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Research article

Complete convergence and complete integral convergence for weighted sums of widely acceptable random variables under the sub-linear expectations

Chengcheng Jia and Qunying Wu*

College of Science, Guilin University of Technology, Guilin 541006, China

* **Correspondence:** Email: wqy666@glut.edu.cn; Tel: +8615877031406.

Abstract: Since the concept of sub-linear expectation space was put forward, it has well supplemented the deficiency of the theoretical part of probability space. In this paper, we establish the complete convergence and complete integration convergence for weighted sums of widely acceptable (abbreviated as WA) random variables under the sub-linear expectations with the different conditions. We extend the complete moment convergence in probability space to sublinear expectation space.

Keywords: sub-linear expectation; complete convergence; complete integral convergence; WA random variables
Mathematics Subject Classification: 60F15

1. Introduction

The classical limit theorem considers additive probability and additive expectation, which is suitable for the case of model determination, but this assumption of additivity is not feasible in many fields of practice. As a mathematical theory, nonlinear expectation can be analyzed and calculated under the uncertainty of the mathematical model. In its research, sub-linear expectation plays a special role and is the most studied. Peng [1–3] put forward the concept of generalization of sublinear expectation space in 2006, which transforms the probability in probability space into the capacity in sublinear expectation space, which enriches the theoretical part of probability space. Then, after Zhang's [4–7] research in sublinear expectation space, some important inequalities are obtained. These inequalities are a powerful tool for us to study sublinear expectation space. In addition, Zhang also studies the law of iterated logarithm and the strong law of large numbers in sublinear expectation space. After further extension, Wu and Jiang [8] obtained the Marcinkiewicz type strong law of numbers and the Chover type iterated logarithm law for more general cases in sublinear expectation space.

In probability space, the complete convergence and complete moment convergence are two very important research parts. The notion of complete convergence was proposed by Hsu and Robbins [9] in 1947. In 1988, Chow [10] introduced the concept of complete moment convergence. The complete moment convergence is stronger than the complete convergence. The complete convergence and complete moment convergence in probability space have been relatively mature. For example, Qiu [11], Wu [12], and Shen [13] respectively obtained the complete convergence and complete moment convergence for independent identically distributed (i.i.d.), negatively associated (NA), extended negatively dependent (END) random variables sequence in probability space. Due to many methods and tools in probability space, sublinear expectation space can not be used, which increases the difficulty of studying sublinear expectation space, but many scholars have done the research, such as Wu [14] pushed the theorem in Wu [12] from probability space to sublinear expectation space. Feng [15] and Liang [16] obtained the complete convergence and complete integral convergence for arrays of row-wise ND and END random variables respectively. Zhong [17] studied the complete convergence and complete integral convergence for the weighted sum of END random variables. Lu [18] obtained more extensive conditions and conclusions than Zhong [17] in sublinear expectation space. The exponential inequality used in this article was proposed by Anna [19] in 2020. In the inequality, it is assumed that the truncated random variable sequence is a WA random variable sequence. Because it was proposed later, there is little research on WA random variable sequence in sublinear expectation space. Hu [20] proved the complete convergence for weighted sums of WA random variables in 2021.

The organizational structure of this paper is as follows. In Section 2, we summarize some basic symbols and concepts, as well as the related properties in sublinear expectation space, and give a preliminary lemma which is helpful to obtain the main results. In Section 3, We deduce [21] from probability space to sublinear expectation space, obtain the corresponding conclusions, and prove the complete convergence and complete integral convergence for the weighted sums of WA random variables in sublinear expectation space.

2. Preliminaries

We use the framework and notations of Peng [1–3]. Let $(\Omega,\mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega,\mathcal{F})$ such that if $X_1, X_2, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\boldsymbol{x})-\varphi(\boldsymbol{y})| \le c\left(1+|\boldsymbol{x}|^m+|\boldsymbol{y}|^m\right)|\boldsymbol{x}-\boldsymbol{y}|,\ \forall \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_n,$$

for some $c>0,\ m\in\mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables. In this case, we denote $X \in \mathcal{H}$.

Definition 2.1. A sub-linear expectation $\hat{\mathbb{E}}$, on $\mathcal{H}$ is a function $\hat{\mathbb{E}}: \mathcal{H} \to [-\infty,+\infty]$ satisfying the following properties: For all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$, then $\hat{\mathbb{E}}X \geq \hat{\mathbb{E}}Y$;

(b) Constant preserving: $\hat{\mathbb{E}}(c) = c$, for $c \in \mathbb{R}$;

(c) Sub-additivity: $\hat{\mathbb{E}}(X+Y) \leq \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ whenever $\hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}X, \lambda \geq 0$. Convention: when $\lambda = 0$ and $\hat{\mathbb{E}}X = +\infty$, $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}X = 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space.

Given a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}X := -\hat{\mathbb{E}}(-X),\ \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that for all $X, Y \in \mathcal{H}$,

$$\hat{\varepsilon}X \leq \hat{\mathbb{E}}X, \hat{\mathbb{E}}(X+c) = \hat{\mathbb{E}}X + c, \left|\hat{\mathbb{E}}(X-Y)\right| \leq \hat{\mathbb{E}}|X-Y| \text{ and } \hat{\mathbb{E}}(X-Y) \geq \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y.$$

If $\hat{\mathbb{E}}Y = \hat{\varepsilon}Y$, then $\hat{\mathbb{E}}(X+aY) = \hat{\mathbb{E}}X + a\hat{\mathbb{E}}Y$ for any $a \in \mathbb{R}$. Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0,1]$ is called a capacity if

$$V(\emptyset) = 0, V(\Omega) = 1; \text{ and } V(A) \leq V(B) \text{ for } \forall A \subseteq B, A, B \in \mathcal{G}.$$

It is called sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. In the sub-linear space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, we denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\left\{\hat{\mathbb{E}}\xi : I(A) \leq \xi,\ \xi \in \mathcal{H}\right\}, \mathcal{V}(A) := 1 - \mathbb{V}(A^c),\ \forall A \in \mathcal{F},$$

where A^c is the complement set of A. By definition of $\mathbb{V}$ and $\mathcal{V}$, it is obvious that $\mathbb{V}$ is sub-additive, and

$$\mathcal{V}(A) \leq \mathbb{V}(A),\ \forall A \in \mathcal{F}.$$

If $f \leq I(A) \leq g$, $f, g \in \mathcal{H}$, then

$$\hat{\mathbb{E}}f \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}g, \hat{\varepsilon}f \leq \mathcal{V}(A) \leq \hat{\varepsilon}g. \tag{2.1}$$

This implies Markov inequality: $\forall X \in \mathcal{H}$,

$$\mathbb{V}(|X| \geq x) \leq \hat{\mathbb{E}}\left(|X|^p\right) / x^p, \forall x > 0, p > 0.$$

From $I(|X| > x) \leq |X|^p / x^p \in \mathcal{H}$. From Lemma 4.1 in Zhang [5], we have Hölder inequality: $\forall X, Y \in \mathcal{H}, p, q > 1$, satisfying $p^{-1} + q^{-1} = 1$,

$$\hat{\mathbb{E}}|XY| \leq \left(\hat{\mathbb{E}}|X|^p\right)^{1/p} \left(\hat{\mathbb{E}}|Y|^q\right)^{1/q},$$

whenever

$$\hat{\mathbb{E}}\left(|X|^p\right)<\infty,\quad \hat{\mathbb{E}}\left(|Y|^q\right)<\infty.$$

Particularly, Jensen inequality:

$$\left(\hat{\mathbb{E}}|X|^r\right)^{1/r}\le\left(\hat{\mathbb{E}}|X|^s\right)^{1/s},\ \text{ for }\ 0<r\le s.$$

We define the Choquet integrals $(C_{\mathbb{V}},C_{\mathcal{V}})$ by

$$C_V(X)=\int_0^\infty V(X\ge x)\,\mathrm{d}x+\int_{-\infty}^0\left[V(X\ge x)-1\right]\mathrm{d}x,$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively.

