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*Research article*

# Analysis of nonlinear coupled Caputo fractional differential equations with boundary conditions in terms of sum and difference of the governing functions

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**Abstract:** In this paper, we introduce a new class of nonlocal multipoint-integral boundary conditions with respect to the sum and difference of the governing functions and analyze a coupled system of nonlinear Caputo fractional differential equations equipped with these conditions. The existence and uniqueness results for the given problem are proved via the tools of the fixed point theory. We also discuss the case of nonlinear Riemann-Liouville integral boundary conditions. The obtained results are well-illustrated with examples.

**Keywords:** fractional differential system; multipoint-integral boundary conditions; existence; fixed point
**Mathematics Subject Classification:** 34A08, 34B15

## 1. Introduction

The topic of fractional differential equations received immense popularity and attraction due to their extensive use in the mathematical modeling of several real world phenomena. Examples include HIV-immune system with memory [1], stabilization of chaotic systems [2], chaotic synchronization [3, 4], ecology [5], infectious diseases [6], economic model [7], fractional neural networks [8, 9], COVID-19 infection [10], etc. A salient feature distinguishing fractional-order differential and integral operators from the classical ones is their nonlocal nature, which can provide the details about the past history of the phenomena and processes under investigation. In the recent years, many researchers contributed to the development of fractional calculus, for example, see [11–24] and the references cited therein. One can also find a substantial material about fractional order coupled systems in the articles [25–34].

In this paper, motivated by [30], we consider a Caputo type coupled system of nonlinear fractional differential equations supplemented with a new set of boundary conditions in terms of the sum and difference of the governing functions given by

$$\begin{cases} {}^{C}D^{\nu}\varphi(t) = f(t,\varphi(t),\psi(t)), \quad t \in J := [0,T], \\ {}^{C}D^{\rho}\psi(t) = g(t,\varphi(t),\psi(t)), \quad t \in J := [0,T], \\ P_1(\varphi+\psi)(0) + P_2(\varphi+\psi)(T) = \displaystyle\sum_{i=1}^{m} a_i(\varphi+\psi)(\sigma_i), \\ \displaystyle\int_0^T (\varphi-\psi)(s)ds - \int_\eta^\zeta (\varphi-\psi)(s)ds = A, \end{cases} \tag{1.1}$$

where ${}^{C}D^{\chi}$ is the Caputo fractional derivative operator of order $\chi \in \{\nu,\rho\}$, $\nu,\rho \in (0,1]$, $0 < \sigma_i < \eta < \zeta < T$, $i = 1,\ldots,m$ (the case $0 < \eta < \zeta < \sigma_i < T$ can be treated in a similar way), $P_1, P_2, a_i, A$ are nonnegative constants, such that $P_1 + P_2 - \sum_{i=1}^{m} a_i \neq 0$, $T - \zeta + \eta \neq 0$, and $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

Here it is imperative to notice that the first condition introduced in the problem (1.1) can be interpreted as the sum of the governing functions $\varphi$ and $\psi$ at the end positions of the interval $[0,T]$ is sum of similar contributions due to arbitrary positions at $\sigma_i \in (0,T), i = 1,...,m$, while the second condition describes that the contribution of the difference of the governing functions $\varphi$ and $\psi$ on the domain $[0,T]$ differs from the one an arbitrary sub-domain $(\eta,\xi)$ by a constant $A$.

We will also study the problem (1.1) by replacing $A$ in the last condition with the one containing nonlinear Riemann-Liouville integral term of the form:

$$\frac{1}{\Gamma(\delta)}\int_0^T (T-s)^{\delta-1} h(s,\varphi(s),\psi(s))ds, \quad \delta > 0, \tag{1.2}$$

where $h : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a given continuous function.

We organize the rest of the paper as follows. In Section 2, we outline the related concepts of fractional calculus and establish an auxiliary lemma for the linear analogue of the problem (1.1). We apply the standard fixed point theorems to derive the existence and uniqueness results for the problem (1.1) in Section 3. The case of nonlinear Riemann-Liouville integral boundary conditions is discussed in Section 4. The paper concludes with some interesting observations and special cases.

## 2. Preliminaries

Let us begin this section with some preliminary concepts of fractional calculus [11].

**Definition 2.1.** *The Riemann-Liouville fractional integral of order $q > 0$ of a function $h : [0,\infty) \to \mathbb{R}$ is defined by*

$$I^q h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds, \quad t > 0,$$

*provided the right-hand side is point-wise defined on $(0,\infty)$, where $\Gamma$ is the Gamma function.*

**Definition 2.2.** *The Caputo fractional derivative of order $q$ for a function $h : [0,\infty] \to \mathbb{R}$ with $h(t) \in AC^n[0,\infty)$ is defined by*

$$ {}^CD^q h(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{q-n+1}} ds = I^{n-q} h^{(n)}(t),\ t>0,\ n-1<q<n. $$

**Lemma 2.1.** *Let $q>0$ and $h(t) \in AC^n[0,\infty)$ or $C^n[0,\infty)$. Then*

$$ (I^q\, {}^CD^q h)(t) = h(t) - \sum_{k=0}^{n-1} \frac{h^{(k)}(0)}{k!} t^k,\ t>0,\ n-1<q<n. \tag{2.1} $$

Now we present an auxiliary lemma related to the linear variant of problem (1.1).

**Lemma 2.2.** *Let $\mathcal{F}, \mathcal{G} \in C[0,T]$, $\varphi, \psi \in AC[0,T]$. Then the solution of the following linear coupled system:*

$$ \begin{cases} {}^CD^\nu \varphi(t) = \mathcal{F}(t), \quad t \in J := [0,T], \\ {}^CD^\rho \psi(t) = \mathcal{G}(t), \quad t \in J := [0,T], \\ P_1(\varphi+\psi)(0) + P_2(\varphi+\psi)(T) = \sum_{i=1}^m a_i(\varphi+\psi)(\sigma_i), \\ \int_0^T (\varphi-\psi)(s)ds - \int_\eta^\zeta (\varphi-\psi)(s)ds = A, \end{cases} \tag{2.2} $$

*is given by*

$$ \begin{aligned} \varphi(t) = {} & \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} \mathcal{F}(s)ds \\ & + \frac{1}{2}\Bigg\{ \frac{A}{\Lambda_2} - \frac{1}{\Lambda_2} \int_0^T \Big( \int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)} \mathcal{F}(x)dx - \int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)} \mathcal{G}(x)dx \Big) ds \\ & - \frac{P_2}{\Lambda_1} \Big( \int_0^T \frac{(T-s)^{\nu-1}}{\Gamma(\nu)} \mathcal{F}(s)ds + \int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)} \mathcal{G}(s)ds \Big) \\ & + \frac{1}{\Lambda_2} \int_\eta^\xi \Big( \int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)} \mathcal{F}(x)dx - \int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)} \mathcal{G}(x)dx \Big) ds \\ & + \frac{\sum_{i=1}^m a_i}{\Lambda_1} \Big( \int_0^{\sigma_i} \frac{(\sigma_i-s)^{\nu-1}}{\Gamma(\nu)} \mathcal{F}(s)ds + \int_0^{\sigma_i} \frac{(\sigma_i-s)^{\rho-1}}{\Gamma(\rho)} \mathcal{G}(s)ds \Big) \Bigg\}, \end{aligned} \tag{2.3} $$

$$ \begin{aligned} \psi(t) = {} & \int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)} \mathcal{G}(s)ds \\ & + \frac{1}{2}\Bigg\{ \frac{-A}{\Lambda_2} + \frac{1}{\Lambda_2} \int_0^T \Big( \int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)} \mathcal{F}(x)dx - \int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)} \mathcal{G}(x)dx \Big) ds \\ & - \frac{P_2}{\Lambda_1} \Big( \int_0^T \frac{(T-s)^{\nu-1}}{\Gamma(\nu)} \mathcal{F}(s)ds + \int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)} \mathcal{G}(s)ds \Big) \\ & - \frac{1}{\Lambda_2} \int_\eta^\xi \Big( \int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)} \mathcal{F}(x)dx - \int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)} \mathcal{G}(x)dx \Big) ds \\ & + \frac{\sum_{i=1}^m a_i}{\Lambda_1} \Big( \int_0^{\sigma_i} \frac{(\sigma_i-s)^{\nu-1}}{\Gamma(\nu)} \mathcal{F}(s)ds + \int_0^{\sigma_i} \frac{(\sigma_i-s)^{\rho-1}}{\Gamma(\rho)} \mathcal{G}(s)ds \Big) \Bigg\}, \end{aligned} \tag{2.4} $$

