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Research article

Existence and multiplicity of positive solutions for a class of Kirchhoff type problems with singularity and critical exponents

Deke Wu, Hongmin Suo*, Linyan Peng, Guaiqi Tian and Changmu Chu

College of Data Science and Information Engineering, Guizhou Minzu University, Guiyang, 550025, China

* **Correspondence:** Email: gzmysxx88@sina.com.

Abstract: In this paper, we study the multiplicity results of positive solutions for a class of Kirchhoff type problems with singularity and critical exponents. Combining with the Nehari method and variational method, we prove the existence of positive ground state solutions. Furthermore, we obtain a relationship between the number of positive solutions and the topology of the global maximum set of $Q(x)$.

Keywords: multiple positive solutions; critical exponents; singularity; Nehari method; variational method
Mathematics Subject Classification: 35B33, 35J75

1. Introduction

In this paper, we are interested in the following Kirchhoff type problem with singularity and critical exponents

$$\begin{cases} -\left(a - b\int_{\mathbb{R}^4} |\nabla u|^2\,dx\right)\Delta u = \lambda f(x)|u|^{-\gamma} + Q(x)|u|^2 u, & \text{in } \mathbb{R}^4, \\ u > 0, & \text{in } \mathbb{R}^4, \end{cases} \tag{1.1}$$

where $a, b > 0$, $0 < \gamma < 1$, $\lambda > 0$ is a parameter, and $f(x)$, $Q(x)$ are nonnegative and continuous functions. Throughout this paper, we make the following assumptions:

(Q_1) $Q(x) \in C(\mathbb{R}^4)$ is bounded on $\mathbb{R}^4$; $Q_M := \max_{x\in\mathbb{R}^4} Q(x)$.

(Q_2) There exist k different points $a_1, a_2, \cdots, a_k$ in $\mathbb{R}^4$ such that $Q(a_j) = Q_M$; moreover, a_j, $j = 1, \cdots, k$ are strict local maximums satisfying $|Q(x) - Q(a_j)| = o(|x - a_j|^{\beta_1})$ with $2 \le \beta_1 < 4$ as $x \to a_j$ uniformly for $j = 1, \cdots, k$.

(f_1) $f \in L^{\frac{4}{3+\gamma}}(\mathbb{R}^4)$ is a positive function.

(f_2) There exist $\delta_1 > 0$, $\frac{5+3\gamma}{2} < \beta_2 < 3+\gamma$ and $\rho_1 > 0$ such that $f(x) \geq \rho_1 |x-a_j|^{-\beta_2}$ for $|x-a_j| < \delta_1$, $j = 1, \cdots, k$, where a_j are defined as in (Q_2).

Problem (1.1) is related to the stationary analogue of the equation that occurs in the study of the vibrations of string or membrane namely,

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} + f_1\left(\frac{\partial u}{\partial t}\right) = \left(p_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} + f_2(x,u), \tag{1.2}$$

which was presented by Kirchhoff in [1], where ρ, h, δ, p_0, L are constants representing some physical meanings, respectively, E is the Young's modulus, $u(x,t)$ is the lateral displacemen, f_1, f_2 are the external forces. When $f_1 = f_2$, it extends the classical D'Alembert wave equation for free vibrations of elastic strings. Due to problem (1.2) being no longer a pointwise identity, so, it is often called nonlocal problem, and has received great attention of many researchers. The following singular Kirchhoff type problem has been considered

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda g(x)\dfrac{u^p}{|x|^s} + \dfrac{h(x)}{u^\gamma}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $3 \leq p < 5 - 2s$, $0 < \gamma < 1$, $\lambda > 0$ and $g,\ h \in C(\overline{\Omega})$ are nonnegative functions. The first to study problem (1.3) were Liu and Sun [2], they proved that problem (1.3) has two positive solutions by using the Nehari method when $N = 3$, $0 \leq s < 1$ and $3 < p < 5 - 2s$. Later, Lei and Liao [3] investigated problem (1.3) with $s = 0$ and $p = 5$, they obtained the existence and multiplicity of positive solutions by using the Nehari method and variational method. The result of [2] was improved in [4] and two positive solutions were obtained under $p = 3$. For more similar works on singular Kirchhoff type problem, one can refer to [5–12] and the references therein.

In 2015, Yin and Liu [13] considered the following new Kirchhoff type problem firstly

$$\begin{cases} -\left(a - b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $a, b > 0$ are parameters. In the case of $f(x,u) = |u|^{p-2}u$ with $2 < p < 2^*$, $2^* = \frac{2N}{N-2}$, they proved the existence and multiplicity of nontrivial solutions for problem (1.4) by using variational method. Such problem presents differently interesting difficulties from Kirchhoff type problem (1.3) due to the absence of new nonlocal term $-b\displaystyle\int_\Omega |\nabla u|^2 dx$. Later on, Lei et al. [14] investigated problem (1.4) with $f(x,u) = \lambda |u|^{-\gamma}$, they obtained the existence and multiplicity of positive solutions by variational method. Particularly, Wang et al. [15] obtained many solutions for problem (1.4) in the case of $f(x,u) = |u|^2 u + \mu g(x)$ by using variational method. For more results related to new Kirchhoff type problem, we refer the readers interested in these to [16, 17]. In fact, for the case $p = p(x)$, some results are given in [18–21]. In particular, Hamdani et al. has considered the following $p(x)$-Kirchhoff type problem in [22]

$$\begin{cases} -\left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx\right)\Delta_{p(x)} u = \lambda |u|^{p(x)-2}u + g(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $a \geq b > 0$ are constants, $\Omega \in \mathbb{R}^N$ is a bounded smooth domain, $p(x) \in C(\overline{\Omega})$ with $1 < p(x) < N$, λ is a real parameter and g(x,u) is a continuous function. Under appropriate hypotheses, the authors used Mountain Pass Theorem and Fountain Theorem to obtain the existence and multiplicity of nontrivial solutions for problem (1.5). In [23], Vetro studied a nonlinear $p(x)$-Kirchhoff type problem with Dirichlet boundary condition, in the case of a reaction term also depends on the gradient. Using a topological approach based on the Galerkin method, the author discussed the existence of two notions of solutions: strong generalized solution and weak solution. Strengthening the bound on the Kirchhoff type term, the author established existence of weak solution, with using the theory of operators of monotone type this time.

To the best of our knowledge, so far few results are known on the relation between the number of the maxima of the coefficient function of the critical term and the number of the positive solutions for new Kirchhoff type problem with singularity. Comparing problem (1.1) with the previous mentioned works, we need to overcome the non-differentiability of the functional of the problem and indirect availability of critical point theory due to the presence of singular term. On the other hand, we try to consider the relationship between the number of positive solutions and the topology of global maximum set of $Q(x)$ by the idea of category. Moreover, we should point out that the appearance of the nonlocal term and the lack of compactness resulted from the nonlinearity with the critical Sobolev growth prevent us from using variational method in a standard way.

The energy functional corresponding to problem (1.1) is defined by

$$I_\lambda(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{4}\int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^4} f(x)\,|u|^{1-\gamma}\,dx, \tag{1.6}$$

for $u \in D^{1,2}(\mathbb{R}^4)$, where $D^{1,2}(\mathbb{R}^4)$ is the completion of $C_0^\infty(\mathbb{R}^4)$ with the norm $\|u\|^2 = \int_{\mathbb{R}^4} |\nabla u|^2 dx$ and the corresponding inner product $(u, v) = \int_{\mathbb{R}^4} \nabla u \cdot \nabla v dx$ for any $u,\ v \in D^{1,2}(\mathbb{R}^4)$. It is well known that the singular term leads to the non-differentiability of I_λ on $D^{1,2}(\mathbb{R}^4)$. Therefore, it is difficult to find out the local minimizer and the mountain pass type solutions of the problem. Here, we say $u \in D^{1,2}(\mathbb{R}^4)$ is a weak solution to problem (1.1), if for any $\varphi \in D^{1,2}(\mathbb{R}^4)$, it holds

$$\begin{aligned} &a\int_{\mathbb{R}^4} \nabla u \nabla \varphi dx - b\left(\int_{\mathbb{R}^4} |\nabla u|^2 dx\right)\int_{\mathbb{R}^4} \nabla u \nabla \varphi dx \\ &- \int_{\mathbb{R}^4} Q(x)\,|u|^2\,u\varphi dx - \lambda\int_{\mathbb{R}^4} f(x)\frac{\varphi}{|u|^\gamma} dx = 0. \end{aligned} \tag{1.7}$$

Our main results are as follows.

Theorem 1.1. *Assume that $a, b > 0$, $0 < \gamma < 1$. If the conditions (Q_1) and (f_1) hold, then there exists Λ_0, such that for each $0 < \lambda < \Lambda_0$, problem (1.1) admits a positive ground state solution.*

Theorem 1.2. *Assume that $a, b > 0$, $0 < \gamma < 1$. If the conditions (Q_1), (Q_2), (f_1) and (f_2) hold, then there exists Λ_{00}, such that for each $0 < \lambda < \Lambda_{00}$, problem (1.1) has at least k positive solutions.*

Throughout this paper, we use the following notations:

• B_r (respectively, ∂B_r) denotes the closed ball (respectively, the sphere) of center zero and radius r, i.e., $B_r = \{u \in D^{1,2}(\mathbb{R}^4) : \|u\| \leq r\}$, $\partial B_r = \{u \in D^{1,2}(\mathbb{R}^4) : \|u\| = r\}$;

- $C, C_0, C_1, C_2, \ldots$ denote various positive constants, which may vary from line to line;
- We use $\to$ ($\rightharpoonup$) to denote the strong (weak) convergence;
- $O(\epsilon^t)$ denotes $\frac{|O(\epsilon^t)|}{\epsilon^t} < C$ as $\epsilon \to 0$, and $o(\epsilon^t)$ denotes $\frac{|o(\epsilon^t)|}{\epsilon^t} \to 0$ as $\epsilon \to 0$;
- Define the best constant $S = \inf\left\{\|u\|^2 : u \in D^{1,2}(\mathbb{R}^4), \int_\Omega |u|^4 dx = 1\right\}$, which is attained by the functions $U_{\varepsilon,x_0}(x) = C_\varepsilon / \left(\varepsilon + |x - x_0|^2\right)$ for all $\varepsilon > 0$, where $C_\varepsilon = (8\varepsilon)^{\frac{1}{2}}$.

2. Preliminaries

In order to prove our results, we first consider the functional on the Nehari manifold:

$$\begin{aligned}\mathcal{N}_\lambda = \Big\{u \in D^{1,2}(\mathbb{R}^4)\backslash\{0\} : a\|u\|^2 - b\|u\|^4 \\ - \int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx - \lambda \int_{\mathbb{R}^4} f(x)|u|^{1-\gamma} dx = 0\Big\},\end{aligned}$$

and split $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^-$ as follows

$$\mathcal{N}_\lambda^+ = \Big\{u \in \mathcal{N}_\lambda : a(1+\gamma)\|u\|^2 - b(3+\gamma)\|u\|^4 - (3+\gamma)\int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx > 0\Big\},$$

$$\mathcal{N}_\lambda^0 = \Big\{u \in \mathcal{N}_\lambda : a(1+\gamma)\|u\|^2 - b(3+\gamma)\|u\|^4 - (3+\gamma)\int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx = 0\Big\},$$

$$\mathcal{N}_\lambda^- = \Big\{u \in \mathcal{N}_\lambda : a(1+\gamma)\|u\|^2 - b(3+\gamma)\|u\|^4 - (3+\gamma)\int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx < 0\Big\}.$$

One can easily see that for $u \in \mathcal{N}_\lambda$,

$$\begin{aligned}& a\|u\|^2 - 3b\|u\|^4 - 3\int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx + \lambda\gamma \int_{\mathbb{R}^4} f(x)|u|^{1-\gamma} dx \\ &= a(1+\gamma)\|u\|^2 - b(3+\gamma)\|u\|^4 - (3+\gamma)\int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx \\ &= -2a\|u\|^2 + (3+\gamma)\lambda \int_{\mathbb{R}^4} f(x)|u|^{1-\gamma} dx \\ &= -2b\|u\|^4 - 2\int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx + (1+\gamma)\lambda \int_{\mathbb{R}^4} f(x)|u|^{1-\gamma} dx.\end{aligned} \tag{2.1}$$