Definition 2.2.

(i) $\hat{\mathbb{E}}$ countably sub-additive: $\hat{\mathbb{E}}$ is called to be countably sub-additive if it satisfies $\hat{\mathbb{E}}(X)\le\sum_{n=1}^{\infty}\hat{\mathbb{E}}(X_n),$ where $X\le\sum_{n=1}^{\infty}X_n,\ X,X_n\in\mathcal{H},X\ge 0,X_n\ge 0,n\ge 1.$

(ii) V is called to be countably sub-additive if

$$V\left(\bigcup_{n=1}^{\infty}A_n\right)\le\sum_{n=1}^{\infty}V(A_n),\ \forall A_n\in\mathcal{F}.$$

Definition 2.3. (Identical distribution) Let X_1 and X_2 be two random variables defined severally in sub-linear expectation space $(\Omega_1,\mathcal{H}_1,\hat{\mathbb{E}}_1)$ and $(\Omega_2,\mathcal{H}_2,\hat{\mathbb{E}}_2)$. They are called identically distributed if

$$\hat{\mathbb{E}}_1(\varphi(X_1))=\hat{\mathbb{E}}_2(\varphi(X_2)),\ \forall\varphi\in C_{l,Lip}(\mathbb{R}),$$

whenever the sub-linear expectations are finite. A sequence $\{X_n;n\ge 1\}$ of random variables is said to be identical distribution if X_i and X_1 are identical distribution for each $i\ge 1$.

Definition 2.4. (WA) Let $\{Y_n;n\ge 1\}$ be a sequence of random variables in a sub-linear expectation space $(\Omega,\mathcal{H},\hat{\mathbb{E}})$. The sequence $\{Y_n;n\ge 1\}$ is called WA if for $t\ge 0$ and for all $n\in\mathbb{N}$

$$\hat{\mathbb{E}}\exp\left(\sum_{i=1}^{n}tY_i\right)\le g(n)\prod_{i=1}^{n}\hat{\mathbb{E}}\exp(tY_i), \tag{2.2}$$

where $0<g(n)<\infty.$

Definition 2.5. [22] A function $L:(0,\infty)\to(0,\infty)$ is:

(i) A slowly varying function (at infinity), if for any $a>0$

$$\lim_{x\to\infty}\frac{L(ax)}{L(x)}=1,$$

(ii) A regularly varying function with index $\alpha>0,$ if for any $a>0$

$$\lim_{x\to\infty}\frac{L(ax)}{L(x)}=a^{\alpha}.$$

Lemma 2.6. [22] Every regularly varying function (with index $\alpha > 0$) $l:(0,\infty)\to(0,\infty)$ is of the form

$$l(x) = x^{\alpha} L(x),$$

where L is a slowly varying function.

In the following, let $\{X_n; n\ge 1\}$ be a sequence of random variables in $(\Omega,\mathcal{H},\hat{\mathbb{E}})$. The symbol c stands for a generic positive constant which may differ from one place to another. Let $a_x \sim b_x$ denote $\lim_{x\to\infty} a_x/b_x = 1$. $a_n \ll b_n$ denote that there exists a constant $c>0$ such that $a_n \le cb_n$ for sufficiently large n, and $I(\cdot)$ denotes an indicator function. $a\vee b$ means to take the maximum value of a and b, while $a\wedge b$ means to take the minimum value of a and b.

To prove our results, we need the following lemmas.

In [17], we can get the following lemma.

Lemma 2.7. [17] Suppose $X\in\mathcal{H}, \alpha>0, p>0,$ and $l(x)$ is a slow varying function.
(i) Then, for $\forall c>0$,

$$C_{\mathbb{V}}\left(|X|^p l\left(|X|^{1/\alpha}\right)\right)<\infty \Leftrightarrow \sum_{n=1}^{\infty} n^{\alpha p-1} l(n)\mathbb{V}\left(|X|>cn^{\alpha}\right)<\infty,$$

taking $l(x)=1$ and $\log x$, respectively, we can get that for $\forall c>0$,

$$C_{\mathbb{V}}\left(|X|^p\right)<\infty \Leftrightarrow \sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{V}\left(|X|>cn^{\alpha}\right)<\infty,$$

$$C_{\mathbb{V}}\left(|X|^p \log|X|\right)<\infty \Leftrightarrow \sum_{n=1}^{\infty} n^{\alpha p-1}\log n\mathbb{V}\left(|X|>cn^{\alpha}\right)<\infty.$$

(ii) If $C_{\mathbb{V}}\left(|X|^p l\left(|X|^{1/\alpha}\right)\right)<\infty$, then for any $\theta>1$ and $c>0$,

$$\sum_{k=1}^{\infty} \theta^{k\alpha p} l(\theta^k)\mathbb{V}\left(|X|>c\theta^{k\alpha}\right)<\infty,$$

taking $l(x)=1$ and $\log x$, respectively, we have

$$C_{\mathbb{V}}\left(|X|^p\right)<\infty \Rightarrow \sum_{k=1}^{\infty} \theta^{k\alpha p}\mathbb{V}\left(|X|>c\theta^{k\alpha}\right)<\infty,$$

$$C_{\mathbb{V}}\left(|X|^p \log|X|\right)<\infty \Rightarrow \sum_{k=1}^{\infty} \theta^{k\alpha p}(\log\theta^k)\mathbb{V}\left(|X|>c\theta^{k\alpha}\right)<\infty.$$

The last one is the exponential inequality for WA random variables, which was can be found in [19].

Lemma 2.8. [19] Let $\{X_1, X_2, \cdots, X_n\}$ be a sequence of random variables in $(\Omega,\mathcal{H},\hat{\mathbb{E}})$, with $\hat{\mathbb{E}}X_i\le 0$ for $1\le i\le n$. Let $d>0$ be a real number, we define $X^{(d)}=\min\{X,d\}$. Assume that $Y_i := X_i^{(d)}$, $1\le i\le n$ satisfy (2.2) for all $t>0$. Then, for all $x>0$, we have

$$\mathbb{V}\left(S_n \geq x\right) \leq \mathbb{V}\left(\max_{1\leq i\leq n} X_i > d\right) + g(n)\exp\left(\frac{x}{d} - \frac{x}{d}\ln\left(1+\frac{xd}{\sum_{i=1}^{n}\hat{\mathbb{E}}X_i^2}\right)\right).$$

3. Results and proof

Next, we give the theorems and proof in this article.

