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Research article

A particular matrix with exponential form, its inversion and some norms

Baijuan Shi*

School of Science, Xi'an University of Posts and Telecommunications, 710121, ShaanXi, China

* **Correspondence:** Email: shibj1992@163.com.

Abstract: In this paper, we study a particular $n \times n$ matrix $A = [a_{k_{ij}}]_{i,j=1}^{n}$ and its Hadamard inverse $A^{\circ(-1)}$, whose entire elements are exponential form $a_k = e(\frac{k}{n}) = e^{\frac{2\pi ik}{n}}$, where $k_{ij} = \min(i, j) + 1$. We study determinants, leading principal minor and inversions of A, $A^{\circ(-1)}$. Then the defined values of Euclidean norms, l_p norms and spectral norms of these matrices are presented, rather than upper and lower bounds, which are different from other articles.

Keywords: particular matrix; exponential form; determinant; inversion; principal minor; norms
Mathematics Subject Classification: 15A15, 15A60

1. Introduction

Matrix theory is widely used in a variety of areas including computing science, applied mathematics, neural network nonlinear system and others. Recently, studying the determinants and some norms of particular matrix has been a hot and important topic in matrix theory[1]. From past to present, different types of matrices have been defined and studied properties of various properties, such as determinants, inverse, norms have been studied by some mathematicians. For example, Akbulak [2] studied Hadamard exponential Hankel matrix of the form $e^{\circ H_n} = [e^{i+j}]_{i,j}$ and presented l_p norm, two upper bounds for spectral norm and eigenvalues of this matrix. By the identities of Gamma function, Bozkurt [3,4,5] determined l_p norm of Cauchy-Toeplitz matrices and Cauchy-Hankel matrices respectively. Civciv [6] established a bound for spectral and l_p norms of the Khatri-Rao products of Cauchy-Hankel matrices. Solak [7] studied the matrix of the form $B = [b_{ij}]$, where $b_{ij} = a + \min(i, j) + 1$ and obtained its Euclidean norm and inversion of B. A few years ago, these matrices attracted the attention of C. Moler, who had experimentally discovered that most of their singular values are clustered near π. The authors in [8] studied some bounds for the spectral norm of particular matrix of the form $A = [a^{\min(i,j)}]_{i,j=0}^{n-1}$, where a is a real positive number.

Inspired by above articles, we study a particular matrix $A = [a_{k_{ij}}]_{i,j=0}^{n-1}$ and its Hadamard inverse $A^{\circ(-1)}$, where $k_{ij} = \min(i, j) + 1$, $a_k = e(\frac{k}{n})$, and $e(x) = e^{2\pi ix}$. Particularly, $e(n) = 1$, and $|e(x)| = 1$.

Hadamard inverse of A is

$$A^{\circ(-1)} = \left[\left(\frac{1}{e(\frac{k}{n})}\right)\right]_{n\times n} = \left[e\left(\frac{-k}{n}\right)\right]_{n\times n}.$$

Two of these matrices are called Min and Hadamard inverse matrices as following, respectively, for more information, we can see reference [3],

$$A = \begin{pmatrix} e(\frac{1}{n}) & e(\frac{1}{n}) & e(\frac{1}{n}) & \cdots & e(\frac{1}{n}) & e(\frac{1}{n}) \\ e(\frac{1}{n}) & e(\frac{2}{n}) & e(\frac{2}{n}) & \cdots & e(\frac{2}{n}) & e(\frac{2}{n}) \\ e(\frac{1}{n}) & e(\frac{2}{n}) & e(\frac{3}{n}) & \cdots & e(\frac{3}{n}) & e(\frac{3}{n}) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ e(\frac{1}{n}) & e(\frac{2}{n}) & e(\frac{3}{n}) & \cdots & e(\frac{n-1}{n}) & e(\frac{n-1}{n}) \\ e(\frac{1}{n}) & e(\frac{2}{n}) & e(\frac{3}{n}) & \cdots & e(\frac{n-1}{n}) & e(\frac{n}{n}) \end{pmatrix}_{n\times n}, \quad (1.1)$$