Lemma 2.1. *Assume that* $0 < \lambda < \Lambda_1$, $u \in D^{1,2}(\mathbb{R}^4)\backslash\{0\}$, *then there exists unique* $0 < t^+ = t^+(u) < t_{\max} < t^- = t^-(u)$ *such that* $t^+u \in \mathcal{N}^+$, $t^-u \in \mathcal{N}^-$, $I_\lambda(t^+u) = \inf_{0<t<t^-} I_\lambda(tu)$ *and* $I_\lambda(t^-u) = \sup_{t>t_{\max}} I_\lambda(tu)$. *Furthermore,* $\mathcal{N}_\lambda^0 = \emptyset$ for all $0 < \lambda < \Lambda_1$, where

$$\Lambda_1 = \frac{2S^{\frac{1-\gamma}{2}}}{\|f\|_{\frac{4}{3+\gamma}}} \left(\frac{a}{3+\gamma}\right)^{\frac{3+\gamma}{2}} \left(\frac{1+\gamma}{b + S^{-2}Q_M}\right)^{\frac{1+\gamma}{2}}.$$

Proof. For any $u \in D^{1,2}(\mathbb{R}^4)\backslash\{0\}$ and $t \geq 0$, we have

$$\begin{aligned} t\frac{dI_\lambda(tu)}{dt} =& at^2\|u\|^2 - bt^4\|u\|^4 - t^4\int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx - \lambda t^{1-\gamma}\int_{\mathbb{R}^4} f(x)|u|^{1-\gamma}dx \\ =& t^{1-\gamma}\Big[at^{1+\gamma}\|u\|^2 - bt^{3+\gamma}\|u\|^4 \\ &- t^{3+\gamma}\int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx - \lambda\int_{\mathbb{R}^4} f(x)|u|^{1-\gamma}dx\Big] \\ =& t^{1-\gamma}\left[h(t) - \lambda\int_{\mathbb{R}^4} f(x)|u|^{1-\gamma}dx\right], \end{aligned} \tag{2.2}$$

where $h(t) = at^{1+\gamma}\|u\|^2 - bt^{3+\gamma}\|u\|^4 - t^{3+\gamma}\int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx$. Easy computations show that $h'(t) > 0$ for all $0 < t < t_{\max}$ and $h'(t) < 0$ for all $t > t_{\max}$. Thus, $h(t)$ achieves its maximum at

$$t_{\max} = \left[\frac{a(1+\gamma)\|u\|^2}{(3+\gamma)\left(b\|u\|^4 + \int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx\right)}\right]^{\frac{1}{2}} \tag{2.3}$$

with

$$\begin{aligned} \max_{t\in[0,+\infty)} h(t) &= h(t_{\max}) \\ &= \frac{2a\|u\|^2}{3+\gamma}\left(\frac{a(1+\gamma)\|u\|^2}{(3+\gamma)\left(b\|u\|^4 + \int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx\right)}\right)^{\frac{1+\gamma}{2}} \\ &= 2\|u\|^{3+\gamma}\left(\frac{a}{3+\gamma}\right)^{\frac{3+\gamma}{2}}\left(\frac{1+\gamma}{b\|u\|^4 + \int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx}\right)^{\frac{1+\gamma}{2}}. \end{aligned} \tag{2.4}$$

Since $0 < \gamma < 1$, Hölder inequality, Sobolev embedding inequality and (f_1) imply

$$\int_{\mathbb{R}^4} f(x)|u|^{1-\gamma}dx \leq \|f\|_{\frac{4}{3+\gamma}}\left(\int_{\mathbb{R}^4}|u|^4dx\right)^{\frac{1-\gamma}{4}} \leq \|f\|_{\frac{4}{3+\gamma}} S^{\frac{\gamma-1}{2}}\|u\|^{1-\gamma}. \tag{2.5}$$

It follows from Sobolev embedding inequality, (2.4) and (2.5) that

$$\begin{aligned} & h(t_{max}) - \lambda\int_{\mathbb{R}^4} f(x)|u|^{1-\gamma}dx \\ \geq & 2\|u\|^{3+\gamma}\left(\frac{a}{3+\gamma}\right)^{\frac{3+\gamma}{2}}\left(\frac{1+\gamma}{b\|u\|^4 + \int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx)}\right)^{\frac{1+\gamma}{2}} \\ & - \lambda\|f\|_{\frac{4}{3+\gamma}} S^{\frac{\gamma-1}{2}}\|u\|^{1-\gamma} \\ \geq & \left[2\left(\frac{a}{3+\gamma}\right)^{\frac{3+\gamma}{2}}\left(\frac{1+\gamma}{b+S^{-2}Q_M}\right)^{\frac{1+\gamma}{2}} - \lambda\|f\|_{\frac{4}{3+\gamma}} S^{\frac{\gamma-1}{2}}\right]\|u\|^{1-\gamma} > 0, \end{aligned} \tag{2.6}$$

for all $0 < \lambda < \Lambda_1$. Consequently, t_0^+ and t_0^- satisfy $0 < t_0^+ < t_{max} < t_0^-$ such that

$$h(t_0^+) = \lambda \int_{\mathbb{R}^4} f(x)|u|^{1-\gamma} dx = h(t_0^-)$$

and

$$h'(t_0^+) > 0 > h'(t_0^-),$$

that is, $t_0^+ u \in \mathcal{N}_\lambda^+$ and $t_0^- u \in \mathcal{N}_\lambda^-$. Hence, $\mathcal{N}_\lambda^\pm \neq \emptyset$ for all $0 < \lambda < \Lambda_1$. We can further obtain from (2.2) that $\frac{dI_\lambda(tu)}{dt} > 0$ for all $t_0^+ < t < t_0^-$, $\frac{dI_\lambda(tu)}{dt} < 0$ for all $0 < t < t_0^+$ and $t > t_0^-$. Thus, $I_\lambda(t_0^+ u) = \inf_{0<t<t_0^-} I_\lambda(tu)$ and $I_\lambda(t_0^- u) = \sup_{t>t_{max}} I_\lambda(tu)$.

Now, we come to show that $\mathcal{N}_\lambda^0 = \emptyset$ for all $0 < \lambda < \Lambda_1$. Arguing by contradiction, assume that there exists $u_0 \in \mathcal{N}_\lambda^0$ and $u_0 \neq 0$. Similarly to (2.6), we can obtain from (2.1) that

$$\begin{aligned} 0 &< \frac{2a\|u_0\|^2}{3+\gamma} \left(\frac{a(1+\gamma)\|u_0\|^2}{(3+\gamma)\left(b\|u_0\|^4 + \int_{\mathbb{R}^4} Q(x)\,|u_0|^4\,dx\right)} \right)^{\frac{1+\gamma}{2}} - \lambda \int_{\mathbb{R}^4} f(x)|u_0|^{1-\gamma} dx \\ &= \frac{2a\|u_0\|^2}{3+\gamma} - \lambda \int_{\mathbb{R}^4} f(x)|u_0|^{1-\gamma} dx = 0, \end{aligned}$$

which is a contradiction. Hence, $\mathcal{N}_\lambda^0 = \emptyset$ for all $0 < \lambda < \Lambda_1$. □

Lemma 2.2. *Assume that* $0 < \lambda < \Lambda_1$*, then there exists a gap structure in* $\mathcal{N}_\lambda$*:*

$$\|U\| > C_{b1} > C_{\lambda 2} > \|u\|, u \in \mathcal{N}_\lambda^+, U \in \mathcal{N}_\lambda^-,$$

where

$$C_{b1} = \left(\frac{a(1+\gamma)}{(3+\gamma)(b + S^{-2} Q_M)} \right)^{\frac{1}{2}}, C_{\lambda 2} = \left(\lambda \frac{3+\gamma}{2a} \|f\|_{\frac{4}{3+\gamma}} S^{\frac{\gamma-1}{2}} \right)^{\frac{1}{\gamma+1}}. \tag{2.7}$$

Proof. Since $0 < \lambda < \Lambda_1$, according to Lemma 2.1, we have $\mathcal{N}_\lambda^\pm \neq \emptyset$. For any $u \in \mathcal{N}_\lambda^+$, it follows from (2.1) and (2.5) that

$$\|u\|^2 < \frac{3+\gamma}{2a} \lambda \int_{\mathbb{R}^4} f(x)|u|^{1-\gamma} dx \leq \frac{3+\gamma}{2a} \lambda \|f\|_{\frac{4}{3+\gamma}} S^{\frac{\gamma-1}{2}} \|u\|^{1-\gamma},$$

which yields $\|u\| < C_{\lambda 2}$.
For any $U \in \mathcal{N}_\lambda^-$, it follows from (2.1) and Sobolev embedding inequality that

$$\begin{aligned} a(1+\gamma)\|U\|^2 &< b(3+\gamma)\|U\|^4 + (3+\gamma) \int_{\mathbb{R}^4} Q(x)\,|U|^4\,dx \\ &\leq (3+\gamma)(b\|U\|^4 + S^{-2} Q_M \|U\|^4), \end{aligned}$$

which yields $\|U\| > C_{b1}$. Using $0 < \lambda < \Lambda_1$, one can further obtain

$$C_{\lambda 2} < \left(\Lambda_1 \frac{3+\gamma}{2a} \|f\|_{\frac{4}{3+\gamma}} S^{\frac{\gamma-1}{2}} \right)^{\frac{1}{\gamma+1}} = C_{b1}.$$

The proof is completed. □

Lemma 2.3. *Assume that* $0 < \lambda < \Lambda_1$*, then* $\mathcal{N}_\lambda^-$ is a closed set in $D^{1,2}(\mathbb{R}^4)$.

Proof. Since $0 < \lambda < \Lambda_1$, by Lemma 2.1, one has $\mathcal{N}_\lambda^- \neq \emptyset$. Let U_n be a sequence in $\mathcal{N}_\lambda^-$ with $U_n \to U_0$ in $D^{1,2}(\mathbb{R}^4)$, then $U_n \to U_0$ in $L^4(\Omega)$. Since $\mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$, one can obtain from (2.1) and (2.5) that

$$\begin{aligned} a\|U_0\|^2 = a\lim_{n\to\infty}\|U_n\|^2 &= \lim_{n\to\infty}\left[b\|U_n\|^4 + \int_{\mathbb{R}^4} Q(x)\,|U_n|^4\,dx + \lambda\int_{\mathbb{R}^4} f(x)|U_n|^{1-\gamma}dx\right] \\ &= b\|U_0\|^4 + \int_{\mathbb{R}^4} Q(x)\,|U_0|^4\,dx + \lambda\int_{\mathbb{R}^4} f(x)|U_0|^{1-\gamma}dx \end{aligned}$$

and

$$\begin{aligned} &-2a\|U_0\|^2 + \lambda(3+\gamma)\int_{\mathbb{R}^4} f(x)|U_0|^{1-\gamma}dx \\ &= \lim_{n\to\infty}\left[-2a\|U_n\|^2 + \lambda(3+\gamma)\int_{\mathbb{R}^4} f(x)|U_n|^{1-\gamma}dx\right] \le 0, \end{aligned}$$

so $U_0 \in \mathcal{N}_\lambda^- \cup \{0\}$. It follows from $U_n \in \mathcal{N}_\lambda^-$ and Lemma 2.2 that

$$a\|U_0\|^2 = a\lim_{n\to\infty}\|U_n\|^2 > aC_{b1}^2 > 0,$$

that is, $U_0 \neq 0$. Hence, $U_0 \in \mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^-$ is a closed set in $D^{1,2}(\mathbb{R}^4)$. □

Lemma 2.4 *Given* $u \in \mathcal{N}_\lambda^\pm$*, there exist* $\varepsilon > 0$ *and a continuous function* $z_\varepsilon : B_\varepsilon(0) \to \mathbb{R}^+$ *defined for* $w \in D^{1,2}(\mathbb{R}^4), w \in B_\varepsilon(0)$ *such that*

$$z_\varepsilon(0) = 1, \quad z_\varepsilon(w)(u+w) \in \mathcal{N}_\lambda^\pm, \quad \forall w \in D^{1,2}(\mathbb{R}^4), \quad \|w\| < \varepsilon.$$

Proof. We only prove the case of $u \in \mathcal{N}_\lambda^-$. The case of $u \in \mathcal{N}_\lambda^+$ can be proved by a similar argument. Define $F : \mathbb{R} \times D^{1,2}(\mathbb{R}^4) \to \mathbb{R}$ by

$$\begin{aligned} F(t,w) =& at^{1+\gamma}\|u+w\|^2 - bt^{3+\gamma}\|u+w\|^4 \\ &- t^{3+\gamma}\int_{\mathbb{R}^4} Q(x)\,|u+w|^4\,dx - \lambda\int_{\mathbb{R}^4} f(x)|u+w|^{1-\gamma}dx. \end{aligned}$$