*where*

$$\Lambda_1 := P_1 + P_2 - \sum_{i=1}^{m} a_i \neq 0, \tag{2.5}$$

$$\Lambda_2 := T - \zeta + \eta \neq 0. \tag{2.6}$$

**Proof.** Applying the operators $I^{\nu}$ and $I^{\rho}$ on the first and second fractional differential equations in (2.2) respectively and using Lemma 2.1, we obtain

$$\varphi(t) = \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(s)ds + c_1, \tag{2.7}$$

$$\psi(t) = \int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(s)ds + c_2, \tag{2.8}$$

where $c_1, c_2 \in \mathbb{R}$. Inserting (2.7) and (2.8) in the condition $P_1(\varphi+\psi)(0)+P_2(\varphi+\psi)(T) = \sum_{i=1}^{m} a_i(\varphi+\psi)(\sigma_i)$, we get

$$\begin{aligned} c_1 + c_2 &= \frac{1}{\Lambda_1}\Big\{\sum_{i=1}^{m} a_i\Big(\int_0^{\sigma_i} \frac{(\sigma_i - s)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(s)ds + \int_0^{\sigma_i} \frac{(\sigma_i - s)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(s)ds\Big) \\ &\quad -P_2\Big(\int_0^T \frac{(T-s)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(s)ds + \int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(s)ds\Big)\Big\}. \end{aligned} \tag{2.9}$$

Using (2.7) and (2.8) in the condition $\int_0^T (\varphi-\psi)(s)ds - \int_\eta^\zeta (\varphi-\psi)(s)ds = A$, we obtain

$$\begin{aligned} c_1 - c_2 &= \frac{1}{\Lambda_2}\Big\{A - \int_0^T \Big(\int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(x)dx - \int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(x)dx\Big)ds \\ &\quad + \int_\eta^\xi \Big(\int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(x)dx - \int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(x)dx\Big)ds\Big\}. \end{aligned} \tag{2.10}$$

Solving (2.9) and (2.10) for $c_1$ and $c_2$, yields

$$\begin{aligned} c_1 &= \frac{1}{2}\Big\{\frac{A}{\Lambda_2} - \frac{1}{\Lambda_2}\int_0^T \Big(\int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(x)dx - \int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(x)dx\Big)ds \\ &\quad -\frac{P_2}{\Lambda_1}\Big(\int_0^T \frac{(T-s)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(s)ds + \int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(s)ds\Big) \\ &\quad +\frac{1}{\Lambda_2}\int_\eta^\xi \Big(\int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(x)dx - \int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(x)dx\Big)ds \\ &\quad +\frac{1}{\Lambda_1}\sum_{i=1}^{m} a_i\Big(\int_0^{\sigma_i} \frac{(\sigma_i - s)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(s)ds + \int_0^{\sigma_i} \frac{(\sigma_i - s)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(s)ds\Big)\Big\}, \end{aligned}$$

and

$$
\begin{aligned}
c_2 \;=\; & \frac{1}{2}\Big\{\frac{-A}{\Lambda_2}+\frac{1}{\Lambda_2}\int_0^T\Big(\int_0^s\frac{(s-x)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(x)dx-\int_0^s\frac{(s-x)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(x)dx\Big)ds\\
& -\frac{P_2}{\Lambda_1}\Big(\int_0^T\frac{(T-s)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(s)ds+\int_0^T\frac{(T-s)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(s)ds\Big)\\
& -\frac{1}{\Lambda_2}\int_\eta^\xi\Big(\int_0^s\frac{(s-x)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(x)dx-\int_0^s\frac{(s-x)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(x)dx\Big)ds\\
& +\frac{1}{\Lambda_1}\sum_{i=1}^m a_i\Big(\int_0^{\sigma_i}\frac{(\sigma_i-s)^{\nu-1}}{\Gamma(\nu)}\mathcal{F}(s)ds+\int_0^{\sigma_i}\frac{(\sigma_i-s)^{\rho-1}}{\Gamma(\rho)}\mathcal{G}(s)ds\Big)\Big\}.
\end{aligned}
$$

Substituting the values of $c_1$ and $c_2$ in (2.7) and (2.8) respectively, we get the solution (2.3) and (2.4). By direct computation, one can obtain the converse of this lemma. The proof is complete. □

## 3. Main results

Let $X = C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$ denote the Banach space endowed with the norm $\|(\varphi,\psi)\| = \|\varphi\|+\|\psi\| = \sup_{t\in[0,T]}|\varphi(t)|+\sup_{t\in[0,T]}|\psi(t)|$, $(\varphi,\psi)\in X$. In view of Lemma 2.2, we define an operator $\Phi : X\to X$ in relation to the problem (1.1) as

$$
\Phi(\varphi,\psi)(t) := (\Phi_1(\varphi,\psi)(t),\Phi_2(\varphi,\psi)(t)), \tag{3.1}
$$

where

$$
\begin{aligned}
&\Phi_1(\varphi,\psi)(t)\\
=\;& \frac{1}{\Gamma(\nu)}\int_0^t(t-s)^{\nu-1}f(s,\varphi(s),\psi(s))ds\\
+\;& \frac{1}{2}\Big\{\frac{A}{\Lambda_2}-\frac{1}{\Lambda_2}\int_0^T\Big(\int_0^s\frac{(s-x)^{\nu-1}}{\Gamma(\nu)}f(x,\varphi(x),\psi(x))dx-\int_0^s\frac{(s-x)^{\rho-1}}{\Gamma(\rho)}g(x,\varphi(x),\psi(x))dx\Big)ds\\
-\;& \frac{P_2}{\Lambda_1}\Big(\int_0^T\frac{(T-s)^{\nu-1}}{\Gamma(\nu)}f(s,\varphi(s),\psi(s))ds+\int_0^T\frac{(T-s)^{\rho-1}}{\Gamma(\rho)}g(s,\varphi(s),\psi(s))ds\Big)\\
+\;& \frac{1}{\Lambda_2}\int_\eta^\xi\Big(\int_0^s\frac{(s-x)^{\nu-1}}{\Gamma(\nu)}f(x,\varphi(x),\psi(x))dx-\int_0^s\frac{(s-x)^{\rho-1}}{\Gamma(\rho)}g(x,\varphi(x),\psi(x))dx\Big)ds\\
+\;& \frac{1}{\Lambda_1}\sum_{i=1}^m a_i\Big(\int_0^{\sigma_i}\frac{(\sigma_i-s)^{\nu-1}}{\Gamma(\nu)}f(s,\varphi(s),\psi(s))ds+\int_0^{\sigma_i}\frac{(\sigma_i-s)^{\rho-1}}{\Gamma(\rho)}g(s,\varphi(s),\psi(s))ds\Big)\Big\},
\end{aligned} \tag{3.2}
$$

and

$$
\begin{aligned}
&\Phi_2(\varphi,\psi)(t)\\
=\;& \frac{1}{\Gamma(\rho)}\int_0^t(t-s)^{\rho-1}g(s,\varphi(s),\psi(s))ds\\
+\;& \frac{1}{2}\Big\{\frac{-A}{\Lambda_2}+\frac{1}{\Lambda_2}\int_0^T\Big(\int_0^s\frac{(s-x)^{\nu-1}}{\Gamma(\nu)}f(x,\varphi(x),\psi(x))dx-\int_0^s\frac{(s-x)^{\rho-1}}{\Gamma(\rho)}g(x,\varphi(x),\psi(x))dx\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{P_2}{\Lambda_1}\Big(\int_0^T \frac{(T-s)^{\nu-1}}{\Gamma(\nu)} f(s,\varphi(s),\psi(s))ds + \int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)} g(s,\varphi(s),\psi(s))ds\Big)\\
&- \frac{1}{\Lambda_2}\int_\eta^\xi \Big(\int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)} f(x,\varphi(x),\psi(x))dx - \int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)} g(x,\varphi(x),\psi(x))dx\Big)ds\\
&+ \frac{1}{\Lambda_1}\sum_{i=1}^m a_i\Big(\int_0^{\sigma_i} \frac{(\sigma_i-s)^{\nu-1}}{\Gamma(\nu)} f(s,\varphi(s),\psi(s))ds + \int_0^{\sigma_i} \frac{(\sigma_i-s)^{\rho-1}}{\Gamma(\rho)} g(s,\varphi(s),\psi(s))ds\Big)\Big\}. \qquad (3.3)
\end{aligned}
$$

In the forthcoming analysis, we need the following assumptions.