Let $\{X_n; n\geq 1\}$ be a sequence of random variables in sub-linear expectation space $(\Omega,\mathcal{H},\hat{\mathbb{E}}), \alpha > 1/2, \alpha p > 1, \varepsilon > 0, \delta > 0$ and $\beta_1 = \frac{[\alpha(p\wedge 2)-1]\varepsilon}{4(\alpha p - 1 + \delta)} > 0.$ For fixed $n \geq 1,$ denote for $1\leq i\leq n$ that

$$Y_i = -\beta_1 n^\alpha I\left(X_i < -\beta_1 n^\alpha\right) + X_i I\left(|X_i| \leq \beta_1 n^\alpha\right) + \beta_1 n^\alpha I\left(X_i > \beta_1 n^\alpha\right). \tag{3.1}$$

Theorem 3.1 Let $\alpha > 1/2, \alpha p > 1$ and $\{X_n; n\geq 1\}$ be a sequence of random variables in $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ with $\hat{\mathbb{E}}X_i = \hat{\varepsilon}X_i = 0$ if $p > 1$ such that sequence $\{Y_i; 1\leq i\leq n\}$ of truncated random variables is WA and control coefficient $g(n)$ in (2.2) is regularly varying function with index δ for some $\delta > 0$. Assume that $\{a_{ni}; 1\leq i\leq n, n\geq 1\}$ is an array of real numbers and there exist some q with $q > \max\{2, p\}$ have

$$\sum_{i=1}^{n}|a_{ni}|^q = O(n), \quad |a_{ni}| \leq c \tag{3.2}$$

and there also exist a random variable $X\in\mathcal{H}$ and a constant c satisfying

$$\hat{\mathbb{E}}[f(X_n)] \leq c\hat{\mathbb{E}}[f(X)],\ n\geq 1,\ 0\leq f\in C_{l,Lip}(\mathbb{R}), \tag{3.3}$$

then

$$\hat{\mathbb{E}}|X|^p \leq C_{\mathbb{V}}\left(|X|^p\right) < \infty, \tag{3.4}$$

implies that for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| > \varepsilon n^\alpha\right) < \infty. \tag{3.5}$$

Let $0 < \beta_2 < \min\left\{\frac{2\wedge(p\vee r)}{2r}, \frac{\alpha[2\wedge(p\vee r)]-1}{2(\alpha p - 1 + \delta)}\right\}.$ For any $1\leq i\leq n, n\geq 1,$ and $t\geq n^{\alpha r},$ denote

$$Y_i' = -\beta_2 t^{1/r} I\left(X_i < -\beta_2 t^{1/r}\right) + X_i I\left(|X_i| \leq \beta_2 t^{1/r}\right) + \beta_2 t^{1/r} I\left(X_i > \beta_2 t^{1/r}\right). \tag{3.6}$$

Theorem 3.2. Let $r > 0, \alpha > 1/2, \alpha(p\vee r) > 1$ and $\{X_n; n\geq 1\}$ be a sequence of random variables in $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ with $\hat{\mathbb{E}}X_i = \hat{\varepsilon}X_i = 0$ if $p\vee r > 1$ such that sequence $\{Y_i'; 1\leq i\leq n\}$ of truncated random variables is WA and control coefficient $g(n)$ in (2.2) is regularly varying function with index δ for some $\delta > 0$. Assume that $\{a_{ni}; 1 \le i \le n, n \ge 1\}$ is an array of real numbers and condition (3.2) holds for $q > \max\{2, p \vee r\}$, moreover, the condition (3.3) is also true, then

$$\begin{cases} \hat{\mathbb{E}}|X|^{p\vee r} \le C_{\mathbb{V}}\left(|X|^{p\vee r}\right) < \infty & \text{if } r \ne p; \\ \hat{\mathbb{E}}|X|^{p}\log|X| \le C_{\mathbb{V}}\left(|X|^{p}\log|X|\right) < \infty & \text{if } r = p; \end{cases} \tag{3.7}$$

implies that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha r - 2} C_{\mathbb{V}}\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right)_+^r < \infty. \tag{3.8}$$

Remark. In Theorem 3.2, we extend the complete moment convergence for the weighted sums of random variables in the probability space of article [21] to the complete integral convergence for the weighted sums of WA random variables in sublinear expectation space.

Proof of Theorem 3.1. Since $\sum_{i=1}^{n} a_{ni}X_i = \sum_{i=1}^{n} a_{ni}^{+}X_i - \sum_{i=1}^{n} a_{ni}^{-}X_i$, we have

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right) \le \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\left|\sum_{i=1}^{n} a_{ni}^{+}X_i\right| > \frac{\varepsilon n^{\alpha}}{2}\right) + \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\left|\sum_{i=1}^{n} a_{ni}^{-}X_i\right| > \frac{\varepsilon n^{\alpha}}{2}\right).$$

So, without loss of generality, we can assume that $a_{ni} \ge 0$ for $1 \le i \le n$ and $n \ge 1$.

If we want to prove (3.5), we just need to prove

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_i > \varepsilon n^{\alpha}\right) < \infty, \forall \varepsilon > 0. \tag{3.9}$$

Because of considering $\{-X_n; n \ge 1\}$ still satisfies the conditions in the Theorem 3.1, we can obtain

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_i < -\varepsilon n^{\alpha}\right) < \infty, \forall \varepsilon > 0. \tag{3.10}$$

Form (3.9) and (3.10), we can get (3.5). The following proves that (3.9) is established. The definition of $\{Y_i; 1 \le i \le n\}$ is (3.1). For fixed $n \ge 1$, denote for $1 \le i \le n$ that

$$Z_i = X_i - Y_i = \left(X_i + \beta_1 n^{\alpha}\right) I\left(X_i < -\beta_1 n^{\alpha}\right) + \left(X_i - \beta_1 n^{\alpha}\right) I\left(X_i > \beta_1 n^{\alpha}\right).$$

It is easily checked that for $\forall \varepsilon > 0$,

$$\left(\sum_{i=1}^{n} a_{ni}X_i > \varepsilon n^{\alpha}\right) \subset \bigcup_{i=1}^{n}\left(|X_i| > \beta_1 n^{\alpha}\right) \cup \left(\sum_{i=1}^{n} a_{ni}Y_i > \varepsilon n^{\alpha}\right). \tag{3.11}$$

So, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_i > \varepsilon n^{\alpha}\right)\\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{V}\left(|X_i|>\beta_1 n^{\alpha}\right)+\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}Y_i > \varepsilon n^{\alpha}\right)\\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{V}\left(|X_i|>\beta_1 n^{\alpha}\right)+\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}(Y_i-\hat{\mathbb{E}}Y_i) > \varepsilon n^{\alpha}-\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}Y_i\right|\right)\\
&:= I_1+I_2.
\end{aligned}
$$

To prove (3.9), it suffices to show $I_1<\infty$ and $I_2<\infty$.

For $0<\mu<1$, let $g(x)\in C_{l,Lip}(\mathbb{R})$ be a decreasing function when $x\ge 0$ such that $0\le g(x)\le 1$ for all x and $g(x)=1$ if $|x|\le\mu$, $g(x)=0$ if $|x|\ge 1$. Then

$$
I(|x|\le\mu)\le g(|x|)\le I(|x|\le 1),\ I(|x|>1)\le 1-g(|x|)\le I(|x|>\mu). \tag{3.12}
$$

By (3.12), Lemma 2.7 (i) and (3.3), we can get that

$$
\begin{aligned}
I_1 &\le \sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\hat{\mathbb{E}}\left(1-g\left(\frac{|X_i|}{\beta_1 n^{\alpha}}\right)\right)\\
&\le c\sum_{n=1}^{\infty} n^{\alpha p-1}\hat{\mathbb{E}}\left(1-g\left(\frac{|X|}{\beta_1 n^{\alpha}}\right)\right)\\
&\le c\sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{V}\left(|X|>cn^{\alpha}\right)\\
&<\infty.
\end{aligned}
$$

In the following, we prove that $I_2<\infty$. Firstly, we will show that

$$
n^{-\alpha}\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}Y_i\right|\to 0,\quad n\to\infty.
$$

