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Research article

On a semiring variety generated by $B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2$

Lili Wang, Aifa Wang*and Peng Li

School of Science, Chongqing University of Technology, Chongqing 400054, China

* **Correspondence:** Email: wangaf@cqut.edu.cn.

Abstract: We study the semiring variety generated by $B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2$. We prove that this variety is finitely based and prove that the lattice of subvarieties of this variety is a distributive lattice of order 2327. Moreover, we deduce this variety is hereditarily finitely based.

Keywords: semiring; variety; lattice; identity; hereditarily finitely based
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1. Introduction

A semiring is an algebra with two associative binary operations $+, \cdot$, in which $+$ is commutative and $\cdot$ distributive over $+$ from the left and right. Such an algebra is a common generalization of both rings and distributive lattices. It has broad applications in information science and theoretical computer science (see [5, 6]). In this paper, we shall investigate some small-order semirings which will play a crucial role in subsequent follows.

The semiring A with addition and multiplication table (see [12])

+	0	a	1
0	0	0	0
a	0	a	0
1	0	0	1

·	0	a	1
0	0	0	0
a	0	1	a
1	0	a	1

The semiring B with addition and multiplication table (see [4])

+	a	b	c
a	a	b	c
b	b	b	b
c	c	b	c

·	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

Eight 2-element semirings with addition and multiplication table (see [2])

Semiring	+		·		Semiring	+		·	
L_2	0	1	0	0	R_2	0	1	0	1
	1	1	1	1		1	1	0	1
M_2	0	1	0	1	D_2	0	1	0	0
	1	1	1	1		1	1	0	1
N_2	0	1	0	0	T_2	0	1	1	1
	1	1	0	0		1	1	1	1
Z_2	0	0	0	0	W_2	0	0	0	0
	0	0	0	0		0	0	0	1

For any semiring S, we denote by S^0 the semiring obtained from S by adding an extra element 0 and where $a = 0 + a = a + 0, 0 = 0a = a0$ for every $a \in S$. For any semiring S, S^* will denote the (multiplicative) left-right dual of S. In 2005, Pastijn et al. [4, 9, 10] studied the semiring variety generated by B^0 and $(B^0)^*$ (Denoted by $\mathbf{Sr}(2, 1)$). They showed that the lattice of subvarieties of this variety is distributive and contains 78 varieties precisely. Moreover, each of these is finitely based. In 2016, Ren et al. [12, 13] studied the variety generated by $B^0, (B^0)^*$ and A^0 (Denoted by $\mathbf{Sr}(3, 1)$). They showed that the lattice of subvarieties of this variety is distributive and contains 179 varieties precisely. Moreover, each of these is finitely based. From [4, 10], we have $\mathbf{HSP}(L_2, R_2, M_2, D_2) \subsetneq \mathbf{HSP}(B^0, (B^0)^*)$. So

$$\mathbf{HSP}(L_2, R_2, M_2, D_2) \subsetneq \mathbf{HSP}(L_2, R_2, M_2, D_2, Z_2, W_2) \subsetneq \mathbf{HSP}(B^0, (B^0)^*, Z_2, W_2).$$

In 2016, Shao and Ren [15] studied the variety $\mathbf{HSP}(L_2, R_2, M_2, D_2, Z_2, W_2)$ (Denoted by $\mathbf{S}_6$). They showed that the lattice of subvarieties of this variety is distributive and contains 64 varieties precisely. Moreover, each of these is finitely based. Recently, Ren and Zeng [14] studied the variety generated by $B^0, (B^0)^*, N_2, T_2$. They proved that the lattice of subvarieties of this variety is a distributive lattice of order 312 and that each of its subvarieties is finitely based. In [16], Wang, Wang and Li studied the variety generated by $B^0, (B^0)^*, A^0, N_2, T_2$. They proved that the lattice of subvarieties of this variety is a distributive lattice of order 716 and that each of its subvarieties is finitely based. It is easy to check

$$\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2) \subsetneq \mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2).$$

So semiring variety $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2)$ is a proper subvariety of the semiring variety $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$. The main purpose of this paper is to study the variety $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$. We show that the lattice of subvarieties of this variety is a distributive lattice of order 2327. Moreover, we show this variety is hereditarily finitely based.

2. Preliminaries

By a *variety* we mean a class of algebras of the same type that is closed under subalgebras, homomorphic images and direct products (see [11]). Let $\mathbf{W}$ be a variety, let $\mathcal{L}(\mathbf{W})$ denote the lattice of subvarieties of $\mathbf{W}$ and let $\mathrm{Id}_{\mathbf{W}}(X)$ denote the set of all identities defining $\mathbf{W}$. If $\mathbf{W}$ can be defined by finitely many identities, then we say that $\mathbf{W}$ is *finitely based* (see [14]). In other words, $\mathbf{W}$ is said to be finitely based if there exists a finite subset Σ of $\mathrm{Id}_{\mathbf{W}}(X)$ such that for any $p \approx q \in \mathrm{Id}_{\mathbf{W}}(X)$, $p \approx q$ can be derived from Σ, i.e., $\Sigma \vdash p \approx q$. Otherwise, we say that $\mathbf{W}$ is *nonfinitely based*. Recall that $\mathbf{W}$ is said to be hereditarily finitely based if all members of $\mathcal{L}(\mathbf{W})$ are finitely based. If a variety $\mathbf{W}$ is finitely based and $\mathcal{L}(\mathbf{W})$ is a finite lattice, then $\mathbf{W}$ is hereditarily finitely based (see [14]).

