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Research article

A piecewise heat equation with constant and variable order coefficients: A new approach to capture crossover behaviors in heat diffusion

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Abstract: While the equation depicting the heat flow within homogeneous has been applied in several fields with some success, it has also faced several difficulties to depict heat diffusion in some non-homogeneous media. For particular behaviours adequate differential operators have been applied, for instance, a long-range behaviour has been depicted using operators based on power law kernel; stochastic behaviours have been included into mathematical equation using random function, some well-defined crossover behaviours have been depicted using the differential operators based on the generalized Mittag-Leffler kernel. Nevertheless, complex crossovers behaviours have not been modelled efficiently due to limitation of existing theories. Nevertheless, very recently piecewise calculus was proposed and applied in some complex world problems with great success. In this paper, heat equation with constant and variable coefficients will be subjected to piecewise numerical analysis. Several cases are considered, and their numerical simulation depicted.

Keywords: piecewise calculus; crossover behaviours; heat flow; complex media; nonlocality; randomness
Mathematics Subject Classification: 37E05, 34K28, 65R10, 26A33

1. Introduction

In many instances, humans have noticed that some real world problems exhibit changes in space and time. Thus, to understand their behaviours researchers have designed some mathematical model using the concept of partial differential equations. Due to multi-faces of natural behaviours, several concepts were introduced. For example deterministic approaches, within which several differential and integrals operators have been introduced like classical differential and integral operators, fractional differential and integral operators; fuzzy approach has been introduced and applied in many real world problems with great success and limitations, stochastic approaches have also been initiated [1–8, 13, 14]. All these approaches have been applied successfully in their respective capabilities, however with some limitations. In particular, changes occurring in these real-world problems could be mathematically evaluated using the concept of differentiation. However those differential operators cannot really capture crossover behaviours, especially those having particular changings in patterns for example a passage from power law behaviour to stochastic with no steady state, or a passage from stochastic to fading memory with not steady state, or a passage from fuzzy patterns to stochastic, many more can be listed here. Indeed, such behaviours are observed in many real world problems for example, underground water flows from purely matrix soils to a geological formation with fractures with self-similar patterns. The water flowing in a purely matrix soil follows clearly fading memory behaviours while, within the fractures with self-similar patterns, water flows with stochastic behaviours. This gives in general the flow to have crossover behaviours with no steady state. The first part can then be modelled using fading memory processes while the second part can be modelled using stochastic behaviours. Indeed none of the mentioned different operators will be able to portray such behaviour. To solve this problem, Atangana and Seda introduced the idea of piecewise differential and integral operators [9]. The concept has been applied very successfully for linear and nonlinear ordinary differential equations. In particular, in epidemiology where the spread of some diseases with crossover behaviours have been modelled successfully, up to date no work has been done on the direction of partial differential equations. In this paper therefore, we will apply this approach on a simple heat equation, to see if such can be useful. The heat equation will be considered for constant and variable coefficients, three patterns will be considered [10–12]. It shall be noted that, already Wu et al. have introduced the concept of short memory derivative where they divided an interval in several sub-intervals within which the fractional order change [15, 16].

2. Piecewise heat equation with no variable coefficients

One of the wider studied partial differential equations within the field of pure mathematics and applied mathematics is perhaps the heat equation. The model was purposely initiated to replicate the diffusion of heat via a given media. This equation can be regarded as caloric function also be considered on the well-known Riemann manifolds as it led to several geometric application. Several versions and modifications have been suggested to suit some specific scenarios, leading to several application in many fields of science and applied mathematics. For example, this equation is associated with the investigation of the well-known Brownian motion and random walks through Fokker-Planck model. In finance theory, used as Black-Scholes model is known to be a modified of the heat equation. This equation has found application in image processing, as it can be used to resolve pixilation and point out edges. In this section, we shall provide a different version of heat equation, that will be able to replicate heat diffusion in a media with three parts with different structures. We will assume that, from 0 to L_1 there is a normal diffusion, from L_1 to L_2 there is fading memory diffusion due to elasticity of the media, from L_2 to L_3, the diffusion follows randomness. The mathematical model able to replicate such scenario is given below in:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - V\frac{\partial C}{\partial x} \quad t \in [0, T_1], x \in [0.L_1], \tag{2.1}$$

$$_{T1}{}^{FC}D_t^{\alpha} C = D\frac{\partial^2 C}{\partial x^2} - V\frac{\partial C}{\partial x} \quad t \in [T_1, T_2], x \in [L_1, L_2], \tag{2.2}$$

$$\partial C = D\frac{\partial^2 C}{\partial x^2} - V\frac{\partial C}{\partial x} + \sigma C \partial B(x,t) \quad t \in [T_2, T], x \in [L_2, L]. \tag{2.3}$$

It is noted that, in Eq (2.2), the Caputo-Fabrizio derivative is used to account for fading memory processes. For readers that are not aware of the Caputo-Fabrizio derivative, we provide its definition as:

$$^{FC}_{T_1}D_t^{\alpha} c(x,t) = \frac{M(\alpha)}{(1-\alpha)} \int_{T_1}^{t} \frac{\partial C(x,y)}{\partial y} \exp\left[-\frac{1-\alpha}{\alpha}(t-y)\right] dy. \tag{2.4}$$

In Eq (2.3) the parameter σ is the density of randomness and the function $B(t)$ is an environmental noise function. The initial condition are defined accordingly to the interval. To solve the above piecewise heat equation, we proceed with an appropriate numerical scheme for each interval. In the first part, we consider simple classical numerical scheme based on forward method. The derivation is presented from Eq (2.5) to Eq (2.7) as follow. First, we apply the divide the interval in small portions then apply the integral on both side between t_j and $t_{(j+1)}$.