$(H_1)$ There exist continuous nonnegative functions $\mu_i, \kappa_i \in C([0,1],\mathbb{R}^+), i = 1,2,3$, such that

$$
|f(t,\varphi,\psi)| \le \mu_1(t) + \mu_2(t)|\varphi| + \mu_3(t)|\psi| \quad \forall (t,\varphi,\psi) \in J\times\mathbb{R}^2;
$$

$$
|g(t,\varphi,\psi)| \le \kappa_1(t) + \kappa_2(t)|\varphi| + \kappa_3(t)|\psi| \quad \forall (t,\varphi,\psi) \in J\times\mathbb{R}^2.
$$

$(H_2)$ There exist positive constants $\alpha_i,\ \beta_i,\ i = 1,2$, such that

$$
|f(t,\varphi_1,\psi_1) - f(t,\varphi_2,\psi_2)| \le \alpha_1|\varphi_1-\varphi_2| + \alpha_2|\psi_1-\psi_2|, \quad \forall\, t\in J,\ \varphi_i,\psi_i \in \mathbb{R},\ i = 1,2;
$$

$$
|g(t,\varphi_1,\psi_1) - g(t,\varphi_2,\psi_2)| \le \beta_1|\varphi_1-\varphi_2| + \beta_2|\psi_1-\psi_2|, \quad \forall\, t\in J,\ \varphi_i,\psi_i \in \mathbb{R},\ i = 1,2.
$$

For computational convenience, we introduce the notation:

$$
\varrho_1 = \frac{1}{2|\Lambda_1|}\Big[\sum_{i=1}^m a_i \frac{\sigma_i^\nu}{\Gamma(\nu+1)} + P_2\frac{T^\nu}{\Gamma(\nu+1)}\Big] + \frac{1}{2|\Lambda_2|}\Big[\frac{\zeta^{\nu+1}-\eta^{\nu+1}}{\Gamma(\nu+2)} + \frac{T^{\nu+1}}{\Gamma(\nu+2)}\Big], \qquad (3.4)
$$

$$
\varrho_2 = \frac{1}{2|\Lambda_1|}\Big[\sum_{i=1}^m a_i \frac{\sigma_i^\rho}{\Gamma(\rho+1)} + P_2\frac{T^\rho}{\Gamma(\rho+1)}\Big] + \frac{1}{2|\Lambda_2|}\Big[\frac{\zeta^{\rho+1}-\eta^{\rho+1}}{\Gamma(\rho+2)} + \frac{T^{\rho+1}}{\Gamma(\rho+2)}\Big], \qquad (3.5)
$$

and

$$
\begin{aligned}
M_0 = \min\Bigg\{&1 - \Bigg[\|\mu_2\|\Big(2\varrho_1 + \frac{T^\nu}{\Gamma(\nu+1)}\Big) + \|\kappa_2\|\Big(2\varrho_2 + \frac{T^\rho}{\Gamma(\rho+1)}\Big)\Bigg],\\
&1 - \Bigg[\|\mu_3\|\Big(2\varrho_1 + \frac{T^\nu}{\Gamma(\nu+1)}\Big) + \|\kappa_3\|\Big(2\varrho_2 + \frac{T^\rho}{\Gamma(\rho+1)}\Big)\Bigg]\Bigg\}.
\end{aligned}
$$

We make use of the following fixed point theorem [35] to prove the existence of solutions for the problem (1.1).

**Lemma 3.1.** *Let $\mathcal{E}$ be the Banach space and $Q : \mathcal{E} \to \mathcal{E}$ be a completely continuous operator. If the set $\Omega = \{x \in \mathcal{E} | x = \mu Q x, 0 < \mu < 1\}$ is bounded, then $Q$ has a fixed point in $\mathcal{E}$.*

**Theorem 3.1.** *Suppose that $f, g : J\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and the condition $(H_1)$ holds. Then there exists at least one solution for the problem (1.1) on $J$ if*

$$
\begin{aligned}
\|\mu_2\|\Big(2\varrho_1 + \tfrac{T^\nu}{\Gamma(\nu+1)}\Big) + \|\kappa_2\|\Big(2\varrho_2 + \tfrac{T^\rho}{\Gamma(\rho+1)}\Big) &< 1,\\
\|\mu_3\|\Big(2\varrho_1 + \tfrac{T^\nu}{\Gamma(\nu+1)}\Big) + \|\kappa_3\|\Big(2\varrho_2 + \tfrac{T^\rho}{\Gamma(\rho+1)}\Big) &< 1,
\end{aligned} \qquad (3.6)
$$

*where $\varrho_i\,(i = 1,2)$ are defined in (3.4)–(3.5).*

**Proof.** Observe that continuity of $\Phi : X \to X$ follows from that of the functions $f$ and $g$. Now we show that the operator $\Phi$ maps any bounded subset of $X$ into a relatively compact subset of $X$. For that, let $\Omega_{\bar{r}} \subset X$ be bounded. Then, for the positive real constants $L_f$ and $L_g$, we have

$$|f(t,\varphi(t),\psi(t))| \le L_f,\ \ |g(t,\varphi(t),\psi(t))| \le L_g,\ \ \forall(\varphi,\psi) \in \Omega_{\bar{r}}.$$

So, for any $(\varphi,\psi) \in \Omega_{\bar{r}}$, $t \in J$, we get

$$\begin{aligned}
|\Phi_1(\varphi,\psi)(t)| \le\ & \frac{L_f}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}ds \\
+\ & \frac{1}{2}\Big\{\frac{A}{|\Lambda_2|} + \frac{1}{\Lambda_2}\int_0^T \Big(L_f\int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)}dx + L_g\int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)}dx\Big)ds \\
+\ & \frac{P_2}{|\Lambda_1|}\Big(L_f\int_0^T \frac{(T-s)^{\nu-1}}{\Gamma(\nu)}ds + L_g\int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)}ds\Big) \\
+\ & \frac{1}{|\Lambda_2|}\int_\eta^\xi \Big(L_f\int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)}dx + L_g\int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)}dx\Big)ds \\
+\ & \frac{1}{|\Lambda_1|}\sum_{i=1}^m a_i\Big(L_f\int_0^{\sigma_i} \frac{(\sigma_i-s)^{\nu-1}}{\Gamma(\nu)}ds + L_g\int_0^{\sigma_i} \frac{(\sigma_i-s)^{\rho-1}}{\Gamma(\rho)}ds\Big)\Big\} \\
\le\ & \frac{L_fT^\nu}{\Gamma(\nu+1)} + \frac{L_f}{2|\Lambda_1|}\Big[\sum_{i=1}^m a_i\frac{\sigma_i^\nu}{\Gamma(\nu+1)} + P_2\frac{T^\nu}{\Gamma(\nu+1)}\Big] + \frac{L_f}{2|\Lambda_2|}\Big[\frac{\zeta^{\nu+1}-\eta^{\nu+1}}{\Gamma(\nu+2)} + \frac{T^{\nu+1}}{\Gamma(\nu+2)}\Big] \\
+\ & \frac{L_g}{2|\Lambda_1|}\Big[\sum_{i=1}^m a_i\frac{\sigma_i^\rho}{\Gamma(\rho+1)} + P_2\frac{T^\rho}{\Gamma(\rho+1)}\Big] + \frac{L_g}{2|\Lambda_2|}\Big[\frac{\zeta^{\rho+1}-\eta^{\rho+1}}{\Gamma(\rho+2)} + \frac{T^{\rho+1}}{\Gamma(\rho+2)}\Big] + \frac{A}{2|\Lambda_2|},
\end{aligned}$$

which, in view of (3.4) and (3.5), takes the form:

$$|\Phi_1(\varphi,\psi)(t)| \le L_f\Big(\frac{T^\nu}{\Gamma(\nu+1)} + \varrho_1\Big) + L_g\varrho_2 + \frac{A}{2|\Lambda_2|}. \tag{3.7}$$