By (3.2) and Hölder inequality, we have for any $0<\rho<q$ that

$$
\sum_{i=1}^{n} a_{ni}{}^{\rho}\le\left(\sum_{i=1}^{n}\left(a_{ni}{}^{q}\right)\right)^{\rho/q}\left(\sum_{i=1}^{n}1\right)^{1-\rho/q}\le cn. \tag{3.13}
$$

For any $\lambda>0$, by (3.12) and C_r inequality, we have

$$
|Y_i|^{\lambda}\ll|X_i|^{\lambda}I(|X_i|\le\beta_1 n^{\alpha})+\beta_1{}^{\lambda}n^{\alpha\lambda}I(|X_i|>\beta_1 n^{\alpha})\le|X_i|^{\lambda}g\left(\frac{\mu|X_i|}{\beta_1 n^{\alpha}}\right)+\beta_1{}^{\lambda}n^{\alpha\lambda}\left(1-g\left(\frac{|X_i|}{\beta_1 n^{\alpha}}\right)\right),
$$

$$
|Z_i|^{\lambda}\ll|X_i+\beta_1 n^{\alpha}|^{\lambda}I(X_i<-\beta_1 n^{\alpha})+|X_i-\beta_1 n^{\alpha}|^{\lambda}I(X_i>\beta_1 n^{\alpha})\le|X_i|^{\lambda}\left(1-g\left(\frac{|X_i|}{\beta_1 n^{\alpha}}\right)\right).
$$

Thus

$$\hat{\mathbb{E}}|Y_i|^\lambda \ll \hat{\mathbb{E}}|X|^\lambda g\left(\frac{\mu|X|}{\beta_1 n^\alpha}\right) + {\beta_1}^\lambda n^{\alpha\lambda}\hat{\mathbb{E}}\left(1 - g\left(\frac{|X|}{\beta_1 n^\alpha}\right)\right)$$

$$\leq \hat{\mathbb{E}}|X|^\lambda g\left(\frac{\mu|X|}{\beta_1 n^\alpha}\right) + {\beta_1}^\lambda n^{\alpha\lambda}\mathbb{V}(|X| > \mu\beta_1 n^\alpha),$$

$$\hat{\mathbb{E}}|Z_i|^\lambda \ll \hat{\mathbb{E}}|X_i|^\lambda \left(1 - g\left(\frac{|X_i|}{\beta_1 n^\alpha}\right)\right) \ll \hat{\mathbb{E}}|X|^\lambda \left(1 - g\left(\frac{|X|}{\beta_1 n^\alpha}\right)\right). \tag{3.14}$$

By Lemma 2.7 (i), we can get that

$$\sum_{n=1}^{\infty} \mathbb{V}\left(|X| > cn^\alpha\right) \leq \sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{V}\left(|X| > cn^\alpha\right) < \infty,$$

and $\mathbb{V}\left(|X| > cn^\alpha\right)\downarrow$, so we get $n\mathbb{V}\left(|X| > cn^\alpha\right) \to 0$ as $n \to \infty$.

When $0 < p \leq 1$. Since $q > \max\{2, p\} > 1$, by (3.13), (3.14) and $\alpha p > 1$, we have that

$$n^{-\alpha}\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}Y_i\right| \leq n^{-\alpha}\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}|Y_i|$$

$$\ll n^{1-\alpha}\left(\hat{\mathbb{E}}|X| g\left(\frac{\mu|X|}{\beta_1 n^\alpha}\right) + \beta_1 n^\alpha \mathbb{V}(|X| > cn^\alpha)\right)$$

$$= n^{1-\alpha}\hat{\mathbb{E}}|X| g\left(\frac{\mu|X|}{\beta_1 n^\alpha}\right) + \beta_1 n \mathbb{V}(|X| > cn^\alpha)$$

$$\leq cn^{1-\alpha p}\hat{\mathbb{E}}|X|^p \to 0, \quad n \to \infty.$$

When $p > 1$. Since $q > p > 1$, by (3.13), (3.14) and $\hat{\mathbb{E}}X_i = 0$, we have

$$n^{-\alpha}\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}Y_i\right| \leq n^{-\alpha}\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}|X_i - Y_i|$$

$$= n^{-\alpha}\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}|Z_i|$$

$$\ll n^{1-\alpha}\hat{\mathbb{E}}|X|\left(1 - g\left(\frac{|X|}{\beta_1 n^\alpha}\right)\right)$$

$$\leq cn^{1-\alpha p}\hat{\mathbb{E}}|X|^p \to 0, n \to \infty.$$

Hence, $n^{-\alpha}\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}Y_i\right| \leq \varepsilon/2$ for all n large enough, which implies that

$$I_2 \leq \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}(Y_i - \hat{\mathbb{E}}Y_i) > \frac{\varepsilon n^\alpha}{2}\right).$$

According to assume that sequence $\{Y_i; 1 \leq i \leq n\}$ of truncated random variables is WA and $a_{ni} \geq 0$, by (2.2), we have

$$\hat{\mathbb{E}}\exp\left(\sum_{i=1}^{n} ta_{ni}Y_i\right) \le g(n)\prod_{i=1}^{n}\hat{\mathbb{E}}\exp\left(ta_{ni}Y_i\right).$$

Because $\exp\left(-\sum_{i=1}^{n} ta_{ni}\hat{\mathbb{E}}Y_i\right) \ge 0,$ we can get that

$$\begin{aligned}\hat{\mathbb{E}}\exp\left(\sum_{i=1}^{n} ta_{ni}(Y_i-\hat{\mathbb{E}}Y_i)\right) &= \exp\left(-\sum_{i=1}^{n} ta_{ni}\hat{\mathbb{E}}Y_i\right)\hat{\mathbb{E}}\exp\left(\sum_{i=1}^{n} ta_{ni}Y_i\right)\\ &\le \prod_{i=1}^{n}\exp\left(-ta_{ni}\hat{\mathbb{E}}Y_i\right)g(n)\prod_{i=1}^{n}\hat{\mathbb{E}}\exp\left(ta_{ni}Y_i\right)\\ &= g(n)\prod_{i=1}^{n}\hat{\mathbb{E}}\exp\left(ta_{ni}(Y_i-\hat{\mathbb{E}}Y_i)\right),\end{aligned}$$

which means that $a_{ni}(Y_i-\hat{\mathbb{E}}Y_i)$ are WA random variables. Without loss of generality, according to (3.2), we assume that $a_{ni} \le 1/2$, then

$$a_{ni}(Y_i-\hat{\mathbb{E}}Y_i) \le a_{ni}(|Y_i|+\hat{\mathbb{E}}|Y_i|) \le 2a_{ni}\beta_1 n^{\alpha} \le \beta_1 n^{\alpha}.$$

We can verify that $a_{ni}(Y_i-\hat{\mathbb{E}}Y_i) = \min\left\{a_{ni}(Y_i-\hat{\mathbb{E}}Y_i),\ \beta_1 n^{\alpha}\right\}.$