$$A^{\circ(-1)} = \begin{pmatrix} e(\frac{-1}{n}) & e(\frac{-1}{n}) & e(\frac{-1}{n}) & \cdots & e(\frac{-1}{n}) & e(\frac{-1}{n}) \\ e(\frac{-1}{n}) & e(\frac{-2}{n}) & e(\frac{-2}{n}) & \cdots & e(\frac{-2}{n}) & e(\frac{-2}{n}) \\ e(\frac{-1}{n}) & e(\frac{-2}{n}) & e(\frac{-3}{n}) & \cdots & e(\frac{-3}{n}) & e(\frac{-3}{n}) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ e(\frac{-1}{n}) & e(\frac{-2}{n}) & e(\frac{-3}{n}) & \cdots & e(\frac{-(n-1)}{n}) & e(\frac{-(n-1)}{n}) \\ e(\frac{-1}{n}) & e(\frac{-2}{n}) & e(\frac{-3}{n}) & \cdots & e(\frac{-(n-1)}{n}) & e(\frac{-n}{n}) \end{pmatrix}_{n\times n}. \quad (1.2)$$

Looking at above matrices we see that these matrices are symmetric, so they are normal matrix, and the positions of the each entries make a "Γ pattern".
For any $m \times n$ matrix A, the l_p norm of A is defined by

$$||A||_p = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^p\right)^{\frac{1}{p}}.$$

For $p = 2$ this norm is called Euclidean norm showed by

$$||A||_E = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}.$$

The spectral norm of matrix A is defined by

$$||A||_2 = \sqrt{\max_{1\le i\le n}\lambda_i(A^HA)},$$

where $\lambda_i(A^HA)$ are the eigenvalues of matrices A^HA and A^H is the conjugate transpose of A.
The spectral radius of A is defined by

$$\rho(A) = \max_i |\lambda_i(A)|, \quad i = 1, 2, \cdots, n.$$

For normal matrix, we know that the spectral norm of A is equal to spectral radius. Different from other articles, the matrix elements are complex numbers $e(\frac{k}{n})$ whose modulus is 1. Therefore, based on the special identities, we can give the defined value of spectral norm rather than upper and lower bound estimates.

First, by some methods, in the beginning of this section we show that determinants of matrix A and its Hadamard inverse $A^{\circ(-1)}$. Then we get some relationship of leading principle minor. Subsequently we present that these two matrices are invertible but not positive definite, and then we present that the inversion of A and $A^{\circ(-1)}$ are tridiagonal matrices. After that we get the defined value of Euclidean norm, l_p norm and spectral norm. All definitions and statements of this paper are available in reference [1,9].

2. Preliminaries and main results

Theorem 1. Let A be a matrix as in (1.1), $n \geq 2$, then

$$\det A = -e\left(\frac{1}{n} + \frac{n}{2}\right)\left(e\left(\frac{1}{n}\right) - 1\right)^{n-1} \neq 0.$$

Proof. By using elementary row operations on (1.1), we have

$$\det A = \det\begin{pmatrix} e(\frac{1}{n}) & e(\frac{1}{n}) & e(\frac{1}{n}) & \cdots & e(\frac{1}{n}) & e(\frac{1}{n}) \\ 0 & e(\frac{2}{n}) - e(\frac{1}{n}) & e(\frac{2}{n}) - e(\frac{1}{n}) & \cdots & e(\frac{2}{n}) - e(\frac{1}{n}) & e(\frac{2}{n}) - e(\frac{1}{n}) \\ 0 & 0 & e(\frac{3}{n}) - e(\frac{2}{n}) & \cdots & e(\frac{3}{n}) - e(\frac{2}{n}) & e(\frac{3}{n}) - e(\frac{2}{n}) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e(\frac{n-1}{n}) - e(\frac{n-2}{n}) & e(\frac{n-1}{n}) - e(\frac{n-2}{n}) \\ 0 & 0 & 0 & \cdots & 0 & e(\frac{n}{n}) - e(\frac{n-1}{n}) \end{pmatrix}_{n\times n}$$

$$= -e\left(\frac{1}{n} + \frac{n}{2}\right)\left(e\left(\frac{1}{n}\right) - 1\right)^{n-1} \neq 0.$$

Therefore, the matrix A is invertible. However, because the matrix elements are complex numbers, we can't confirm that all leading principal minor or all eigenvalues of A are positive, and then matrix A isn't positive definite. Now we give the inversion of matrix A.