In a similar fashion, one can obtain

$$|\Phi_2(\varphi,\psi)(t)| \le L_f\varrho_1 + L_g\Big(\frac{T^\rho}{\Gamma(\rho+1)} + \varrho_2\Big) + \frac{A}{2|\Lambda_2|}. \tag{3.8}$$

From (3.7) and (3.8), we get

$$\begin{aligned}
\|\Phi(\varphi,\psi)\| &= \|\Phi_1(\varphi,\psi)\| + \|\Phi_2(\varphi,\psi)\| \\
&\le L_f\Big(\frac{T^\nu}{\Gamma(\nu+1)} + 2\varrho_1\Big) + L_g\Big(\frac{T^\rho}{\Gamma(\rho+1)} + 2\varrho_2\Big) + \frac{A}{|\Lambda_2|}.
\end{aligned}$$

From the foregoing inequality, we deduce that the operator $\Phi$ is uniformly bounded.
In order to show that $\Phi$ maps bounded sets into equicontinuous sets of $X$, let $t_1, t_2 \in [0,T],\ t_1 < t_2$, and $(\varphi,\psi) \in \Omega_{\bar{r}}$. Then

$$\begin{aligned}
|\Phi_1(\varphi,\psi)(t_2) - \Phi_1(\varphi,\psi)(t_1)| \le\ & \Big|\frac{1}{\Gamma(\nu)}\Big(\int_0^{t_1} [(t_2-s)^{\nu-1} - (t_1-s)^{\nu-1}]f(s,\varphi(s),\psi(s))ds \\
& + \int_{t_1}^{t_2} (t_2-s)^{\nu-1} f(s,\varphi(s),\psi(s))ds\Big)\Big|
\end{aligned}$$

$$\leq \ L_f\Big(\frac{2(t_2-t_1)^\nu + t_2^\nu - t_1^\nu}{\Gamma(\nu+1)}\Big).$$

Analogously, we can obtain

$$|\Phi_2(\varphi,\psi)(t_2) - \Phi_2(u,v)(t_1)| \ \leq \ L_g\Big(\frac{2(t_2-t_1)^\rho + t_2^\rho - t_1^\rho}{\Gamma(\rho+1)}\Big).$$

Clearly the right-hand sides of the above inequalities tend to zero when $t_1 \to t_2$, independently of $(\varphi,\psi) \in \Omega_{\bar{r}}$. Thus it follows by the Arzelá-Ascoli theorem that the operator $\Phi : X \to X$ is completely continuous.

Next we consider the set $\mathcal{E} = \{(\varphi,\psi) \in X|(\varphi,\psi) = \lambda\Phi(\varphi,\psi), 0 < \lambda < 1\}$ and show that it is bounded. Let $(\varphi,\psi) \in \mathcal{E}$, then $(\varphi,\psi) = \lambda\Phi(\varphi,\psi)$, $0 < \lambda < 1$. For any $t \in J$, we have

$$\varphi(t) = \lambda\Phi_1(\varphi,\psi)(t), \ \ \psi(t) = \lambda\Phi_2(\varphi,\psi)(t).$$

As in the previous step, using $\varrho_i$ $(i = 1,2)$ given by (3.4)-(3.5), we find that

$$\begin{aligned} |\varphi(t)| = \lambda|\Phi_1(\varphi,\psi)(t)| \ &\leq \ \Big(\|\mu_1\| + \|\mu_2\|\|\varphi\| + \|\mu_3\|\|\psi\|\Big)\Big(\frac{T^\nu}{\Gamma(\nu+1)} + \varrho_1\Big) \\ &+ \ \Big(\|\kappa_1\| + \|\kappa_2\|\|\varphi\| + \|\kappa_3\|\|\psi\|\Big)\varrho_2 + \frac{A}{2|\Lambda_2|}, \end{aligned}$$

$$\begin{aligned} |\psi(t)| = \lambda|\Phi_2(\varphi,\psi)(t)| \ &\leq \ \Big(\|\mu_1\| + \|\mu_2\|\|\varphi\| + \|\mu_3\|\|\psi\|\Big)\varrho_1 \\ &+ \ \Big(\|\kappa_1\| + \|\kappa_2\|\|\varphi\| + \|\kappa_3\|\|\psi\|\Big)\Big(\frac{T^\rho}{\Gamma(\rho+1)} + \varrho_2\Big) + \frac{A}{2|\Lambda_2|}. \end{aligned}$$

In consequence, we get

$$\begin{aligned} \|\varphi\| + \|\psi\| \ &\leq \ \|\mu_1\|\Big(2\varrho_1 + \frac{T^\nu}{\Gamma(\nu+1)}\Big) + \|\kappa_1\|\Big(2\varrho_2 + \frac{T^\rho}{\Gamma(\rho+1)}\Big) + \frac{A}{|\Lambda_2|} \\ &+ \ \Big[\|\mu_2\|\Big(2\varrho_1 + \frac{T^\nu}{\Gamma(\nu+1)}\Big) + \|\kappa_2\|\Big(2\varrho_2 + \frac{T^\rho}{\Gamma(\rho+1)}\Big)\Big]\|\varphi\| \\ &+\Big[\|\mu_3\|\Big(2\varrho_1 + \frac{T^\nu}{\Gamma(\nu+1)}\Big) + \|\kappa_3\|\Big(2\varrho_2 + \frac{T^\nu}{\Gamma(\nu+1)}\Big)\Big]\|\psi\|. \end{aligned}$$

Thus, by the condition (3.6), we have

$$\|(\varphi,\psi)\| \ \leq \ \frac{1}{M_0}\Big\{\|\mu_1\|\Big(2\varrho_1 + \frac{T^\nu}{\Gamma(\nu+1)}\Big) + \|\kappa_1\|\Big(2\varrho_2 + \frac{T^\rho}{\Gamma(\rho+1)}\Big) + \frac{A}{|\Lambda_2|}\Big\},$$

which shows that $\|(\varphi,\psi)\|$ is bounded for $t \in J$. In consequence, the set $\mathcal{E}$ is bounded. Thus it follows by the conclusion of Lemma 3.1 that the operator $\Phi$ has at least one fixed point, which is indeed a solution of the problem (1.1). □

Letting $\mu_2(t) = \mu_3(t) \equiv 0$ and $\kappa_2(t) = \kappa_3(t) \equiv 0$, the statement of Theorem 3.1 takes the following form.

**Corollary 3.1.** *Let $f, g : J \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions such that*

$$|f(t,\varphi,\psi)| \le \mu_1(t), \quad |g(t,\varphi,\psi)| \le \kappa_1(t), \quad \forall (t,\varphi,\psi) \in J \times \mathbb{R}^2,$$

*where $\mu_1, \kappa_1 \in C([0,T], \mathbb{R}^+)$. Then there exists at least one solution for the problem (1.1) on J.*

**Corollary 3.2.** *If $\mu_i(t) = \lambda_i$, $\kappa_i(t) = \varepsilon_i, i = 1,2,3,$ then the condition $(H_1)$ becomes:*

$(H_1')$ *there exist real constants $\lambda_i, \varepsilon_i > 0,\ i = 1,2,$ such that*

$$|f(t,\varphi,\psi)| \le \lambda_1 + \lambda_2|\varphi| + \lambda_3|\psi| \quad \forall (t,\varphi,\psi) \in J \times \mathbb{R}^2;$$

$$|f(t,\varphi,\psi)| \le \varepsilon_1 + \varepsilon_2|\varphi| + \varepsilon_3|\psi| \quad \forall (t,\varphi,\psi) \in J \times \mathbb{R}^2;$$

*and (3.6) takes the form:*

$$\begin{aligned} \lambda_2\Big(2\varrho_1 + \frac{T^\nu}{\Gamma(\nu+1)}\Big) + \varepsilon_2\Big(2\varrho_2 + \frac{T^\rho}{\Gamma(\rho+1)}\Big) &< 1, \\ \lambda_3\Big(2\varrho_1 + \frac{T^\nu}{\Gamma(\nu+1)}\Big) + \varepsilon_3\Big(2\varrho_2 + \frac{T^\rho}{\Gamma(\rho+1)}\Big) &< 1. \end{aligned}$$

*Then there exists at least one solution for the problem (1.1) on J.*

The next result is concerned with the existence of a unique solution for the problem (1.1) and is reliant on the contraction mapping principle due to Banach.