So $\left\{a_{ni}(Y_i-\hat{\mathbb{E}}Y_i); 1\le i\le n, n\ge 1\right\}$ satisfy the conditions in Lemma 2.8 with $\hat{\mathbb{E}}\left(a_{ni}(Y_i-\hat{\mathbb{E}}Y_i)\right)=0.$ Taking $x=\frac{\varepsilon n^{\alpha}}{2}, d=\beta_1 n^{\alpha}=\frac{[\alpha(p\wedge 2)-1]\varepsilon n^{\alpha}}{4(\alpha p-1+\delta)}$ in Lemma 2.8, we obtain

$$\begin{aligned}I_2 &\le \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}(Y_i-\hat{\mathbb{E}}Y_i) > \frac{\varepsilon n^{\alpha}}{2}\right)\\ &\le \sum_{n=1}^{\infty} n^{\alpha p-2}\left[\mathbb{V}\left(\max_{1\le i\le n}\left(a_{ni}(Y_i-\hat{\mathbb{E}}Y_i)\right) > d\right) + g(n)\exp\left(\frac{\varepsilon n^{\alpha}}{2d}-\frac{\varepsilon n^{\alpha}}{2d}\ln\left(1+\frac{\frac{\varepsilon n^{\alpha}}{2}d}{\sum_{i=1}^{n}\hat{\mathbb{E}}\left|a_{ni}(Y_i-\hat{\mathbb{E}}Y_i)\right|^2}\right)\right)\right]\\ &\le \sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{V}\left(\left|a_{ni}(Y_i-\hat{\mathbb{E}}Y_i)\right| > \beta_1 n^{\alpha}\right) + c\sum_{n=1}^{\infty} n^{\alpha p-2} g(n)\left(n^{-2\alpha}\sum_{i=1}^{n}\hat{\mathbb{E}}\left|a_{ni}(Y_i-\hat{\mathbb{E}}Y_i)\right|^2\right)^{\frac{\varepsilon}{2\beta_1}}\\ &:= I_{21}+I_{22}.\end{aligned}$$

Let $\beta>0$, $g_{j\beta}(x)\in C_{l,Lip}(\mathbb{R})$, $j\ge 1$, suppose $g_{j\beta}(x)$ is an even function, such that $0\le g_{j\beta}(x)\le 1$ for all x and $g_{j\beta}(x)=1$ if $\beta 2^{(j-1)\alpha}/\mu \le |x| \le \beta 2^{j\alpha}/\mu$, $g_{j\beta}(x)=0$ if $|x|<\beta 2^{(j-1)\alpha}$ or $|x|>(1+\mu)\beta 2^{j\alpha}/\mu$. Then for any $l>0$,

$$\begin{aligned}&g_{j\beta}(X) \le I\left(\beta 2^{\alpha(j-1)} < |X| \le (1+\mu)\beta 2^{\alpha j}/\mu\right),\\ &|X|^l g\left(\frac{\mu|X|}{\beta 2^{\alpha k}}\right) \le \frac{\beta^l}{\mu^l}+\sum_{j=1}^{k}|X|^l g_{j\beta}(X).\end{aligned} \tag{3.15}$$

The truncation that defines Y as X is as follows

$$Y=-\beta_1 n^{\alpha} I\left(X<-\beta_1 n^{\alpha}\right)+XI\left(|X|\le\beta_1 n^{\alpha}\right)+\beta_1 n^{\alpha} I\left(X>\beta_1 n^{\alpha}\right).$$

According to Markov inequality, C_r inequality, (3.2), (3.3), (3.14), (3.15), Lemma 2.7, $q>p$ and $g(x)\downarrow$ when $x\ge 0$. Then

$$\begin{aligned}
I_{21} &\ll \sum_{n=1}^{\infty} n^{\alpha p-2}\cdot n^{-\alpha q}\sum_{i=1}^{n} a_{ni}{}^{q}\hat{\mathbb{E}}|Y_i|^q \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\hat{\mathbb{E}}|Y|^q \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\hat{\mathbb{E}}|X|^q\, g\left(\frac{\mu|X|}{\beta_1 n^{\alpha}}\right)+\sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{V}\left(|X|>cn^{\alpha}\right) \\
&\ll \sum_{k=1}^{\infty}\sum_{n=2^{k-1}}^{2^k-1} n^{\alpha p-\alpha q-1}\hat{\mathbb{E}}|X|^q\, g\left(\frac{\mu|X|}{\beta_1 n^{\alpha}}\right) \\
&\ll \sum_{k=1}^{\infty} 2^{k(p-q)\alpha}\,\hat{\mathbb{E}}|X|^q\, g\left(\frac{\mu|X|}{\beta_1 2^{k\alpha}}\right) \\
&\le \sum_{k=1}^{\infty} 2^{k(p-q)\alpha}\,\hat{\mathbb{E}}\left(\frac{\beta_1^{\,q}}{\mu^q}+\sum_{j=1}^{k}|X|^q\, g_{j\beta_1}(X)\right) \\
&\le c\sum_{k=1}^{\infty} 2^{k(p-q)\alpha}+\sum_{k=1}^{\infty} 2^{k(p-q)\alpha}\sum_{j=1}^{k}\hat{\mathbb{E}}|X|^q\, g_{j\beta_1}(X) \\
&\ll \sum_{j=1}^{\infty}\hat{\mathbb{E}}|X|^q\, g_{j\beta_1}(X)\sum_{k=j}^{\infty} 2^{k(p-q)\alpha} \\
&\ll \sum_{j=1}^{\infty} 2^{j(p-q)\alpha}\,\hat{\mathbb{E}}|X|^q\, g_{j\beta_1}(X) \\
&\ll \sum_{j=1}^{\infty} 2^{\alpha pj}\mathbb{V}\left(|X|>c2^{j\alpha}\right)<\infty.
\end{aligned}$$

Next, we prove $I_{22}<\infty$. If $p\ge 2$, then $d=\dfrac{(2\alpha-1)\varepsilon n^{\alpha}}{4(\alpha p-1+\delta)}$, by (3.3), (3.4), (3.13), $\alpha p>1$, C_r inequality, the condition of $g(n)$, there exist a slowly varying function $L(n)$, such that $g(n)=n^{\delta}L(n)$, we have

$$\begin{aligned}
I_{22} &\ll \sum_{n=1}^{\infty} n^{\alpha p-2} g(n)\left(n^{-2\alpha}\sum_{i=1}^{n} a_{ni}{}^{2}\hat{\mathbb{E}}Y_i^2\right)^{\frac{\varepsilon}{2\beta_1}}\\
&\le c\sum_{n=1}^{\infty} n^{\alpha p-2} g(n)\left(n^{1-2\alpha}\hat{\mathbb{E}}X^2\right)^{\frac{\varepsilon}{2\beta_1}}\\
&\le c\sum_{n=1}^{\infty} n^{\alpha p-2+\delta-2(\alpha p-1+\delta)} L(n)\\
&\le c\sum_{n=1}^{\infty} n^{-\alpha p} L(n) < \infty.
\end{aligned}$$

If $p<2$, then $d=\frac{(\alpha p-1)\varepsilon n^{\alpha}}{4(\alpha p-1+\delta)}$, by (3.3), (3.4), (3.13), $\alpha p>1$, C_r inequality, we have

$$\begin{aligned}
I_{22} &\ll \sum_{n=1}^{\infty} n^{\alpha p-2} g(n)\left(n^{-2\alpha}\sum_{i=1}^{n} a_{ni}{}^{2}\hat{\mathbb{E}}Y_i^2\right)^{\frac{\varepsilon}{2\beta_1}}\\
&\le c\sum_{n=1}^{\infty} n^{\alpha p-2} g(n)\left[n^{-2\alpha}\sum_{i=1}^{n} a_{ni}{}^{2}\cdot\left(n^{\alpha(2-p)}\hat{\mathbb{E}}|X|^p\right)\right]^{\frac{\varepsilon}{2\beta_1}}\\
&\le c\sum_{n=1}^{\infty} n^{\alpha p-2+\delta-2(\alpha p-1+\delta)} L(n)\\
&\le c\sum_{n=1}^{\infty} n^{-\alpha p} L(n) < \infty.
\end{aligned}$$

Hence, the proof of Theorem 3.1 is completed.