A semiring is called an *additively idempotent semiring* (ai-semiring for short) if its additive reduct is a semilattice, i.e., a commutative idempotent semigroup. It is also called a *semilattice-ordered semigroup* (see [3, 8, 12]). The variety of all semirings (resp. all ai-semirings) is denoted by **SR** (resp. **AI**). Let X denote a fixed countably infinite set of variables and X^+ the free semigroup on X (see [8]). A semiring identity (**SR**-identity for short) is an expression of the form $u \approx v$, where u and v are terms with $u = u_1 + \cdots + u_k$, $v = v_1 + \cdots + v_\ell$, where $u_i, v_j \in X^+$. Let $\underline{k}$ denote the set $\{1, 2, \ldots, k\}$ for a positive integer k, Σ be a set of identities which include the identities determining **AI** (Each identity in Σ is called an **AI**-identity) and $u \approx v$ be an **AI**-identity. It is easy to check that the ai-semiring variety defined by $u \approx v$ coincides with the ai-semiring variety defined by the identities $u \approx u + v_j, v \approx v + u_i, i \in \underline{k}, j \in \underline{\ell}$. Thus, in order to show that $u \approx v$ is derivable from Σ, we only need to show that $u \approx u + v_j, v \approx v + u_i, i \in \underline{k}, j \in \underline{\ell}$ can be derived from Σ (see [9]).

To solve the word problem for the variety $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$, the following notions and notations are needed. Let q be an element of X^+. Then

- the *head* of q, denoted by $h(q)$, is the first variable occurring in q;
- the *tail* of q, denoted by $t(q)$, is the last variable occurring in q;
- the *content* of q, denoted by $c(q)$, is the set of variables occurring in q;
- the *length* of q, denoted by $|q|$, is the number of variables occurring in q counting multiplicities;
- the *initial part* of q, denoted by $i(q)$, is the word obtained from q by retaining only the first occurrence of each variable;
- the *final part* of q, denoted by $f(q)$, is the word obtained from q by retaining only the last occurrence of each variable;
- $r(q)$ denotes set $\{x \in X \mid \text{the number of occurrences of } x \text{ in } q \text{ is odd}\}$.

By [13, Lemma 1.2], $\mathbf{Sr}(3, 1)$ satisfies the identity $p \approx q$ if and only if $(i(p), f(p), r(p)) = (i(q), f(q), r(q))$. This result will be used later without any further notice. The basis for each one of N_2, T_2, Z_2, W_2 can be found from [2] (See Table 1).

Table 1. Bases for N_2, T_2, Z_2, W_2.

Semiring	Equational basis	Semiring	Equational basis
N_2	$xy \approx zt,\ x + x^2 \approx x$	T_2	$xy \approx zt,\ x + x^2 \approx x^2$
Z_2	$x + y \approx z + u, xy \approx x + y$	W_2	$x + y \approx z + u, x^2 \approx x, xy \approx yx$

By [15, Lemma 1.1] and the Table 1, we have

Lemma 2.1. *Let $u \approx v$ be a nontrivial* **SR**-*identity, where $u = u_1 + u_2 + \cdots + u_m$, $v = v_1 + v_2 + \cdots + v_n$, $u_i, v_j \in X^+$, $i \in \underline{m}$, $j \in \underline{n}$. Then*
(i) $N_2 \models u \approx v$ *if and only if* $\{u_i \in u \mid |u_i| = 1\} = \{v_i \in v \mid |v_i| = 1\}$;
(ii) $T_2 \models u \approx v$ *if and only if* $\{u_i \in u \mid |u_i| \geq 2\} \neq \phi$, $\{v_i \in v \mid |v_i| \geq 2\} \neq \phi$;
(iii) $Z_2 \models u \approx v$ *if and only if* $(\forall x \in X)\, u \neq x,\ v \neq x$;
(iv) $W_2 \models u \approx v$ *if and only if* $m = n = 1, c(u_1) = c(v_1)$ *or* $m, n \geq 2$.

Suppose that $u = u_1 + \cdots + u_m, u_i \in X^+, i \in \underline{m}$. Let 1 be a symbol which is not in X and Y an arbitrary subset of $\bigcup_{i=1}^{i=m} c(u_1)$. For any u_i in u, if $c(u_i) \subseteq Y$, put $h_Y(u_i) = 1$. Otherwise, we shall denote by $h_Y(u_i)$ the first variable occurring in the word obtained from u_i by deleting all variables in Y. The set $\{h_Y(u_i)|u_i \in u\}$ is written $H_Y(u)$. Dually, we have the notations $t_Y(u_i)$ and $T_Y(u_i)$. In particular, if $Y = \emptyset$, then $h_Y(u_i) = h(u_i)$ and $t_Y(u_i) = t(u_i)$. Moreover, if $c(u_i) \cap Y \neq \emptyset$ for every u_i in u, then we write $D_Y(u) = \emptyset$. Otherwise, $D_Y(u)$ is the sum of all terms u_i in u such that $c(u_i) \cap Y = \emptyset$. By [13, Lemma 2.3 and 2.11] and [4, Lemma 2.4 and its dual, Lemma 2.5 and 2.6], we have

Lemma 2.2. *Let $u \approx u + q$ be an **AI**-identity, where $u = u_1 + \cdots + u_m, u_i, q \in X^+, i \in \underline{m}$. If $u \approx u + q$ holds in* $\mathbf{Sr}(3, 1)$*, then*