$$\begin{aligned} \int_{t_j}^{t_{j+1}} \frac{\partial C}{\partial t}(x,t)dt =& - V \int_{t_j}^{t_{j+1}} \frac{\partial C}{\partial x}(x,t)dt + D \int_{t_j}^{t_{j+1}} \frac{\partial^2 C}{\partial x^2}(x,t)dt, \\ C(x_i, t_{j+1}) - C(x_i, t_j) =& - V \int_{t_j}^{t_{j+1}} \left(\frac{C(x+h,t) - C(x-h,t)}{2h} \right) dt + O(h^2) \\ & + D \int_{t_j}^{t_{j+1}} \left\{ \frac{C(x+h,t) - 2C(x,t) + C(x-h,t)}{h^2} + O(h^2) \right\} dt, \end{aligned} \tag{2.5}$$

$$\begin{aligned} C(x_i, t_{j+1}) - C(x_i, t_j) =& -\frac{V}{2h}\left[C(x+h,t) - C(x-h,t)\right]\Delta t \\ & + \frac{D}{h^2}\left[C(x+h,t) - 2C(x,t) + C(x-h,t)h^2\right]\Delta t + O(h^2), \\ C(x_i, t_{j+1}) = C_i^{j+1} =& -\frac{V}{2h}\Delta t\left(C_{i+1}^j - C_{i-1}^j\right) + D\frac{\Delta t}{h^2}\left(C_{i+1}^j - 2C_i^j + C_i^j\right) + O(\Delta t h^2), \end{aligned} \tag{2.6}$$

$$C_i^{j+1} = C_i^j - V\frac{\Delta t}{h}(C_{i+1}^j - C_{i-1}^j) + D\frac{\Delta t}{h^2}(C_{i+1}^j - 2C_i^j + C_{i+1}^j). \tag{2.7}$$

Here we proceed with the fading memory part; again, we transform the differential equation into an integral equation

$$_{T1}{}^{FC}D_t^{\alpha} C = D\frac{\partial^2 C}{\partial x^2} - V\frac{\partial C}{\partial x}, \tag{2.8}$$

$$
\begin{aligned}
C(x_i,t_{k+1})-C(x_i,t_k)=&\frac{(1-\alpha)}{M(\alpha)}\left(-V\frac{\partial C}{\partial x}(x_i,t_{k+1})+D\frac{\partial^2 C}{\partial x^2}(x_i,t_{k+1})\right)\\
&+\frac{\alpha}{M(\alpha)}\int_{t_k}^{t_{k+1}}\left(-V\frac{\partial C}{\partial x}(x_i,\tau)+D\frac{\partial^2 C}{\partial x^2}(x_i,\tau)\right)d\tau,\\
C(x_i,t_{k+1})-C(x_i,t_k)=&\frac{(1-\alpha)}{M(\alpha)}\left(-V\frac{\partial C}{\partial x}(x_i,t_{k+1})+D\frac{\partial^2 C}{\partial x^2}(x_i,t_{k+1})\right)\\
&+\frac{\alpha}{M(\alpha)}\sum_{J=0}^{k}\int_{t_j}^{t_{j+1}}\left[-V\frac{\partial C(x_i,\tau)}{\partial x}+D\frac{\partial^2 C(x_i,\tau)}{\partial x^2}\right]d\tau,
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
C(x_i,t_{k+1})=&C(x_i,t_k)+\frac{(1-\alpha)}{M(\alpha)}\left[\frac{V}{2h}\left[C(x_{i+1},t_k)-C(x_{i-1},t_k)\right]\right]\\
&+\frac{D}{h^2}\left[C(x_{i+1},t_k)-2C(x_i,t_k)+C(x_{i-1},t_k)\right]+O(h^2)\\
&+\frac{\alpha}{M(\alpha)}\sum_{j=0}^{k}\left[D\frac{C(x_{i+1},t_j)-2C(x_i,t_j)+C(x_{i-1},t_j)}{h^2}\right.\\
&\left.-\frac{V}{2h}\Delta t\Big(C(x_{i+1},t_j)-C(x_{i-1},t_j)\Big)\right]+O(h^2\Delta t).
\end{aligned}
\tag{2.10}
$$

The last part, we proceed with stochastic part of the piecewise heat equation. We again apply the integral on both sides, and then apply the numerical technique on the deterministic and stochastic part to obtain:

$$
\begin{aligned}
C_i^{k+1}=&C_i^{T_1}+\frac{(1-\alpha)}{M(\alpha)}\left[-\frac{V}{2h}(C_{i+1}^k-C_{i-1}^k)\frac{Dt}{h^2}(C_{i+1}^k-2C_i^k+C_{i-1}^k)\right.\\
&\left.+\frac{\alpha}{M(\alpha)}\sum_{j=0}^{k}\left[-\frac{V}{2h}(C_{i+1}^j-C_{i-1}^j)\frac{Dt}{h^2}(C_{i+1}^j-2C_i^j+C_{i-1}^j)\right]\right],
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
&C(x_i,t_{t_{k+1}}-C(x_i,t_k)\\
&\quad=\int_{t_k}^{k+1}\left[-V\frac{\partial C}{\partial x}(x_i,\tau)+D\frac{\partial^2 C}{\partial x^2}\right]d\tau+\sigma\int_{t_k}^{t_{k+1}}C(x_i,\tau)dB(x_i,\tau)d\tau,
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
C_i^{k+1}=&C_i^k-V\left(\frac{C_{i+1}^k-C_i^k}{2h}\right)+D\left(\frac{C_{i+1}^k-2C_i^k+C_{i-1}^k}{h^2}\right)\\
&+\sigma C_i^j(B_i^{k+1}-B_i^k)+O(\Delta t)+O(h^2)+O(\Delta t h^2).
\end{aligned}
\tag{2.13}
$$

2.1. Piecewise heat model with classical, power law and stochastic behavior

In this section, we assume that the second process observed in the second part of the material long-range behavior, therefore the classical time differential operator is replaced by the power-law Caputo derivative, since long range can be well depicted via power law. The piecewise heat diffusion model associated to this is given below. Since we have presented the derivation of the numerical solution for part one and three, we will only present full detail of discretization of the second part that involve the Caputo fractional derivative. The derivation of this solution is presented from (2.13) to (2.24).

$$
\frac{\partial C}{\partial t}=D\frac{\partial^2 C}{\partial x^2}-V\frac{\partial C}{\partial x}\quad t\in[0,T_1],x\in[0.L_1],
\tag{2.14}
$$

$$ {}_{T1}{}^{C}D_t^{\alpha}C = D\frac{\partial^2 C}{\partial x^2} - V\frac{\partial C}{\partial x} \quad t \in [T_1, T_2], x \in [L_1, L_2], \tag{2.15} $$

$$ \partial C = D\frac{\partial^2 C}{\partial x^2} - V\frac{\partial C}{\partial x} + \sigma C \partial B(x,t) \quad t \in [T_2, T], x \in [0.L_1], \tag{2.16} $$

$$ \begin{aligned} &C(x_i, t_{k+1}) - C(x_i, t_k) \\ &\quad = \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_{k+1}} \left[-V\frac{\partial C}{\partial x}(x_i,\tau) + D\frac{\partial^2 C}{\partial x^2}(x_i,\tau) \right] (t_{k+1}-\tau)^{\alpha-1} d\tau, \end{aligned} \tag{2.17} $$

