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Research article

Existence of fixed point results in orthogonal extended b-metric spaces with application

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Abstract: In this paper, we investigate the conditions for the existence of fixed-point for generalized contractions in the orthogonal extended b-metric spaces endowed with an arbitrary binary relation. We establish some unique fixed-point theorems. The obtained results generalize and improve many earlier fixed point results. We also provide some nontrivial examples to corroborate our results. As an application, we investigate solution for the system of boundary value problem.

Keywords: extended b-metric; orthogonal metric; fixed point; unique fixed point
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1. Introduction

As we know, fixed point theorems play a vital role in proving the existence and uniqueness of the solutions to various mathematical models such integral and partial equations, variational inequalities and approximation theory.

Since the axiomatic appearance of metric space, this structure has been generalized in many different ways. One such generalization of metric space, which has been center of interest for many researchers is the notion of b-metric space. The idea of b-metric was initiated by Bourbaki [1]. Czerwik [2] gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with the view of generalizing the Banach contraction principal. Later, Kamran et al. [3] generalized the notion of b-metric space by introducing the concept of extended b-metric space. T. Abdeljawadet et al. [4], extended the concept of b-metric to extended b-metric space and solved some nonlinear integral equation and fractional differential equation. T. Abdeljawad et al. [5] also extended his work onto Double controlled metric type spaces and investigated some fixed point results.

Recently, Gordji et al. [6], introduced the notion of orthogonal sets and gave an extension of Banach fixed point theorem and also extended his work in [7] to investigate fixed point in the setting of generalized orthogonal metric spaces. In this article, we have generalized the notions introduced Gordji et al., and many more related work by applying the structure of orthogonal sets on extended b-metric spaces, which is a generalization of b-metric spaces and metric spaces. The superiority of our results can be seen in the non-trivial example illustrated in the main results.

Through out this paper, O is used for Orthogonal that is, O-Sequence (Orthogonal Sequence), O-Cauchy (Orthogonal Cauchy), O-Complete (Orthogonal Complete), O-Continuous (Orthogonal Continuous), O-Extended (Orthogonal Extended)and $\perp$ is used for binary relation.

2. Preliminaries

Definition 2.1. [2] Let Ω be a non empty set and $s \geq 1$. A mapping $d : \Omega \times \Omega \to \mathbb{R}^+$ is called a b-metric if for all $\omega, \varpi, z \in \Omega$, it satisfies:

(a) $d(\omega, \varpi) = 0$ if and only if $\omega = \varpi$;
(b) $d(\omega, \varpi) = d(\varpi, \omega)$;
(c) $d(\omega, \varpi) \leq s[d(\omega, z) + d(z, \varpi)]$.
Then the triplet (Ω, d, s) is called a b-metric space.

Definition 2.2. [3] Let Ω be a non empty set and $\theta : \Omega \times \Omega \to [1, \infty)$. A mapping $d_\theta : \Omega \times \Omega \to \mathbb{R}^+$ is called an extended $b-$metric if for all $\omega, \varpi, z \in \Omega$ it satisfies:

$(d_\theta 1)$ $d_\theta(\omega, \varpi) = 0$ if and only if $\omega = \varpi$;
$(d_\theta 2)$ $d_\theta(\omega, \varpi) = d_\theta(\varpi, \omega)$;
$(d_\theta 3)$ $d_\theta(\omega, z) \leq \theta(\omega, z)[d_\theta(\omega, \varpi) + d_\theta(\varpi, z)]$.
Then the pair (Ω, d_θ) is called an extended b-metric space.

Remark 2.3. [3] If $\theta(\omega, \varpi) = s$ for $s \geq 1$ then it is a b-metric space.

Example 2.4. Let $\Omega = \{1, 2, 3\}$ and two mappings $\theta : \Omega \times \Omega \to [1, \infty)$ and $d_\theta : \Omega \times \Omega \to \mathbb{R}^+$defined by

$$\begin{aligned} \theta(\omega, \varpi) &= 1 + \omega + \varpi, \\ d_\theta(1,1) &= d_\theta(2,2) = d_\theta(3,3) = 0 \\ d_\theta(1,2) &= d_\theta(2,1) = 80, d_\theta(1,3) = d_\theta(3,1) = 1000, \\ d_\theta(2,3) &= d_\theta(3,2) = 400. \end{aligned}$$