(i) *for every $Z \subseteq \bigcup_{i=1}^{i=m} c(u_i)\backslash c(q)$, there exists p_1 in X^+ with $r(p_1) = r(q)$ and $c(q) \subseteq c(p_1) \subseteq \bigcup_{i=1}^{i=k} c(u_i)$ such that $D_Z(u) \approx D_Z(u) + p_1$ holds in* $\mathbf{Sr}(3, 1)$*, where $D_Z(u) = u_1 + \cdots + u_k$.*

(ii) *for every $Y \subseteq Z = \bigcup_{i=1}^{i=m} c(u_i)\backslash c(q)$, $H_Y(D_Z(u)) = H_Y(D_Z(u) + p_1)$ and $T_Y(D_Z(u)) = T_Y(D_Z(u) + p_1)$.*

Throughout this paper, $u \overset{(3.1),(3.2),\cdots}{\approx} v$ denotes the identity $u \approx v$ can be derived from the identities $(3.1), (3.2), \cdots$ and the identities determining **SR**. For other notations and terminology used in this paper, the reader is referred to [1, 4, 7, 13, 15].

3. Equational basis of $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$

In this section, we shall show that the variety $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$ is finitely based. Indeed, we have

Theorem 3.1. *The semiring variety $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$ is determined by (3.1)–(3.12),*

$$x^3y \approx xy; \quad (3.1)$$
$$xy^3 \approx xy; \quad (3.2)$$
$$(xy)^2 \approx x^2y^2; \quad (3.3)$$
$$(xy)^3 \approx xy; \quad (3.4)$$
$$x^2yx \approx xyx^2; \quad (3.5)$$
$$xyzx \approx xyx^2zx; \quad (3.6)$$
$$xy + z \approx xy + z + xyz^2; \quad (3.7)$$
$$xy + z \approx xy + z + z^2xy; \quad (3.8)$$
$$xy + z \approx xy + z + xz^2y; \quad (3.9)$$
$$xy + z \approx xy + z + z^3; \quad (3.10)$$
$$x + y + zt \approx x + y + zt + xzty; \quad (3.11)$$
$$x + y \approx x + y + y. \quad (3.12)$$

Proof. From [13] and Lemma 2.1, we know that both $\mathbf{Sr}(3, 1)$ and $\mathbf{HSP}(N_2, T_2, Z_2, W_2)$ satisfy identities (3.1)–(3.12) and so does $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$.

Next, we shall show that every identity that holds in $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$ can be derived from (3.1)–(3.12) and the identities determining **SR**. Let $u \approx v$ be such an identity, where $u = u_1 + u_2 + \cdots + u_m$, $v = v_1 + v_2 + \cdots + v_n$, $u_i, v_j \in X^+$, $1 \leq i \leq m, 1 \leq j \leq n$. By Lemma 2.1 (iv), we only need to consider the following two cases:

Case 1. $m = n = 1$ and $c(u_1) = c(v_1)$. From $\mathbf{Sr}(3,1), T_2, Z_2 \models u_1 \approx v_1$, it follows that $(i(u_1), f(u_1), r(u_1)) = (i(v_1), f(v_1), r(v_1))$, $|u_1| \geq 2$ and $|v_1| \geq 2$. Hence $u_1 \overset{(3.1)\sim(3.6)}{\approx} v_1$.

Case 2. $m, n \geq 2$. It is easy to verify that $u \approx v$ and the identity (3.12) imply the identities $u \approx u + v_j$, $v \approx v + u_i$ for all i, j such that $1 \leq i \leq m, 1 \leq j \leq n$. Conversely, the latter $m + n$ identities imply $u \approx u + v \approx v$. Thus, to show that $u \approx v$ is derivable from (3.1)–(3.12) and the identities determining **SR**, we need only show that the simpler identities $u \approx u + v_j$, $v \approx v + u_i$ for all i, j such that $1 \leq i \leq m, 1 \leq j \leq n$. Hence we need to consider the following two cases:

Case 2.1. $u \approx u + q$, where $|q| = 1$. Since $N_2 \models u \approx u + q$, there exists $u_s = q$. Thus $u + q \approx u' + u_s + q \approx u' + u_s + u_s \overset{(3.12)}{\approx} u' + u_s \approx u$.

Case 2.2. $u \approx u + q$, where $|q| \geq 2$. Since $u \approx u + q$ holds in T_2, it follows from Lemma 2.1 (ii) that there exists u_i in u such that $u_i > 1$. Put $Z = (\bigcup_{i=1}^{i=m} c(u_i)) \backslash c(q)$. Assume that $D_Z(u) = u_1 + \cdots + u_k$. Then $\bigcup_{i=1}^{i=k} c(u_i) = c(q)$. By Lemma 2.2 (i), there exists $p_1 \in X^+$ such that $r(p_1) = r(q)$ and $c(q) \subseteq c(p_1) \subseteq \bigcup_{i=1}^{i=k} c(u_i)$. Moreover,

$$\begin{aligned} u &\approx u + u_i + D_Z(u) \\ &\approx u + u_i + p_1 + D_Z(u) \\ &\approx u + u_i + p_1 + D_Z(u) + p_1^3 && \text{(by (3.10))} \\ &\approx u + u_i + p_1 + D_Z(u) + p_1^3 + p_1^3 u_1^2 u_2^2 \cdots u_k^2. && \text{(by (3.7))} \end{aligned}$$

Write $p = p_1^3 u_1^2 u_2^2 \cdots u_k^2$. Thus $c(p) = c(q)$, $r(p) = r(q)$ and we have derived the identity

$$u \approx u + p. \tag{3.13}$$

Due to $|p| > 1$, it follows that (3.4) implies the identity

$$p^3 \approx p. \tag{3.14}$$

Suppose that $i(q) = x_1 x_2 \cdots x_\ell$. We shall show by induction on j that for every $1 \leq j \leq \ell$, $u \approx u + x_1^2 x_2^2 \cdots x_\ell^2 p$ is derivable from (3.1)–(3.11) and the identities defining **SR**.