Proof of Theorem 3.2. Without loss of generality, we also can assume that $a_{ni}\ge 0$ for $1\le i\le n$ and $n\ge 1$. Where, the definitions of $g(x)$ and $g_{j\beta}(x)$ are the same as in the proof of Theorem 3.1. For $\forall\varepsilon>0$, we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} C_{\mathbb{V}}\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right|-\varepsilon n^{\alpha}\right)_+^r\\
&=\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\int_0^{\infty}\mathbb{V}\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right|-\varepsilon n^{\alpha}>t^{1/r}\right)\mathrm{d}t\\
&=\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\int_0^{n^{\alpha r}}\mathbb{V}\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right|-\varepsilon n^{\alpha}>t^{1/r}\right)\mathrm{d}t+\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}\mathbb{V}\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right|-\varepsilon n^{\alpha}>t^{1/r}\right)\mathrm{d}t\\
&\le\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right|>\varepsilon n^{\alpha}\right)+\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}\mathbb{V}\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right|>t^{1/r}\right)\mathrm{d}t\\
&:=J_1+J_2.
\end{aligned}$$

According to Theorem 3.1, we have $J_1<\infty$. So if we want to prove (3.8), we just need to prove $J_2<\infty$. Hence, we first to prove

$$H := \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} X_i > t^{1/r} \right) \mathrm{d}t. \tag{3.16}$$

The definition of $\{Y_i'; 1 \le i \le n\}$ is (3.6). For any $1 \le i \le n, n \ge 1,$ and $t \ge n^{\alpha r}$, denote

$$Z_i' = \left(X_i + \beta_2 t^{1/r} \right) I\left(X_i < -\beta_2 t^{1/r} \right) + \left(X_i - \beta_2 t^{1/r} \right) I\left(X_i > \beta_2 t^{1/r} \right). \tag{3.17}$$

We have

$$\begin{aligned} H &\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \sum_{i=1}^{n} \mathbb{V}\left(|X_i| > \beta_2 t^{1/r} \right) \mathrm{d}t + \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} Y_i' > t^{1/r} \right) \mathrm{d}t \\ &\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \sum_{i=1}^{n} \mathbb{V}\left(|X_i| > \beta_2 t^{1/r} \right) \mathrm{d}t + \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} \left(Y_i' - \hat{\mathbb{E}} Y_i' \right) > t^{1/r} - \left| \sum_{i=1}^{n} a_{ni} \hat{\mathbb{E}} Y_i' \right| \right) \mathrm{d}t \\ &:= H_1 + H_2. \end{aligned}$$

In order to prove $H < \infty$, it suffices to show $H_1 < \infty$ and $H_2 < \infty$. Firstly, we prove $H_1 < \infty$, by (3.7), (3.12), Lemma 2.7 (i), $g(x) \downarrow$ when $x \ge 0$, we have

$$\begin{aligned} H_1 &\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \sum_{i=1}^{n} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X_i|}{\beta_2 t^{1/r}} \right) \right) \mathrm{d}t \\ &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1} \int_{n^{\alpha r}}^{\infty} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X|}{\beta_2 t^{1/r}} \right) \right) \mathrm{d}t \\ &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1} \sum_{m=n}^{\infty} \int_{m^{\alpha r}}^{(m+1)^{\alpha r}} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X|}{\beta_2 t^{1/r}} \right) \right) \mathrm{d}t \\ &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1} \sum_{m=n}^{\infty} m^{\alpha r-1} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X|}{\beta_2 m^{\alpha}} \right) \right) \\ &= \sum_{m=1}^{\infty} m^{\alpha r-1} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X|}{\beta_2 m^{\alpha}} \right) \right) \sum_{n=1}^{m} n^{\alpha p-\alpha r-1} \\ &\ll \begin{cases} \sum_{m=1}^{\infty} m^{\alpha p-1} \mathbb{V}\left(|X| > \mu \beta_2 m^{\alpha} \right) & \text{if } r < p; \\ \sum_{m=1}^{\infty} m^{\alpha p-1} \log m \mathbb{V}\left(|X| > \mu \beta_2 m^{\alpha} \right) & \text{if } r = p; \\ \sum_{m=1}^{\infty} m^{\alpha r-1} \mathbb{V}\left(|X| > \mu \beta_2 m^{\alpha} \right) & \text{if } r > p; \end{cases} \\ &= \begin{cases} \sum_{m=1}^{\infty} m^{\alpha (p \vee r)-1} \mathbb{V}\left(|X| > c m^{\alpha} \right) < \infty & \text{if } r \ne p; \\ \sum_{m=1}^{\infty} m^{\alpha p-1} \log m \mathbb{V}\left(|X| > c m^{\alpha} \right) < \infty & \text{if } r = p. \end{cases} \end{aligned}$$

Then, we prove $H_2<\infty$. Firstly, we will show that

$$\sup_{t\ge n^{\alpha r}} t^{-1/r}\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}Y_i'\right|\to 0,\quad n\to\infty.$$

Similar to (3.14), by (3.12), (3.17), C_r inequality, for any $\lambda>0$, we can get that

$$\begin{aligned}
&\hat{\mathbb{E}}|Y_i'|^{\lambda}\ll \hat{\mathbb{E}}|X|^{\lambda} g\left(\frac{\mu|X|}{\beta_2 t^{1/r}}\right)+\beta_2{}^{\lambda} t^{\lambda/r}\mathbb{V}\left(|X|>\mu\beta_2 t^{1/r}\right),\\
&\hat{\mathbb{E}}|Z_i'|^{\lambda}\ll \hat{\mathbb{E}}|X_i|^{\lambda}\left(1-g\left(\frac{|X_i|}{\beta_2 t^{1/r}}\right)\right)\ll \hat{\mathbb{E}}|X|^{\lambda}\left(1-g\left(\frac{|X|}{\beta_2 t^{1/r}}\right)\right).
\end{aligned}\tag{3.18}$$

The truncation that defines Y' as X is as follows

$$Y'=-\beta_2 t^{1/r} I\left(X<-\beta_2 t^{1/r}\right)+XI\left(|X|\le \beta_2 t^{1/r}\right)+\beta_2 t^{1/r} I\left(X>\beta_2 t^{1/r}\right).$$

When $0<p\vee r\le 1$. Since $t\ge n^{\alpha r}$, $\hat{\mathbb{E}}|X|^{p\vee r}<\infty$, and $\alpha(p\vee r)>1$, we get

$$\begin{aligned}
\sup_{t\ge n^{\alpha r}} t^{-1/r}\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}Y_i'\right| &\ll \sup_{t\ge n^{\alpha r}} t^{-1/r} n\hat{\mathbb{E}}|Y'|\\
&\ll \sup_{t\ge n^{\alpha r}} t^{-1/r} n\left(\hat{\mathbb{E}}|X| g\left(\frac{\mu|X|}{\beta_2 t^{1/r}}\right)+\beta_2 t^{1/r}\mathbb{V}\left(|X|>\mu\beta_2 t^{1/r}\right)\right)\\
&=\sup_{t\ge n^{\alpha r}} t^{-1/r} n\left(\hat{\mathbb{E}}|X|^{(p\vee r)}\cdot|X|^{1-(p\vee r)} g\left(\frac{\mu|X|}{\beta_2 t^{1/r}}\right)+\beta_2 t^{1/r}\mathbb{V}\left(|X|>\mu\beta_2 t^{1/r}\right)\right)\\
&\le cn^{1-\alpha(p\vee r)}\hat{\mathbb{E}}|X|^{p\vee r}+\beta_2 n\mathbb{V}\left(|X|>cn^{\alpha}\right)\to 0,\quad n\to\infty.
\end{aligned}$$

