_0^\infty(\Omega), \psi \ge 0$ and let $x_1 \in \Omega \setminus B_R(x_0)$ and $\varepsilon > 0$ such that $\psi(x) > 0$ for any $x \in B_\varepsilon(x_1) \subset \Omega \setminus B_R(x_0)$. For any $t > 1$, we have

$$\begin{aligned}
J(t\psi) = {} & a\int_\Omega \frac{t^{p(x)}}{p(x)}|\Delta \psi|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{t^{p(x)}}{p(x)}|\Delta \psi|^{p(x)}dx\right)^2 \\
& - \lambda \int_\Omega \frac{V(x)t^{q(x)}}{q(x)}|\psi|^{q(x)}dx \\
\le {} & \frac{at^{\alpha p^+}}{p^-}\int_\Omega |\Delta\psi|^{p(x)}dx - \frac{bt^{2\alpha p^-}}{2(p^+)^2}\left(\int_\Omega |\Delta\psi|^{p(x)}dx\right)^2 \\
& - \lambda t^{q_2^-}\int_{\Omega\setminus B_R(x_0)} \frac{V(x)}{q(x)}|\psi|^{q(x)}dx.
\end{aligned}$$

Since $\alpha p^+ < q_2^-$ we infer that $\lim\limits_{t\to\infty} J(t\psi) = -\infty$.

(iii) Let $\omega \in C_0^\infty(\Omega), \omega \ge 0$ and there exist $x_2 \in B_r(x_0)$ and $\varepsilon > 0$ such that for any $x \in B_\varepsilon(x_2) \subset B_r(x_0)$

we have $\omega(x) > 0$. Let $0 < t < 1$, we obtain

$$\begin{aligned}
J(t\omega) &= a\int_\Omega \frac{t^{p(x)}}{p(x)}|\Delta\omega|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{t^{p(x)}}{p(x)}|\Delta\omega|^{p(x)}dx\right)^2 \\
&- \lambda\int_\Omega \frac{V(x)t^{q(x)}}{q(x)}|\omega|^{q(x)}dx \\
&\leq \frac{at^{\alpha p^-}}{p^-}\int_\Omega |\Delta\omega|^{p(x)}dx - \frac{bt^{2\alpha p^+}}{2(p^+)^2}\left(\int_\Omega |\Delta\omega|^{p(x)}dx\right)^2 \\
&- \lambda t^{q_1^+}\int_{B_r(x_0)} \frac{V(x)}{q(x)}|\omega|^{q(x)}dx. \\
&\leq \frac{at^{\alpha p^-}}{p^-}\int_\Omega |\Delta\omega|^{p(x)}dx - \lambda t^{q_1^+}\int_{B_r(x_0)} \frac{V(x)}{q(x)}|\omega|^{q(x)}dx.
\end{aligned}$$

Thus, we have $J(t\omega) < 0$ for any $0 < t < \delta^{\frac{1}{\alpha p^- - q_1^+}}$; where

$$0 < \delta < \min\left\{1, \frac{p^-\lambda\int_{B_r(x_0)}\frac{V(x)}{q(x)}|\omega|^{q(x)}dx}{a\int_\Omega |\Delta\omega|^{p(x)}dx}\right\}.$$

By using Ekeland variational principle in reference [19], we can similarly reach the following conclusions:

There exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$J(u_0) \to \underline{c} = \inf_{u\in\overline{B}_\rho(0)} J(u) < 0 \text{ and } J'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty. \tag{3.3}$$

It has been proved by Lemma 2.2, the sequence $\{u_n\}$ converges strongly to some u_2 as $n \to \infty$. Moreover, since $J \in C^1(X, \mathbb{R})$, by (3.3) it follows that $J'(u_2) = 0$. Thus, u_2 is a non-trivial weak solution of problem (1.1) with negative energy $J(u_2) = \underline{c} < 0$.

Finally, since $J(u_1) = \overline{c} > 0 > \underline{c} = J(u_2)$, we can point out the fact that $u_1 \neq u_2$.

Let

$$M = \sup_{u\in X} J(u),$$

then $M < +\infty$, hence

$$-M = \inf_{u\in X} -J(u).$$

Applying Ekland's variational principle on space X for $-J(u)$, there exists the $(PS)_{-M}$ sequence of $-J(u)$, so there is the $(PS)_M$ sequence of $J(u)$.