From Lemma 2.1 (ii), there exists u_{i_1} in $D_Z(u)$ with $c(u_{i_1}) \subseteq c(q)$ such that $h(u_{i_1}) = h(q) = x_1$. Furthermore,

$$\begin{aligned} u &\approx u + u_{i_1} + p && \text{(by (3.13))} \\ &\approx u + u_{i_1} + p + u_{i_1}^2 p && \text{(by (3.8))} \\ &\approx u + u_{i_1} + p + x_1^2 u_{i_1}^2 p && \text{(by (3.1))} \\ &\approx u + u_{i_1} + p + x_1^2 u_{i_1}^2 p + x_1^2 p^2 u_{i_i}^2 p && \text{(by (3.9))} \\ &\approx u + u_{i_1} + p + x_1^2 u_{i_1}^2 p + x_1^2 p. && \text{(by (3.6),(3.14))} \end{aligned}$$

Therefore

$$u \approx u + x_1^2 p. \tag{3.15}$$

Assume that for some $1 < j \leq \ell$,

$$u \approx u + x_1^2 x_2^2 \cdots x_{j-1}^2 p \tag{3.16}$$

is derivable from (3.1–3.12) and the identities defining **SR**. By Lemma 2.1 (ii), there exists u_i in $D_Z(u)$ with $c(u_i) \subseteq c(q)$ such that $u_i = u_{i_1} x_j u_{i_2}$ and $c(u_{i_1}) \subseteq \{x_1, x_2, \ldots, x_{j-1}\}$. It follows that

$$\begin{aligned} u &\approx u + u_i + p \\ &\approx u + u_i + p + u_i^2 p && \text{(by (3.8))} \\ &\approx u + u_i + p + u_{i_1}^2 x_j^2 u_{i_2}^2 p && \text{(by (3.3))} \\ &\approx u + u_i + p + u_{i_1}^2 x_j^2 u_{i_2}^2 p + u_{i_1}^2 x_j^2 p^2 u_{i_2}^2 p && \text{(by (3.9))} \\ &\approx u + u_i + p + u_{i_1}^2 x_j^2 u_{i_2}^2 p + u_{i_1}^2 x_j^2 p. && \text{(by (3.6),(3.14))} \end{aligned}$$

Consequently

$$u \approx u + u_{i_1}^2 x_j^2 p. \tag{3.17}$$

Moreover, we have

$$\begin{aligned} u &\approx u + x_1^2 x_2^2 \cdots x_{j-1}^2 p + u_{i_1}^2 x_j^2 p && \text{(by (3.16),(3.17))} \\ &\approx u + x_1^2 x_2^2 \cdots x_{j-1}^2 p + u_{i_1}^2 x_j^2 p + x_1^2 x_2^2 \cdots x_{j-1}^2 (u_{i_1}^2 x_j^2 p)^2 p && \text{(by (3.9))} \\ &\approx u + x_1^2 x_2^2 \cdots x_{j-1}^2 p + u_{i_1}^2 x_j^2 p + x_1^2 x_2^2 \cdots x_{j-1}^2 x_j^2 p. && \text{(by (3.3),(3.6),(3.14)))} \end{aligned}$$

Hence $u \approx u + x_1^2 x_2^2 \cdots x_{j-1}^2 x_j^2 p$. Using induction we have

$$u \approx u + i^2(q)p. \tag{3.18}$$

Dually,

$$u \approx u + pf^2(q). \tag{3.19}$$

Thus

$$\begin{aligned} u &\approx u + p + i^2(q)p + pf^2(q) && \text{(by (3.13),(3.18),(3.19))} \\ &\approx u + p + i^2(q)p + pf^2(q) + i^2(q)pppf^2(q) && \text{(by (3.11))} \\ &\approx u + p + i^2(q)p + pf^2(q) + i^2(q)pf^2(q) && \text{(by (3.14))} \\ &\approx u + p + i^2(q)p + pf^2(q) + q. && \text{(by (3.1)–(3.6))} \end{aligned}$$

It follows that $u \approx u + q$. □

4. The lattice $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2))$

In this section we characterize the lattice $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2))$. Throughout this section, $t(x_1, \ldots, x_n)$ denotes the term t which contains no other variables than $x_1, \ldots, x_n$ (but not necessarily all of them). Let $S \in \mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$ and let $E^+(S)$ denote the set $\{a \in S \mid a + a = a\}$, where any element of $E^+(S)$ is said to be an *additive idempotent* of $(S, +)$. Notice that $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$ satisfies the identities

$$(x + y) + (x + y) \approx (x + x) + (y + y), \tag{4.1}$$

$$xy + xy \approx (x + x)(y + y). \tag{4.2}$$

By (4.1) and (4.2), it is easy to verify that $E^+(S) = \{a + a \mid a \in S\}$ forms a subsemiring of S. To characterize the lattice $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2))$, we need to consider the following mapping

$$\begin{aligned} \varphi : \mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)) &\to \mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2)), \\ \mathbf{W} &\mapsto \mathbf{W} \cap \mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2). \end{aligned} \tag{4.3}$$