**Theorem 3.2.** *Let $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions and the assumption $(H_2)$ holds. Then the problem (1.1) has a unique solution on J if*

$$\alpha\Big(\frac{T^\nu}{\Gamma(\nu+1)} + 2\varrho_1\Big) + \beta\Big(\frac{T^\rho}{\Gamma(\rho+1)} + 2\varrho_2\Big) < 1, \tag{3.9}$$

*where $\alpha = \max\{\alpha_1, \alpha_2\}$, $\beta = \max\{\beta_1, \beta_2\}$ and $\varrho_i,\ i = 1,2,$ are defined in (3.4)-(3.5).*

**Proof.** Consider the operator $\Phi : X \to X$ defined by (3.1) and take

$$r > \frac{M_1\Big(\frac{T^\nu}{\Gamma(\nu+1)} + 2\varrho_1\Big) + M_2\Big(\frac{T^\rho}{\Gamma(\rho+1)} + 2\varrho_2\Big) + \frac{A}{|\Lambda_2|}}{1 - \Big(\alpha\Big(\frac{T^\nu}{\Gamma(\nu+1)} + 2\varrho_1\Big) + \beta\Big(\frac{T^\rho}{\Gamma(\rho+1)} + 2\varrho_2\Big)\Big)},$$

where $M_1 = \sup_{t\in[0,T]} |f(t,0,0)|$, and $M_2 = \sup_{t\in[0,T]} |g(t,0,0)|$. Then we show that $\Phi B_r \subset B_r$, where $B_r = \{(\varphi,\psi) \in X : \|(\varphi,\psi)\| \le r\}$. By the assumption $(H_1)$, for $(\varphi,\psi) \in B_r,\ t \in [0,T]$, we have

$$\begin{aligned} |f(t,\varphi(t),\psi(t))| &\le |f(t,\varphi(t),\psi(t)) - f(t,0,0)| + |f(t,0,0)| \\ &\le \alpha(|\varphi(t)| + |\psi(t)|) + M_1 \\ &\le \alpha(\|\varphi\| + \|\psi\|) + M_1. \end{aligned}$$

In a similar manner, one can find that

$$|g(t,\varphi(t),\psi(t))| \le \beta(\|\varphi\| + \|\psi\|) + M_2.$$

In consequence, for $(\varphi,\psi)\in B_r$, we obtain

$$\begin{aligned}
|\Phi_1(\varphi,\psi)(t)| \ \le\ & \frac{T^{\nu}}{\Gamma(\nu+1)}\Big(\alpha(\|\varphi\|+\|\psi\|)+M_1\Big)\\
&+\frac{1}{2}\Bigg[\frac{A}{|\Lambda_2|}+\frac{1}{|\Lambda_2|}\Big(\frac{T^{\nu+1}}{\Gamma(\nu+2)}\Big(\alpha(\|\varphi\|+\|\psi\|)+M_1\Big)+\frac{T^{\rho+1}}{\Gamma(\rho+2)}\Big(\beta(\|\varphi\|+\|\psi\|)+M_2\Big)\Big)\\
&+\frac{P_2}{|\Lambda_1|}\Big(\frac{T^{\nu}}{\Gamma(\nu+1)}\Big(\alpha(\|\varphi\|+\|\psi\|)+M_1\Big)+\frac{T^{\rho}}{\Gamma(\rho+1)}\Big(\beta(\|\varphi\|+\|\psi\|)+M_2\Big)\Big)\\
&+\frac{1}{|\Lambda_2|}\Big(\frac{\zeta^{\nu+1}-\eta^{\nu+1}}{\Gamma(\nu+2)}\Big(\alpha(\|\varphi\|+\|\psi\|)+M_1\Big)+\frac{\zeta^{\rho+1}-\eta^{\rho+1}}{\Gamma(\rho+2)}\Big(\beta(\|\varphi\|+\|\psi\|)+M_2\Big)\Big)\\
&+\frac{1}{|\Lambda_1|}\sum_{i=1}^{m}a_i\Big(\frac{\sigma_i^{\nu}}{\Gamma(\nu+1)}\Big(\alpha(\|\varphi\|+\|\psi\|)+M_1\Big)+\frac{\sigma_i^{\rho}}{\Gamma(\rho+1)}\Big(\beta(\|\varphi\|+\|\psi\|)+M_2\Big)\Big)\Bigg],
\end{aligned}$$

which, on taking the norm for $t\in J$, yields

$$\|\Phi_1(\varphi,\psi)\|\le\Big(\alpha\Big(\frac{T^{\nu}}{\Gamma(\nu+1)}+\varrho_1\Big)+\beta\varrho_2\Big)(\|\varphi\|+\|\psi\|)+M_1\Big(\frac{T^{\nu}}{\Gamma(\nu+1)}+\varrho_1\Big)+M_2\varrho_2+\frac{A}{2|\Lambda_2|}.$$

In the same way, for $(\varphi,\psi)\in B_r$, one can obtain

$$\|\Phi_2(\varphi,\psi)\|\le\Big(\alpha\varrho_1+\beta\Big(\frac{T^{\rho}}{\Gamma(\rho+1)}+\varrho_2\Big)\Big)(\|\varphi\|+\|\psi\|)+M_1\varrho_1+M_2\Big(\frac{T^{\rho}}{\Gamma(\rho+1)}+\varrho_2\Big)+\frac{A}{2|\Lambda_2|}.$$

Therefore, for any $(\varphi,\psi)\in B_r$, we have

$$\begin{aligned}
\|\Phi(\varphi,\psi))\| \ =\ & \|\Phi_1(\varphi,\psi)\|+\|\Phi_2(\varphi,\psi)\|\\
\le\ & \Big(\alpha\Big(\frac{T^{\nu}}{\Gamma(\nu+1)}+2\varrho_1\Big)+\beta\Big(\frac{T^{\rho}}{\Gamma(\rho+1)}+2\varrho_2\Big)\Big)(\|\varphi\|+\|\psi\|)\\
+\ & M_1\Big(\frac{T^{\nu}}{\Gamma(\nu+1)}+2\varrho_1\Big)+M_2\Big(\frac{T^{\rho}}{\Gamma(\rho+1)}+2\varrho_2\Big)+\frac{A}{|\Lambda_2|}<r,
\end{aligned}$$

which shows that $\Phi$ maps $B_r$ into itself.