Since $u \in \mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$, we obtain $F(1,0) = 0$ and

$$F_t(1,0) = a(1+\gamma)\|u\|^2 - b(3+\gamma)\|u\|^4 - (3+\gamma)\int_{\mathbb{R}^4} Q(x)\,|u|^4\,dx < 0.$$

Using implicit function theorem to F at the point $(1,0)$, we may deduce that there is $\overline{\varepsilon} > 0$ satisfying for $w \in D^{1,2}(\mathbb{R}^4)$, $\|w\| < \overline{\varepsilon}$, the equation $F(t,w) = 0$ has a unique continuous solution $t = z_\varepsilon(w) > 0$ with $z_\varepsilon(0) = 1$. Since $F(z_\varepsilon(w), w) = 0$ for $w \in D^{1,2}(\mathbb{R}^4)$, $\|w\| < \overline{\varepsilon}$, we get

$$\begin{aligned} 0 =& az_\varepsilon^{1+\gamma}(w)\|u+w\|^2 - bz_\varepsilon^{3+\gamma}(w)\|u+w\|^4 \\ &- z_\varepsilon^{3+\gamma}(w)\int_{\mathbb{R}^4} Q(x)\,|u+w|^4\,dx - \lambda\int_{\mathbb{R}^4} f(x)|u+w|^{1-\gamma}dx \\ =& z_\varepsilon(w)^{\gamma-1}\Big[az_\varepsilon^2(w)\|u+w\|^2 - bz_\varepsilon^4(w)\|u+w\|^4 \end{aligned}$$

$$-z_\varepsilon^4(w)\int_{\mathbb{R}^4}Q(x)\,|u+w|^4\,dx-\lambda z_\varepsilon^{1-\gamma}(w)\int_{\mathbb{R}^4}f(x)|u+w|^{1-\gamma}dx\Big],$$

that is, $z_\varepsilon(w)(u+w)\in\mathcal{N}_\lambda$ for all $w\in D^{1,2}(\mathbb{R}^4)$ and $\|w\|<\overline{\varepsilon}$. Since $F_t(1,0)<0$ and

$$\begin{aligned}F_t(z_\varepsilon(w),w)=&a(1+\gamma)z_\varepsilon(w)^\gamma\|u+w\|^2+b(3+\gamma)z_\varepsilon^{2+\gamma}(w)\|u+w\|^2\\&-(3+\gamma)z_\varepsilon^{2+\gamma}(w)\int_{\mathbb{R}^4}Q(x)|u+w|^4dx\\=&\Big[a(1+\gamma)\|z_\varepsilon(w)(u+w)\|^2+b(3+\gamma)\|z_\varepsilon(w)(u+w)\|^4\\&-(3+\gamma)\int_{\mathbb{R}^4}Q(x)|z_\varepsilon(w)(u+w)|^4dx\Big]/z_\varepsilon^{2-\gamma}(w),\end{aligned}$$

we can choose $\epsilon>0$ suitably small($\epsilon<\overline{\epsilon}$) such that $\|w\|<\epsilon$, for $w\in D^{1,2}(\mathbb{R}^4)$,

$$\begin{aligned}a(1+\gamma)\|z_\varepsilon(w)(u+w)\|^2+b(3+\gamma)\|z_\varepsilon(w)(u+w)\|^4\\-(3+\gamma)\int_{\mathbb{R}^4}Q(x)|z_\varepsilon(w)(u+w)|^4dx<0,\end{aligned}$$

which yields $z_\varepsilon(w)(u+w)\in\mathcal{N}_\lambda^-$, for any $w\in D^{1,2}(\mathbb{R}^4)$, $\|w\|<\epsilon$. This completes the proof. □

Lemma 2.5. *The functional I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. Moreover,*
(i) If $0<\lambda<\Lambda_1$, then $\inf_{\mathcal{N}_\lambda^+}I_\lambda<0$;
(ii) If $0<\lambda<\Lambda_2$, then $\inf_{\mathcal{N}_\lambda^-}I_\lambda\geq M_0>0$ for some constants, where

$$M_0=M_0\Big(\lambda,\gamma,S,Q_M,\|f\|_{\frac{4}{3+\gamma}}\Big).$$

and

$$\Lambda_2=\tfrac{1-\gamma}{2}\Lambda_1.$$

Proof. For any $u\in\mathcal{N}_\lambda$, from (1.6), (2.1) and (2.5) we can obtain that

$$\begin{aligned}I_\lambda(u)&=\frac{a}{2}\|u\|^2-\frac{b}{4}\|u\|^4-\frac{1}{4}\int_{\mathbb{R}^4}Q(x)\,|u|^4\,dx-\frac{\lambda}{1-\gamma}\int_{\mathbb{R}^4}f(x)\,|u|^{1-\gamma}\,dx\\&\geq\frac{a}{4}\|u\|^2-\frac{\lambda(3+\gamma)}{4(1-\gamma)}\int_{\mathbb{R}^4}f(x)\,|u|^{1-\gamma}\,dx\\&\geq\frac{a}{4}\|u\|^2-\frac{\lambda(3+\gamma)}{4(1-\gamma)}\|f\|_{\frac{4}{3+\gamma}}S^{\frac{\gamma-1}{2}}\|u\|^{1-\gamma}.\end{aligned}\tag{2.8}$$

Due to $0<\gamma<1$, I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.

(i) When $0<\lambda<\Lambda_1$, we have $\mathcal{N}_\lambda^\pm\neq0$, from Lemma 2.1, $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are two closed sets in $D^{1,2}(\mathbb{R}^4)$ from Lemma 2.3. So $\inf_{\mathcal{N}_\lambda^+}I_\lambda$ and $\inf_{\mathcal{N}_\lambda^-}I_\lambda$ are well defined. For any $u\in\mathcal{N}_\lambda^+\subset\mathcal{N}_\lambda$, we can get from $0<\gamma<1$, $\lambda>0$ and (2.5) that

$$\begin{aligned}I_\lambda(u)&=\frac{a}{2}\|u\|^2-\frac{b}{4}\|u\|^4-\frac{1}{4}\int_{\mathbb{R}^4}Q(x)\,|u|^4\,dx-\frac{\lambda}{1-\gamma}\int_{\mathbb{R}^4}f(x)\,|u|^{1-\gamma}\,dx\\&=\frac{a}{4}\|u\|^2-\frac{\lambda(3+\gamma)}{4(1-\gamma)}\int_{\mathbb{R}^4}f(x)\,|u|^{1-\gamma}\,dx\\&<-\frac{a(1+\gamma)}{4(1-\gamma)}\|u\|^2<0,\end{aligned}$$

which yields $\inf_{\mathcal{N}_\lambda^+} I_\lambda < 0$.

(ii) Let $u \in \mathcal{N}_\lambda^-$, it follows from Lemma 2.2 that $\|u\| > C_{b1}$ (see (2.7)). For all $0 < \lambda < \Lambda_2$, one can get from above inequality and (2.8) that

$$\begin{aligned} I_\lambda(u) \geq & \frac{a}{4}\|u\|^2 - \frac{\lambda(3+\gamma)}{4(1-\gamma)}\|f\|_{\frac{4}{3+\gamma}} S^{\frac{\gamma-1}{2}} \|u\|^{1-\gamma} \\ \geq & \|u\|^{1-\gamma}\left[\frac{a}{4}\|u\|^{1+\gamma} - \frac{\lambda(3+\gamma)}{4(1-\gamma)}\|f\|_{\frac{4}{3+\gamma}} S^{\frac{\gamma-1}{2}}\right] \\ \geq & \left(\frac{a(1+\gamma)}{(3+\gamma)(b+S^{-2}Q_M)}\right)^{\frac{1-\gamma}{2}} \Big[\frac{a}{4}\left(\frac{a(1+\gamma)}{(3+\gamma)(b+S^{-2}Q_M)}\right)^{\frac{1+\gamma}{2}} \\ & - \frac{\lambda(3+\gamma)}{4(1-\gamma)}\|f\|_{\frac{4}{3+\gamma}} S^{\frac{\gamma-1}{2}}\Big], \end{aligned}$$

which implies that there exists a constant $M_0 = M_0\left(\lambda, \gamma, S, Q_M, \|f\|_{\frac{4}{3+\gamma}}\right)$ such that $\inf_{\mathcal{N}_\lambda^-} I_\lambda \geq M_0 > 0$ for all $0 < \lambda < \Lambda_2$. □

According to Lemmas 2.1 and (2.3), for $0 < \lambda < \Lambda_1$, $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are two closed sets in $D^{1,2}(\mathbb{R}^4)$, then we can apply Ekeland's variational principle to find the minimums of functional I_λ on both $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$. Let $\{u_n\} \subset \mathcal{N}_\lambda^\pm$ be a minimizing sequence for I_λ on $\mathcal{N}_\lambda^\pm$. That is, $\{u_n\} \subset \mathcal{N}_\lambda^\pm$ satisfy

$$\alpha_\lambda^\pm < I_\lambda(u_n) < \alpha_\lambda^\pm + \frac{1}{n} \tag{2.9}$$

and

$$I_\lambda(z) \geq I_\lambda(u_n) - \frac{1}{n}\|u_n - z\|, \quad \forall z \in \mathcal{N}_\lambda^\pm, \tag{2.10}$$

where

$$\alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u), \ \alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} I_\lambda(u) \text{ and } \alpha_\lambda = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u).$$

From $I_\lambda(|u_n|) = I_\lambda(u_n)$, we can assume that $u_n \geq 0$. Moreover, Lemma 2.5 shows that $\|u_n\| \leq C_0$ for some suitable positive constant C_0, so there exists a nonnegative function $u_\lambda \in D^{1,2}(\mathbb{R}^4)$ such that

$$\begin{cases} u_n \rightharpoonup u_\lambda, & in\ D^{1,2}(\mathbb{R}^4) \text{ ÉÏ}, \\ u_n \to u_\lambda, & in\ L^p_{loc}\left(\mathbb{R}^4\right)\ (2 \leq p < 4), \\ u_n(x) \to u_\lambda(x), & \text{a.e. } in\ \mathbb{R}^4. \end{cases} \tag{2.11}$$

By Vitali theorem, as done similarly in the proof of [11], we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^4} f(x)\,|u_n|^{1-\gamma}\,dx = \int_{\mathbb{R}^4} f(x)\,|u_\lambda|^{1-\gamma}\,dx, \tag{2.12}$$

when $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^4)$.

Lemma 2.6. *Assume that* $0 < \lambda < \Lambda_1$ *and* $\{u_n\} \subset \mathcal{N}_\lambda^\pm$ *satisfy* (2.11) *with* $u_\lambda \not\equiv 0$, *then there exists a constant* $C_1 > 0$ *such that the following alternative holds true:*
(i) If $\{u_n\} \subset \mathcal{N}_\lambda^+$, *we have*

$$a(1+\gamma)\|u_n\|^2 - b(3+\gamma)\|u_n\|^4 - (3+\gamma)\int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx \geq C_1;$$

(ii) If $\{u_n\} \subset \mathcal{N}_\lambda^-$, *we have*

$$a(1+\gamma)\|u_n\|^2 - b(3+\gamma)\|u_n\|^4 - (3+\gamma)\int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx \leq -C_1.$$

Proof. We just prove (i), since (ii) follows similarly. Since $\{u_n\} \subset \mathcal{N}_\lambda^+$, (2.1), (2.12) and $u_\lambda \not\equiv 0$, it is enough to show that

$$(3+\gamma)\lambda\int_{\mathbb{R}^4} f(x)|u_\lambda|^{1-\gamma}dx > \liminf_{n\to\infty}(2a\|u_n\|^2). \tag{2.13}$$

Arguing by contradiction, assume that

$$(3+\gamma)\lambda\int_{\mathbb{R}^4} f(x)|u_\lambda|^{1-\gamma}dx = \liminf_{n\to\infty}(2a\|u_n\|^2). \tag{2.14}$$

Since $\{u_n\} \subset \mathcal{N}_\lambda^+$, one has

$$(3+\gamma)\lambda\int_{\mathbb{R}^4} f(x)|u_n|^{1-\gamma}dx > 2a\|u_n\|^2.$$

According to (2.12), we can further obtain

$$(3+\gamma)\lambda\int_{\mathbb{R}^4} f(x)|u_\lambda|^{1-\gamma}dx = \limsup_{n\to\infty}(2a\|u_n\|^2) = \liminf_{n\to\infty}(2a\|u_n\|^2). \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$(3+\gamma)\lambda\int_{\mathbb{R}^4} f(x)|u_\lambda|^{1-\gamma}dx = \lim_{n\to\infty}(2a\|u_n\|^2). \tag{2.16}$$