Lemma 1. Let A is an $n \times n$ nonsingular matrix and b is an $n \times 1$ matrix, c is a real number. If

$$M = \begin{pmatrix} A & b \\ b^T & c \end{pmatrix},$$

then the inversion of M is

$$N = \begin{pmatrix} A^{-1} + \frac{1}{l}A^{-1}bb^TA^{-1} & -\frac{1}{l}A^{-1}b \\ -\frac{1}{l}b^TA^{-1} & \frac{1}{l} \end{pmatrix},$$

where $l = c - b^TA^{-1}b$.

Proof. By the definition of M and N we have $M \cdot N = E_{n+1}$. Thus $M^{-1} = N$.

Theorem 2. Let A be as in (1.1), then A is invertible and the inversion of A is a symmetric tridiagonal matrix of the form

$$A^{-1}=\begin{pmatrix} c_0 & -c_1 & 0 & 0 & \cdots & 0 \\ -c_1 & b_2 & -c_2 & 0 & \cdots & 0 \\ 0 & -c_2 & b_3 & -c_3 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -c_{n-2} & b_{n-1} & -c_{n-1} \\ 0 & 0 & \cdots & 0 & -c_{n-1} & c_{n-1} \end{pmatrix}_{n\times n},$$

where

$$b_k=\frac{e(\frac{1}{n})+1}{e(\frac{k}{n})(e(\frac{1}{n})-1)},\quad c_k=\frac{1}{e(\frac{k}{n})(e(\frac{1}{n})-1)}.$$

Proof. Using elementary row operations to solve the inversion of A, namely

$$(A\quad I)\to(I\quad A^{-1}).$$

By mathematical induction on n, for $n=2$, we have

$$A=\begin{pmatrix} e(\frac{1}{n}) & e(\frac{1}{n}) \\ e(\frac{1}{n}) & e(\frac{2}{n}) \end{pmatrix}_{2\times 2}$$

$$\begin{aligned}\begin{pmatrix} e(\frac{1}{n}) & e(\frac{1}{n}) & 1 & 0 \\ e(\frac{1}{n}) & e(\frac{2}{n}) & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} e(\frac{1}{n}) & e(\frac{1}{n}) & 1 & 0 \\ 0 & e(\frac{2}{n})-e(\frac{1}{n}) & -1 & 1 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 1 & \frac{1}{e(\frac{1}{n})} & 0 \\ 0 & 1 & -\frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} & \frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 0 & \frac{1}{e(\frac{1}{n})}+\frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} & -\frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} \\ 0 & 1 & -\frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} & \frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} \end{pmatrix},\end{aligned}$$

so we have, for 2×2 matrix A,

$$A^{-1}=\begin{pmatrix} \frac{1}{e(\frac{1}{n})}+\frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} & -\frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} \\ -\frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} & \frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} \end{pmatrix}.$$

In this way, for 3×3 matrix A, we can get

$$A^{-1}=\begin{pmatrix} \frac{1}{e(\frac{1}{n})}+\frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} & -\frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} & 0 \\ -\frac{1}{e(\frac{2}{n})-e(\frac{1}{n})} & \frac{1}{e(\frac{2}{n})-e(\frac{1}{n})}+\frac{1}{e(\frac{3}{n})-e(\frac{2}{n})} & -\frac{1}{e(\frac{3}{n})-e(\frac{2}{n})} \\ 0 & -\frac{1}{e(\frac{3}{n})-e(\frac{2}{n})} & \frac{1}{e(\frac{3}{n})-e(\frac{2}{n})} \end{pmatrix},$$