$$ \begin{aligned} &C(x_i, t_{k+1}) - C(x_i, T_1) \\ &\quad = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \left[-V\frac{\partial C}{\partial x}(x_i,\tau) + D\frac{\partial^2}{\partial x^2}(x_i,\tau) \right] (t_{k+1}-\tau)^{\alpha-1} d\tau, \end{aligned} \tag{2.18} $$

$$ \begin{aligned} C_i^{k+1} =& C_i^{T_i} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \left[-V\frac{\partial C}{\partial x}(x_i,\tau) + D\frac{\partial^2 C}{\partial x^2}(x_i,\tau) \right] (t_{k+1}-\tau)^{\alpha-1} d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_{T_k}^{t_{k+1}} \left[-V\frac{\partial C}{\partial x}(x_i,\tau) + D\frac{\partial^2 C}{\partial x^2}(x_i,\tau) \right] (t_{k+1}-\tau)^{\alpha-1} d\tau, \end{aligned} \tag{2.19} $$

$$ \begin{aligned} C_i^{k+1} =& C_i^{T_1} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \left[-\frac{V}{2h}\left[C(x_{i+1},\tau) - C(x_{i-1},\tau)\right] \right. \\ &\left. + \frac{D}{h^2}\left[C(x_{i+1},\tau) - 2C(x_i,\tau) + C(x_{i-1},\tau)\right] + O(h^2) \right] (t_{k+1}-\tau)^{\alpha-1} d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_{k+1}} \left[-\frac{V}{2h}\left[C(x_{i+1},\tau) - C(x_{i-1},\tau)\right] \right. \\ &\left. + \frac{D}{h^2}\left[C(x_{i+1},\tau) - 2C(x_i,\tau) + C(x_{i-1},\tau)\right] + O(h^2) \right] (t_{k+1}-\tau)^{\alpha-1} d\tau, \end{aligned} \tag{2.20} $$

$$ \begin{aligned} I_1 =& \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \left[-\frac{V}{2h}\left[C(x_{i+1},\tau) - C(x_{i-1},\tau)\right] \right. \\ &\left. + \frac{D}{h^2}\left[C(x_{i+1},\tau) - 2C(x_i,\tau) + C(x_{i-1},\tau)\right] + O(h^2) \right] (t_{k+1}-\tau)^{\alpha-1} d\tau, \end{aligned} \tag{2.21} $$

$$ \begin{aligned} I_2 =& \int_{t_k}^{t_{k+1}} \left[-\frac{V}{2h}\left[C(x_{i+1},\tau) - C(x_{i-1},\tau)\right] \right. \\ &\left. + \frac{D}{h^2}\left[C(x_{i+1},\tau) - 2C(x_i,\tau) + C(x_{i-1},\tau)\right] + O(h^2) \right] (t_{k+1}-\tau)^{\alpha-1} d\tau, \end{aligned} \tag{2.22} $$

$$ \begin{aligned} I_2 =& \left(-\frac{V}{2h}\left(C_{i+1}^k - C_{i-1}^k\right) + \frac{D}{h^2}\left[C_{i+1}^k - 2C_i^k + C_{i-1}^k\right] \right) \int_{t_k}^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-1} d\tau \\ =& \left(-\frac{V}{2h}\left(C_{i+1}^k - C_{i-1}^k\right) + \frac{D}{h^2}\left[C_{i+1}^k - 2C_i^k + C_{i-1}^k\right] \right) \frac{\Delta t^{\alpha}}{\alpha}, \end{aligned} \tag{2.23} $$

$$ g_1(\tau) = \frac{C(x_{i+1},\tau) - C(x_{i-1},\tau)}{2h}, \tag{2.24} $$

$$ g_i(\tau) = \frac{C(x_{i+1},\tau) - 2C(x_i,\tau) + C(x_{i-1},\tau)}{h^2}, \tag{2.25} $$

$$
\begin{aligned}
C_i^{k+1} =& C_i^{T_1} + \sum_{j=0}^{k-1} \left\{ -V \frac{C_{i+1}^{j+1} - C_{i-1}^{j+1}}{2h} \right\} \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} \left[(k-j+1)^{\alpha+1} - (k-j)^{\alpha} \right] \\
&+ \left(-V \frac{C_{i+1}^{j} - C_{i-1}^{j}}{2h} \right\} \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} \left[(k-j)^{\alpha+1} - (k-j+1)^{\alpha}(k-j-\alpha) \right] \\
&+ D \frac{\left(C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1} \right)}{h^2} \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} \left[(k-j+1)^{\alpha+1} - (k-j)^{\alpha} \right] (k-j+\alpha+1) \\
&+ D \frac{\left(C_{i+1}^{j} - 2C_i^{j} + C_{i-1}^{j} \right)}{h^2} \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} \left[(k-j)^{\alpha+1} - (k-j+1)^{\alpha}(k-j-\alpha) \right] \\
&+ \frac{\Delta t^{\alpha}}{\Gamma(\alpha+1)} \left(-\frac{V}{2h} (C_{i+1}^{k} - C_{i-1}^{k}) \right) + \frac{D}{h^2} (C_{i+1}^{k} - 2C_i^{j} + C_{i-1}^{j}).
\end{aligned} \tag{2.26}
$$

2.2. Piecewise heat model with classical, Mittag-Leffler and stochastic processes

In this case, we will assume that the second part has a crossover behavior from stretched exponential to power law, the rest stay the same. Thus to include into mathematical formulation these processes, we consider the first part to be modeled using the classical differential operator, the second part will be modeled using the Atangana-Baleanu fractional derivative and the last part we considered the inclusion of random function. For this part the Atangana-Baleanu fractional derivative. The numerical solution is derived below as:

$$
\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} - V \frac{\partial C}{\partial x} \quad t \in [0, T_1], x \in [0.L_1], \tag{2.27}
$$

$$
{}_{T1}^{ABC} D_t^{\alpha} C = D \frac{\partial^2 C}{\partial x^2} - V \frac{\partial C}{\partial x} \quad t \in [T_1, T_2], x \in [L_1, L_2], \tag{2.28}
$$

$$
\partial C = D \frac{\partial^2 C}{\partial x^2} - V \frac{\partial C}{\partial x} + \sigma C \partial B(x,t) \quad t \in [T_2, T], x \in [L_2, L], \tag{2.29}
$$

$$
\begin{aligned}
&C(x_i, t_{k+1}) - C(x_i, T_1) \\
&\quad = \frac{(1-\alpha)}{AB(\alpha)} \left[-V \frac{\partial C}{\partial x}(x_i, t_{k+1}) + D \frac{\partial^2 C}{\partial x^2}(x_i, t_{k+1}) \right] \\
&\quad + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_{T_1}^{t_{k+1}} \left[-V \frac{\partial C}{\partial x}(x_i, \tau) + D \frac{\partial^2 C}{\partial x^2}(x_i, \tau) \right] (t_{k+1} - \tau)^{\alpha-1} (t_{k+1} - \tau)^{\alpha} d\tau,
\end{aligned} \tag{2.30}
$$