Since there is a global minimum, we can define a minimization sequence and prove its convergence. And then we get our third solution.

4. Final comments

When $\lambda V(x)|u|^{q(x)-2}u$ is replaced by $\lambda f(x,u)$, the problem (1.1) becomes

$$\begin{cases}\left(a-b\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\Delta^2_{p(x)}u=\lambda f(x,u), & \text{in } \Omega,\\ u=\Delta u=0, & \text{on } \partial\Omega,\end{cases} \tag{4.1}$$

where f satisfies the following properties:
(f_1) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathédory condition and there exists a constant $C_1\geq 0$ such that

$$|f(x,t)|\leq C_1(1+|t|^{q(x)-1})$$

for all $(x,t)\in\Omega\times\mathbb{R}$ with $q(x)\in C_+(\overline{\Omega})$ and $q(x)<p_2^*(x)$ for all $x\in\overline{\Omega}$ where $p_2^*(x):=\frac{Np(x)}{N-2p(x)}$ if $p(x)<\frac{N}{2}$, $p_2^*(x)=\infty$ if $p(x)\geq\frac{N}{2}$.
(f_2) There exists $k>0,\theta>\frac{2(p^+)^2}{p^-}$ such that for all $x\in\Omega$ and all $t\in\mathbb{R}$ with $|t|\geq k$,

$$0<\theta F(x,t)\leq tf(x,t),$$

where $F(x,t)=\int_0^t f(x,s)ds$.
(f_3) $f(x,t)=o(|t|^{2p^+-1})$ as $t\to 0$ uniformly with respect to $x\in\Omega$, with $q^->2(p^+)^2$.
(f_4) $f(x,-t)=-f(x,t)$ for all $x\in\Omega$ and $t\in\mathbb{R}$.
Theorem 4.1. *When $\lambda V(x)|u|^{q(x)-2}u$ is replaced by $\lambda f(x,u)$, if (f_1),(f_2),(f_4) hold, and $q^->2(p^+)^2>2p^+$, then problem (4.1) has a sequence of weak solutions $\{\pm u_k\}$ such that $I(\pm u_k)\to-\infty$ as $k\to+\infty$.*

By referring to the methods and conditions in [17], we can obtain the existence of two related theorems for this problem, the main conclusion is that by using the symmetric mountain pass theorem, we can conclude that there are a series of weak solutions for the problem (4.1). The method of proving $(PS)_c$ condition is similar to Lemma 2.2.
Definition 4.1. A function $u\in X$ is a weak solution of (4.1), if

$$\left(a-b\int_\Omega\frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\int_\Omega|\Delta u|^{p(x)-2}\Delta u\Delta\varphi dx=\lambda\int_\Omega f(x,u)\varphi dx,$$

where $\varphi\in X$.

Define the energy functional $I:X\to\mathbb{R}$ associated with problem (4.1) by

$$I(u)=a\int_\Omega\frac{1}{p(x)}|\Delta u|^{p(x)}dx-\frac{b}{2}\left(\int_\Omega\frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)^2-\lambda\int_\Omega F(x,u)dx, \tag{4.2}$$

for all $u\in X$, $I(u)$ is well defined and of C^1 class on X. Moreover, we have

$$\begin{aligned}\langle I'(u),v\rangle=&\left(a-b\int_\Omega\frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\int_\Omega|\Delta u|^{p(x)-2}\Delta u\Delta v dx\\&-\lambda\int_\Omega f(x,u)vdx,\end{aligned} \tag{4.3}$$

for all $u,v\in X$.
Lemma A. *(Symmetric mountain pass theorem, See [47]) Let $X=Y\bigoplus Z$ be an infinite dimensional*

Banach space, where Y is finite dimensional, and $I \in C^1(X, \mathbb{R})$ satisfies the $(PS)_c$ condition as well as the following properties:
(i) $I(u) = 0$ and there exist two constants $r, \alpha > 0$ such that $I|_{\partial B_r} \geq \alpha$.
(ii) I is even.
(iii) for all finite dimensional subspace $\widetilde{X} \subset X$ there exists $R = R(\widetilde{X}) > 0$ such that $I(u) \leq 0$ for all $u \in X \setminus B_R(\widetilde{X})$, where $B_R(\widetilde{X}) = \{u \in \widetilde{X} : \|u\| < R\}$.
Then I has an unbounded sequence of critical points.