It is easy to prove that $\varphi(\mathbf{W}) = \{E^+(S) \mid S \in \mathbf{W}\}$ for each member $\mathbf{W}$ of $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2))$. If $\mathbf{W}$ is the subvariety of $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2)$ determined by the identities

$$u_i(x_{i_1}, \ldots, x_{i_n}) \approx v_i(x_{i_1}, \ldots, x_{i_n}), i \in \underline{k},$$

then $\widehat{\mathbf{W}}$ denotes the subvariety of $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$ determined by the identities

$$u_i(x_{i_1} + x_{i_1}, \ldots, x_{i_n} + x_{i_n}) \approx v_i(x_{i_1} + x_{i_1}, \ldots, x_{i_n} + x_{i_n}), i \in \underline{k}. \tag{4.4}$$

Lemma 4.1. *[16] The ai-semiring variety* $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2)$ *is determined by the identities (3.1)–(3.11) and* $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2))$ *is a distributive lattice of order 716.*

Lemma 4.2. *Let* $\mathbf{W}$ *be a member of* $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2))$. *Then,* $\widehat{\mathbf{W}} = \mathbf{W} \vee \mathbf{HSP}(Z_2, W_2)$.

Proof. Since $\mathbf{W}$ satisfies the identities (4.4), it follows that $\mathbf{W}$ is a subvariety of $\widehat{\mathbf{W}}$. Both Z_2 and W_2 are members of $\widehat{\mathbf{W}}$ and so $\mathbf{W} \vee \mathbf{HSP}(Z_2, W_2) \subseteq \widehat{\mathbf{W}}$. To show the converse inclusion, it suffices to show that every identity that is satisfied by $\mathbf{W} \vee \mathbf{HSP}(Z_2, W_2)$ can be derived by the identities holding in $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$ and

$$u_i(x_{i_1} + x_{i_1}, \ldots, x_{i_n} + x_{i_n}) \approx v_i(x_{i_1} + x_{i_1}, \ldots, x_{i_n} + x_{i_n}),\ i \in \underline{k},$$

if $\mathbf{W}$ is the subvariety of $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2))$ determined by $u_i(x_{i_1}, \ldots, x_{i_n}) \approx v_i(x_{i_1}, \ldots, x_{i_n})$, $i \in \underline{k}$. Let $u \approx v$ be such an identity, where $u = u_1 + u_2 + \cdots + u_m, v = v_1 + v_2 + \cdots + v_n, u_i, v_j \in X^+, 1 \leq i \leq m, 1 \leq j \leq n$. By Lemma 2.1 (8), we only need to consider the following two cases.

Case 1. $m, n \geq 2$. By identity (3.12), $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$ satisfies the identities

$$u + u \approx u, \tag{4.5}$$

$$v + v \approx v. \tag{4.6}$$

Since $u \approx v$ holds in $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2)$, we have that it is derivable from the collection Σ of $u_i \approx v_i, i \in \underline{k}$ and the identities determining $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2)$. From [1, Exercise II.14.11], it follows that there exist $t_1, t_2, \ldots, t_\ell \in P_f(X^+)$ such that

- $t_1 = u, t_\ell = v$;
- For any $i = 1, 2, \ldots, \ell - 1$, there exist $p_i, q_i, r_i \in P_f(X^+)$ (where p_i, q_i and r_i may be empty words), a semiring substitution φ_i and an identity $u_i' \approx v_i' \in \Sigma$ such that

$$t_i = p_i\varphi_i(w_i)q_i + r_i, t_{i+1} = p_i\varphi_i(s_i)q_i + r_i,$$
$$\text{where either } w_i = u_i', s_i = v_i' \text{ or } w_i = v_i', s_i = u_i'.$$

Let Σ' denote the set $\{u+u \approx v+v \mid u \approx v \in \Sigma\}$. For any $i = 1, 2, \ldots, \ell-1$, we shall show that $t_i + t_i \approx t_{i+1} + t_{i+1}$ is derivable from Σ' and the identities holding in $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$. Indeed, we have

$$\begin{aligned} t_i + t_i &= p_i\varphi_i(w_i)q_i + r_i + p_i\varphi_i(w_i)q_i + r_i \\ &\approx p_i\varphi_i(w_i)q_i + p_i\varphi_i(w_i)q_i + r_i + r_i \\ &\approx p_i(\varphi_i(w_i + w_i))q_i + r_i + r_i \\ &\approx p_i(\varphi_i(s_i + s_i))q_i + r_i + r_i \\ &\quad \text{(since } w_i + w_i \approx s_i + s_i \in \Sigma' \text{ or } s_i + s_i \approx w_i + w_i \in \Sigma'\text{)} \\ &\approx p_i\varphi_i(s_i)q_i + p_i\varphi_i(s_i)q_i + r_i + r_i \\ &\approx p_i\varphi_i(s_i)q_i + r_i + p_i\varphi_i(s_i)q_i + r_i \\ &= t_{i+1} + t_{i+1}. \end{aligned}$$

Further,

$$u + u = t_1 + t_1 \approx t_2 + t_2 \approx \cdots \approx t_\ell + t_\ell = v + v.$$

This implies the identity

$$u + u \approx v + v. \tag{4.7}$$

We now have

$$u \overset{(4.6)}{\approx} u + u \overset{(4.7)}{\approx} v + v \overset{(4.6)}{\approx} v. \tag{4.8}$$

Case 2. $m = n = 1$ and $c(u) = c(v)$. Since $Z_2 \models u_1 \approx v_1$, $u_1 \neq x, v_1 \neq x$, for every $x \in X$. Since $u_1 \approx v_1$ holds in $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2)$, we have that it is derivable from the collection Σ of $u_i \approx v_i, i \in \underline{k}$ and the identities defining $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2)$. From [1, Exercise II.14.11], it follows that there exist $t_1, t_2, \ldots, t_\ell \in P_f(X^+)$ such that