Passing to the limit as $n \to \infty$ and using (2.1) and (2.16), we get

$$\lim_{n\to\infty}\left[b\|u_n\|^4 + \int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx\right] = \lambda\frac{1+\gamma}{2}\int_{\mathbb{R}^4} f(x)|u_\lambda|^{1-\gamma}dx. \tag{2.17}$$

Therefore, it follows from (2.16) and (2.17) that

$$\frac{1+\gamma}{3+\gamma}\lim_{n\to\infty}\left[\frac{a\|u_n\|^2}{b\|u_n\|^4 + \int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx}\right] = 1, \tag{2.18}$$

For $0 < \lambda < \Lambda_1$, passing to the limit as $n \to \infty$ in (2.6), using (2.12), (2.16) and (2.18), we have

$$\begin{aligned}
0 &< \lim_{n\to\infty}\left[2\left(\frac{a}{3+\gamma}\right)^{\frac{3+\gamma}{2}}\left(\frac{1+\gamma}{b+S^{-2}Q_M}\right)^{\frac{1+\gamma}{2}} - \lambda\|f\|_{\frac{4}{3+\gamma}}S^{\frac{\gamma-1}{2}}\right]\|u_n\|^{1-\gamma} \\
&\le \lim_{n\to\infty}\frac{2a\|u_n\|^2}{3+\gamma}\left(\frac{a(1+\gamma)\|u_n\|^2}{(3+\gamma)\left(b\|u_n\|^4+\int_{\mathbb{R}^4}Q(x)\,|u_n|^4\,dx\right)}\right)^{\frac{1+\gamma}{2}} \\
&\quad - \lim_{n\to\infty}\lambda\int_{\mathbb{R}^4}f(x)|u_n|^{1-\gamma}dx \\
&= \lim_{n\to\infty}\frac{2a\|u_n\|^2}{3+\gamma} - \lambda\int_{\mathbb{R}^4}f(x)|u_\lambda|^{1-\gamma}dx = 0,
\end{aligned}$$

which is impossible. Thus, (2.13) holds and the proof of Lemma 2.6 is completed. □

Given $s > 0$ small enough and $0 \le \psi \in D^{1,2}(\mathbb{R}^4)$, we set $u = u_n$ and $w = s\psi$ in Lemma 2.4, then we obtain that there exists a continuous function $z_n : B_\varepsilon(0) \to \mathbb{R}^+$ such that $z_n(0) = 1$ and $z_n(s\psi)(u_n + s\psi) \in \mathcal{N}_\lambda^\pm$. For simplicity, we denote $z_{n,\psi}(s) = z_n(s\psi)$ in the following proof. However, we have no idea whether or not $z_{n,\psi}(s)$ is differentiable. For the sake of proof, we set

$$z'_{n,\psi}(0) = \lim_{s\to 0^+}\frac{z_{n,\psi}(s)-1}{s} \in [-\infty, +\infty].$$

Lemma 2.7. *Assume that* $0 < \lambda < \Lambda_1$. *Suppose* $\{u_n\} \subset \mathcal{N}_\lambda^\pm$ *satisfy* (2.10) *and* (2.11) *with* $u_\lambda \not\equiv 0$, *then* $z'_{n,\psi}(0)$ *is uniformly bounded for any* $0 \le \psi \in D^{1,2}(\mathbb{R}^4)$.

Proof. We only consider the case of $u_n,\ z_{n,\psi}(s)(u_n + s\psi) \in \mathcal{N}_\lambda^+$ since the situation on $\mathcal{N}_\lambda^-$ can be proved similarly. Since $u_n,\ z_{n,\psi}(s)(u_n + s\psi) \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we have

$$a\|u_n\|^2 - b\|u_n\|^4 - \int_{\mathbb{R}^4}Q(x)\,|u_n|^4\,dx - \lambda\int_{\mathbb{R}^4}f(x)|u_n|^{1-\gamma}dx = 0$$

and

$$\begin{aligned}
&az_{n,\psi}^2(s)\|u_n+s\psi\|^2 - bz_{n,\psi}^4(s)\|u_n+s\psi\|^4 \\
&- z_{n,\psi}^4(s)\int_{\mathbb{R}^4}Q(x)\,|u_n+s\psi|^4\,dx - \lambda z_{n,\psi}^{1-\gamma}(s)\int_{\mathbb{R}^4}f(x)|u_n+s\psi|^{1-\gamma}dx = 0.
\end{aligned}$$

Using $0 < \gamma < 1$ and $\lambda > 0$, the above two equalities yield

$$\begin{aligned}
0 =& a\left[z_{n,\psi}^2(s)-1\right]\|u_n+s\psi\|^2 - b\left[z_{n,\psi}^4(s)-1\right]\|u_n+s\psi\|^4\\
&-\left[z_{n,\psi}^4(s)-1\right]\int_{\mathbb{R}^4} Q(x)\,|u_n+s\psi|^4\,dx\\
&-\lambda\left[z_{n,\psi}^{1-\gamma}(s)-1\right]\int_{\mathbb{R}^4} f(x)|u_n+s\psi|^{1-\gamma}dx\\
&+a\left[\|u_n+s\psi\|^2-\|u_n\|^2\right]-b\left[\|u_n+s\psi\|^4-\|u_n\|^4\right]\\
&-\int_{\mathbb{R}^4} Q(x)\left[|u_n+s\psi|^4-|u_n|^4\right]dx-\lambda\int_{\mathbb{R}^4} f(x)\left[|u_n+s\psi|^{1-\gamma}-|u_n|^{1-\gamma}\right]dx\\
\le&\left[z_{n,\psi}(s)-1\right]\Big\{a\left[z_{n,\psi}(s)+1\right]\|u_n+s\psi\|^2-b\frac{z_{n,\psi}^4(s)-1}{z_{n,\psi}(s)-1}\|u_n+s\psi\|^4\\
&-\frac{z_{n,\psi}^4(s)-1}{z_{n,\psi}(s)-1}\int_{\mathbb{R}^4} Q(x)\,|u_n+s\psi|^4\,dx\\
&-\lambda\frac{z_{n,\psi}^{1-\gamma}(s)-1}{z_{n,\psi}(s)-1}\int_{\mathbb{R}^4} f(x)|u_n+s\psi|^{1-\gamma}dx\Big\}+a\left[\|u_n+s\psi\|^2-\|u_n\|^2\right].
\end{aligned}$$

Dividing by $s > 0$ and passing to the limit as $s \to 0^+$, using (2.1), we obtain

$$\begin{aligned}
0 \le& z'_{n,\psi}(0)\Big\{2a\|u_n\|^2-4b\|u_n\|^4-4\int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx\\
&-(1-\gamma)\lambda\int_{\mathbb{R}^4} f(x)|u_n|^{1-\gamma}dx\Big\}+2a\int_{\mathbb{R}^4}\nabla u_n\nabla\psi dx\\
\le& z'_{n,\psi}(0)\Big\{a(1+\gamma)\|u_n\|^2-b(3+\gamma)\|u_n\|^4\\
&-(3+\gamma)\int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx\Big\}+2a\int_{\mathbb{R}^4}\nabla u_n\nabla\psi dx,
\end{aligned}\tag{2.19}$$

which implies that $z'_{n,\psi}(0) \neq -\infty$ according to Lemma 2.6 and the boundedness of $\{u_n\}$. Now we show that $z'_{n,\psi}(0) \neq +\infty$. Arguing by contradiction, we assume that $z'_{n,\psi}(0) = +\infty$ and so $z'_{n,\psi}(0) > 1$ for n sufficiently large and $s > 0$ small. Applying condition (2.10) with $z = z_{n,\psi}(s)(u_n + s\psi)$, we have

$$\begin{aligned}
&\frac{1}{n}\left[z_{n,\psi}(s)-1\right]\|u_n\|+\frac{s}{n}z_{n,\psi}(s)\|\psi\|\\
&\ge\frac{1}{n}\|u_n-z_{n,\psi}(s)(u_n+s\psi)\|\\
&\ge I_\lambda(u_n)-I_\lambda(z_{n,\psi}(s)(u_n+s\psi)).
\end{aligned}\tag{2.20}$$

Since $u_n \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, then one can get from (1.6) and (2.20) that

$$\begin{aligned}
\frac{\|\psi\|}{n} z_{n,\psi}(s) \geq & \frac{z_{n,\psi}(s)-1}{s}\Big\{-\frac{\|u_n\|}{n} - a(\frac{1}{2}-\frac{1}{1-\gamma})\|u_n+s\psi\|^2 \\
& + b(\frac{1}{4}-\frac{1}{1-\gamma})\frac{z_{n,\psi}^4(s)-1}{z_{n,\psi}(s)-1}\|u_n+s\psi\|^4 \\
& + (\frac{1}{4}-\frac{1}{1-\gamma})\frac{z_{n,\psi}^4(s)-1}{z_{n,\psi}(s)-1}\int_{\mathbb{R}^4} Q(x)\,|u_n+s\psi|^4\,dx\Big\} \\
& - a(\frac{1}{2}-\frac{1}{1-\gamma})\frac{\|u_n+s\psi\|^2-\|u_n\|^2}{s} \\
& + b(\frac{1}{4}-\frac{1}{1-\gamma})\frac{\|u_n+s\psi\|^4-\|u_n\|^4}{s} \\
& + (\frac{1}{4}-\frac{1}{1-\gamma})\int_{\mathbb{R}^4} Q(x)\frac{|u_n+s\psi|^4-|u_n|^4}{s}dx.
\end{aligned}$$

Letting $s \to 0^+$, using the continuity of $z_{n,\psi}(s)$, Lemma 2.6 and $\|u_n\| \leq C_0$, we obtain

$$\begin{aligned}
\frac{\|\psi\|}{n} \geq & z_{n,\psi}'(0)\Big\{-\frac{\|u_n\|}{n} - a(1-\frac{2}{1-\gamma})\|u_n\|^2 + b(1-\frac{4}{1-\gamma})\|u_n\|^4 \\
& + (1-\frac{4}{1-\gamma})\int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx\Big\} - a(1-\frac{2}{1-\gamma})\int_{\mathbb{R}^4}\nabla u_n\nabla\psi dx \\
& + b(1-\frac{4}{1-\gamma})\int_{\mathbb{R}^4}|\nabla u_n|^2\int_{\mathbb{R}^4}\nabla u_n\nabla\psi dx \\
& + (1-\frac{4}{1-\gamma})\int_{\mathbb{R}^4} Q(x)\,|u_n|^3\,\psi dx \\
= & z_{n,\psi}'(0)\Big\{-\frac{\|u_n\|}{n} + \frac{1}{1-\gamma}\Big(a(1-\gamma)\|u_n\|^2 - b(3-\gamma)\|u_n\|^4 \\
& - (3-\gamma)\int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx\Big)\Big\} - a(1-\frac{2}{1-\gamma})\int_{\mathbb{R}^4}\nabla u_n\nabla\psi dx \\
& + b(1-\frac{4}{1-\gamma})\int_{\mathbb{R}^4}|\nabla u_n|^2\int_{\mathbb{R}^4}\nabla u_n\nabla\psi dx \\
& + (1-\frac{4}{1-\gamma})\int_{\mathbb{R}^4} Q(x)\,|u_n|^3\,\psi dx \\
\geq & z_{n,\psi}'(0)\left(-\frac{C_0}{n} + \frac{C_1}{1-\gamma}\right) - a(1-\frac{2}{1-\gamma})\int_{\mathbb{R}^4}\nabla u_n\nabla\psi dx \\
& + b(1-\frac{4}{1-\gamma})\int_{\mathbb{R}^4}|\nabla u_n|^2\int_{\mathbb{R}^4}\nabla u_n\nabla\psi dx \\
& + (1-\frac{4}{1-\gamma})\int_{\mathbb{R}^4} Q(x)\,|u_n|^3\,\psi dx
\end{aligned} \tag{2.21}$$

which is impossible because $z_{n,\psi}'(0) = +\infty$ and $-\frac{C_0}{n} + \frac{C_1}{1-\gamma} > 0$ for n large enough. Hence, $z_{n,\psi}'(0) \neq +\infty$. To sum up, $|z_{n,\psi}'(0)| < +\infty$. Moreover, three inequalities (2.19), Lemma 2.6 and (2.21) with $\|u_n\| \leq C_0$ also imply that

$$z_{n,\psi}'(0) \leq C_2, \tag{2.22}$$

for n sufficiently large and a suitable positive constant C_2. □

Lemma 2.8. *Assume that* $0 < \lambda < \Lambda_1$*. Suppose* $\{u_n\} \subset \mathcal{N}_\lambda^\pm$ *satisfy* (2.10) *and* (2.11) *with* $u_\lambda \not\equiv 0$*, then for any* $0 \le \psi \in D^{1,2}(\mathbb{R}^4)$*, we have, as* $n \to \infty$*,*

$$\begin{aligned} &a\int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx - b\left(\int_{\mathbb{R}^4} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx \\ &- \int_{\mathbb{R}^4} Q(x)\,|u_n|^2\, u_n \psi dx - \lambda \int_{\mathbb{R}^4} f(x)\,|u_n|^{-\gamma}\,\psi dx = o(1). \end{aligned} \tag{2.23}$$