Therefore, for $n \times n$ matrix A, we can get

$$A^{-1} = \begin{pmatrix} \frac{1}{a_1} + \frac{1}{a_2 - a_1} & -\frac{1}{a_2 - a_1} & 0 & \cdots & 0 \\ -\frac{1}{a_2 - a_1} & \frac{1}{a_2 - a_1} + \frac{1}{a_3 - a_2} & -\frac{1}{a_3 - a_2} & \cdots & 0 \\ 0 & -\frac{1}{a_3 - a_2} & \frac{1}{a_3 - a_2} + \frac{1}{a_4 - a_3} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & -\frac{1}{a_{n-1} - a_{n-2}} & \frac{1}{a_{n-1} - a_{n-2}} + \frac{1}{a_n - a_{n-1}} & -\frac{1}{a_n - a_{n-1}} \\ 0 & 0 & & -\frac{1}{a_n - a_{n-1}} & \frac{1}{a_n - a_{n-1}} \end{pmatrix}_{n \times n},$$

where $a_k = e(\frac{k}{n})$.
In addition,

$$\frac{1}{e(\frac{k+1}{n}) - e(\frac{k}{n})} = \frac{1}{e(\frac{k}{n})(e(\frac{1}{n}) - 1)},$$
$$\frac{1}{e(\frac{k}{n}) - e(\frac{k-1}{n})} + \frac{1}{e(\frac{k+1}{n}) - e(\frac{k}{n})} = \frac{e(\frac{1}{n}) + 1}{e(\frac{k}{n})(e(\frac{1}{n}) - 1)}.$$

In fact, by mathematical induction on n.

The result is true for $n = 2$, now assume that the result is true for n, that is $A = [a_{k_{ij}}]_{n \times n}, A^{-1} = [a_{k_{ij}}]^{-1}_{n \times n}$.

Thus by taking $b = (e(\frac{1}{n}), e(\frac{2}{n}), \cdots, e(\frac{n}{n}))^T, b^T = (e(\frac{1}{n}), e(\frac{2}{n}), \cdots, e(\frac{n}{n}))$, and $c = e(\frac{n+1}{n})$ along with Lemma 1 the proof is completed for $n + 1$. Therefore the result is true for each n. For convenience, we denote

$$b_k = \frac{e(\frac{1}{n}) + 1}{e(\frac{k}{n})(e(\frac{1}{n}) - 1)}, \quad c_k = \frac{1}{e(\frac{k}{n})(e(\frac{1}{n}) - 1)}.$$

Hence, we get

$$A^{-1} = \begin{pmatrix} c_0 & -c_1 & 0 & 0 & \cdots & 0 \\ -c_1 & b_2 & -c_2 & 0 & \cdots & 0 \\ 0 & -c_2 & b_3 & -c_3 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -c_{n-2} & b_{n-1} & -c_{n-1} \\ 0 & 0 & \cdots & 0 & -c_{n-1} & c_{n-1} \end{pmatrix}_{n \times n}.$$

The proof of Theorem 2 provides a method to solve the inversion of "Γ" matrix which is tridiagonal matrix.

The next part, Euclidean norm, l_p norm and spectral norms of matrix A, are considered, we can get concise results.

Theorem 3. Let A be a matrix as in (1.1), then the l_p norm of A is

$$||A||_p = n^{\frac{2}{p}}.$$

Proof. By definition of the Euclidean norm, by $|e\left(\frac{k}{n}\right)| = 1$, we have

$$||A||_p^p = \sum_{k=1}^{n} (2n - 2k + 1)|e\left(\frac{k}{n}\right)|^p = \sum_{k=1}^{n} (2n - 2k + 1) = n^2.$$

Theorem 4. Let A be a matrix as in (1.1), then we get the spectral norm of A,

$$\|A\|_2 = \max\left(1, 2\sin\left(\frac{\pi}{n}\right)\right).$$

Proof. Since matrix A is Hermite matrix which are also normal matrix. So the spectral norm of A is equal to its spectral radius. By Theorem 1, we have the eigenvalues of A are respectively:

$$e\left(\frac{1}{n}\right), e\left(\frac{2}{n}\right) - e\left(\frac{1}{n}\right), e\left(\frac{3}{n}\right) - e\left(\frac{2}{n}\right), \cdots, e\left(\frac{n}{n}\right) - e\left(\frac{n-1}{n}\right).$$