Proof of Theorem 4.1. The method and steps of proving that problem (4.1) satisfies the condition of $(PS)_c$ are consistent with Lemma 2.2, from (f_1), Propositions 2.1 and 2.5, we deduce easily that

$$\int_\Omega f(x, u_n)(u - u_n)dx \to 0 \text{ as } n \to \infty.$$

Hence, we can deduce that I satisfies the $(PS)_c$ condition.

Due to (f_4), and I is an even functional and satisfies the $(PS)_c$ condition. We will show that I satisfies the conditions of Lemma A.

(i) Obviously, $I(0) = 0$. Since $p^+ < (p^+)^2 < q^- < q(x) < p_2^*(x)$, $X \hookrightarrow L^{2p^+}(\Omega)$, $X \hookrightarrow L^{q(x)}(\Omega)$ and then there exist $C_3, C_4 > 0$ such that

$$|u|_{2p^+} \leq C_3\|u\|, \ |u|_{q(x)} \leq C_4\|u\|.$$

By (f_1) and (f_3), we have

$$|F(x, u)| \leq \varepsilon|u|^{2p^+} + C_\varepsilon|u|^{q(x)}, \text{ for all } (x, u) \in \Omega \times R. \tag{4.4}$$

Let $r \in (0, 1)$ and $u \in X$ be such that $\|u\| = r$. Thus, by considering (4.4), Propositions 2.5 and 2.7, hypothesis that $1 < p^- < p(x) < p^+ < 2p^- < q^- < q(x) < p^*(x)$, we have

$$\begin{aligned}
I(u) &= a \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)^2 - \lambda \int_\Omega F(x, u)dx \\
&\geq a \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)^2 \\
&- \lambda\varepsilon \int_\Omega |u|^{2p^+}dx - \lambda C_\varepsilon \int_\Omega |u|^{q(x)}dx \\
&\geq a \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)^2 \\
&- \lambda\varepsilon C_3^{2p^+}\|u\|^{2p^+} - \lambda C_\varepsilon C_4\|u\|^{q^-} \\
&\geq \frac{a}{p^+}\|u\|^{p^+} - \frac{b}{2(p^-)^2}\|u\|^{2p^-} - \lambda\varepsilon C_3^{2p^+}\|u\|^{2p^+} - \lambda C_\varepsilon C_4\|u\|^{q^-} \\
&= \|u\|^{p^+}\left(\frac{a}{p^+} - \frac{b}{2(p^-)^2}\|u\|^{2p^- - p^+} - \lambda\varepsilon C_3^{2p^+}\|u\|^{2p^+ - p^+} - \lambda C_\varepsilon C_4\|u\|^{q^- - p^+}\right) \\
&= r^{p^+}\left(\frac{a}{p^+} - \frac{b}{2(p^-)^2}r^{2p^- - p^+} - \lambda\varepsilon C_3^{2p^+}r^{p^+} - \lambda C_\varepsilon C_4 r^{q^- - p^+}\right).
\end{aligned}$$

Set

$$\lambda^* = \frac{\frac{a}{p^+} - \frac{b}{2(p^-)^2} r^{2p^- - p^+}}{2\left(\varepsilon C_3^{2p^+} r^{p^+} - C_\varepsilon C_4 r^{q^- - p^+}\right)} \text{ and } \alpha = \lambda^* r^{p^+},$$

so for any $\lambda \in (0, \lambda^*)$, there exists $\alpha > 0$ such that $u \in X$ with $\|u\| = r$, we have $I(u) \geq \alpha > 0$.

(ii) It is clear that I is even.