Proof. For any $0 \le \psi \in D^{1,2}(\mathbb{R}^4)$, applying condition (2.10) with $z = z_{n,\psi}(s)(u_n + s\psi)$, it leads to

$$\begin{aligned} &\frac{\left[z_{n,\psi}(s) - 1\right]}{s}\frac{\|u_n\|}{n} + \frac{z_{n,\psi}(s)\|\psi\|}{n} \ge \frac{1}{ns}\|u_n - z_{n,\psi}(s)(u_n + s\psi)\| \\ \ge&\frac{1}{s}\left[I_\lambda(u_n) - I_\lambda(z_{n,\psi}(s)(u_n + s\psi))\right]\frac{\left[z_{n,\psi}(s) - 1\right]}{s}\Big\{ - a\frac{\left[z_{n,\psi}(s) + 1\right]}{2}\|u_n + s\psi\|^2 \\ &+ b\Big(\frac{z_{n,\psi}^4(s) - 1}{4[z_{n,\psi}(s) - 1]}\Big)\|u_n + s\psi\|^4 \\ &+ \frac{z_{n,\psi}^4(s) - 1}{4[z_{n,\psi}(s) - 1]}\int_{\mathbb{R}^4} Q(x)\,|u_n + s\psi|^4\, dx \\ &+ \frac{z_{n,\psi}^{1-\gamma}(s) - 1}{(1-\gamma)[z_{n,\psi}(s) - 1]}\int_{\mathbb{R}^4} Q(x)\,|u_n + s\psi|^{1-\gamma}\, dx\Big\} \\ &- \frac{a}{2}\frac{\|u_n + s\psi\|^2 - \|u_n\|^2}{s} + \frac{b}{4}\frac{\|u_n + s\psi\|^4 - \|u_n\|^4}{s} \\ &+ \frac{1}{4}\int_{\mathbb{R}^4} Q(x)\frac{|u_n + s\psi|^4 - |u_n|^4}{s}dx \\ &+ \frac{1}{1-\gamma}\int_{\mathbb{R}^4} Q(x)\frac{|u_n + s\psi|^{1-\gamma} - |u_n|^{1-\gamma}}{s}dx. \end{aligned}$$

Passing to the limit as $s \to 0^+$, according to the continuity of $z_{n,\psi}(s)$, Fatou's Lemma and $0 < \gamma < 1$, we have

$$\begin{aligned} &\frac{|z'_{n,\psi}(0)| \cdot \|u_n\|}{n} + \frac{\|\psi\|}{n} \\ \ge& z'_{n,\psi}(0)\Big\{ - a\|u_n\|^2 + b\|u_n\|^4 + \int_{\mathbb{R}^4} Q(x)\,|u_n|^4\, dx + \lambda\int_{\mathbb{R}^4} f(x)|u_n|^{1-\gamma}dx\Big\} \\ &- a\int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx + b\left(\int_{\mathbb{R}^4} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx \\ &+ \int_{\mathbb{R}^4} Q(x)\,|u_n|^2\, u_n \psi dx + \lambda\int_{\mathbb{R}^4} f(x)\,|u_n|^{-\gamma}\,\psi dx \\ =& - a\int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx + b\left(\int_{\mathbb{R}^4} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx \\ &+ \int_{\mathbb{R}^4} Q(x)\,|u_n|^2\, u_n \psi dx + \lambda\int_{\mathbb{R}^4} f(x)\,|u_n|^{-\gamma}\,\psi dx, \end{aligned}$$

since $u_n \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$. Thanks to $\|u_n\| \leq C_0$, $|z'_{n,\psi}(0)| \leq C_2$ and (2.22), we passing to the limit, we have

$$\begin{aligned} &a \int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx - b\left(\int_{\mathbb{R}^4} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx \\ &- \int_{\mathbb{R}^4} Q(x)\,|u_n|^2\, u_n \psi dx - \lambda \int_{\mathbb{R}^4} f(x)\,|u_n|^{-\gamma}\, \psi dx \geq o(1). \end{aligned} \tag{2.24}$$

Now, we come to show that (2.24) holds for every $\psi \in D^{1,2}(\mathbb{R}^4)$. For any $\psi \in D^{1,2}(\mathbb{R}^4)$ and $\varepsilon > 0$, set $\psi_\varepsilon = u_n + \varepsilon\psi$ and $\Omega_\varepsilon = \{x \in \mathbb{R}^4 : \psi_\varepsilon \leq 0\}$. Since $u_n \in \mathcal{N}_\lambda$, by applying inequality (2.24) with $\psi = \psi_\varepsilon^+$, we have

$$\begin{aligned} o(1) \leq& \frac{1}{\varepsilon}\Big\{a \int_{\mathbb{R}^4} \nabla u_n \nabla \psi_\varepsilon^+ dx - b\left(\int_{\mathbb{R}^4} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^4} \nabla u_n \nabla \psi_\varepsilon^+ dx \\ &- \int_{\mathbb{R}^4} Q(x)\,|u_n|^2\, u_n \psi_\varepsilon^+ dx - \lambda \int_{\mathbb{R}^4} f(x)\,|u_n|^{-\gamma}\, \psi_\varepsilon^+ dx\Big\} \\ =& \frac{1}{\varepsilon}\Big\{a \int_{\mathbb{R}^4\backslash\Omega_\varepsilon} \nabla u_n \nabla (u_n + \varepsilon\psi) dx \\ &- b \int_{\mathbb{R}^4\backslash\Omega_\varepsilon} |\nabla u_n|^2 \int_{\Omega\backslash\Omega_\varepsilon} \nabla u_n \nabla (u_n + \varepsilon\psi) dx \\ &- \int_{\mathbb{R}^4\backslash\Omega_\varepsilon} Q(x)\,|u_n|^2\, u_n (u_n + \varepsilon\psi) dx \\ &- \lambda \int_{\mathbb{R}^4\backslash\Omega_\varepsilon} f(x)\,|u_n|^{-\gamma}\, (u_n + \varepsilon\psi) dx\Big\} \\ =& \frac{1}{\varepsilon}\Big\{a\|u_n\|^2 - b\|u_n\|^4 - \int_{\mathbb{R}^4} Q(x)\,|u_n|^4\, dx - \lambda \int_{\mathbb{R}^4} f(x)|u_n|^{1-\gamma} dx\Big\} \\ &+ \Big\{a \int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx - b\left(\int_{\mathbb{R}^4} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx \\ &- \int_{\mathbb{R}^4} Q(x)\,|u_n|^2\, u_n \psi dx - \lambda \int_{\mathbb{R}^4} f(x)\,|u_n|^{-\gamma}\, \psi dx\Big\} \\ &- \frac{1}{\varepsilon}\Big\{a \int_{\Omega_\varepsilon} \nabla u_n \nabla (u_n + \varepsilon\psi) dx \\ &- b \int_{\Omega_\varepsilon} |\nabla u_n|^2 \int_{\Omega_\varepsilon} \nabla u_n \nabla (u_n + \varepsilon\psi) dx \\ &- \int_{\Omega_\varepsilon} Q(x)\,|u_n|^2\, u_n (u_n + \varepsilon\psi) dx - \lambda \int_{\Omega_\varepsilon} f(x)\,|u_n|^{-\gamma}\, (u_n + \varepsilon\psi) dx\Big\} \\ \leq& \Big\{a \int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx - b\left(\int_{\mathbb{R}^4} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx \\ &- \int_{\mathbb{R}^4} Q(x)\,|u_n|^2\, u_n \psi dx - \lambda \int_{\mathbb{R}^4} f(x)\,|u_n|^{-\gamma}\, \psi dx\Big\} - a \int_{\Omega_\varepsilon} \nabla u_n \nabla \psi dx. \end{aligned} \tag{2.25}$$

Letting $\varepsilon \to 0^+$ to the above inequality and using the fact that $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0^+$, we have

$$a\int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx - b\left(\int_{\mathbb{R}^4} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^4} \nabla u_n \nabla \psi dx$$
$$-\int_{\mathbb{R}^4} Q(x)\,|u_n|^2\, u_n \psi dx - \lambda \int_{\mathbb{R}^4} f(x)\,|u_n|^{-\gamma}\,\psi dx \geq o(1), \quad \forall \psi \;\in D^{1,2}(\mathbb{R}^4).$$

This inequality also holds for $-\psi$, so we get that (2.23) holds for every $\psi \in D^{1,2}(\mathbb{R}^4)$. □

Lemma 2.9. *Assume that* $0 < \lambda < \Lambda_1$. *Suppose* $\{u_n\} \subset \mathcal{N}_\lambda^\pm$ *satisfy* (2.10) *and* (2.11) *and*

$$I_\lambda(u_n) \to c < c_\lambda, \;\; as\; n \to \;\infty, \tag{2.26}$$

where $c \neq 0$ *and* $c_\lambda = \frac{a^2 s^2}{4(bs^2+Q_M)} - D\lambda^{\frac{4}{3+\gamma}}$ *with*

$$D = \frac{3+\gamma}{4}\left[\frac{16S^2(bs^2+Q_M)}{bQ_M(1-\gamma)}\right]^{\frac{1-\gamma}{3+\gamma}}\left[\frac{(1+\gamma)\|f\|_{\frac{4}{3+\gamma}}}{2(1-\gamma)}\right]^{\frac{4}{3+\gamma}},$$

then $u_\lambda \not\equiv 0$ *and* $\{u_n\}$ *possesses a subsequence strongly convergent to* u_λ *in* $D^{1,2}(\mathbb{R}^4)$.

Proof. We claim that $u_\lambda \not\equiv 0$. Arguing by contradiction, we assume $u_\lambda \equiv 0$. Then, by $u_n \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$ and (2.12), we have

$$a\|u_n\|^2 - b\|u_n\|^4 - \int_{\mathbb{R}^4} Q(x)\,|u_n|^4 = o(1). \tag{2.27}$$

It follows from (1.6), (2.27) and $I_\lambda \to c \neq 0$ that

$$c = I_\lambda(u_n) + o(1) = \frac{a}{4}\|u_n\|^2 + o(1). \tag{2.28}$$

If $c < 0$, we get a contradiction from the last equality. If $c > 0$, there exists $n_0 \in \mathcal{N}$ such that $\|u_n\| \geq c$ for $n \geq n_0$. This together with (2.28) leads to $\lim_{n\to\infty} \|u_n\|^2 \geq \frac{as^2}{(bs^2+Q_M)}$. Then, by (2.26), (2.28) and the above equality, we obtain that

$$c < c_\lambda = \frac{a^2 s^2}{4(bs^2+Q_M)} - D\lambda^{\frac{4}{3+\gamma}} < \frac{a^2 s^2}{4(bs^2+Q_M)} \leq \frac{a}{4}\lim_{n\to\infty}\|u_n\|^2 = c,$$

which is a contradiction. Therefore $u_\lambda \not\equiv 0$. We will prove that $u_n \to u_\lambda$ in $D^{1,2}(\mathbb{R}^4)$. Write $v_n = u_n - u_\lambda$ and we claim that $\|v_n\| \to 0$. Otherwise, up to a subsequence (still denoted by v_n), we may suppose $\|v_n\| \to l$ with $l > 0$. From (2.23), we have that $\langle I'_\lambda(u_n), u_\lambda\rangle = o(1)$ and hence

$$\begin{aligned} o(1) =& a\|u_\lambda\|^2 - b(l^2 + \|u_\lambda\|^2)\|u_\lambda\|^2 \\ &- \int_{\mathbb{R}^4} Q(x)\,|u_\lambda|^4\,dx - \lambda\int_{\mathbb{R}^4} f(x)|u_\lambda|^{1-\gamma}dx. \end{aligned} \tag{2.29}$$