When $p\vee r>1$. Since $\hat{\mathbb{E}}X_i=0$ and $t\ge n^{\alpha r}$, we can get that

$$\begin{aligned}
\sup_{t\ge n^{\alpha r}} t^{-1/r}\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}Y_i'\right| &\le \sup_{t\ge n^{\alpha r}} t^{-1/r}\sum_{i=1}^{n} a_{ni}\left|\hat{\mathbb{E}}X_i-\hat{\mathbb{E}}Y_i'\right|\\
&\le \sup_{t\ge n^{\alpha r}} t^{-1/r}\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}|Z_i'|\\
&\le cn^{1-\alpha}\hat{\mathbb{E}}|X|\left(1-g\left(\frac{|X|}{\beta_2 n^{\alpha}}\right)\right)\\
&\le cn^{1-\alpha}\cdot\frac{\hat{\mathbb{E}}|X|\cdot|X|^{p\vee r-1}}{n^{\alpha(p\vee r-1)}}\left(1-g\left(\frac{|X|}{\mu\beta_2 n^{\alpha}}\right)\right)\\
&\le cn^{1-\alpha(p\vee r)}\hat{\mathbb{E}}|X|^{p\vee r}\to 0,\quad n\to\infty.
\end{aligned}$$

It follows that for all n large enough,

$$\sup_{t\ge n^{\alpha r}} t^{-1/r}\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}Y_i'\right| < \frac{1}{2},$$

which imply that

$$H_2 \le \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(Y_i' - \hat{\mathbb{E}}Y_i'\right) > \frac{t^{1/r}}{2}\right)\mathrm{d}t.$$

For fixed $t \ge n^{\alpha r}$ and $n \ge 1$, through the definition in Theorem 3.2 and assume that $a_{ni} \le 1/2$, we know that $\left\{a_{ni}(Y_i' - \hat{\mathbb{E}}Y_i'); 1 \le i \le n, n \ge 1\right\}$ are WA random variables with $\hat{\mathbb{E}}\left(a_{ni}(Y_i' - \hat{\mathbb{E}}Y_i')\right) = 0$ and $a_{ni}(Y_i' - \hat{\mathbb{E}}Y_i') = \min\left\{a_{ni}(Y_i' - \hat{\mathbb{E}}Y_i'),\ \beta_2 t^{1/r}\right\}$. Use Lemma 2.8 for $\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(Y_i' - \hat{\mathbb{E}}Y_i'\right) > t^{1/r}/2\right)$, taking $0 < \beta_2 < \min\left\{\frac{2\wedge(p\vee r)}{2r}, \frac{\alpha[2\wedge(p\vee r)]-1}{2(\alpha p - 1 + \delta)}\right\}$, $d = \beta_2 t^{1/r}$, $x = t^{1/r}/2$, we have

$$\begin{aligned}
&\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(Y_i' - \hat{\mathbb{E}}Y_i'\right) > \frac{t^{1/r}}{2}\right),\\
&\le \mathbb{V}\left(\max_{1\le i\le n}\left(a_{ni}(Y_i' - \hat{\mathbb{E}}Y_i')\right) > d\right) + g(n)\exp\left(\frac{x}{d} - \frac{x}{d}\ln\left(1 + \frac{xd}{\sum_{i=1}^{n}\hat{\mathbb{E}}\left|a_{ni}(Y_i' - \hat{\mathbb{E}}Y_i')\right|^2}\right)\right)\\
&\le \sum_{i=1}^{n}\mathbb{V}\left(\left|a_{ni}(Y_i' - \hat{\mathbb{E}}Y_i')\right| > ct^{1/r}\right) + cg(n)\left(t^{-2/r}\sum_{i=1}^{n}\hat{\mathbb{E}}\left|a_{ni}(Y_i' - \hat{\mathbb{E}}Y_i')\right|^2\right)^{\frac{1}{2\beta_2}},
\end{aligned}$$

thus

$$\begin{aligned}
H_2 &\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}\sum_{i=1}^{n}\mathbb{V}\left(\left|a_{ni}(Y_i' - \hat{\mathbb{E}}Y_i')\right| > ct^{1/r}\right)\mathrm{d}t\\
&\quad + c\sum_{n=1}^{\infty}{}^{\alpha p-\alpha r-2} g(n)\int_{n^{\alpha r}}^{\infty}\left(t^{-2/r}\sum_{i=1}^{n}\hat{\mathbb{E}}\left|a_{ni}(Y_i' - \hat{\mathbb{E}}Y_i')\right|^2\right)^{\frac{1}{2\beta_2}}\mathrm{d}t\\
&:= H_{21} + H_{22}.
\end{aligned}$$

So, to prove $H_2 < \infty$, we first need to prove $H_{21} < \infty$. By Markov inequality, C_r inequality, (3.12), (3.15), (3.17), Lemma 2.7 (ii), $q > p \vee r$ and $H_1 < \infty$, we have that