Next it will be shown that the operator $\Phi$ is a contraction. For $(\varphi_1,\psi_1)$, $(\varphi_2,\psi_2)\in\mathcal{E}$, $t\in[0,T]$, it follows by $(H_2)$ that

$$\begin{aligned}
&|\Phi_1(\varphi_1,\psi_1)(t)-\Phi_1(\varphi_2,\psi_2)(t)|\\
\le\ & \int_0^t\frac{(t-s)^{\nu-1}}{\Gamma(\nu)}\Big|f(s,\varphi_1(s),\psi_1(s))-f(s,\varphi_2(s),\psi_2(s))\Big|ds\\
+\ & \frac{1}{2}\Bigg\{\frac{1}{|\Lambda_2|}\int_0^T\Big(\int_0^s\frac{(s-x)^{\nu-1}}{\Gamma(\nu)}\Big|f(x,\varphi_1(x),\psi_1(x))-f(x,\varphi_2(x),\psi_2(x))\Big|dx\\
+\ & \int_0^s\frac{(s-x)^{\rho-1}}{\Gamma(\rho)}\Big|g(x,\varphi_1(x),\psi_1(x))-g(x,\varphi_2(x),\psi_2(x))\Big|dx\Big)ds\\
+\ & \frac{P_2}{|\Lambda_1|}\Big(\int_0^T\frac{(T-s)^{\nu-1}}{\Gamma(\nu)}\Big|f(s,\varphi_1(s),\psi_1(s))-f(s,\varphi_2(s),\psi_2(s))\Big|ds\\
+\ & \int_0^T\frac{(T-s)^{\rho-1}}{\Gamma(\rho)}\Big|g(s,\varphi_1(s),\psi_1(s))-g(s,\varphi_2(s),\psi_2(s))\Big|ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{|\Lambda_2|}\int_{\eta}^{\xi}\Big(\int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)}\Big|f(x,\varphi_1(x),\psi_1(x)) - f(x,\varphi_2(x),\psi_2(x))\Big|dx \\
&+ \int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)}\Big|g(x,\varphi_1(x),\psi_1(x)) - g(x,\varphi_2(x),\psi_2(x))\Big|dx\Big)ds \\
&+ \frac{1}{|\Lambda_1|}\sum_{i=1}^{m} a_i\Big(\int_0^{\sigma_i} \frac{(\sigma_i-s)^{\nu-1}}{\Gamma(\nu)}\Big|f(s,\varphi_1(s),\psi_1(s)) - f(s,\varphi_2(s),\psi_2(s))\Big|ds \\
&+ \int_0^{\sigma_i} \frac{(\sigma_i-s)^{\rho-1}}{\Gamma(\rho)}\Big|g(s,\varphi_1(s),\psi_1(s)) - g(s,\varphi_2(s),\psi_2(s))\Big|ds\Big)\Big\} \\
&\leq \Big\{\alpha\Big(\frac{T^{\nu}}{\Gamma(\nu+1)} + \varrho_1\Big) + \beta\varrho_2\Big\}(\|\varphi_1-\varphi_2\| + \|\psi_1-\psi_2\|),
\end{aligned}
$$

and

$$
\begin{aligned}
&|\Phi_2(\varphi_1,\psi_1)(t) - \Phi_2(\varphi_2,\psi_2)(t)| \\
&\leq \int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)}\Big|g(s,\varphi_1(s),\psi_1(s)) - g(s,\varphi_2(s),\psi_2(s))\Big|ds \\
&+ \frac{1}{2}\Big\{\frac{1}{|\Lambda_2|}\int_0^T\Big(\int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)}\Big|f(x,\varphi_1(x),\psi_1(x)) - f(x,\varphi_2(x),\psi_2(x))\Big|dx \\
&+ \int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)}\Big|g(x,\varphi_1(x),\psi_1(x)) - g(x,\varphi_2(x),\psi_2(x))\Big|dx\Big)ds \\
&+ \frac{P_2}{|\Lambda_1|}\Big(\int_0^T \frac{(T-s)^{\nu-1}}{\Gamma(\nu)}\Big|f(s,\varphi_1(s),\psi_1(s)) - f(s,\varphi_2(s),\psi_2(s))\Big|ds \\
&+ \int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)}\Big|g(s,\varphi_1(s),\psi_1(s)) - g(s,\varphi_2(s),\psi_2(s))\Big|ds\Big) \\
&+ \frac{1}{|\Lambda_2|}\int_{\eta}^{\xi}\Big(\int_0^s \frac{(s-x)^{\nu-1}}{\Gamma(\nu)}\Big|f(x,\varphi_1(x),\psi_1(x)) - f(x,\varphi_2(x),\psi_2(x))\Big|dx \\
&+ \int_0^s \frac{(s-x)^{\rho-1}}{\Gamma(\rho)}\Big|g(x,\varphi_1(x),\psi_1(x)) - g(x,\varphi_2(x),\psi_2(x))\Big|dx\Big)ds \\
&+ \frac{1}{|\Lambda_1|}\sum_{i=1}^{m} a_i\Big(\int_0^{\sigma_i} \frac{(\sigma_i-s)^{\nu-1}}{\Gamma(\nu)}\Big|f(s,\varphi_1(s),\psi_1(s)) - f(s,\varphi_2(s),\psi_2(s))\Big|ds \\
&+ \int_0^{\sigma_i} \frac{(\sigma_i-s)^{\rho-1}}{\Gamma(\rho)}\Big|g(s,\varphi_1(s),\psi_1(s)) - g(s,\varphi_2(s),\psi_2(s))\Big|ds\Big)\Big\} \\
&\leq \Big\{\alpha\varrho_1 + \beta\Big(\frac{T^{\rho}}{\Gamma(\rho+1)} + \varrho_2\Big)\Big\}(\|\varphi_1-\varphi_2\| + \|\psi_1-\psi_2\|).
\end{aligned}
$$

In view of the foregoing inequalities, it follows that

$$
\begin{aligned}
\|\Phi(\varphi_1,\psi_1) - \Phi(\varphi_2,\psi_2)\| &= \|\Phi_1(\varphi_1,\psi_1) - \Phi_1(\varphi_2,\psi_2)\| + \|\Phi_2(\varphi_1,\psi_1) - \Phi_2(\varphi_2,\psi_2)\| \\
&\leq \Big\{\alpha\Big(\frac{T^{\nu}}{\Gamma(\nu+1)} + 2\varrho_1\Big) + \beta\Big(\frac{T^{\rho}}{\Gamma(\rho+1)} + 2\varrho_2\Big)\Big\}\|(\varphi_1-\varphi_2,\psi_1-\psi_2)\|.
\end{aligned}
$$

Using the condition (3.9), we deduce from the above inequality that $\Phi$ is a contraction mapping. Consequently $\Phi$ has a unique fixed point by the application of contraction mapping principle. Hence there exists a unique solution for the problem (1.1) on $J$. The proof is finished. □

**Example 3.1.** *Consider the following problem*

$$\begin{cases} {}^{C}D^{1/2}\varphi(t) = f(t,\varphi(t),\psi(t)), \ \ t \in J := [0,2], \\ {}^{C}D^{4/5}\psi(t) = g(t,\varphi(t),\psi(t)), \ \ t \in J := [0,2], \\ (\varphi+\psi)(0) + 5/2(\varphi+\psi)(2) = 1/2(\varphi+\psi)(1/4) + 3/2(\varphi+\psi)(1/2), \\ \int_0^2 (\varphi-\psi)(s)ds - \int_{2/3}^{3/4}(\varphi-\psi)(s)ds = 1, \end{cases} \tag{3.10}$$

*where* $\nu = 1/2$, $\rho = 4/5$, $\eta = 2/3$, $\zeta = 3/4$, $a_1 = 1/2$, $a_2 = 3/2$, $P_1 = 1$, $P_2 = 5/2$, $\sigma_1 = 1/4, \sigma_2 = 1/2$, $A = 1$, $T = 2$, *and* $f(t,\varphi,\psi)$ *and* $g(t,\varphi,\psi)$ *will be fixed later.*

Using the given data, we find that $\Lambda_1 = 1.5, \Lambda_2 = 1.91666667$, $\varrho_1 = 2.110627579$, $\varrho_2 = 2.494392906$, where $\Lambda_1$, $\Lambda_2$, $\varrho_1$ and $\varrho_2$ are respectively given by (2.5), (2.6), (3.4) and (3.5). For illustrating theorem 3.1, we take

$$f(t,\varphi,\psi) = \frac{e^{-t}}{5\sqrt{16+t^2}}\Big(\tan^{-1}\varphi + \psi + \cos t\Big) \text{ and } g(t,\varphi,\psi) = \frac{1}{(t+2)^6}\Big(\frac{|\varphi|}{1+|\psi|} + t\psi + e^{-t}\Big). \tag{3.11}$$

Clearly $f$ and $g$ are continuous and satisfy the condition $(H_1)$ with $\mu_1(t) = \frac{e^{-t}\cos t}{5\sqrt{16+t^2}}$, $\mu_2(t) = \frac{e^{-t}}{5\sqrt{16+t^2}}$, $\mu_3(t) = \frac{e^{-t}}{10\sqrt{16+t^2}}$, $\kappa_1(t) = \frac{e^{-t}}{(t+2)^6}$, $\kappa_2(t) = \frac{1}{(t+2)^6}$, and $\kappa_3(t) = \frac{1}{2(t+2)^6}$. Also

$$\|\mu_2\|\Big(2\varrho_1 + \frac{T^{\nu}}{\Gamma(\nu+1)}\Big) + \|\kappa_2\|\Big(2\varrho_2 + \frac{T^{\rho}}{\Gamma(\rho+1)}\Big) \approx 0.398009902,$$

and

$$\|\mu_3\|\Big(2\varrho_1 + \frac{T^{\nu}}{\Gamma(\nu+1)}\Big) + \|\kappa_3\|\Big(2\varrho_2 + \frac{T^{\rho}}{\Gamma(\rho+1)}\Big) \approx 0.199004951 < 1.$$

Thus all the conditions of theorem 3.1 hold true and hence the problem (3.10) with $f(t,\varphi,\psi)$ and $g(t,\varphi,\psi)$ given by (3.11) has at least one solution on $[0,2]$.