Moreover, by $\langle I'_\lambda(u_n), u_n\rangle = o(1)$, we can use Brézis-Lieb's Lemma to obtain

$$\begin{aligned} o(1) =& a(\|u_\lambda\|^2 + \|v_n\|^2) - b(\|v_n\|^4 + 2\|v_n\|^2\|u_\lambda\|^2 + \|u_\lambda\|^4) \\ &- \int_{\mathbb{R}^4} Q(x)\,|u_\lambda|^4\,dx - \int_{\mathbb{R}^4} Q(x)\,|v_n|^4\,dx - \lambda\int_{\mathbb{R}^4} f(x)|u_\lambda|^{1-\gamma}dx. \end{aligned} \tag{2.30}$$

Combing with (2.29) and (2.30), we obtain

$$o(1) = a\|v_n\|^2 - bv_n\|^4 - b\|v_n\|^2\|u_\lambda\|^2 - \int_{\mathbb{R}^4} Q(x)\,|v_n|^4\,dx \tag{2.31}$$

and so, by the Sobolev inequality, we have

$$\begin{aligned} &a\|v_n\|^2 - b\|v_n\|^4 - b\|v_n\|^2\|u_\lambda\|^2 \\ &= \int_{\mathbb{R}^4} Q(x)\,|v_n|^4\,dx + o(1) \le S^{-2}Q_M\|v_n\|^4 + o(1). \end{aligned}$$

As $n \to \infty$, we obtain that

$$l^2 \ge \frac{S^2(a - b\|u_\lambda\|^2)}{S^2 b + Q_M} \ge 0. \tag{2.32}$$

By (1.6), (2.29), (2.32) and Hölder's inequality, we obtain

$$\begin{aligned} I_\lambda(u_\lambda) =& \frac{a}{2}\|u_\lambda\|^2 - \frac{b}{4}\|u_\lambda\|^4 - \frac{1}{4}\int_{\mathbb{R}^4} Q(x)\,|u_\lambda|^4\,dx \\ &- \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^4} f(x)\,|u_\lambda|^{1-\gamma}\,dx \\ =& \frac{b}{4}\|u_\lambda\|^4 + \frac{b}{2}l^2\|u_\lambda\|^2 \\ &- \lambda\left(\frac{1}{1-\gamma} - \frac{1}{2}\right)\int_{\mathbb{R}^4} f(x)\,|u_\lambda|^{1-\gamma}\,dx + \frac{1}{4}\int_{\mathbb{R}^4} Q(x)\,|u_\lambda|^4\,dx \\ \ge& \frac{b}{4}\|u_\lambda\|^4 + \frac{b}{2}\frac{S^2(a - b\|u_\lambda\|^2)}{S^2 b + Q_M}\|u_\lambda\|^2 \\ &- \lambda\left(\frac{1}{1-\gamma} - \frac{1}{2}\right)\int_{\mathbb{R}^4} f(x)\,|u_\lambda|^{1-\gamma}\,dx \\ =& \frac{b(S^2 b + Q_M)\|u_\lambda\|^4}{4(S^2 b + Q_M)} - \frac{b^2 S^2\|u_\lambda\|^4}{2(S^2 b + Q_M)} \\ &+ \frac{abS^2\|u_\lambda\|^2}{2(S^2 b + Q_M)} - \lambda\frac{1+\gamma}{2(1-\gamma)}\|f\|_{\frac{4}{3+\gamma}} S^{\frac{\gamma-1}{2}}\|u\|^{1-\gamma} \\ =& \frac{abS^2\|u_\lambda\|^2}{2(S^2 b + Q_M)} - \frac{b^2 S^2\|u_\lambda\|^4}{4(S^2 b + Q_M)} \\ &+ \frac{bQ_M\|u_\lambda\|^4}{4(S^2 b + Q_M)} - \lambda\frac{1+\gamma}{2(1-\gamma)}\|f\|_{\frac{4}{3+\gamma}} S^{\frac{\gamma-1}{2}}\|u\|^{1-\gamma} \\ \ge& \frac{abS^2\|u_\lambda\|^2}{2(S^2 b + Q_M)} - \frac{b^2 S^2\|u_\lambda\|^4}{4(S^2 b + Q_M)} \\ &- \lambda^{\frac{4}{3+\gamma}}\frac{3+\gamma}{4}\left[\frac{16S^2(bS^2 + Q_M)}{bQ_M(1-\gamma)}\right]^{\frac{1-\gamma}{3+\gamma}}\left[\frac{(1+\gamma)\|f\|_{\frac{4}{3+\gamma}}}{2(1-\gamma)}\right]^{\frac{4}{3+\gamma}}. \end{aligned} \tag{2.33}$$

Furthermore, by using (2.29)–(2.33), we deduce that

$$
\begin{aligned}
c+o(1) &= I_\lambda(u_n)\\
&=\frac{a}{2}\|u_n\|^2-\frac{b}{4}\|u_n\|^4-\frac{1}{4}\int_{\mathbb{R}^4}Q(x)|u_n|^4\,dx-\frac{\lambda}{1-\gamma}\int_{\mathbb{R}^4}f(x)|u_n|^{1-\gamma}\,dx\\
&=\frac{a}{2}\|u_\lambda\|^2-\frac{b}{4}\|u_\lambda\|^4-\frac{1}{4}\int_{\mathbb{R}^4}Q(x)|u_\lambda|^4\,dx-\frac{\lambda}{1-\gamma}\int_{\mathbb{R}^4}f(x)|u_\lambda|^{1-\gamma}\,dx\\
&\quad+\frac{a}{2}\|v_n\|^2-\frac{b}{4}\|v_n\|^4-\frac{b}{2}\|v_n\|^2\|u_\lambda\|^2-\frac{1}{4}\int_{\mathbb{R}^4}Q(x)|v_n|^4\,dx+o(1)\\
&=I_\lambda(u_\lambda)+\frac{a}{4}\|v_n\|^2-\frac{b}{4}\|v_n\|^2\|u_\lambda\|^2+o(1)\\
&=I_\lambda(u_\lambda)+\frac{a-b\|u_\lambda\|^2}{4}l^2+o(1)\\
&\geq I_\lambda(u_\lambda)+\frac{a^2S^2}{4(bS^2+Q_M)}-\frac{abS^2\|u_\lambda\|^2}{2(S^2b+Q_M)}+\frac{b^2S^2\|u_\lambda\|^4}{4(S^2b+Q_M)}+o(1)\\
&\geq\frac{a^2S^2}{4(bS^2+Q_M)}-D\lambda^{\frac{4}{3+\gamma}}=c_\lambda,
\end{aligned}
$$

which contradicts the assumption $c<\frac{a^2S^2}{4(bS^2+Q_M)}-D\lambda^{\frac{4}{3+\gamma}}$. Therefore, the claim holds, namely, $u_n\to u_\lambda$ in $D^{1,2}(\mathbb{R}^4)$. This completes the proof of Lemma 2.9. □

3. Proof of Theorem 1.1

In this section, we want to prove Theorem 1.1 by a minimization argument on $\mathcal{N}_\lambda^+$.

Proof. There exists a constant $\Lambda_3=\left(\frac{a^2S^2}{4D(bS^2+Q_M)}\right)^{\frac{3+\gamma}{4}}$ such that $c_\lambda>0$ for $\lambda<\Lambda_3$. Set $0<\lambda<\Lambda_0=\min\{\Lambda_1,\Lambda_3\}$, then Lemmas 2.1–2.9 hold for all $0<\lambda<\Lambda_0=\min\{\Lambda_1,\Lambda_3\}$. Due to Lemmas 2.1 and 2.3 and Ekeland's variational principle, we can obtain the minimizing sequence $\{u_n\}\subset\mathcal{N}_\lambda^+$ satisfying (2.9)–(2.11). Obviously, $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^4)$, going if necessary to a subsequence, still denoted by itself, there exists $u_*\in D^{1,2}(\mathbb{R}^4)$ such that

$$
\begin{cases}
u_n\rightharpoonup u_*, & \text{in } D^{1,2}(\mathbb{R}^4)\ \text{ÉÏ},\\
u_n\to u_*, & \text{in } L^p_{loc}\left(\mathbb{R}^4\right)\ (2\leq p<4),\\
u_n(x)\to u_*(x), & \text{a.e. in } \mathbb{R}^4.
\end{cases}
$$

Now, we will prove that u_* is a positive ground state solution of problem (1.1). According to (2.9) and Lemma 2.5, we have

$$I_\lambda(u_n)\to\alpha_\lambda^+<0<c_\lambda,$$

so Lemma 2.9 with $c=\alpha_\lambda^+$ results in $u_*\not\equiv0$ and $u_n\to u_*$ in $D^{1,2}(\mathbb{R}^4)$. One can further obtain from the above relation, $u_n\in\mathcal{N}_\lambda^+\subset\mathcal{N}_\lambda$ and Lemma 2.6 (i) imply that $u_*\in\mathcal{N}_\lambda$ and

$$a(1+\gamma)\|u_*\|^2-b(3+\gamma)\|u_*\|^4-(3+\gamma)\int_{\mathbb{R}^4}Q(x)|u_*|^4\,dx>0.$$

Hence, $u_* \in \mathcal{N}_\lambda^+$. Furthermore, passing to the limit as $n \to \infty$ in (2.23) and using Fatou's Lemma, we get

$$\begin{aligned}\lambda \int_{\mathbb{R}^4} f(x)\,|u_*|^{-\gamma}\,\psi dx \le & \liminf_{n\to\infty} \lambda \int_{\mathbb{R}^4} f(x)\,|u_n|^{-\gamma}\,\psi dx \\ =& a \int_{\mathbb{R}^4} \nabla u_* \nabla \psi dx - b \int_{\mathbb{R}^4} |\nabla u_*|^2 \int_{\mathbb{R}^4} \nabla u_* \nabla \psi dx \\ & - \int_{\mathbb{R}^4} Q(x)\,|u_*|^3\,\psi dx,\end{aligned} \tag{3.1}$$

for any $0 \le \psi \in D^{1,2}(\mathbb{R}^4)$. We can repeat the arguments used in (2.24) and (2.25) to derive that (3.1) holds for any $\psi \in D^{1,2}(\mathbb{R}^4)$. Consequently, u_* verifies (1.7) by the arbitrariness of $\psi \in D^{1,2}(\mathbb{R}^4)$ in (3.1), hence u_* is a weak solution of problem (1.1). Furthermore, similar to the proof of [24] Theorem 1, we have $u_* \in C^2_{loc}(\mathbb{R}^4)$. Since $u_* \ge 0$, $u_* \not\equiv 0$, and u_* satisfies (1.7), by the strong maximum principle, it suggests that $u_* > 0$ in $D^{1,2}(\mathbb{R}^4)$. Furthermore, there holds

$$I_\lambda(u_*) = \lim_{n\to\infty} I_\lambda(u_n) = \alpha_\lambda^+.$$

Finally, we want to show that $u_* \in \mathcal{N}_\lambda^+$ and $I_\lambda(u_*) = \alpha_\lambda^+$. For any $u \in \mathcal{N}_\lambda^-$, according to Lemma 2.1, there exists unique $0 < t^+(u) < t_{\max} < t^-(u)$ such that $t^+(u)u \in \mathcal{N}_\lambda^+$, $t^-(u)u \in \mathcal{N}_\lambda^-$, $I_\lambda(t^+(u)u) = \inf_{0<t\le t^-(u)} I_\lambda(tu)$ and $I_\lambda(t^-(u)u) = \sup_{t\ge t_{\max}} I_\lambda(tu)$. Then $t^-(u) = 1$ and there exists $\bar{t}(u) \in (t_{\max}, t^-(u))$ such that $I_\lambda(t^+(u)u) < I_\lambda(\bar{t}(u)u)$. So

$$\alpha_\lambda^+ \le I_\lambda(t^+(u)u) < I_\lambda(\bar{t}(u)u) \le I_\lambda(t^-(u)u) = I_\lambda(u).$$

By the arbitrariness of $u \in \mathcal{N}_\lambda^-$ and the definitions of $\alpha_\lambda^\pm$ and α, we have $\alpha_\lambda^+ < \alpha_\lambda^-$ and so $\alpha_\lambda = \alpha_\lambda^+$ thanks to $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ by Lemma 2.1. Therefore, $I_\lambda(u_*) = \alpha_\lambda = \alpha_\lambda^+$ and thus u_* is a ground state solution of problem (1.1). This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