$$
\begin{aligned}
&C(x_i, t_{k+1}) - C(x_i, T_1) \\
&\quad = \frac{(1-\alpha)}{AB(\alpha)} \left\{ -V \left(\frac{C(x_{i+1}, t_{k+1}) - C(x_{i-1}, t_{k+1})}{2h} \right) \right\} \\
&\quad + D \left(\frac{C(x_{i+1}, t_{k+1}) - 2C(x_i, t_{k+1}) + C(x_{i-1}, t_{k+1})}{h^2} \right) + O(h^2) \\
&\quad + \frac{\alpha}{AB(\alpha)} \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_{k+1}} \left[-V \frac{\partial C}{\partial x}(x_i, \tau) + D \frac{\partial^2 C}{\partial x^2}(x_i, \tau) \right] (t_{k+1} - \tau)^{\alpha-1} (t_{k+1} - \tau)^{\alpha} d\tau,
\end{aligned} \tag{2.31}
$$

$$
\mu_1 = \frac{V(1-\alpha)}{AB(\alpha)2h},
$$

$$\mu_2 = \frac{D(1-\alpha)}{Ab(\alpha)h^2},$$

$$\begin{aligned}
&C_i^{k+1} + \mu_1\left(C_{i+1}^{k+1} - C_{i-1}^{k+1}\right) - \mu_2\left(C_{i+1}^{k+1} - 2C_i^{k+1} + C_{i-1}^{k+1}\right)\\
&\quad C_i^{T_1} + \sum_{j=0}^{k-1} \frac{\alpha\Delta t^\alpha}{AB(\alpha)\Gamma(\alpha+2)}\left\{-V\frac{C_{i+1}^{j+1} - C_{i-1}^{j+1}}{2h} + D\frac{C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1}}{h^2}\right\}\\
&\quad \times\left((k-j+1)^{\alpha+1} - (k-j)^\alpha(k-j+\alpha+1)\right)\\
&\quad +\left(-V\frac{C_{i+1}^{j+1} - C_{i-1}^{j+1}}{2h} + D\frac{C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1}}{h^2}\right)\\
&\quad \times\left((k-j)^{\alpha+1} - (k-j)^\alpha(k-j+\alpha+1)\right),
\end{aligned} \tag{2.32}$$

$$\begin{aligned}
&-(\mu_1+\mu_2)C_{i-1}^{k+1} + (1+2\mu_2)C_i^{k+1} + (\mu_1-\mu_2)C_{i+1}^{k+1}\\
&\quad = C_i^{T_1} + \frac{\alpha}{AB(\alpha)}\frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{k-1} A_j + \frac{\alpha}{AB(\alpha)}\frac{(\Delta t)^\alpha}{\Gamma(\alpha+1)}\dot{R},
\end{aligned} \tag{2.33}$$

$$\begin{aligned}
A_j =& \left(-V\frac{C_{i+1}^{j+1} - C_{i-1}^{j+1}}{2h} + D\frac{C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1}}{h^2}\right)\\
&\times\left((k-j+1)^{\alpha+1} - (k-j)^\alpha(k-j+\alpha+1)\right)\\
&+\left(-V\frac{C_{i+1}^{j+1} - C_{i-1}^{j+1}}{2h} + D\frac{C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1}}{h^2}\right)\\
&\times\left((k-j)^{\alpha+1} - (k-j)^\alpha(k-j+\alpha+1)\right),
\end{aligned} \tag{2.34}$$

$$R_k = \left[-\frac{V}{2h}(C_{i+1}^k - C_{i-1}^k) + \frac{D}{h^2}\left(C_{i+1}^k - 2C_i^k + C_{i-1}^k\right)\right], \tag{2.35}$$

$$O(h^2(\Delta t)^{\alpha+1})$$

$$H = \begin{pmatrix}
\Lambda+2\mu_2 & (\mu_1-\mu_2) & 0 & --- & 0 & 0\\
-(\mu_1+\mu_2) & \Lambda+2\mu_2 & (\mu_1-\mu_2) & 0 & --- & 0\\
0 & -(\mu_1+\mu_2) & 0 & 0 & --- & 0\\
| & | & | & | & | & 0\\
| & | & | & | & | & 0\\
| & | & | & | & | & (\mu_1-\mu_2)\\
0 & 0 & 0 & 0 & -(\mu_1+\mu_2) & \Lambda+2\mu_2
\end{pmatrix}. \tag{2.36}$$

3. Piece wise heat equation with variable coefficients

As a variant of the heat equation, a model with heat flow with variable coefficients have been a centre several studies in the last decades, in pure and applied mathematics. Variable coefficients are included in the mathematical model to capture complexities of geological formation. For example, attention has been devoted to the study underpinning the conditions under which this equation has unique solutions and how solutions can be obtained. Analytical and numerical methods have been developed to address this problem and application of this equation in different fields has been paid attention. With the same aim to include into mathematical formation the complexities effects of the geological formation, we shall convert the heat equation with variable coefficients into piecewise heat equation with variable coefficients. Three cases will be considered as presented before where coefficients were constants as we presented before

$$\frac{\partial C(x,t)}{\partial t} = D(x,t)\frac{\partial^2 C}{\partial x^2} - V(x,t)\frac{\partial C(x,t)}{\partial x} \quad t \in [0, T_1], x \in [0.L_1], \tag{3.1}$$

$$_{T1}{}^{FC}D_t^{\alpha} C(x,t) = D(x,t)\frac{\partial^2 C(x,t)}{\partial x^2} - V(x,t)\frac{\partial C(x,t)}{\partial x} \quad t \in [T_1, T_2], x \in [L_1, L_2], \tag{3.2}$$

$$\partial C(x,t) = D(x,t)\frac{\partial^2 C}{\partial x^2} - V(x,t)\frac{\partial C}{\partial x} + \sigma C(x,t)\partial B(x,t) \quad t \in [T_2, T], x \in [L_2, L], \tag{3.3}$$