(iii) By(f_2), we have that

$$F(x,t) \geq C|t|^\theta - C. \tag{4.5}$$

Let $R = R(\widetilde{X}) > 1$, for all $u \in \widetilde{X}$ with $\|u\| > R$. By (4.5), we have

$$\begin{aligned} I(u) &= a \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx - \frac{b}{2} \left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx \right)^2 - \lambda \int_\Omega F(x,u) dx \\ &\leq \frac{a}{p^-} \int_\Omega |\Delta u|^{p(x)} dx - \frac{b}{2(p^+)^2} \left(\int_\Omega |\Delta u|^{p(x)} dx \right)^2 - \lambda C \int_\Omega |u|^\theta dx + \lambda C \int_\Omega dx \\ &= \frac{a}{p^-} \int_\Omega |\Delta u|^{p(x)} dx - \frac{b}{2(p^+)^2} \left(\int_\Omega |\Delta u|^{p(x)} dx \right)^2 - \lambda C \int_\Omega |u|^\theta dx + \lambda C |\Omega|. \end{aligned}$$

Thus, all norms on the finite dimensional space $\widetilde{X}$ are equivalent, so there exists $C_W > 0$ such that

$$\int_\Omega |u|^\theta dx \geq C_W \|u\|^\theta.$$

Therefore, we obtain

$$I(u) \leq \frac{a}{p^-} \|u\|^{p^+} - \frac{b}{2(p^+)^2} \|u\|^{2p^-} - \lambda C C_W \|u\|^\theta + \lambda C |\Omega|.$$

Due to $\theta > p^+$, it follows that for some $\|u\| > R$ large enough, we can infer that $I(u) \leq 0$. Therefore, the conclusion of Theorem 4.1 can be obtained by using the symmetric mountain pass theorem.

This paper can be regarded as an extension of literature [26]. In addition, for this kind of negative nonlocal term problem, in recent literature [46], the author studied a class of variable exponent $p(x)$-Kirchhoff type problem with convection. In this paper, Galerkin method, pseudomonotone operators and a fixed-point argument were used to study the existence of the solution to this problem. In [34], the author studied the multiplicity results of a class of $p(x)$-Choquard equations with nonlocal and nondegenerate Kirchhoff terms by using truncation argument and Krasnoselskiis genus method.

5. Conclusions

In this paper, we study a class of $p(x)$-biharmonic problems with negative nonlocal terms and weight function. Using the mountain pass theorem and the Ekeland's variational principle, we proved the existence of at least three solutions. We also proved the existence of infinite solutions when the nonlinear term is a general function.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (No. 11661021, 11861021); Science and Technology Foundation of Guizhou Province (No. KY[2017]1084) and Natural Science Research Project of Guizhou Minzu University (No. GZMUZK[2021]YB20). Authors are grateful to the referees for their very constructive comments and valuable suggestions.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. E. Acerbi, G. Mingione, Gradient estimates for the $p(x)$-Laplacean system, *J. Reine Angew. Math.,* **584** (2005), 117–148. http://dx.doi.org/10.1515/crll.2005.2005.584.117
2. R. Aboulaich, D. Meskine, A. Souissi, New diffusion models in image processing, *Comput. Math. Appl.,* **56** (2008), 874–882. http://dx.doi.org/10.1016/j.camwa.2008.01.017
3. G. A. Afrouzi, N. T. Chung, M. Mirzapour, Existence and multiplicity of solutions for Kirchhoff type problems involving $p(x)$-biharmonic operators, *J. Anal. Appl.,* **33** (2014), 289–303. http://dx.doi.org/10.4171/ZAA/3445
4. R. Alsaedi, A. Dhifli, A. Ghanmi, Low perturbations of p-biharmonic equations with competing nonlinearities, *Complex Var. Elliptic,* **66** (2021), 642–657. http://dx.doi.org/10.1080/17476933.2020.1747057
5. C. O. Alves, J. P. Barreiro, Multiple solutions for a class of quasilinear problems involving variable exponents, *Asymptot. Anal.,* **96** (2016), 161–184. http://dx.doi.org/10.3233/ASY-151340
6. A. Amrouss, El. F. Moradi, M. Moussaoui, Existence of solutions for fourth-order PDES with variable exponent, *Electron. J. Differ. Eq.,* **153** (2009), 1–13. http://dx.doi.org/10.1016/j.dam.2009.09.005
7. S. Antontsev, M. Chipot, Y. Xie, Uniqueness results for equations of the $p(x)$-Laplacian type, *Adv. Math. Sci. Appl.,* **17** (2007), 287–304.
8. S. N. Antontsev, J. F. Rodrigues, On stationary thermo-rheological viscous flows, *Ann. Univ. Ferrara Sez. VII Sci. Mat.,* **52** (2006), 19–36. http://dx.doi.org/10.1007/s11565-006-0002-9
9. M. Berger, A new approach to the analysis of large deflection of plate, *J. Appl. Mech.,* **22** (1955), 465–472. http://dx.doi.org/10.1115/1.4011138
10. S. S. Byun, O. Jihoon, On $W^{1,q(\cdot)}$-estimates for elliptic equations of $p(x)$-Laplacian type, *J. Math. Pures Appl.,* **106** (2016), 512–545. http://dx.doi.org/10.1016/j.matpur.2016.03.002
11. J. Chabrowski, Y. Fu, Existence of solutions for $p(x)$-Laplacian problems on a bounded domain, *J. Math. Anal. Appl.,* **306** (2005), 604–618. http://dx.doi.org/10.1016/j.jmaa.2004.10.028
12. Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.,* **66** (2006), 1383–1406. http://dx.doi.org/10.2307/4096243