It is well known that the function

$$U_{\varepsilon,x_0}(x) = \frac{(8\varepsilon)^{\frac{1}{2}}}{(\varepsilon + |x - x_0|^2)}, \quad \varepsilon > 0, \quad x \in \mathbb{R}^4. \tag{4.1}$$

solves

$$-\Delta u = u^3, \quad x \in \mathbb{R}^4.$$

For $j = 1, 2, \cdots, k$ fixed, we define a cut off function $\varphi_j(x) \in C_0^\infty(\mathbb{R}^4)$ such that $0 \le \varphi_j(x) \le 1$, $\varphi_j(x) = 1$ for $|x - a_j| < \delta_1$ and $\varphi_j(x) = 0$ for $|x - a_j| > 2\delta_1$ with $0 < \delta_1 < \delta/2$ where δ_1 is given in (f_2). Let $u_{\varepsilon,j}(x) = \varphi_j(x - a_j)U_{\varepsilon,a_j}(x)$. The following estimates can be easily established from similar estimates in [25]:

$$\Delta u_{\varepsilon,j}(x) = \frac{\Delta\varphi_j(x - a_j)(8\varepsilon)^{\frac{1}{2}}}{\left(\varepsilon + |x - a_j|^2\right)} - \frac{2|x - a_j|\varphi_j(x - a_j)(8\varepsilon)^{\frac{1}{2}}}{\left(\varepsilon + |x - a_j|^2\right)}.$$

Since $\varphi_j(x - a_j) = 1$ near a_j, it follows that

$$||u_{\varepsilon,j}||^2 = S^2 + O(\varepsilon) \quad |u_{\varepsilon,j}|_4^4 = S^2 + O(\varepsilon^2). \tag{4.2}$$

Lemma 4.1. *Assume that* (Q_1), (Q_2), (f_1) *and* (f_2) *hold, then there exists* $0 < \Lambda_4 < \Lambda_0$, *such that for* $0 < \lambda < \Lambda_4$ *and* $\varepsilon > 0$ *small, we have*

$$\sup_{t\geq 0} I_\lambda(tu_{\varepsilon,j}) < c_\lambda,$$

where c_λ *is given in Lemma 2.9.*

Proof. For $j = 1, 2, \cdots, k$, according to (Q_2), for any $\rho > 0$, there exists a $\delta_0 > 0$ such that

$$|Q(x) - Q(a_j)| < \rho|x - a_j|^{\beta_1}, \quad \forall 0 < |x - a_j| < \delta_0 < \delta. \tag{4.3}$$

For $0 < \lambda < \Lambda_2$, by Lemmas 2.1 and 2.5 (ii), there exists $t_\varepsilon > t_{\max} > 0$ such that $t_\varepsilon u_{\varepsilon,j} \in \mathcal{N}_\lambda^-$ and $I_\lambda(t_\varepsilon u_{\varepsilon,j}) = \sup_{t\geq 0} I_\lambda(tu_{\varepsilon,j}) \geq \beta_0 > 0$. On the other side, $I_\lambda(t_\varepsilon u_{\varepsilon,j}) \to -\infty$ as $t \to +\infty$ suggest that there exists $t_0 > 0$ such that $t_\varepsilon < t_0$. Hence, $t_{\max} < t_\varepsilon < t_0$. To proceed, set

$$p(t) = \frac{at^2}{2}\|u_{\varepsilon,j}\|^2 - \frac{bt^4}{4}\|u_{\varepsilon,j}\|^4 - \frac{t^4}{4}\int_{\mathbb{R}^4} Q_M|u_{\varepsilon,j}|^4 dx.$$

We easily see that $p(t)$ achieves its maximum at

$$\begin{aligned}
T_{\max} &= \left(\frac{a\|u_{\varepsilon,j}\|^2}{b\|u_{\varepsilon,j}\|^4 + \int_{\mathbb{R}^4} Q_M|u_{\varepsilon,j}|^4 dx}\right)^{1/2} \\
&= \left(\frac{aS^2 + O(\varepsilon)}{bS^4 + Q_M S^2 + O(\varepsilon)}\right)^{1/2} \\
&= \left(\frac{aS^2}{bS^4 + Q_M S^2}\right)^{1/2} + O(\varepsilon),
\end{aligned}$$

with

$$p(t_\varepsilon) < p(T_{\max}) = \frac{aS^2}{bS^2 + Q_M} + O(\varepsilon). \tag{4.4}$$

Since $t_{\max} < t_\varepsilon < t_0$, by the definition of $u_{\varepsilon,j}$, (4.3) and using a change of variables, we obtain that for ε small enough

$$\begin{aligned}
\frac{t_\varepsilon}{4}\int_{\mathbb{R}^4} |Q(x) - Q(a_j)||u_{\varepsilon,j}|^4 dx =& \frac{t_\varepsilon}{4}\int_{|x-a_j|<\delta} |Q(x) - Q(a_j)||u_{\varepsilon,j}|^4 dx \\
=& \frac{t_\varepsilon}{4}\int_{|x-a_j|<\delta_0} \frac{\rho|x-a_j|^{\beta_1}(8\varepsilon)^2}{(\varepsilon + |x - x_0|^2)^4} dx \\
&+ 2Q(M)\frac{t_\varepsilon}{4}\int_{\delta_0\leq|x-a_j|<\delta} \frac{(8\varepsilon)^2}{(\varepsilon + |x - x_0|^2)^4} dx \\
\leq& C_3\rho\varepsilon^2 \int_0^{\delta_0} \frac{r^{3+\beta_1}}{(\varepsilon + r^2)^4} dr + C_4\varepsilon^2 \int_{\delta_0}^{\delta} \frac{r^3}{(\varepsilon + r^2)^4} dr \\
\leq& C_3\rho\varepsilon^{\frac{\beta_1}{2}} \int_0^{\frac{\delta_0}{\sqrt{\varepsilon}}} \frac{r^{3+\beta_1}}{(1 + r^2)^4} dr + C_4\varepsilon^2 \int_{\delta_0}^{\delta} r^{-5} dr \\
\leq& C_5\varepsilon^{\frac{\beta_1}{2}} + C_6\varepsilon^2.
\end{aligned} \tag{4.5}$$

Similarly, by (f_2) and $\frac{5+3\gamma}{2} < \beta_2 < 3+\gamma$, for any $\varepsilon > 0$ satisfying $0 < \varepsilon < \delta_1^2$, we have

$$\begin{aligned}
&\lambda\frac{t_\varepsilon^{1-\gamma}}{1-\gamma}\int_{|x-a_j|<\delta} f(x)|u_{\varepsilon,j}|^{1-\gamma}dx\\
&= \lambda\frac{t_\varepsilon^{1-\gamma}}{1-\gamma}\left[\int_{|x-a_j|<\delta_1} f(x)|u_{\varepsilon,j}|^{1-\gamma}dx + \int_{\delta_1<|x-a_j|<\delta} f(x)|u_{\varepsilon,j}|^{1-\gamma}dx\right]\\
&\geq \lambda\frac{t_{\max}^{1-\gamma}}{1-\gamma}\int_{|x-a_j|<\delta_1} \frac{\rho_1|x-a_j|^{-\beta_2}(8\varepsilon)^{\frac{1-\gamma}{2}}}{(\varepsilon+|x-x_0|^2)^{1-\gamma}}dx \qquad (4.6)\\
&= C_7\rho_1\varepsilon^{\frac{1-\gamma}{2}}\lambda\int_0^{\delta_1}\frac{r^3}{r^{\beta_2}(\varepsilon+r^2)^{1-\gamma}}dr = C_7\rho_1\varepsilon^{\frac{3+\gamma-\beta_2}{2}}\lambda\int_0^{\frac{\delta_1}{\sqrt{\varepsilon}}}\frac{r^3}{r^{\beta_2}(1+r^2)^{1-\gamma}}dr\\
&\geq C_7\rho_1\varepsilon^{\frac{3+\gamma-\beta_2}{2}}\lambda\int_0^1\frac{r^3}{r^{\beta_2}2^{1-\gamma}}dr = \lambda C_8\varepsilon^{\frac{3+\gamma-\beta_2}{2}}.
\end{aligned}$$

Therefore, from (4.4)–(4.6) and $2 \leq \beta_1 < 4$, it holds

$$\begin{aligned}
I_\lambda(t_\varepsilon u_{\varepsilon,j}) =& p(t) + \frac{t_\varepsilon}{4}\int_{\mathbb{R}^4}|Q(x)-Q(a_j)||u_{\varepsilon,j}|^4dx\\
&- \lambda\frac{t_\varepsilon^{1-\gamma}}{1-\gamma}\int_{|x-a_j|<\delta} f(x)|u_{\varepsilon,j}|^{1-\gamma}dx\\
\leq& \frac{aS^2}{bS^2+Q_M} + O(\varepsilon) + C_5\varepsilon^{\frac{\beta_1}{2}} + C_6\varepsilon^2 - \lambda C_8\varepsilon^{\frac{3+\gamma-\beta_2}{2}}\\
\leq& \frac{aS^2}{bS^2+Q_M} + C_9\varepsilon - \lambda C_8\varepsilon^{\frac{3+\gamma-\beta_2}{2}}.
\end{aligned}$$

Set $\varepsilon = \lambda^{\frac{4}{3+\gamma}}$ and $\Lambda_5 = \left(\frac{C_8}{C_9+D}\right)^{\frac{3+\gamma}{2\beta_2-5-3\gamma}}$, since $\frac{5+3\gamma}{2} < \beta_2 < 3+\gamma$, we have

$$C_9\varepsilon - \lambda C_8\varepsilon^{\frac{3+\gamma-\beta_2}{2}} = \lambda^{\frac{4}{3+\gamma}}\left(C_9 - \lambda C_8\lambda^{\frac{5+3\gamma-2\beta_2}{3+\gamma}}\right) < -D\lambda^{\frac{4}{3+\gamma}},$$

and so

$$I_\lambda(t_\varepsilon u_{\varepsilon,j}) = \sup_{t\geq 0} I_\lambda(t_\varepsilon u_{\varepsilon,j}) < \frac{aS^2}{bS^2+Q_M} - D\lambda^{\frac{4}{3+\gamma}} = c_\lambda,$$

for all $0 < \lambda < \Lambda_4 = \min\{\Lambda_2, \Lambda_3, \Lambda_5\}$. □

Now we choose R_0 satisfying $R_0 > \Sigma_j^k|a_i|$ and set

$$\eta(x) = \begin{cases} 1, & 0 \leq t \leq R_0,\\ \frac{R_0}{t}, & R_0 \leq t. \end{cases}$$

Similar to [26], we define a map of "barycenter type" $\beta : D^{1,2}(\mathbb{R}^4)\setminus\{0\} \to \mathbb{R}^4$ as

$$\beta(u) = \frac{\int_{\mathbb{R}^4}\eta(|x|)x|u|^4dx}{\int_{\mathbb{R}^4}|u|^4dx}.$$

For $j = 1, 2, \cdots, k$, we also define

$$\Upsilon_\lambda^j = \left\{u \in \mathcal{N}_\lambda^- : |\beta(u) - a_j| < r_0\right\}, \quad \Phi_\lambda^j = \left\{u \in \mathcal{N}_\lambda^- : |\beta(u) - a_j| = r_0\right\}, \tag{4.7}$$

where $r_0 > 0$ such that

$$\overline{B_{r_0}(a_i)} \cap \overline{B_{r_0}(a_j)} = \emptyset, \quad j = 1, 2, \cdots, k.$$

By Lemma 2.1, there exists a unique $t_\varepsilon > 0$ such that $t_\varepsilon u_{\varepsilon,j} \in \mathcal{N}_\lambda^-$. Furthermore, as $\varepsilon \to 0^+$, we obtain by using a change of variables that

$$\begin{aligned}
\beta(t_\varepsilon u_{\varepsilon,j}) &= \frac{\displaystyle\int_{\mathbb{R}^4} \eta(|x|) x \varphi_j^4(x) \frac{(8\varepsilon)^2}{\left(\varepsilon + |x - a_j|^2\right)^4} dx}{\displaystyle\int_{\mathbb{R}^4} \varphi_j^4(x) \frac{(8\varepsilon)^2}{\left(\varepsilon + |x - a_j|^2\right)^4} dx} \\
&= \frac{\displaystyle\int_{\mathbb{R}^4} \eta(|\sqrt{\varepsilon}x + a_j|)(\sqrt{\varepsilon}x + a_j)\varphi_j^4(\sqrt{\varepsilon}x + a_j) \frac{(8\varepsilon)^2}{(1 + |x|^2)^4} dx}{\displaystyle\int_{\mathbb{R}^4} \varphi_j^4(\sqrt{\varepsilon}x + a_j) \frac{(8\varepsilon)^2}{(1 + |x|^2)^4} dx} \\
&\to a_j.
\end{aligned}$$

From this, we deduce the following fact.