$$\begin{aligned}
&\int_{t_j}^{t_{j+1}} \frac{\partial C(x,t)}{\partial t}dt = -\int_{t_j}^{t_{j+1}} V(x,t)\frac{\partial C(x,t)}{\partial x}dt + \int_{t_j}^{t_{j+1}} \frac{\partial^2 C(x,t)}{\partial x^2}D(x,t)dt,\\
&C(x_i,t_{j+1}) - C(x_i,t_j) = -\int_{t_j}^{t_{j+1}} \frac{[C(x+h,t) - C(x-h,t)]}{2h}V(x,t)dt + O(h)\\
&+\int_{t_j}^{t_{j+1}} \left\{\frac{C(x+h,t) - 2C(x,t) + C(x-h,t)}{h^2}D(x,t)dt + O(h^2)\right\},
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
C(x_i,t_{j+1}) &- C(x_i,t_j)\\
&= -\frac{V(x_i,t_j)}{2h}\left[C(x+h,t) - C(x-h,t)\right]\Delta t\\
&+ \frac{D(x_i,t_j)}{h^2}\left[C(x+h,t) - 2C(x,t) + C(x-h,t)h^2\right]\Delta t + O(h^2),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
C(x_i,t_{j+1}) =&C_i^{j+1}\\
=& -\frac{V(x_i,t_j)}{2h}\Delta t\left(C_{i+1}^j - C_{i-1}^j\right) + D(x_i,t_j)\frac{\Delta t}{h^2}\left(C_{i+1}^j - 2C_i^j + C_i^j\right) + O(\Delta t h^2),
\end{aligned}$$

$$C_i^{j+1} = C_i^j - V(x_i,t_j)\frac{\Delta t}{h}(C_{i+1}^j - C_{i-1}^j) + D(x_i,t_j)\frac{\Delta t}{h^2}(C_{i+1}^j - 2C_i^j + C_{i+1}^j), \tag{3.6}$$

$$_{T1}{}^{FC}D(x,t)_t^{\alpha} C = D(x,t)\frac{\partial^2 C(x,t)}{\partial x^2} - V(x,t)\frac{\partial C(x,t)}{\partial x}, \tag{3.7}$$

$$\begin{aligned}
C(x_i,t_{k+1}) - C(x_i,T_1) =&\frac{(1-\alpha)}{M(\alpha)}\left(-V(x_i,t_j)\frac{\partial C(x_i,t_{k+1})}{\partial x} + D(x_i,t_j)\frac{\partial^2 C(x_i,t_{k+1})}{\partial x^2}\right)\\
&+ \frac{\alpha}{M(\alpha)}\int_{t_k}^{t_{k+1}}\left(-V(x_i,t_j)\frac{\partial C(x_i,\tau)}{\partial x} + D(x_i,t_j)\frac{\partial^2 C(x_i,\tau)}{\partial x^2}\right)dt,\\
C(x_i,t_{k+1}) - C(x_i,T_1) =&\frac{(1-\alpha)}{M(\alpha)}\left(-V(x_i,t_j)\frac{\partial C(x_i,t_{k+1})}{\partial x} + D(x_i,t_j)\frac{\partial^2 C(x_i,t_{k+1})}{\partial x^2})\right)\\
&+ \frac{\alpha}{M(\alpha)}\sum_{J=0}^{k}\int_{t_j}^{t_{j+1}}\left[-V(x_i,t_j)\frac{\partial C(x,\tau)}{\partial x} + D(x_i,t_j)\frac{\partial^2 C(x,\tau)}{\partial x^2}\right]d\tau,
\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
C(x_i,t_{k+1}) =& C(x_i,T_1) + \frac{(1-\alpha)}{M(\alpha)}\left[\frac{V(x_i,t_j)}{2h}\left[C(x_{i+1},t_k) - C(x_{i-1},t_k)\right]\right] \\
&+ \frac{D(x_i,t_j)}{h^2}\left[C(x_{i+1},t_k) - 2C(x_i,t_k) + C(x_{i-1},t_k)h^2\right] + O(h^2) \\
&+ \frac{\alpha}{M(\alpha)}\sum_{j=0}^{k}\left[D(x_i,t_j)\frac{C(x_{i+1},t_j) - 2C(x_i,t_j) + C(x_{i-1},t_j)}{h^2}\right. \\
&\left.-\frac{V(x_i,t_j)}{2h}\Delta t\left(C(x_{i+1},t_j) - C(x_{i-1},t_j)\right) + O(h^2\Delta t)\right],
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
C_i^{k+1} =& C_i^{T_1} + \frac{(1-\alpha)}{M(\alpha)}\left[-\frac{V(x_i,t_j)}{2h}(C_{i+1}^k - C_{i-1}^k)\frac{D(x_i,t_j)t}{h^2}(C_{i+1}^k - 2C_i^k + C_{i-1}^k)\right. \\
&\left.+\frac{\alpha}{M(\alpha)}\sum_{j=0}^{k}\left[-\frac{V(x_i,t_j)}{2h}(C_{i+1}^j - C_{i-1}^j)\frac{D(x_i,t_j)t}{h^2}(C_{i+1}^j - 2C_i^j + C_{i-1}^j)\right]\right],
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
C(x_i,t_{t_{k+1}}) - C(x_i,t_k) =& \int_{t_k}^{k+1}\left[-V(x_i,t_j)\frac{\partial C}{\partial x}(x_i,\tau) + D(x_i,t_j)\frac{\partial^2 C}{\partial x^2}\right]d\tau \\
&+ \sigma\int_{t_k}^{t_{k+1}} C(x_i,\tau)dB(x,\tau)d\tau,
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
C_i^{k+1} =& C_i^k - V(x_i,t_j)\left(\frac{C_{i+1}^k - C_i^k}{2h}\right) + D(x_i,t_j)\left(\frac{C_{i+1}^k - 2C_i^k + C_{i-1}^k}{h^2}\right) \\
&+ \sigma C_i^j(B_i^{k+1} - B_i^k) + O(\Delta t) + O(h^2) + O(\Delta t h^2).
\end{aligned} \tag{3.12}
$$

4. Piecewise heat model with classical, power law and stochastic behavior with variable coefficient

In this section, we shall consider a piecewise heat equation with variable coefficients. The first part shall be with classical derivative, second with Caputo derivative and last stochastic.