For the modulus of these eigenvalues, we have $|e\left(\frac{1}{n}\right)| = 1$, and

$$\begin{aligned} |e\left(\frac{k+1}{n}\right) - e\left(\frac{k}{n}\right)| &= |e\left(\frac{k}{n}\right)(e\left(\frac{1}{n}\right) - 1)| = |e\left(\frac{1}{n}\right) - 1| \\ &= |\cos\left(\frac{2\pi}{n}\right) - 1 + i\sin\left(\frac{2\pi}{n}\right)| \\ &= 2\sin\left(\frac{\pi}{n}\right). \end{aligned}$$

Hence, spectral radius of A is $\rho(A) = \max\left(1, 2\sin\left(\frac{\pi}{n}\right)\right)$.
Namely,

$$\|A\|_2 = \rho(A) = \max\left(1, 2\sin\left(\frac{\pi}{n}\right)\right).$$

Theorem 5. Let A be a matrix as in (1.1) and $\triangle_n$ denotes the leading principal minor of A, then we have

$$\begin{aligned} &(1)\quad \triangle_1 = e\left(\frac{1}{n}\right),\ \ \triangle_n = -e\left(\frac{1}{n} + \frac{n}{2}\right)\left(e\left(\frac{1}{n}\right) - 1\right)^{n-1} = \det A; \\ &(2)\quad \frac{\triangle_{k+1}}{\triangle_k} = e\left(\frac{k+1}{n}\right) - e\left(\frac{k}{n}\right); \\ &(3)\quad \triangle_n \triangle_{n-2} = e\left(\frac{1}{n}\right)\triangle_{n-1}^2; \\ &(4)\quad \triangle_1\,\triangle_2 \cdots \triangle_n = e\left(\frac{n^2}{3} + \frac{n}{2} + \frac{1}{6}\right)\left(e\left(\frac{1}{n}\right) - 1\right)^{\frac{n(n-1)}{2}}. \end{aligned}$$

Proof. By Theorem 1, we can get (1)

$$\begin{aligned} &\triangle_1 = e\left(\frac{1}{n}\right), \\ &\triangle_2 = e\left(\frac{1}{n}\right)e\left(\frac{1}{n}\right)\left(e\left(\frac{1}{n}\right) - 1\right), \\ &\triangle_3 = e\left(\frac{1}{n}\right)e\left(\frac{1+2}{n}\right)\left(e\left(\frac{1}{n}\right) - 1\right)^2, \end{aligned}$$

$$\triangle_4 = e\left(\frac{1}{n}\right)e\left(\frac{1+2+3}{n}\right)\left(e\left(\frac{1}{n}\right)-1\right)^3, \cdots$$

$$\triangle_{n-2} = e\left(\frac{1}{n}\right)e\left(\frac{1+2+3+\cdots+(n-3)}{n}\right)\left(e\left(\frac{1}{n}\right)-1\right)^{n-1}$$
$$= -e\left(\frac{1}{n}+\frac{n}{2}\right)\left(e\left(\frac{1}{n}\right)-1\right)^{n-3},$$

$$\triangle_{n-1} = e\left(\frac{1}{n}\right)e\left(\frac{1+2+3+\cdots+(n-2)}{n}\right)\left(e\left(\frac{1}{n}\right)-1\right)^{n-1}$$
$$= -e\left(\frac{1}{n}+\frac{n}{2}\right)\left(e\left(\frac{1}{n}\right)-1\right)^{n-2},$$

$$\triangle_n = e\left(\frac{1}{n}\right)e\left(\frac{1+2+3+\cdots+(n-1)}{n}\right)\left(e\left(\frac{1}{n}\right)-1\right)^{n-1}$$
$$= -e\left(\frac{1}{n}+\frac{n}{2}\right)\left(e\left(\frac{1}{n}\right)-1\right)^{n-1} = \det A.$$