$$
\begin{aligned}
H_{21} &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}\left(t^{-q/r}\sum_{i=1}^{n} a_{ni}{}^{q}\hat{\mathbb{E}}|Y_i|^q\right)\mathrm{d}t\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1}\int_{n^{\alpha r}}^{\infty} t^{-q/r}\left(\hat{\mathbb{E}}|X|^q g\left(\frac{\mu|X|}{\beta_2 t^{1/r}}\right)+\beta_2{}^{q}t^{q/r}\hat{\mathbb{E}}\left(1-g\left(\frac{|X|}{\mu\beta_2 t^{1/r}}\right)\right)\right)\mathrm{d}t\\
&\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1}\int_{n^{\alpha r}}^{\infty} t^{-q/r}\hat{\mathbb{E}}|X|^q g\left(\frac{\mu|X|}{\beta_2 t^{1/r}}\right)\mathrm{d}t+c\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1}\int_{n^{\alpha r}}^{\infty}\hat{\mathbb{E}}\left(1-g\left(\frac{|X|}{\mu\beta_2 t^{1/r}}\right)\right)\mathrm{d}t\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha r}\sum_{m=n}^{\infty}\int_{m^{\alpha r}}^{(m+1)^{\alpha r}} t^{-q/r}\hat{\mathbb{E}}|X|^q g\left(\frac{\mu|X|}{\beta_2 t^{1/r}}\right)\mathrm{d}t\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha r}\sum_{m=n}^{\infty} m^{\alpha r-\alpha q-1}\hat{\mathbb{E}}|X|^q g\left(\frac{\mu|X|}{\beta_2(m+1)^{\alpha}}\right)\\
&= \sum_{m=1}^{\infty} m^{\alpha r-\alpha q-1}\hat{\mathbb{E}}|X|^q g\left(\frac{\mu|X|}{\beta_2(m+1)^{\alpha}}\right)\sum_{n=1}^{m} n^{\alpha p-1-\alpha r}\\
&\ll \begin{cases}\displaystyle\sum_{m=1}^{\infty} m^{\alpha(p\vee r)-\alpha q-1}\hat{\mathbb{E}}|X|^q g\left(\frac{\mu|X|}{\beta_2(m+1)^{\alpha}}\right) & \text{if } r\ne p;\\ \displaystyle\sum_{m=1}^{\infty} m^{\alpha r-\alpha q-1}\log m\hat{\mathbb{E}}|X|^q g\left(\frac{\mu|X|}{\beta_2(m+1)^{\alpha}}\right) & \text{if } r=p;\end{cases}\\
&= \begin{cases}\displaystyle\sum_{k=1}^{\infty}\sum_{m=2^{k-1}}^{2^k-1} m^{\alpha(p\vee r)-\alpha q-1}\hat{\mathbb{E}}|X|^q g\left(\frac{\mu|X|}{\beta_2(m+1)^{\alpha}}\right) & \text{if } r\ne p;\\ \displaystyle\sum_{k=1}^{\infty}\sum_{m=2^{k-1}}^{2^k-1} m^{\alpha r-\alpha q-1}\log m\hat{\mathbb{E}}|X|^q g\left(\frac{\mu|X|}{\beta_2(m+1)^{\alpha}}\right) & \text{if } r=p;\end{cases}\\
&\ll \begin{cases}\displaystyle\sum_{k=1}^{\infty} 2^{k[\alpha(p\vee r)-\alpha q]}\hat{\mathbb{E}}|X|^q g\left(\frac{\mu|X|}{\beta_2 2^{k\alpha}}\right) & \text{if } r\ne p;\\ \displaystyle\sum_{k=1}^{\infty} 2^{k(\alpha r-\alpha q)}\log 2^k\,\hat{\mathbb{E}}|X|^q g\left(\frac{\mu|X|}{\beta_2 2^{k\alpha}}\right) & \text{if } r=p;\end{cases}\\
&\ll \begin{cases}\displaystyle\sum_{k=1}^{\infty} 2^{k[\alpha(p\vee r)-\alpha q]}\hat{\mathbb{E}}\left(\frac{\beta_2{}^{q}}{\mu^q}+\sum_{j=1}^{k}|X|^q g_{j\beta_2}(X)\right) & \text{if } r\ne p;\\ \displaystyle\sum_{k=1}^{\infty} 2^{k(r-q)\alpha}\log 2^k\,\hat{\mathbb{E}}\left(\frac{\beta_2{}^{q}}{\mu^q}+\sum_{j=1}^{k}|X|^q g_{j\beta_2}(X)\right) & \text{if } r=p;\end{cases}\\
&\ll \begin{cases}\displaystyle\sum_{k=1}^{\infty} 2^{k[\alpha(p\vee r)-\alpha q]}+\sum_{k=1}^{\infty} 2^{k[\alpha(p\vee r)-\alpha q]}\sum_{j=1}^{k}\hat{\mathbb{E}}|X|^q g_{j\beta_2}(X) & \text{if } r\ne p;\\ \displaystyle\sum_{k=1}^{\infty} 2^{k(r-q)\alpha}\log 2^k+\sum_{k=1}^{\infty} 2^{k(r-q)\alpha}\log 2^k\sum_{j=1}^{k}\hat{\mathbb{E}}|X|^q g_{j\beta_2}(X) & \text{if } r=p;\end{cases}\\
&\ll \begin{cases}\displaystyle\sum_{j=1}^{\infty}\hat{\mathbb{E}}|X|^q g_{j\beta_2}(X)\sum_{k=j}^{\infty} 2^{k[\alpha(p\vee r)-\alpha q]} & \text{if } r\ne p;\\ \displaystyle\sum_{j=1}^{\infty}\hat{\mathbb{E}}|X|^q g_{j\beta_2}(X)\sum_{k=j}^{\infty} 2^{k(r-q)\alpha}\log 2^k & \text{if } r=p;\end{cases}\\
&\ll \begin{cases}\displaystyle\sum_{j=1}^{\infty} 2^{\alpha(p\vee r)j}\mathbb{V}\left(|X|>c2^{j\alpha}\right)<\infty & \text{if } r\ne p;\\ \displaystyle\sum_{j=1}^{\infty} 2^{\alpha rj}\log 2^j\,\mathbb{V}\left(|X|>c2^{j\alpha}\right)<\infty & \text{if } r=p.\end{cases}
\end{aligned}
$$

Then, we prove $H_{22} < \infty$. Similar to previous proof, we consider the following two situations.

If $(p \vee r) \geq 2$. By $\beta_2 < \frac{1}{r}, \alpha p - 2 + \delta + \frac{1-2\alpha}{2\beta_2} < -1$, (3.3), (3.7), (3.13), C_r inequality, we have

$$
\begin{aligned}
H_{22} &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} g(n) \int_{n^{\alpha r}}^{\infty} \left(t^{-2/r} \sum_{i=1}^{n} a_{ni}{}^2 \hat{\mathbb{E}} Y_i^2 \right)^{\frac{1}{2\beta_2}} \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2+\frac{1}{2\beta_2}} g(n) \int_{n^{\alpha r}}^{\infty} \left(t^{-2/r} \hat{\mathbb{E}} X^2 \right)^{\frac{1}{2\beta_2}} \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2+\frac{1}{2\beta_2}} g(n) \int_{n^{\alpha r}}^{\infty} t^{-\frac{1}{r\beta_2}} \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} n^{\alpha p-2+\delta+\frac{1-2\alpha}{2\beta_2}} L(n) < \infty.
\end{aligned}
$$

If $(p \vee r) < 2$. By $\beta_2 < \frac{p \vee r}{2r}, \alpha p - 2 + \delta + \frac{1-(p \vee r)\alpha}{2\beta_2} < -1$, $\hat{\mathbb{E}}|X|^{p \vee r} < \infty$, C_r inequality, we have

$$
\begin{aligned}
H_{22} &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} g(n) \int_{n^{\alpha r}}^{\infty} \left(t^{-2/r} \sum_{i=1}^{n} a_{ni}{}^2 \hat{\mathbb{E}} Y_i^2 \right)^{\frac{1}{2\beta_2}} \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} g(n) \int_{n^{\alpha r}}^{\infty} \left(t^{-\frac{p \vee r}{r}} \sum_{i=1}^{n} a_{ni}{}^2 \hat{\mathbb{E}} |X|^{p \vee r} \right)^{\frac{1}{2\beta_2}} \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2+\frac{1}{2\beta_2}} g(n) \int_{n^{\alpha r}}^{\infty} t^{-\frac{p \vee r}{2r\beta_2}} \mathrm{d}t \\
&\leq c \sum_{n=1}^{\infty} n^{\alpha p-2+\delta+\frac{1-(p \vee r)\alpha}{2\beta_2}} L(n) < \infty.
\end{aligned}
$$

We have proved (3.16). Because of considering $\{-X_n; n \geq 1\}$ instead of $\{X_n; n \geq 1\}$ in Theorem 3.2, Theorem 3.2 still holds. Then we can obtain

$$
\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} X_i < -t^{1/r} \right) \mathrm{d}t < \infty. \tag{3.19}
$$

According to (3.16) and (3.19), we can get $J_2 < \infty$. Hence, the finishes the proof of Theorem 3.2.

In conclusion, we prove the complete convergence and complete integral convergence for weighted sums of WA random variables under the sub-linear expectations.

4. Conclusions

In this paper, we extend the conclusion in probability space to sublinear expectation space and obtain the complete convergence and complete integral convergence for weighted sums of WA random variables under the sub-linear expectations, which enriches the limit theory research of WA random variable sequence in sublinear expectation space. In the future work, we will establish the corresponding inequalities in the sublinear expectation space according to the existing important inequalities and moment inequalities in the probability space, overcome the problems caused by the sub-additive of V and $\hat{\mathbb{E}}$, and generalize the complete convergence and complete integral convergence in the sublinear expectation to obtain a conclusion similar to that in the original probability space.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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