$$
\frac{\partial C(x,t)}{\partial t} = D(x,t)\frac{\partial^2 C(x,t)}{\partial x^2} - V(x,t)\frac{\partial C(x,t)}{\partial x} \quad t \in [0,T_1], x \in [0.L_1], \tag{4.1}
$$

$$
{}_{T1}{}^{C}D_t^{\alpha} C(x,t) = D(x,t)\frac{\partial^2 C(x,t)}{\partial x^2} - V(x,t)\frac{\partial C(x,t)}{\partial x} \quad t \in [T_1,T_2], x \in [L_1,L_2], \tag{4.2}
$$

$$
\partial C = D(x,t)\frac{\partial^2 C(x,t)}{\partial x^2} - V(x,t)\frac{\partial C(x,t)}{\partial x} + \sigma C(x,t)\partial B(x,t) \quad t \in [T_2,T], x \in [0.L_1], \tag{4.3}
$$

$$
\begin{aligned}
&C(x_i,t_{k+1}) - C(x_i,t_k) \\
&= \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t_{k+1}}\left(-V(x_i,\tau)\frac{\partial}{\partial x}C(x_i,\tau) + D(x_i,\tau)\frac{\partial^2 C(x_i,\tau)}{\partial x^2}\right)(t_{k+1}-\tau)^{\alpha-1}d\tau,
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
&C(x_i,t_{k+1}) - C(x_i,t_k) \\
&= \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\left(-V(x_i,\tau)\frac{\partial C(x_i,\tau)}{\partial x} + D(x_i,\tau)\frac{\partial^2 C(x_i,\tau)}{\partial x^2}\right)(t_{k+1}-\tau)^{\alpha-1}d\tau,
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
C_i^{k+1} =& C_i^{T_i} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \left[-V(x_i,\tau) \frac{\partial C(x_i,\tau)}{\partial x} + D(x_i,\tau) \frac{\partial^2 C(x_i,\tau)}{\partial x^2} \right] (t_{k+1}-\tau)^{\alpha-1} d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_{k+1}} \left[-V(x_i,\tau) \frac{\partial C(x_i,\tau)}{\partial x} + D(x_i,\tau) \frac{\partial^2 C(x_i,\tau)}{\partial x^2} \right] (t_{k+1}-\tau)^{\alpha-1} d\tau,
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
C_i^{k+1} =& C_i^{T_1} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \left[-\frac{V(x_i,\tau)}{2h} \left[C(x_{i+1},\tau) - C(x_{i-1},\tau) \right] \right. \\
& \left. + \frac{D(x_i,\tau)}{h^2} \left[C(x_{i+1},\tau) - 2C(x_i,\tau) + C(x_{i-1},\tau) \right] + O(h^2) \right] (t_{k+1}-\tau)^{\alpha-1} d\tau \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_{k+1}} \left[-\frac{V(x_i,\tau)}{2h} \left[C(x_{i+1},\tau) - C(x_{i-1},\tau) \right] \right. \\
& \left. + \frac{D(x_i,\tau)}{h^2} \left[C(x_{i+1},\tau) - 2C(x_i,\tau) + C(x_{i-1},\tau) \right] + O(h^2) \right] (t_{k+1}-\tau)^{\alpha-1} d\tau,
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
I_1 =& \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \left[-\frac{V(x_i,\tau)}{2h} \left[C(x_{i+1},\tau) - C(x_{i-1},\tau) \right] \right. \\
& \left. + \frac{D(x_i,\tau)}{h^2} \left[C(x_{i+1},\tau) - 2C(x_i,\tau) + C(x_{i-1},\tau) \right] + O(h^2) \right] (t_{k+1}-\tau)^{\alpha-1} d\tau,
\end{aligned} \tag{4.8}
$$

$$
\begin{aligned}
I_2 =& \int_{t_k}^{t_{k+1}} \left[-\frac{V(x_i,\tau)}{2h} \left[C(x_{i+1},\tau) - C(x_{i-1},\tau) \right] \right. \\
& \left. + \frac{D(x_i,\tau)}{h^2} \left[C(x_{i+1},\tau) - 2C(x_i,\tau) + C(x_{i-1},\tau) \right] + O(h^2) \right] (t_{k+1}-\tau)^{\alpha-1} d\tau,
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
I_2 =& \left(-\frac{V(x_i,t_j)}{2h} \left(C_{i+1}^k - C_{i-1}^k \right) + \frac{D(x_i,t_j)}{h^2} \left[C_{i+1}^k - 2C_i^k + C_{i-1}^k \right] \right) \int_{t_k}^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-1} d\tau \\
=& \left(-\frac{V(x_i,t_j)}{2h} \left(C_{i+1}^k - C_{i-1}^k \right) + \frac{D(x_i,t_j)}{h^2} \left[C_{i+1}^k - 2C_i^k + C_{i-1}^k \right] \right) \frac{\Delta t^\alpha}{\alpha},
\end{aligned} \tag{4.10}
$$

$$
g_1(\tau) = \frac{C(x_{i+1},\tau) - C(x_{i-1},\tau)}{2h}, \tag{4.11}
$$

$$
g_i(\tau) = \frac{C(x_{i+1},\tau) - 2C(x_i,\tau) + C(x_{i-1},\tau)}{h^2}, \tag{4.12}
$$

$$
\begin{aligned}
C_i^{k+1} =& C_i^{T_1} + \sum_{j=0}^{k-1} \left\{ -V(x_i,t_j) \frac{C_{i+1}^{j+1} - C_{i-1}^{j+1}}{2h} \right\} \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} \left[(k-j+1)^{\alpha+1} - (k-j)^{\alpha} \right] \\
&+ \left\{ -V(x_i,t_j) \frac{C_{i+1}^{j} - C_{i-1}^{j}}{2h} \right\} \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} \left[(k-j)^{\alpha+1} - (k-j+1)^{\alpha}(k-j-\alpha) \right] \\
&+ D(x_i,t_j) \frac{\left(C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1} \right)}{h^2} \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} \left[(k-j+1)^{\alpha+1} - (k-j)^{\alpha} \right] (k-j+\alpha+1) \\
&+ D(x_i,t_j) \frac{\left(C_{i+1}^{j} - 2C_i^{j} + C_{i-1}^{j} \right)}{h^2} \frac{\Delta t^{\alpha}}{\Gamma(\alpha+2)} \left[(k-j)^{\alpha+1} - (k-j+1)^{\alpha}(k-j-\alpha) \right] \\
&+ \frac{\Delta t^{\alpha}}{\Gamma(\alpha+1)} \left(-\frac{V(x_i,t_j)}{2h} (C_{i+1}^{k} - C_{i-1}^{k}) \right) + \frac{D(x_i,t_j)}{h^2} (C_{i+1}^{k} - 2C_i^{j} + C_{i-1}^{j}).
\end{aligned} \tag{4.13}
$$

5. Piecewise heat model with classical, Mittag-Leffler and stochastic processes with variable Coefficient

In this section, the aim is to derive numerical solution of a piecewise heat equation with variable where the first part is the classical heat equation, then the second part is display crossover behaviors from exponential decay law to power law with no steady state, and the last part is stochastic behavior. The mathematical equation underpinning is given below as: In this section, we will be interested to discritize the fractional part as we have already presented the numerical solutions of the other equation.