Hence, we have

$$\frac{\triangle_{k+1}}{\triangle_k} = e\left(\frac{k}{n}\right)\left(e\left(\frac{1}{n}\right)-1\right) = e\left(\frac{k+1}{n}\right)-e\left(\frac{k}{n}\right). \tag{2}$$

$$\triangle_n\triangle_{n-2} = e\left(\frac{5}{n}\right)\left(e\left(\frac{1}{n}\right)-1\right)^{2n-4}, \triangle_{n-1}^2 = e\left(\frac{4}{n}\right)\left(e\left(\frac{1}{n}\right)-1\right)^{2n-4}. \tag{3}$$

So we get

$$\triangle_n\triangle_{n-2} = e\left(\frac{1}{n}\right)\triangle_{n-1}^2 .$$

By $e(n) = 1$, and $\frac{n(n-1)}{2}$ is always integer number, that is to say $e\left(\frac{n(n-1)}{2}\right) = 1$, then we have

$$\begin{aligned} (4) \qquad & \triangle_1\,\triangle_2\cdots\triangle_n \\ = \;& e\left(\frac{1+(1+2)+(1+2+3)+\cdots+(1+2+\cdots+(n-1))}{n}\right)\left(e\left(\frac{1}{n}\right)-1\right)^{\frac{n(n-1)}{2}} \\ = \;& e\left(\frac{n^2}{3}+\frac{n}{2}+\frac{1}{6}\right)\left(e\left(\frac{1}{n}\right)-1\right)^{\frac{n(n-1)}{2}}. \end{aligned}$$

Next we're going to discuss the determinant, inversion, Euclidean norm, l_p norm and spectral norm of $A^{\circ(-1)}$.

Theorem 6. Let $A^{\circ(-1)}$ be a matrix as in (1.2), $n \geq 2$, then

$$\det A^{\circ(-1)} = -e\left(-\frac{n}{2}\right)\left(1-e\left(\frac{1}{n}\right)\right)^{n-1} \neq 0.$$

Proof. By using elementary row operations on (1.2), by $e(\frac{1}{2}) = -1$, we have

$$\begin{aligned}
&\det A^{\circ(-1)} \\
= &\det\begin{pmatrix}
e(\frac{-1}{n}) & e(\frac{-1}{n}) & e(\frac{-1}{n}) & \cdots & e(\frac{-1}{n}) & e(\frac{-1}{n}) \\
0 & e(\frac{-2}{n}) - e(\frac{-1}{n}) & e(\frac{-2}{n}) - e(\frac{-1}{n}) & \cdots & e(\frac{-2}{n}) - e(\frac{-1}{n}) & e(\frac{-2}{n}) - e(\frac{-1}{n}) \\
0 & 0 & e(\frac{-3}{n}) - e(\frac{-2}{n}) & \cdots & e(\frac{-3}{n}) - e(\frac{-2}{n}) & e(\frac{-3}{n}) - e(\frac{-2}{n}) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & e(\frac{1-n}{n}) - e(\frac{2-n}{n}) & e(\frac{1-n}{n}) - e(\frac{2-n}{n}) \\
0 & 0 & 0 & \cdots & 0 & e(\frac{-n}{n}) - e(\frac{1-n}{n})
\end{pmatrix} \\
= &-e\left(-\frac{n}{2}\right)\left(1 - e\left(\frac{1}{n}\right)\right)^{n-1} \neq 0.
\end{aligned}$$

Therefore, the matrix $A^{\circ(-1)}$ is invertible. Now we give the inversion of matrix $A^{\circ(-1)}$.

Theorem 7. Let $A^{\circ(-1)}$ be a matrix as in (1.2), $n \geq 2$, then the inversion of $A^{\circ(-1)}$ is a tridiagonal matrix as following:

$$[A^{\circ(-1)}]^{-1} = \begin{pmatrix}
h_0 & -h_1 & 0 & 0 & \cdots & 0 \\
-h_1 & f_2 & -h_2 & 0 & \cdots & 0 \\
0 & -h_2 & f_3 & -h_3 & \cdots & 0 \\
\vdots & & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -h_{n-2} & f_{n-1} & -h_{n-1} \\
0 & 0 & \cdots & 0 & -h_{n-1} & h_{n-1}
\end{pmatrix}_{n\times n},$$

where

$$f_k = \frac{e(\frac{k}{n})(1 + e(\frac{1}{n}))}{1 - e(\frac{1}{n})}, \quad h_k = \frac{e(\frac{k}{n})}{1 - e(\frac{1}{n})}.$$

Proof. In similar way, by using elementary row operations to solve the inversion of $A^{\circ(-1)}$, namely $(A^{\circ(-1)} \quad I) \to (I \quad [A^{\circ(-1)}]^{-1})$.