- $t_1 = u_1, t_\ell = v_1$;
- For any $i = 1, 2, \ldots, \ell - 1$, there exist $p_i, q_i \in P_f(X^+)$ (where p_i and q_i may be empty words), a semiring substitution φ_i and an identity $u_i' \approx v_i' \in \Sigma$ (where u_i' and v_i' are words) such that

$$t_i = p_i\varphi_i(w_i)q_i, t_{i+1} = p_i\varphi_i(s_i)q_i,$$
$$\text{where either } w_i = u_i', s_i = v_i' \text{ or } w_i = v_i', s_i = u_i'.$$

By Lemma 4.1, we have that $u_1 \approx v_1$ can be derived from (3.1)–(3.6), so, by Theorem 3.1, it can be derived from monomial identities holding in $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$. This completes the proof. □

Lemma 4.3. *The following equality holds*

$$\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)) = \bigcup_{\mathbf{W} \in \mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2))} [\mathbf{W}, \widehat{\mathbf{W}}]. \tag{4.9}$$

There are 716 *intervals in* $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2))$*, and each interval is a congruence class of the kernel of the complete epimorphism* φ *in* (4.3).

Proof. Firstly, we shall show that equality (4.9) holds. It is easy to see that

$$\mathcal{L}(\mathbf{HSP}(B^0,(B^0)^*,A^0,N_2,T_2,Z_2,W_2)) = \bigcup_{\mathbf{W}\in\mathcal{L}(\mathbf{HSP}(B^0,(B^0)^*,A^0,N_2,T_2))} \varphi^{-1}(\mathbf{W}).$$

So it suffices to show that

$$\varphi^{-1}(\mathbf{W}) = [\mathbf{W}, \widehat{\mathbf{W}}], \tag{4.10}$$

for each member $\mathbf{W}$ of $\mathcal{L}(\mathbf{HSP}(B^0,(B^0)^*,A^0,N_2,T_2))$. If $\mathbf{W}_1$ is a member of $[\mathbf{W}, \widehat{\mathbf{W}}]$, then it is routine to verify that $\mathbf{W} \subseteq \{E^+(S) \mid S \in \mathbf{W}_1\} \subseteq \mathbf{W}$. This implies that $\{E^+(S) \mid S \in \mathbf{W}_1\} = \mathbf{W}$ and so $\varphi(\mathbf{W}_1) = \mathbf{W}$. Hence, $\mathbf{W}_1$ is a member of $\varphi^{-1}(\mathbf{W})$ and so $[\mathbf{W}, \widehat{\mathbf{W}}] \subseteq \varphi^{-1}(\mathbf{W})$. Conversely, if $\mathbf{W}_1$ is a member of $\varphi^{-1}(\mathbf{W})$, then $\mathbf{W} = \varphi(\mathbf{W}_1) = \{E^+(S) \mid S \in \mathbf{W}_1\}$ and so $\varphi^{-1}(\mathbf{W}) \subseteq [\mathbf{W}, \widehat{\mathbf{W}}]$. This shows that (4.9) holds.

From Lemma 4.1, we know that $\mathcal{L}(\mathbf{HSP}(B^0,(B^0)^*,A^0,N_2,T_2))$ is a lattice of order 716. So there are 716 intervals in $\mathcal{L}(\mathbf{HSP}(B^0,(B^0)^*,A^0,N_2,T_2,Z_2,W_2))$. Next, we show that φ a complete epimorphism. On one hand, it is easy to see that φ is a complete $\wedge$-epimorphism. On the other hand, let $(\mathbf{W}_i)_{i\in I}$ be a family of members of $\mathcal{L}(\mathbf{HSP}(B^0,(B^0)^*,A^0,N_2,T_2,Z_2, W_2))$. Then, by (4.3), we have that $\varphi(\mathbf{W}_i) \subseteq \mathbf{W}_i \subseteq \widehat{\varphi(\mathbf{W}_i)}$ for each $i \in I$. Further,

$$\bigvee_{i\in I}\varphi(\mathbf{W}_i) \subseteq \bigvee_{i\in I}\mathbf{W}_i \subseteq \bigvee_{i\in I}\widehat{\varphi(\mathbf{W}_i)} \subseteq \widehat{\bigvee_{i\in I}\varphi(\mathbf{W}_i)}.$$

This implies that $\varphi(\bigvee_{i\in I}\mathbf{W}_i) = \bigvee_{i\in I}\varphi(\mathbf{W}_i)$. Thus, φ is a complete $\vee$-homomorphism and so φ is a complete $\vee$-epimorphism. By (4.10), we deduce that each interval in (4.3) is a congruence class of the kernel of the complete epimorphism φ. □

In order to characterize the lattice $\mathcal{L}(\mathbf{HSP}(B^0,(B^0)^*,A^0,N_2,T_2,Z_2,W_2))$, by Lemma 4.3, we only need to describe the interval $[\mathbf{W}, \widehat{\mathbf{W}}]$ for each member $\mathbf{W}$ of $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*,A^0,N_2,T_2))$. Next, we have

Lemma 4.4. *Let* $\mathbf{W}$ *be a member of* $\mathcal{L}(\mathbf{HSP}(B^0,(B^0)^*,A^0,N_2,T_2))$. *Then,* $\mathbf{W}\vee\mathbf{HSP}(Z_2)$ *is the subvariety of* $\widehat{\mathbf{W}}$ *determined by the identity*

$$x^3 \approx x^3 + x^3. \tag{4.11}$$

Proof. It is easy to see that both, $\mathbf{W}$ and $\mathbf{HSP}(Z_2)$ satisfy the identity (4.11) and so does $\mathbf{W} \vee \mathbf{HSP}(Z_2)$. In the following we prove that every identity that is satisfied by $\mathbf{W} \vee \mathbf{HSP}(Z_2)$ is derivable from (4.11) and the identities holding in $\widehat{\mathbf{W}}$. Let $u \approx v$ be such an identity, where $u = u_1 + u_2 + \cdots + u_m, v = v_1 + v_2 + \cdots + v_n, u_i, v_j \in X^+, 1 \le i \le m, 1 \le j \le n$. We only need to consider the following cases.