Condition $(d_\theta 1)$ and $(d_\theta 2)$ are obviously. For $(d_\theta 3)$ we have

$$\begin{aligned} d_\theta(1,2) &= 80, \quad \theta(1,2)[d_\theta(1,3) + d_\theta(3,2)] = 4(1000 + 400) = 6400, \\ d_\theta(1,3) &= 1000, \quad \theta(1,3)[d_\theta(1,2) + d_\theta(2,3)] = 5(80 + 400) = 2400. \end{aligned}$$

Similarly for $d_\theta(2,3)$. So for all $\omega, \varpi, z \in \Omega$,

$$d_\theta(\omega, z) \leq \theta(\omega, z)[d_\theta(\omega, \varpi) + d_\theta(\varpi, z)].$$

Hence (Ω, d_θ) is an extended b-metric space.

Definition 2.5. [6] Let Ω be a non empty set. A mapping $d_{\perp} : \Omega \times \Omega \to \mathbb{R}^{+}$defined on orthogonal set $(\Omega, \perp)$ is called an orthogonal-metric if for all $\omega, \varpi, z \in \Omega$ it satisfies:

(i) $d_{\perp}(\omega, \varpi) = 0$ if and only if $\omega = \varpi$;
(ii) $d_{\perp}(\omega, \varpi) = d_{\perp}(\varpi, \omega)$;
(iii) $d_{\perp}(\omega, z) \leq d_{\perp}(\omega, \varpi) + d_{\perp}(\varpi, z)$.
Then the triplet $(\Omega, d_{\perp}, \perp)$ is called an orthogonal-metric space.

Definition 2.6. [7] Let $(\Omega, \perp)$ be an Orthogonal set then a sequence $\{\omega_n\}$ is called an O-sequence if

$$\omega_n \perp \omega_{n+1} \text{ or } \omega_{n+1} \perp \omega_n$$

for all $n \in \mathbb{N}$.

Definition 2.7. [7] Let $(\Omega, \perp)$ be an Orthogonal set then a Cauchy sequence $\{\omega_n\}$ is called an Cauchy O-sequence if

$$\omega_n \perp \omega_{n+1} \text{ or } \omega_{n+1} \perp \omega_n$$

for all $n \in \mathbb{N}$.

Definition 2.8. [7] Let $(\Omega, d_{\perp}, \perp)$ be an orthogonal metric space. A mapping $Q : \Omega \to \Omega$ is O-continuous at $\omega \in \Omega$ if for each O-sequence $\{\omega_n\}_{n \in \mathbb{N}}$ converges to ω implies $Q(\omega_n) \to \omega$ as $n \to \infty$. Also, Q is said to be O-continuous at Ω if Q is O-continuous at each $\omega \in \Omega$.

Definition 2.9. [7] Let $(\Omega, d_{\perp}, \perp)$ be an orthogonal metric space. Then Ω is said to be complete O-metric space (O-complete) if every O-Cauchy sequence is convergent in Ω, briefly $(\Omega, d_{\perp}, \perp)$ is an complete O-metric space.

Remark 2.10. (i) Every convergent O-sequence is an O-Cauchy sequence.

(ii) Every continuous mapping Q is O-continuous, but the converse is not true.

Definition 2.11. [7] A mapping $Q : \Omega \to \Omega$ is said to be an O-contraction with the Lipschitz constant $0 < k < 1$ if

$$d(Q\omega, Q\varpi) \leq kd(\omega, \varpi) \tag{2.1}$$

for all $\omega, \varpi \in \Omega$ with $\omega \perp \varpi$.

Definition 2.12. [7] Let $Q : \Omega \to \Omega$ be a mapping. Q is said to be O-preserving if $\omega \perp \varpi$, then $Q\omega \perp Q\varpi$ for all $\omega, \varpi \in \Omega$.

We have the following lemma about convergence of sequences in O-extended b-metric space.