For $n = 2$, we have

$$A^{\circ(-1)} = \begin{pmatrix} e(\frac{-1}{n}) & e(\frac{-1}{n}) \\ e(\frac{-1}{n}) & e(\frac{-2}{n}) \end{pmatrix}_{2\times 2}$$

$$\begin{aligned}
\begin{pmatrix} e(\frac{-1}{n}) & e(\frac{-1}{n}) & 1 & 0 \\ e(\frac{-1}{n}) & e(\frac{-2}{n}) & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} e(\frac{-1}{n}) & e(\frac{-1}{n}) & 1 & 0 \\ 0 & e(\frac{-2}{n}) - e(\frac{-1}{n}) & -1 & 1 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 1 & e(\frac{1}{n}) & 0 \\ 0 & 1 & -\frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} & \frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 0 & e(\frac{1}{n}) + \frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} & -\frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} \\ 0 & 1 & -\frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} & \frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} \end{pmatrix},
\end{aligned}$$

so for 2×2 matrix $A^{\circ(-1)}$,

$$[A^{\circ(-1)}]^{-1} = \begin{pmatrix} e(\frac{1}{n}) + \frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} & -\frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} \\ -\frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} & \frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} \end{pmatrix}.$$

In this way, for 3×3 matrix $A^{\circ(-1)}$, we can get

$$[A^{\circ(-1)}]^{-1} = \begin{pmatrix} e(\frac{1}{n}) + \frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} & -\frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} & 0 \\ -\frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} & \frac{1}{e(\frac{-2}{n})-e(\frac{-1}{n})} + \frac{1}{e(\frac{-3}{n})-e(\frac{-2}{n})} & -\frac{1}{e(\frac{-3}{n})-e(\frac{-2}{n})} \\ 0 & -\frac{1}{e(\frac{-3}{n})-e(\frac{-2}{n})} & \frac{1}{e(\frac{-3}{n})-e(\frac{-2}{n})} \end{pmatrix}.$$

Therefore, for $n \times n$ matrix $A^{\circ(-1)}$, we can get

$$[A^{\circ(-1)}]^{-1} = \begin{pmatrix} \frac{1}{a_1} + \frac{1}{a_2-a_1} & -\frac{1}{a_2-a_1} & 0 & \cdots & 0 \\ -\frac{1}{a_2-a_1} & \frac{1}{a_2-a_1} + \frac{1}{a_3-a_2} & -\frac{1}{a_3-a_2} & \cdots & 0 \\ 0 & -\frac{1}{a_3-a_2} & \frac{1}{a_3-a_2} + \frac{1}{a_4-a_3} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & -\frac{1}{a_{n-1}-a_{n-2}} & \frac{1}{a_{n-1}-a_{n-2}} + \frac{1}{a_n-a_{n-1}} & -\frac{1}{a_n-a_{n-1}} \\ 0 & 0 & & -\frac{1}{a_n-a_{n-1}} & \frac{1}{a_n-a_{n-1}} \end{pmatrix}_{n\times n},$$

where $a_k = e(\frac{-k}{n})$.
Actually,

$$\frac{1}{e(\frac{-k}{n}) - e(-\frac{k-1}{n})} = \frac{1}{e(\frac{-k}{n})(1 - e(\frac{1}{n}))} = \frac{e(\frac{k}{n})}{1 - e(\frac{1}{n})},$$
$$\frac{1}{e(\frac{-k}{n}) - e(-\frac{k-1}{n})} + \frac{1}{e(-\frac{k+1}{n}) - e(-\frac{k}{n})} = \frac{e(\frac{k}{n})(e(\frac{1}{n}) + 1)}{(1 - e(\frac{1}{n}))}.$$