Case 1. $m = n = 1$. Since Z_2 satisfies $u_1 \approx v_1$, it follows that $|u_1| \neq 1$ and $|v_1| \neq 1$. By Lemma 4.2, $\widehat{\mathbf{W}}$ satisfies the identity $u_1^3 + u_1^3 \approx v_1^3 + v_1^3$. Hence $u_1 \overset{(3.4)}{\approx} u_1^3 \overset{(4.11)}{\approx} u_1^3 + u_1^3 \approx v_1^3 + v_1^3 \overset{(4.11)}{\approx} v_1^3 \overset{(3.4)}{\approx} v_1$.

Case 2. $m = 1, n \ge 2$. Since Z_2 satisfies $u_1 \approx v$, it follows that $|u_1| \neq 1$. By Lemma 4.2, $\widehat{\mathbf{W}}$ satisfies the identity $u_1^3 + u_1^3 \approx v + v$. Hence $u_1 \overset{(3.4)}{\approx} u_1^3 \overset{(4.11)}{\approx} u_1^3 + u_1^3 \approx v + v \overset{(3.11)}{\approx} v$.

Case 3. $m \ge 2, n = 1$. Similar to Case 2.

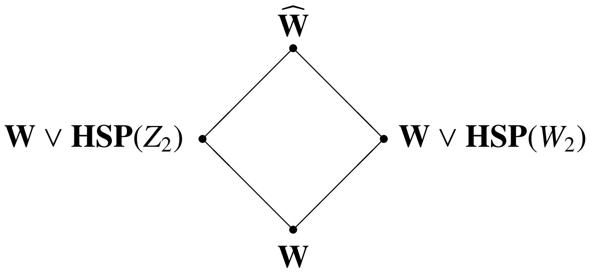
Case 4. $m, n \ge 2$. By Lemma 4.2, $\widehat{\mathbf{W}}$ satisfies the identity $u + u \approx v + v$. Hence $u \overset{(3.11)}{\approx} u + u \approx v + v \overset{(3.11)}{\approx} v$. □

Lemma 4.5. *Let* $\mathbf{W}$ *be a member of* $\mathcal{L}(\mathbf{Sr}(3,1))$. *Then* $\mathbf{W} \vee \mathbf{HSP}(W_2)$ *is the subvariety of* $\widehat{\mathbf{W}}$ *determined by the identities*

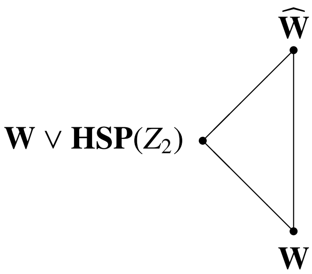
$$x^3 \approx x. \tag{4.12}$$

Proof. It is easy to see that both, $\mathbf{W}$ and $\mathbf{HSP}(W_2)$ satisfy the identity (4.12) and so does $\mathbf{W} \vee \mathbf{HSP}(W_2)$. So it suffices to show that every identity that is satisfied by $\mathbf{W} \vee \mathbf{HSP}(W_2)$ is derivable from (4.12) and the identities holding in $\widehat{\mathbf{W}}$. Let $u \approx v$ be such an identity, where $u = u_1 + u_2 + \cdots + u_m, v = v_1 + v_2 + \cdots + v_n, u_i, v_j \in X^+, 1 \le i \le m, 1 \le j \le n$. By Lemma 4.2, $\widehat{\mathbf{W}}$ satisfies the identity $u^3 \approx v^3$. Hence, $u \overset{(4.12)}{\approx} u^3 \approx v^3 \overset{(4.12)}{\approx} v$. □

Lemma 4.6. *Let* $\mathbf{W}$ *be a member of* $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2))$. *Then the interval* $[\mathbf{W}, \widehat{\mathbf{W}}]$ *of* $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2))$ *is given in Figure 1.*



Case.1 $N_2, T_2 \notin \mathbf{W}$



Case.2 $N_2 \in \mathbf{W}$ or $T_2 \in \mathbf{W}$

Figure 1. The interval $[\mathbf{W}, \widehat{\mathbf{W}}]$.

Proof. Suppose that $\mathbf{W}_1$ is a member of $[\mathbf{W}, \widehat{\mathbf{W}}]$ such that $\mathbf{W}_1 \neq \widehat{\mathbf{W}}$ and $\mathbf{W}_1 \neq \mathbf{W}$. Then, there exists a nontrivial identity $u \approx v$ holding in $\mathbf{W}_1$ such that it is not satisfied by $\widehat{\mathbf{W}}$. Also, we have that $\mathbf{W}_1$ does not satisfy the identity $x + x \approx x$. By Lemma 4.2, we only need to consider the following two cases.

Case 1. $\mathbf{HSP}(Z_2) \models u \approx v, \mathbf{HSP}(W_2) \not\models u \approx v$. Then, $u \approx v$ satisfies one of the following three cases:

- $m = n = 1$, $c(u_1) \neq c(v_1)$, $|u_1| \neq 1$ and $|v_1| \neq 1$;
- $m = 1, n > 1$ and $|u_1| \neq 1$;
- $m > 1, n = 1$ and $|v_1| \neq 1$.