Next we demonstrate the application of Theorem 3.2. Let us choose

$$f(t,\varphi,\psi) = \frac{e^{-t}\tan^{-1}\varphi + \cos\psi}{5\sqrt{16+t^2}} \text{ and } g(t,\varphi,\psi) = \frac{1}{(2+t)^6}\Big(\frac{|\varphi|}{2+|\varphi|} + \sin\psi\Big). \tag{3.12}$$

It is easy to show that the condition $(H_2)$ is satisfied with $\alpha_1 = \alpha_2 = 1/20 = \alpha$ and $\beta_1 = 1/64$, $\beta_2 = 1/128$ and so, $\beta = 1/64$. Also $\alpha\Big(\frac{T^{\nu}}{\Gamma(\nu+1)} + 2\varrho_1\Big) + \beta\Big(\frac{T^{\rho}}{\Gamma(\rho+1)} + 2\varrho_2\Big) \approx 0.39800990 < 1$. Thus the hypothesis of Theorem 3.2 holds and hence its conclusion implies that the problem (3.10) with $f(t,\varphi,\psi)$ and $g(t,\varphi,\psi)$ given by (3.12) has a unique solution on $[0,2]$.

## 4. Nonlinear Riemann-Liouville integral boundary conditions case

In this section, we consider a variant of the problem (1.1) involving a nonlinear Riemann-Liouville integral term in the last boundary condition given by

$$\begin{cases} {}^C D^{\nu}\varphi(t) = f(t,\varphi(t),\psi(t)), \quad t \in J := [0,T], \\ {}^C D^{\rho}\psi(t) = g(t,\varphi(t),\psi(t)), \quad t \in J := [0,T], \\ P_1(\varphi+\psi)(0) + P_2(\varphi+\psi)(T) = \sum_{i=1}^{m} a_i(\varphi+\psi)(\sigma_i), \\ \int_0^T (\varphi-\psi)(s)ds - \int_\eta^\zeta (\varphi-\psi)(s)ds = \frac{1}{\Gamma(\delta)}\int_0^T (T-s)^{\delta-1} h(s,\varphi(s),\psi(s))ds, \ \delta > 0. \end{cases} \tag{4.1}$$

Now we state a uniqueness result for the problem (4.1). We do not provide the proof of this result as it is similar to that of Theorem 3.2.

**Theorem 4.1.** *Let $f, g, h : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be continuous functions and the following assumption holds:*

$(\overline{H}_2)$ *There exist positive constants $\alpha_i$, $\beta_i, \gamma_i$, $i = 1,2$, such that*

$$|f(t,\varphi_1,\psi_1) - f(t,\varphi_2,\psi_2)| \le \alpha_1|\varphi_1-\varphi_2| + \alpha_2|\psi_1-\psi_2|, \quad \forall\, t \in J,\ \varphi_i,\psi_i \in \mathbb{R},\ i = 1,2;$$

$$|g(t,\varphi_1,\psi_1) - g(t,\varphi_2,\psi_2)| \le \beta_1|\varphi_1-\varphi_2| + \beta_2|\psi_1-\psi_2|, \quad \forall\, t \in J,\ \varphi_i,\psi_i \in \mathbb{R},\ i = 1,2;$$

$$|h(t,\varphi_1,\psi_1) - h(t,\varphi_2,\psi_2)| \le \gamma_1|\varphi_1-\varphi_2| + \gamma_2|\psi_1-\psi_2|, \quad \forall\, t \in J,\ \varphi_i,\psi_i \in \mathbb{R},\ i = 1,2.$$

*Then the problem (4.1) has a unique solution on J if*

$$\frac{\gamma T^{\delta}}{|\Lambda_2|\Gamma(\delta+1)} + \alpha\Big(\frac{T^{\nu}}{\Gamma(\nu+1)} + 2\varrho_1\Big) + \beta\Big(\frac{T^{\rho}}{\Gamma(\rho+1)} + 2\varrho_2\Big) < 1, \tag{4.2}$$

*where $\alpha = \max\{\alpha_1,\alpha_2\}$, $\beta = \max\{\beta_1,\beta_2\}, \gamma = \max\{\gamma_1,\gamma_2\}$, and $\varrho_i$, $i = 1,2$ are defined in (3.4)-(3.5).*

**Example 4.1.** *Let us consider the data given in Example 3.1 for the problem (4.1) with* (3.12)*, $h(t,\varphi,\psi) = (\sin\varphi + \cos\psi + 1/2)/\sqrt{t^2+49}$ and $\delta = 3/2$. Then $\gamma = 1/7$ and*

$$\frac{\gamma T^{\delta}}{|\Lambda_2|\Gamma(\delta+1)} + \alpha\Big(\frac{T^{\nu}}{\Gamma(\nu+1)} + 2\varrho_1\Big) + \beta\Big(\frac{T^{\rho}}{\Gamma(\rho+1)} + 2\varrho_2\Big) \approx 0.5565956 < 1.$$

*Clearly the assumptions of Theorem 4.1 are satisfied. Hence, by the conclusion of Theorem 4.1, the problem (4.1) with the given data has a unique solution on* $[0,2]$.

## 5. Conclusions

We have studied a coupled system of nonlinear Caputo fractional differential equations supplemented with a new class of nonlocal multipoint-integral boundary conditions with respect to the sum and difference of the governing functions by applying the standard fixed point theorems. The existence and uniqueness results presented in this paper are not only new in the given configuration but also provide certain new results by fixing the parameters involved in the given problem. For example, our results correspond to the ones with initial-multipoint-integral and terminal-multipoint-integral boundary conditions by fixing $P_2 = 0$ and $P_1 = 0$ respectively in the present results. By taking $A = 0$ in the present study, we obtain the results for the given coupled system of fractional differential equations with the boundary conditions of the form:

$$P_1(\varphi+\psi)(0) + P_2(\varphi+\psi)(T) = \sum_{i=1}^{m} a_i(\varphi+\psi)(\sigma_i), \quad \int_0^T (\varphi-\psi)(s)ds = \int_\eta^\zeta (\varphi-\psi)(s)ds,$$

where the second (integral) condition means that the contribution of the difference of the unknown functions $(\varphi-\psi)$ on the domain $(0, T)$ is equal to that on the sub-domain $(\eta, \zeta)$. Such a situation arises in heat conduction problems with sink and source. In the last section, we discussed the uniqueness of solutions for a variant of the problem (1.1) involving nonlinear Riemann-Liouville integral term in the last boundary condition of (1.1). This consideration further enhances the scope of the problem at hand. As a special case, the uniqueness result (Theorem 4.1) for the problem (4.1) corresponds to nonlinear integral boundary conditions for $\delta = 1$.

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## Conflict of interest

The authors declare that they have no competing interests.