By mathematical induction on n. The result is true for $n = 2$. Now assume that the result is true for n, Thus by taking $b = (e(\frac{-1}{n}), e(\frac{-2}{n}), \cdots, e(\frac{-n}{n}))^T, b^T = (e(\frac{-1}{n}), e(\frac{-2}{n}), \cdots, e(\frac{-n}{n}))$, and $c = e(-\frac{n+1}{n})$ along with Lemma 1 the proof is completed for $n+1$. Therefore the result is true for each n. For convenience, we denote

$$f_k = \frac{e(\frac{k}{n})(1 + e(\frac{1}{n}))}{1 - e(\frac{1}{n})}, \quad h_k = \frac{e(\frac{k}{n})}{1 - e(\frac{1}{n})}.$$

Hence, we have

$$[A^{\circ(-1)}]^{-1} = \begin{pmatrix} h_0 & -h_1 & 0 & 0 & \cdots & 0 \\ -h_1 & f_2 & -h_2 & 0 & \cdots & 0 \\ 0 & -h_2 & f_3 & -h_3 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -h_{n-2} & f_{n-1} & -h_{n-1} \\ 0 & 0 & \cdots & 0 & -h_{n-1} & h_{n-1} \end{pmatrix}.$$

Theorem 8. Let $A^{\circ(-1)}$ be a matrix as in (1.2), then its l_p norm is:

$$||A^{\circ(-1)}||_p = n^{\frac{2}{p}}.$$

Proof. By the definition of l_p norm, we have

$$||A^{\circ(-1)}||_p^p = \sum_{k=1}^{n}(2n - 2k + 1)|e\left(\frac{-k}{n}\right)|^p = \sum_{k=1}^{n}(2n - 2k + 1) = n^2.$$

When $n = 2$, we can get the Euclidean norm$\|A^{\circ(-1)}\|_E = \sqrt{n}$.
Theorem 9. Let $A^{\circ(-1)}$ be a matrix as in (1.2), $n \geq 2$, then the spectral norm is

$$\|A^{\circ(-1)}\|_2 = \max\left(1, 2\sin\left(\frac{\pi}{n}\right)\right).$$

Proof. By Theorem 6, we have the eigenvalues of $A^{\circ(-1)}$ are respectively:

$$e\left(\frac{-1}{n}\right), e\left(\frac{-2}{n}\right) - e\left(\frac{-1}{n}\right), \cdots, e\left(\frac{-n}{n}\right) - e\left(-\frac{n-1}{n}\right).$$

Since matrix $A^{\circ(-1)}$ is normal matrix whose spectral norm is equal to its spectral radius. So we find the modulus of above eigenvalues: $|e\left(\frac{-1}{n}\right)| = 1$,

$$\begin{aligned}
|e\left(\frac{-k}{n}\right) - e\left(-\frac{k-1}{n}\right)| &= |e\left(\frac{-k}{n}\right)\left(1 - e\left(\frac{1}{n}\right)\right)| \\
&= |1 - e\left(\frac{1}{n}\right)| = |1 - \cos\left(\frac{2\pi}{n}\right) - i\sin\left(\frac{2\pi}{n}\right)| \\
&= 2\sin\left(\frac{\pi}{n}\right).
\end{aligned}$$

Hence,

$$\|A^{\circ(-1)}\|_2 = \rho(A) = \max\left(1, 2\sin\left(\frac{\pi}{n}\right)\right) = \begin{cases} 1, & n \geq 6; \\ 2\sin(\frac{\pi}{n}), & n < 6. \end{cases}$$

So far, we've proved all the theorems. We can easily find that matrix A and $A^{\circ(-1)}$ have same Euclidean norm, l_p norm and spectral norm.

3. Conclusions

In this paper, we attempt to compute the determinant, inverse, l_p-norm, and some other properties including those of its Hadamard inverse of particular matrices involving exponential forms and trigonometric functions. The computation complexity of this paper is lower than the previous work. Based on the special properties of exponential form, we get the defined values of l_p norms and spectral norms of particular matrix whose entries are complex numbers $e\left(\frac{k}{n}\right)$. These results will expand the application range of matrix norm and enrich the system of matrix theory.

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Conflict of interest

The author declares no conflict of interest.

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