Lemma 4.2. *There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $1 \le j \le k$, we have $\beta(t_\varepsilon u_{\varepsilon,j}) \in B_{r_0}(a_j)$.*

Lemma 4.2 implies that $\Upsilon_\lambda^j \neq \{0\}$ for all $1 \le j \le k$. By Lemma 2.5, we can define

$$c_\lambda^j = \inf_{u \in \Upsilon_\lambda^j} I_\lambda(u), \quad \widetilde{c}_\lambda^j = \inf_{u \in \Phi_\lambda^j} I_\lambda(u),$$

and so $c_\lambda^j \ge \alpha_\lambda^- \ge M_0 > 0$ if $0 < \lambda < \Lambda_2$.

Lemma 4.3. *Assume that (Q_2) holds, then there exists Λ_6 such that*

$$\widetilde{c}_\lambda^j > \frac{a^2 S^2}{4(bS^2 + Q_M)},$$

for $j = 1, 2, \cdots, k$ and $0 < \lambda < \Lambda_6$.

Proof. Let us argue by contradiction and suppose that there exist sequences $\lambda_n \to 0$, and $\{u_n\} \in \Phi_\lambda^j$ satisfying

$$I_{\lambda_n}(u_n) \to c \le \frac{a^2 S^2}{4(bS^2 + Q_M)},$$

and

$$a\|u_n\|^2 - b\|u_n\|^4 - \int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx - \lambda_n \int_{\mathbb{R}^4} f(x)|u_n|^{1-\gamma} dx = 0. \tag{4.8}$$

Then $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^4)$, since I_λ is coercive according to Lemma 2.5. Moreover, since $\{u_n\} \in \Phi_\lambda^j \subset \mathcal{N}_\lambda^-$, by the Sobolev inequality, we can further get that

$$\begin{aligned}
a(1 + \gamma)\|u_n\|^2 &< b(3 + \gamma)\|u_n\|^4 + (3 + \gamma) \int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx \\
&\le b(3 + \gamma)\|u_n\|^4 + (3 + \gamma) Q_M S^{-2} \|u_n\|^4,
\end{aligned}$$

from which we infer that

$$\|u_n\| > C_{10} > 0,$$

where $C_{10} = \sqrt{\frac{a(1+\gamma)S^2}{(3+\gamma)(bS^2+Q_M)}}$. Noting that $\lambda_n \to 0$, we then deduce from (4.8) that there is a constant $C_{11} > 0$ such that

$$\int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx + b\|u_n\|^4 = a\|u_n\|^2 > C_{11} > 0,$$

for all $n \in \mathcal{N}$. Thus, we are able to choose $t_n > 0$ such that $v_n = t_n u_n$ satisfies

$$a\|v_n\|^2 = \int_{\mathbb{R}^4} Q_M\,|v_n|^4\,dx + b\|v_n\|^4. \tag{4.9}$$

This and Sobolev inequality give that $\|v_n\|^2 \geq \frac{aS^2}{bS^2+Q_M}$. Moreover,

$$t_n = \left(\frac{b\|u_n\|^4 + \int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx + \lambda_n \int_{\mathbb{R}^4} f(x)|u_n|^{1-\gamma}dx}{\int_{\mathbb{R}^4} Q_M\,|u_n|^4\,dx + b\|u_n\|^4} \right)^{\frac{1}{2}}.$$

It follows that t_n is uniformly bounded. Then, we can assume that $\lim_{n\to\infty} t_n = t_0$. By $Q(x) \leq Q_M$, $\lambda_\infty \to 0$ and the boundedness of $\{u_n\}$, we see that $t_0 \leq 1$. We show next that the case $t_0 \leq 1$ leads to a contradiction. Since for $t_0 \leq 1$, we have

$$\begin{aligned}
\frac{a^2S^2}{4(bS^2+Q_M)} &\leq \lim_{n\to\infty} \frac{1}{4} a\|v_n\|^2 = \lim_{n\to\infty} \frac{1}{4} a t_n^2 \|u_n\|^2 \\
&= \lim_{n\to\infty} t_n^2 \Big[(\frac{1}{2} - \frac{1}{4}) \left(a\|u_n\|^2 - b\|u_n\|^4 - \lambda_n \int_{\mathbb{R}^4} f(x)|u_n|^{1-\gamma}dx \right) \\
&\quad + (\frac{1}{2} - \frac{1}{4}) b\|u_n\|^4 + (\frac{1}{2} - \frac{1}{1-\gamma}) \lambda_n \int_{\mathbb{R}^4} f(x)|u_n|^{1-\gamma}dx \Big] \\
&= \lim_{n\to\infty} t_n^2 I_{\lambda_n}(u_n) = t_0^2 c \leq \frac{a^2S^2}{4(bS^2+Q_M)}.
\end{aligned}$$

These inequalities above also provide

$$c = \frac{a^2S^2}{4(bS^2+Q_M)} \quad \text{and} \quad \lim_{n\to\infty} \|v_n\|^2 = \lim_{n\to\infty} \|u_n\|^2 = \frac{aS^2}{bS^2+Q_M}. \tag{4.10}$$

Let $\vartheta_n = \frac{v_n}{|v_n|_4}$, then $\|\vartheta_n\|_4 = 1$. Moreover, by (4.9) and (4.10),

$$\lim_{n\to\infty} \|\vartheta_n\|^2 = \lim_{n\to\infty} \frac{\|v_n\|^2}{|v_n|_4^2} = \lim_{n\to\infty} \frac{\|v_n\|^2}{(a\|v_n\|^2 - b\|v_n\|^4/Q_M)^{\frac{1}{2}}} = S,$$

namely, ϑ_n is a minimizing sequence for S. We now use a result of [27] to find a point $y_0 \in \mathbb{R}^4$ such that

$$|\nabla\vartheta_n|^2 \rightharpoonup d\mu = S\,\delta_{y_0} \quad \text{and} \quad |\vartheta_n|^2 \rightharpoonup d\nu = \delta_{y_0}, \tag{4.11}$$

with the above convergence holding weakly in the sense of measure, where δ_{y_0} is a Dirac mass at y_0. Then

$$\beta(u_n) = \frac{\int_{\mathbb{R}^4} \eta(|x|)x|u_n|^4 dx}{\int_{\mathbb{R}^4} |u_n|^4 dx} = \frac{\int_{\mathbb{R}^4} \eta(|x|)x|\vartheta_n|^4 dx}{\int_{\mathbb{R}^4} |\vartheta_n|^4 dx} \to \eta(|y_0|)y_0, \ as\ n \to \infty.$$

From the definition of $\eta(x)$, $\beta(u_n) \in \partial B_{r_0}(a_j)$ and $\overline{B_{r_0}(a_i)} \cap \overline{B_{r_0}(a_j)} = \emptyset$, $\ j = 1, 2, \cdots, k$, we know that $a_j \neq y_0$ and $y_0 \in \partial B_{r_0}(a_j)$ for any $j = 1, 2, \cdots, k$. Thus, from (4.9) and (4.11), we conclude that

$$\begin{aligned}
\lim_{n\to\infty} I_{\lambda_n}(u_n) &= \lim_{n\to\infty} t_n^2\Big[(\frac{1}{2} - \frac{1}{4})\left(a\|u_n\|^2 - b\|u_n\|^4 - \lambda_n \int_{\mathbb{R}^4} f(x)|u_n|^{1-\gamma}dx\right) \\
&\quad + (\frac{1}{2} - \frac{1}{4})b\|u_n\|^4 + (\frac{1}{2} - \frac{1}{1-\gamma})\lambda_n \int_{\mathbb{R}^4} f(x)|u_n|^{1-\gamma}dx\Big] \\
&= \lim_{n\to\infty} \frac{1}{4} \int_{\mathbb{R}^4} Q(x)\,|u_n|^4\,dx + \lim_{n\to\infty} \frac{1}{4} b\|u_n\|^4 \\
&= \lim_{n\to\infty} \frac{1}{4} \int_{\mathbb{R}^4} Q(x)\,|v_n|^4\,dx + \lim_{n\to\infty} \frac{1}{4} b\|v_n\|^4 \\
&= \frac{Q(y_0)}{4Q_M} \lim_{n\to\infty} \frac{1}{4} \int_{\mathbb{R}^4} Q_M\,|v_n|^4\,dx + \lim_{n\to\infty} \frac{1}{4} b\|v_n\|^4 \\
&= \frac{Q(y_0)}{4Q_M} \lim_{n\to\infty} (a\|v_n\|^2 - b\|v_n\|^4) + \lim_{n\to\infty} \frac{1}{4} b\|v_n\|^4 \\
&= \frac{Q(y_0)}{4Q_M} \frac{a^2S^2}{(bS^2 + Q_M)} + \lim_{n\to\infty} \frac{1}{4} b(1 - \frac{Q(y_0)}{Q_M})\|v_n\|^4 \\
&< \frac{a^2S^2}{4(bS^2 + Q_M)},
\end{aligned}$$

which contradicts (4.10). This completes the proof. □

We show the existence of k solutions in $\mathcal{N}_\lambda^-$.

Proof. Fix $0 < \lambda < \Lambda_{00} = \min\{\Lambda_4, \Lambda_6\}$, where Λ_4 and Λ_6 are defined in Lemmas 4.1 and 4.3, respectively. For any $j = 1, 2, \cdots, k$, since $\mathcal{N}_\lambda^-$ is a closed set by Lemma 2.3, we have, $\overline{\Upsilon_\lambda^j} = \Upsilon_\lambda^j \cup \Phi_\lambda^j$ from the same ideas presented in Lemma (3.1) of [28]. According to Lemmas (4.1) and (4.2) and the definition of c_λ in Lemma 2.9, we have

$$0 < c_\lambda^j \leq \sup_{t\geq 0} I_\lambda(tu_{\varepsilon,j}) < c_\lambda < \frac{a^2S^2}{4(bS^2 + Q_M)}. \tag{4.12}$$

This together with Lemma (4.3) leads to

$$\widetilde{c}_\lambda^j > \frac{a^2S^2}{4(bS^2 + Q_M)} > c_\lambda^j. \tag{4.13}$$

Thus, $c_\lambda^j = \inf_{u\in\overline{\Upsilon_\lambda^j}} I_\lambda(u)$. By Ekeland's variational principle, we can obtain the minimizing sequence $\{u_{n,j}\} \in \overline{\Upsilon_\lambda^j}$ satisfying

$$c_\lambda^j < I_\lambda(u_{n,j}) < c_\lambda^j + \frac{1}{n} \text{ and } I_\lambda(z) \geq I_\lambda(u_{n,j}) - \frac{1}{n}\|u_{n,j} - z\|, \forall z \in \overline{\Upsilon_\lambda^j}$$

with $u_{n,j} \rightharpoonup u_{\lambda,j}$ in $D^{1,2}(\mathbb{R}^4)$ for any $j \in [1,k]$. Therefore, by Lemma 2.9 and (4.13), we have $u_{\lambda,j} \not\equiv 0$ and $u_{n,j} \to u_{\lambda,j}$, up to a subsequence. Thus, replicating the same argument of Section 3, we have $u_{\lambda,j} \in \overline{\Upsilon_\lambda^j} \subset \mathcal{N}_\lambda^-$ is a positive solution of problem (1.1). Moreover, since $\overline{B_{r_0}(a_i)} \cap \overline{B_{r_0}(a_j)} = \emptyset$, for $i \neq j$, we conclude that $u_{\lambda,j} \neq u_{\lambda,i}$ if $i \neq j$. This implies that problem (1.1) has at least k positive solutions $u_{\lambda,j} \in \mathcal{N}_\lambda^-$ $(j = 1, 2, \cdots, k)$ for all $0 < \lambda < \Lambda_{00}$. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, by Lemma 2.2, we can further obtain that u_* and $u_{\lambda,j}(j = 1, 2, \cdots, k)$ are distinct and so problem (1.1) admits at least $k + 1$ positive solutions. □

5. Conclusions

In this paper, we study the multiplicity results of positive solutions for a class of Kirchhoff type problems with singularity and critical exponents. Under suitable assumptions on Q(x) and $f(x)$, by the variational method and delicate estimates, we prove that problem admits $k + 1$ positive solutions for $\lambda > 0$ sufficiently small. The related results in [29–32] are improved and generalized.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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