$$
\frac{\partial C(x,t)}{\partial t} = D(x,t) \frac{\partial^2 C(x,t)}{\partial x^2} - V(x,t) \frac{\partial C(x,t)}{\partial x} \quad t \in [0,T_1], x \in [0.L_1], \tag{5.1}
$$

$$
{}_{T1}^{ABC}D_t^{\alpha} C(x,t) = D(x,t) \frac{\partial^2 C(x,t)}{\partial x^2} - V(x,t) \frac{\partial C(x,t)}{\partial x} \quad t \in [T_1,T_2], x \in [L_1,L_2], \tag{5.2}
$$

$$
\partial C(x,t) = D \frac{\partial^2 C(x,t)}{\partial x^2} - V(x,t) \frac{\partial C}{\partial x} + \sigma C(x,t) \partial B(x,t) \quad t \in [T_2,T], x \in [0.L_1], \tag{5.3}
$$

$$
\begin{aligned}
&C(x_i,t_{k+1}) - C(x_i,T_1) \\
&\quad = \frac{(1-\alpha)}{AB(\alpha)} \left[-V(x_i,t_j) \frac{\partial C(x_i,t_{k+1})}{\partial x} + D(x_i,t_j) \frac{\partial^2 C(x_i,t_{k+1})}{\partial x^2} \right] + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \\
&\quad \times \int_{t_k}^{t_{k+1}} \left[-V(x_i,t_j) \frac{\partial C(x_i,\tau) + D(x_i,t_j)}{\partial x} \frac{\partial^2 C(x_i,\tau)}{\partial x^2} \right] (t_{k+1}-\tau)^{\alpha-1} (t_{k+1}-\tau)^{\alpha} d\tau,
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
&C(x_i,t_{k+1}) - C(x_i,T_1) \\
&\quad = \frac{(1-\alpha)}{AB(\alpha)} \left\{ -V(x_i,t_j) \left(\frac{C(x_{i+1},t_{k+1}) - C(x_{i-1},t_{k+1})}{2h} \right) \right\} \\
&\quad + D(x_i,t_j) \left(\frac{C(x_{i+1},t_{k+1}) - 2C(x_i,t_{k+1}) + C(x_{i-1},t_{k+1})}{h^2} \right) + O(h^2) \\
&\quad + \frac{\alpha}{AB(\alpha)} \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_{k+1}} \left[-V(x_i,t_j) \frac{\partial C(x_i,\tau)}{\partial x} + D(x_i,t_j) \frac{\partial^2 C(x_i,\tau)}{\partial x^2} \right] (t_{k+1}-\tau)^{\alpha-1} d\tau,
\end{aligned} \tag{5.5}
$$

$$\mu_1 = \frac{V(x_i,t_j)(1-\alpha)}{AB(\alpha)2h},$$

$$\mu_2 = \frac{D(x_i,t_j)(1-\alpha)}{AB(\alpha)h^2},$$

$$\begin{aligned}
&C_i^{k+1} + \mu_1\left(C_{i+1}^{k+1} - C_{i-1}^{k+1}\right) - \mu_2\left(C_{i+1}^{k+1} - 2C_i^{k+1} + C_{i-1}^{k+1}\right)C_i^{T_1}\\
&\quad + \sum_{j=0}^{k-1}\frac{\alpha\Delta t^\alpha}{AB(\alpha)\Gamma(\alpha+2)}\left\{-V(x_i,t_j)\frac{C_{i+1}^{j+1} - C_{i-1}^{j+1}}{2h} + D(x_i,t_j)\frac{C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1}}{h^2}\right\}\\
&\quad \times\left((k-j+1)^{\alpha+1} - (k-j)^\alpha(k-j+\alpha+1)\right)\\
&\quad + \left(-V(x_i,t_j)\frac{C_{i+1}^{j+1} - C_{i-1}^{j+1}}{2h} + D(x_i,t_j)\frac{C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1}}{h^2}\right)\\
&\quad \times\left((k-j)^{\alpha+1} - (k-j)^\alpha(k-j+\alpha+1)\right),
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
&-(\mu_1+\mu_2)C_{i-1}^{k+1} + (1+2\mu_2)C_i^{k+1} + (\mu_1-\mu_2)C_{i+1}^{k+1}\\
&\quad = C_i^{T_1} + \frac{\alpha}{AB(\alpha)}\frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{k-1}A_j + \frac{\alpha}{AB(\alpha)}\frac{(\Delta t)^\alpha}{\Gamma(\alpha+1)}R_k,
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
A_j =& \left(-V(x_i,t_j)\frac{C_{i+1}^{j+1} - C_{i-1}^{j+1}}{2h} + D(x_i,t_j)\frac{C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1}}{h^2}\right)\\
&\times\left((k-j+1)^{\alpha+1} - (k-j)^\alpha(k-j+\alpha+1)\right)\\
&+ \left(-V(x_i,t_j)\frac{C_{i+1}^{j+1} - C_{i-1}^{j+1}}{2h} + D(x_i,t_j)\frac{C_{i+1}^{j+1} - 2C_i^{j+1} + C_{i-1}^{j+1}}{h^2}\right)\\
&\times\left((k-j)^{\alpha+1} - (k-j)^\alpha(k-j+\alpha+1)\right),
\end{aligned} \tag{5.8}$$

$$R_k = \left[-\frac{V(x_i,t_j)}{2h}(C_{i+1}^k - C_{i-1}^k) + \frac{D(x_i,t_j)}{h^2}\left(C_{i+1}^k - 2C_i^k + C_{i-1}^k\right)\right], \tag{5.9}$$

$$O(h^2(\Delta t)^{\alpha+1})$$

$$H = \begin{pmatrix}
\Lambda+2\mu_2 & (\mu_1-\mu_2) & 0 & --- & 0 & 0\\
-(\mu_1+\mu_2) & \Lambda+2\mu_2 & (\mu_1-\mu_2) & 0 & --- & 0\\
0 & -(\mu_1+\mu_2) & 0 & 0 & --- & 0\\
| & | & | & | & | & 0\\
| & | & | & | & | & 0\\
| & | & | & | & | & (\mu_1-\mu_2)\\
0 & 0 & 0 & 0 & -(\mu_1+\mu_2) & \Lambda+2\mu_2
\end{pmatrix}. \tag{5.10}$$

6. Numerical simulation

To obtain the numerical solution, we consider the following theoretical parameters: $T_1 = 2, , T_2 = 4, T = 6$, and $L_1 = 4, L_2 = 8, L = 12$, the initial concentration is $C_0 = 100$. Variable coefficients are given as:

$$V(x) = \frac{V}{1+x}, \quad D(x) = Dexp[-\gamma x].$$

Where $V = 0.5$ and $D = 0.075$. The numerical simulations are depicted in Figures 1–5. It is clear that the concept of piecewise is able to replicate some complex behaviours that existing differential operators could not.