## References

1. Y. Ding, Z. Wang, H. Ye, Optimal control of a fractional-order HIV-immune system with memory, *IEEE Trans. Contr. Sys. Techn.*, **20** (2012), 763–769. https://doi.org/10.1109/TCST.2011.2153203
2. M. Faieghi, S. Kuntanapreeda, H. Delavari, D. Baleanu, LMI-based stabilization of a class of fractional-order chaotic systems, *Nonlinear Dynam.*, **72** (2013), 301–309. https://doi.org/10.1007/s11071-012-0714-6
3. F. Zhang, G. Chen, C. Li, J. Kurths, Chaos synchronization in fractional differential systems, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, **371** (2013), 20120155, 26. https://doi.org/10.1098/rsta.2012.0155
4. Y. Xu, W. Li, Finite-time synchronization of fractional-order complex-valued coupled systems, *Phys. A*, **549** (2020), 123903. https://doi.org/10.1016/j.physa.2019.123903
5. M. Javidi, B. Ahmad, Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system, *Ecol. Model.*, **318** (2015), 8–18.

6. A. Carvalho, C. M. A. Pinto, A delay fractional order model for the co-infection of malaria and HIV/AIDS, *Int. J. Dyn. Control*, **5** (2017), 168–186. https://doi.org/10.1007/s40435-016-0224-3
7. V. V. Tarasova, V. E. Tarasov, Logistic map with memory from economic model, *Chaos Soliton. Fract.,* **95** (2017), 84–91. https://doi.org/10.1515/9783110627459
8. Y. Xu, Y. Li, W. Li, Adaptive finite-time synchronization control for fractional-order complex-valued dynamical networks with multiple weights, *Commun. Nonlinear Sci. Numer. Simul.*, **85** (2020), 105239. https://doi.org/10.1016/j.cnsns.2020.105239
9. M. S. Ali, G. Narayanan, V. Shekher, A. Alsaedi, B. Ahmad, Global Mittag-Leffler stability analysis of impulsive fractional-order complex-valued BAM neural networks with time varying delays, *Commun. Nonlinear Sci. Numer. Simul.*, **83** (2020), 105088, 22 pp.
10. A. N. Chatterjee, B. Ahmad, A fractional-order differential equation model of COVID-19 infection of epithelial cells, *Chaos Soliton. Fract.,* **147** (2021), Paper No. 110952, 6 pp. https://doi.org/10.1016/j.chaos.2021.110952
11. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.
12. K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer-verlag Berlin Heidelberg, 2010.
13. B. Ahmad, A. Alsaedi, S. K. Ntouyas, J. Tariboon, *Hadamard-type Fractional Differential Equations, Inclusions and Inequalities*, Springer, Cham, 2017.
14. J. Tariboon, S. K. Ntouyas, S. Asawasamrit, C. Promsakon, Positive solutions for Hadamard differential systems with fractional integral conditions on an unbounded domain, *Open Math.*, **15** (2107), 645–666.
15. A. Alsaedi, B. Ahmad, M. Alghanmi, Extremal solutions for generalized Caputo fractional differential equations with Steiltjes-type fractional integro-initial conditions, *Appl. Math. Lett.*, **91** (2019), 113–120. https://doi.org/10.1016/j.aml.2018.12.006
16. L. Ren, J. Wang, M. Feckan, Periodic mild solutions of impulsive fractional evolution equations, *AIMS Math.,* **5** (2020), 497–506. https://doi.org/10.3934/math.2020033
17. Z. Cen, L. B. Liu, J. Huang, A posteriori error estimation in maximum norm for a two-point boundary value problem with a Riemann-Liouville fractional derivative, *Appl. Math. Lett.*, **102** (2020), 106086, 8 pp. https://doi.org/10.1016/j.aml.2019.106086
18. Z. Cen, L.-B. Liu, J. Huang, A posteriori error estimation in maximum norm for a two-point boundary value problem with a Riemann-Liouville fractional derivative, *Appl. Math. Lett.* **102** (2020), 106086, 8. https://doi.org/10.1016/j.aml.2019.106086
19. L. Zhang, W. Hou, B. Ahmad, G. Wang, Radial symmetry for logarithmic Choquard equation involving a generalized tempered fractional $p$-Laplacian, *Discrete Contin. Dyn. Syst. Ser. S*, **14** (2021), 3851–3863. https://doi.org/10.3934/dcdss.2020445
20. C. S. Goodrich, Nonlocal differential equations with convolution coefficients and applications to fractional calculus, *Adv. Nonlinear Stud.*, **21** (2021), 767–787. https://doi.org/10.1515/ans-2021-2145
21. C. Promsakon, S. K. Ntouyas, J. Tariboon, Hilfer-Hadamard nonlocal integro-multipoint fractional boundary value problems, *J. Funct. Spaces*, **2021** (2021), Art. ID 8031524, 9 pp. https://doi.org/10.1155/2021/8031524

22. H. Fazli, H. Sun, J. J. Nieto, On solvability of differential equations with the Riesz fractional derivative, *Math. Methods Appl. Sci.*, **45** (2022), 197–205. https://doi.org/10.1002/mma.7773
23. A. Salim, M. Benchohra, J. R. Graef, J. E. Lazreg, Initial value problem for hybrid $\psi$-Hilfer fractional implicit differential equations, *J. Fixed Point Theory Appl.*, **24** (2022), Paper No. 7, 14 pp. https://doi.org/10.1007/s11784-021-00920-x
24. J. W. He, Y. Zhou, L. Peng, B. Ahmad, On well-posedness of semilinear Rayleigh-Stokes problem with fractional derivative on $\mathbb{R}^N$, *Adv. Nonlinear Anal.*, **11** (2022), 580–597.
25. J. A. Gallegos, N. Aguila-Camacho, M. A. Duarte-Mermoud, Smooth solutions to mixed-order fractional differential systems with applications to stability analysis, *J. Integral Equations Appl.*, **31** (2019), 59–84. https://doi.org/10.1216/JIE-2019-31-1-59
26. K. M. Owolabi, B. Karaagac, Dynamics of multi-pulse splitting process in one-dimensional Gray-Scott system with fractional order operator, *Chaos Soliton. Fract.*, **136** (2020), 109835. https://doi.org/10.1016/j.chaos.2020.109835
27. A. Samadi, S. K. Ntouyas, Solvability for infinite systems of fractional differential equations in Banach sequence spaces $\ell_p$ and $c_0$, *Filomat*, **34** (2020), 3943–3955.
28. S. Hristova, R. Agarwal, D. O'Regan, Explicit solutions of initial value problems for systems of linear Riemann-Liouville fractional differential equations with constant delay, *Adv. Difference Equ.*, **2020** (2020), Paper No. 180, 18 pp. https://doi.org/10.1186/s13662-020-02643-8
29. B. Ahmad, S. Hamdan, A. Alsaedi, S. K. Ntouyas, On a nonlinear mixed-order coupled fractional differential system with new integral boundary conditions, *AIMS Math.*, **6** (2021), 5801–5816. https://doi.org/10.3934/math.2021343
30. B. Ahmad, M. Alghanmi, A. Alsaedi, J. J. Nieto, Existence and uniqueness results for a nonlinear coupled system involving Caputo fractional derivatives with a new kind of coupled boundary conditions, *Appl. Math. Lett.*, **116** (2021), 107018, 10 pp. https://doi.org/10.1016/j.aml.2021.107018
31. R. Luca, On a system of Riemann-Liouville fractional differential equations with coupled nonlocal boundary conditions, *Adv. Difference Equ.*, **2021** (2021), Paper No. 134, 25 pp.
32. B. Ahmad, J. Henderson, R. Luca, *Boundary Value Problems for Fractional Differential Equations and Systems*, Trends in Abstract and Applied Analysis, 9, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2021. https://doi.org/10.1186/s13662-021-03303-1
33. S. Abbas, M. Benchohra, A. Petrusel, Coupled Hilfer and Hadamard fractional differential systems in generalized Banach spaces, *Fixed Point Theory*, **23** (2022), 21–34. https://doi.org/10.3390/math7030285
34. B. Ahmad, B. Alghamdi, R. P. Agarwal, A. Alsaedi, Riemann–Liouville fractional integro-differential equations with fractional nonlocal multi-point boundary conditions, *Fractals*, **30** (2022), 2240002, 11 pp. https://doi.org/10.1142/S0218348X22400023
35. D. R. Smart, *Fixed Point Theorems*, Cambridge Tracts in Mathematics, No. 66, Cambridge University Press, London-New York, 1974.

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