13. C. M. Chu, C. Y. Lei, H. M. Suo, Positive solutions for a nonlocal problem with singularity, *Electron. J. Differ. Eq.,* **85** (2017), 1–9.

14. N. T. Chung, Multiple solutions for a class of $p(x)$-Laplacian problems involving concave-convex nonlinearities, *Electron. J. Qual. Theory Differ. Equ.,* **26** (2013), 1–17. http://dx.doi.org/10.14232/ejqtde.2013.1.26

15. N. T. Chung, On some $p(x)$-Kirchhoff type equations with weights, *J. Appl. Math. Inform.,* **32** (2014), 113–128. http://dx.doi.org/10.14317/jami.2014.113

16. F. Colasuonno, P. Pucci, Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations, *Nonlinear Anal.,* **74** (2011), 5962–5974. http://dx.doi.org/10.1016/j.na.2011.05.073

17. O. Darhouche, Existence and multiplicity results for a class of Kirchhoff type problems involving the $p(x)$-biharmonic operator, *Bol. Soc. Parana. Mat.,* **37** (2017), 23–33. http://dx.doi.org/10.5269/bspm.v37i2.32100

18. D. Edmunds, J. Rakosnik, Sobolev embeddings with variable exponent, *Stud. Math.,* **143** (2000), 267–293. http://dx.doi.org/10.4064/sm-143-3-267-293

19. I. Ekeland, On the variational principle, *J. Math. Anal. Appl.,* **47** (1974), 324–353. http://dx.doi.org/10.1016/0022-247X(74)90025-0

20. X. L. Fan, X. Han, Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, *Nonlinear Anal.,* **59** (2004), 173–188. http://dx.doi.org/10.1016/j.na.2004.07.009

21. X. L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.,* **263** (2001), 424–446. http://dx.doi.org/10.1006/jmaa.2000.7617

22. A. Ferrero, G. Warnault, On solutions of second and fourth order elliptic equations with power-type nonlinearities, *Nonlinear Anal.,* **70** (2009), 2889–2902. http://dx.doi.org/10.1016/j.na.2008.12.041

23. Y. Fu, A. Zang, Interpolation inequalities for derivatives in variable exponent Lebesgue-Sobolev spaces, *Nonlinear Anal.,* **69** (2008), 3629–3636. http://dx.doi.org/10.1016/j.na.2007.10.001

24. B. Ge, Y. H. Wu, Q. Zhou, Eigenvalue of the $p(x)$-biharmonic operator with indefinite weight, *Z. Angew. Math. Phys.,* **66** (2014), 1007–1021. http://dx.doi.org/10.1007/s00033-014-0465-y

25. A. Ghanmi, Nontrivial solutions for Kirchhoff-type problems involving the $p(x)$-Laplace operator, *Rocky MT J. Math.,* **48** (2018), 1145–1158. http://dx.doi.org/10.1216/RMJ-2018-48-4-1145

26. M. K. Hamdani, A. Harrabi, F. Mtiri, D. D. Repovš, Existence and multiplicity results for a new $p(x)$-Kirchhoff problem, *Nonlinear Anal.,* **190** (2020), 1–15. http://dx.doi.org/10.1016/j.na.2019.111598

27. O. Kavacik, J. Rakosnik, On spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, *Czech. Math. J.,* **41** (1991), 592–618.