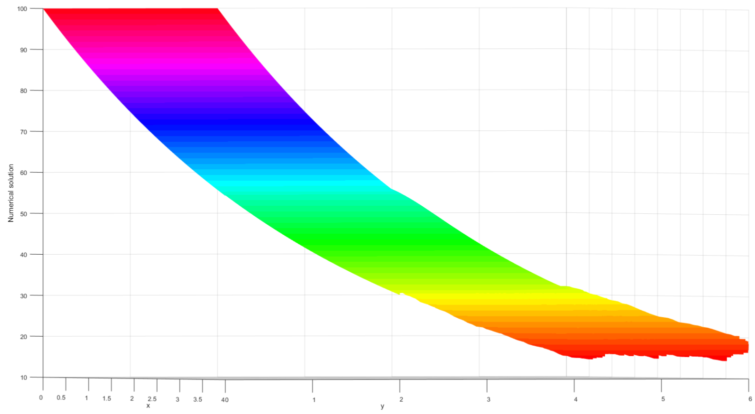

Figure 1. Numerical simulation of a piecewise heat equation.

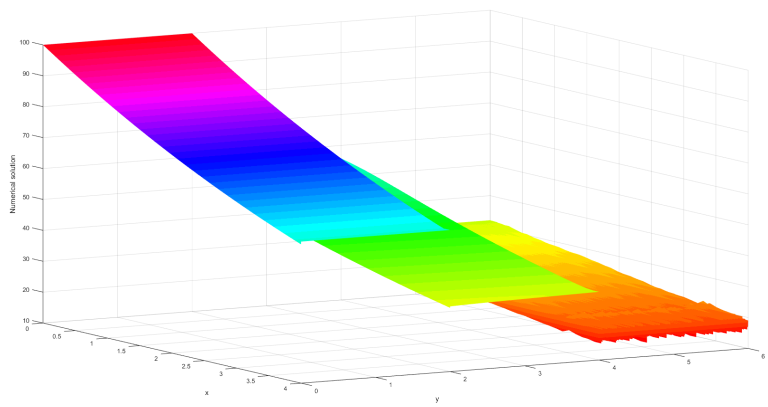

Figure 2. Numerical simulation of a piecewise heat equation.

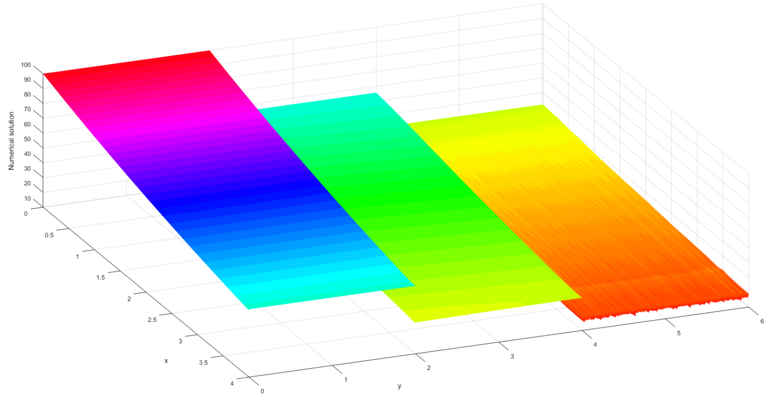

Figure 3. Numerical simulation of a piecewise heat equation.

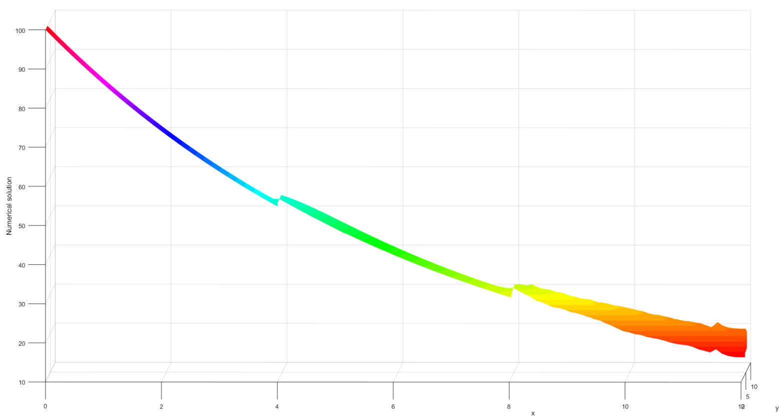

Figure 4. Numerical simulation of a piecewise heat equation.

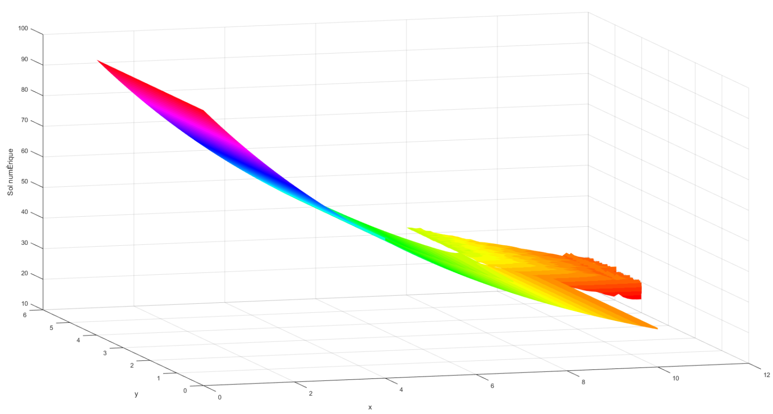

Figure 5. Numerical simulation of a piecewise heat equation.

7. Conclusions

Several real-world problems exhibit behaviours with crossover, although several mathematical models have been suggested to model complex processes, it is worth noting that, processes with crossover effects are very complex to be modelled efficiently using exiting differential and integral operators. This is also true in functional analysis, where some sharps cannot be obtained using single function, in this field, concept of piecewise functions have been introduced and applied on several occasions with great success. Likewise, in the field of differential and integral calculus, the concept of piecewise differential and integral operators and has been found to be powerful tool to handle problems with crossover effect. We have applied the concept on a heat equation and obtained simulation resembling real world scenarios.

Conflict of interest

Both authors declare that there is no conflict of interest for this paper.

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