It is easy to see that, in each of the above cases, $u \approx v$ can imply the identity $x^3 \approx x^3 + x^3$. By Lemma 4.4, we have that $\mathbf{W}_1$ is a subvariety of $\mathbf{W} \vee \mathbf{HSP}(Z_2)$. On the other hand, since $\mathbf{W}_1 \models x^3 \approx x^3 + x^3$ and $\mathbf{W}_1 \not\models x + x \approx x$, it follows that Z_2 is a member of $\mathbf{W}_1$ and so $\mathbf{W} \vee \mathbf{HSP}(Z_2)$ is a subvariety of $\mathbf{W}_1$. Thus, $\mathbf{W}_1 = \mathbf{W} \vee \mathbf{HSP}(Z_2)$.

Case 2. $\mathbf{HSP}(Z_2) \not\models u \approx v, \mathbf{HSP}(W_2) \models u \approx v$. Then, $u \approx v$ satisfies one of the following two cases:

- $m = n = 1$, $c(u_1) = c(v_1)$ and $|u_1| = 1$;
- $m = n = 1$, $c(u_1) = c(v_1)$ and $|v_1| = 1$.

If $N_2, T_2 \notin \mathbf{W}$, then, in each of the above cases, $u \approx v$ can imply the identity $x \approx x^3$. By Lemma 4.5, $\mathbf{W}_1$ is a subvariety of $\mathbf{W} \vee \mathbf{HSP}(W_2)$. On the other hand, since $\mathbf{W}_1 \models x \approx x^3$ and $\mathbf{W}_1 \not\models x \approx x + x$, it follows that W_2 is a member of $\mathbf{W}_1$ and so $\mathbf{W} \vee \mathbf{HSP}(W_2)$ is a subvariety of $\mathbf{W}_1$. Thus, $\mathbf{W}_1 = \mathbf{W} \vee \mathbf{HSP}(W_2)$.

If $N_2 \in \mathbf{W}$, then, by Lemma 2.1 (i), $|u_1| = |v_1| = 1$, a contradiction. Thus, $\mathbf{V}_1 = \widehat{\mathbf{V}}$.

If $T_2 \in \mathbf{W}$, then, by Lemma 2.1 (ii), $|u_1| \geq 2, |v_1| \geq 2$, a contradiction. Thus, $\mathbf{V}_1 = \widehat{\mathbf{V}}$. □

By Lemma 4.3 and 4.6, we can show that the lattice $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2))$ of subvarieties of the variety $\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$ contains 2327 elements. In fact, we have

Theorem 4.7. *$\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2))$ is a distributive lattice of order* 2327.

Proof. We recall from [16] that $\mathbf{Sr}(3, 1) \vee T_2$ [$\mathbf{Sr}(3, 1) \vee N_2$] contains 358 subvarieties since $\mathbf{Sr}(3, 1)$ contains 179 subvarieties. By Lemma 4.3 and 4.6, we can show that $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2))$ has exactly 2327 (where $2327 = 179 \times 4 + 358 \times 3 \times 2 - 179 \times 3$) elements. Suppose that $\mathbf{W}_1, \mathbf{W}_2$ and $\mathbf{W}_3$ are members of $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2))$ such that $\mathbf{W}_1 \vee \mathbf{W}_2 = \mathbf{W}_1 \vee \mathbf{W}_3$ and $\mathbf{W}_1 \wedge \mathbf{W}_2 = \mathbf{W}_1 \wedge \mathbf{W}_3$. Then, by Lemma 4.3

$$\varphi(\mathbf{W}_1) \vee \varphi(\mathbf{W}_2) = \varphi(\mathbf{W}_1) \vee \varphi(\mathbf{W}_3)$$

and

$$\varphi(\mathbf{W}_1) \wedge \varphi(\mathbf{W}_2) = \varphi(\mathbf{W}_1) \wedge \varphi(\mathbf{W}_3).$$

Since $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2)$ is distributive, it follows that $\varphi(\mathbf{W}_2) = \varphi(\mathbf{W}_3)$. Write $\mathbf{W}$ for $\varphi(\mathbf{W}_2)$. Then both $\mathbf{W}_2, \mathbf{W}_3$ are members of $[\mathbf{W}, \widehat{\mathbf{W}}]$. Suppose that $\mathbf{W}_2 \neq \mathbf{W}_3$. Then, by Lemma 4.6, $\mathbf{W}_1 \vee \mathbf{W}_2 = \mathbf{W}_1 \vee \mathbf{W}_3$ and $\mathbf{W}_1 \wedge \mathbf{W}_2 = \mathbf{W}_1 \wedge \mathbf{W}_3$ can not hold at the same time. This implies that $\mathbf{W}_2 = \mathbf{W}_3$. □

By Theorem 4.1, 4.7 and [14, Corollary 1.2], we now immediately deduce

Corollary 4.8. *$\mathbf{HSP}(B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2)$ is hereditarily finitely based.*

5. Conclusions

This article considers a semiring variety generated by $B^0, (B^0)^*, A^0, N_2, T_2, Z_2, W_2$. The finite basis problem for semirings is an interesting developing topic, with plenty of evidence of a high level of complexity along the lines of the more well-developed area of semigroup varieties. This article is primarily a contribution toward the property of being hereditarily finite based, meaning that all subvarieties are finitely based. This property is of course useful because it guarantees the finite basis property of a large number of examples.

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Conflict of interest

The authors declare that they do not have any conflict of interests regarding this paper.

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