28. K. Kefi, V. D. Rădulescu, On a $p(x)$-biharmonic problem with singular weights, *Z. Angew. Math. Phys.,* **68** (2017), 1–15. http://dx.doi.org/10.1007/s00033-017-0827-3

29. K. Khaled, For a class of $p(x)$-biharmonic operators with weights, *Rev. Acad. Colombiana Cienc. Exact. Fís. Natur.,* **113** (2018), 1–16. http://dx.doi.org/10.1007/s13398-018-0567-z

30. G. Kirchhoff, *Mechanik*, Teubner: Leipzig, Germany, 1883.

31. L. Kong, On a fourth order elliptic problem with a $p(x)$-biharmonic operator, *Appl. Math. Lett.,* **27** (2014), 21–25. http://dx.doi.org/10.1016/j.aml.2013.08.007

32. C. Y. Lei, J. F. Liao, H. M. Suo, Multiple positive solutions for nonlocal problems involving a sign-changing potential, *Electron. J. Differ. Eq.,* **9** (2017), 1–8.
33. C. Y. Lei, H. M. Suo, Y. Wang, Multiple positive solutions for a nonlocal problem involving critical exponent, *Electron. J. Differ. Eq.,* **275** (2017), 1–11.
34. B. B. V. Maia, On a class of $p(x)$-Choquard equations with sign-changing potential and upper critical growth, *Rend. Circ. Mat. Palerm.,* **70** (2021), 1175–1199. http://dx.doi.org/10.1007/s12215-020-00553-y
35. L. Mbarki, Existence results for perturbed weighted $p(x)$-biharmonic problem with Navier boundary conditions, *Complex Var. Elliptic Equ.,* **66** (2020), 569–582. http://dx.doi.org/10.1080/17476933.2020.1729140
36. G. Mingione, V. D. Rădulescu, Recent developments in problems with nonstandard growth andnonuniform elipicity, *J. Math. Anal. Appl.,* **501** (2021), 1–41. http://dx.doi.org/10.1016/j.jmaa.2021.125197
37. J. Musielak, W. Orlicz, On modular spaces, *Stud. Math.,* **18** (1959), 9–65. http://dx.doi.org/10.4064/sm-18-1-49-65
38. H. Nakano, *Modulared semi-ordered linear spaces*, Maruzen Co., 1950.
39. H. Nakano, *Topology and modular spaces*, Springer-Verlag, Berlin, 1951, 15–32.
40. W. Orlicz, Uber konjugierte exponentenfolgen, *Stud. Math.,* **3** (1931), 200–212. http://dx.doi.org/10.4064/sm-3-1-200-211
41. M. A. Ragusa, A. Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents, *Adv. Nonlinear Anal.,* **9** (2020), 710–728. http://dx.doi.org/10.1515/anona-2020-0022
42. K. Rajagopal, M. Růžička, On the modeling of electrorheological materials, *Mech. Res. Commun.,* **23** (1996), 401–407. http://dx.doi.org/10.1016/0093-6413(96)00038-9
43. K. Rajagopal, M. Růžička, Mathematical modeling of electrorheological materials, *Continuum Mech. Therm.,* **13** (2001), 59–78. http://dx.doi.org/10.1007/s001610100034
44. X. H. Tang, B. T. Cheng, Ground state sign-changing solutions for Kirchhoff type problems in bounded domains, *J. Differ. Equations,* **261** (2016), 2384–2402. http://dx.doi.org/10.1016/j.jde.2016.04.032
45. X. H. Tang, S. T. Chen, Ground state solutions of Nehari-Pohozaev type for Kirchhoff-type problems with general potentials, *Calc. Var. Partial Dif.,* **56** (2017), 110. http://dx.doi.org/10.1007/s00526-017-1214-9
46. C. Vetro, Variable exponent $p(x)$-Kirchhoff type problem with convection, *J. Math. Anal. Appl.,* **506** (2022), 1–16. http://dx.doi.org/10.1016/j.jmaa.2021.125721
47. M. Willem, *Minimax theorems, progress in nonlinear differential equations and their applications,* Birkhauser Boston, Inc., Boston, MA, 1996. http://dx.doi.org/10.1007/978-1-4612-4146-1