Lemma 2.13. *Let $(\Omega, d_{\theta_{\perp}}, \perp)$ be a orthogonal-extended b-metric space with continues control function $\theta_{\perp}$ and suppose that $\{\omega_n\}$ and $\{\varpi_n\}$ are convergent to ω and ϖ respectively. Then we have*

$$\frac{1}{[\theta_{\perp}(\omega, \varpi)]^2} d_{\theta_{\perp}}(\omega, \varpi) \leq \liminf_{n \to \infty} d_{\theta_{\Re}}(\omega_n, \varpi_n) \leq \limsup_{n \to \infty} d_{\theta_{\perp}}(\omega_n, \varpi_n)$$
$$\leq [\theta_{\perp}(\omega, \varpi)]^2 d_{\theta_{\perp}}(\omega, \varpi).$$

In particular if $d_{\theta_{\perp}}(\omega, \varpi) = 0$, then we have $\lim_{n \to \infty} d_{\theta_{\perp}}(\omega_n, \varpi_n) = 0$.

Moreover, for all $\omega, \varpi \in \Omega$ we have

$$\frac{1}{\theta_{\perp}(\omega, \varpi)} d_{\theta_{\perp}}(\omega, \varpi) \leq \liminf_{n \to \infty} d_{\theta_{\perp}}(\omega_n, \varpi) \leq \limsup_{n \to \infty} d_{\theta_{\perp}}(\omega_n, \varpi) \leq \theta_{\perp}(\omega, \varpi) d_{\theta_{\perp}}(\omega, \varpi).$$

Proof. Using the triangle inequality in a orthogonal-extended b-metric space it is easy to see that

$$\begin{aligned} d_{\theta_\perp}(\omega,\varpi) &\leq \theta_\perp(\omega,\varpi)d_{\theta_\perp}(\omega,\omega_n)+\theta_\perp(\omega,\varpi)\theta_\perp(\omega_n,\varpi)d_{\theta_\perp}(\omega_n,\varpi_n) \\ &\quad +\theta_\perp(\omega,\varpi)\theta_\perp(\omega_n,\varpi)d_{\theta_\perp}(\varpi_n,\varpi) \end{aligned}$$

and

$$\begin{aligned} d_{\theta_\perp}(\omega_n,\varpi_n) &\leq \theta_\perp(\omega_n,\varpi_n)d_{\theta_\perp}(\omega_n,\omega)+\theta_\perp(\omega_n,\varpi_n)\theta_\perp(\omega,\varpi_n)d_{\theta_\perp}(\omega,\varpi) \\ &\quad +\theta_\perp(\omega_n,\varpi_n)\theta_\perp(\omega,y_n)d_{\theta_\perp}(\varpi,\varpi_n). \end{aligned}$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the first desired result.

Also,

$$d_{\theta_\perp}(\omega,\varpi) \leq \theta_\perp(\omega,\varpi)d_{\theta_\perp}(\omega,\omega_n)+\theta_\perp(\omega,\varpi)d_{\theta_\perp}(\omega_n,\varpi)$$

and

$$d_{\theta_\perp}(\omega_n,\varpi) \leq \theta_\perp(\omega_n,\varpi)d_{\theta_\perp}(\omega_n,\omega)+\theta_\perp(\omega_n,\varpi)d_{\theta_\perp}(\omega,\varpi).$$

□

3. Main results

Now, we introduce the notion of orthogonal-extended b-metric spaces and utilize this concept to investigate some fixed point results. Motivated by the work of Gordji et al. [6]. We introduce the notion of Banach contraction in the sense of O-extended b-metric space.

Definition 3.1. Let Ω be a non empty set and $\theta_\perp : \Omega\times\Omega \to [1,\infty)$. A mapping $d_{\theta\perp} : \Omega\times\Omega \to \mathbb{R}^+$ defined on orthogonal set $(\Omega, \perp)$ is called orthogonal-extended b-metric if for all $\omega, \varpi, z \in \Omega$ it satisfies:
(1) $d_{\theta\perp}(\omega,\varpi) = 0$ if and only if $\omega = \varpi$;
(2) $d_{\theta_\perp}(\omega,\varpi) = d_{\theta_\perp}(\varpi,\omega)$;
(3) $d_{\theta_\perp}(\omega,z) \leq \theta_\perp(\omega,z)\left[d_{\theta_\perp}(\omega,\varpi)+d_{\theta_\perp}(\varpi,z)\right]$.
Then the triplet $(\Omega,\ d_{\theta_{\perp},}\ \perp)$ is called an orthogonal-extended b-metric space.

Remark 3.2. Every extended b-metric is an orthogonal extended b-metric, but the converse is not true general.

Let See the example for above remark.

Example 3.3. Let $\Omega = \{0,1,2\}$ and $d_{\theta_\perp} : \Omega\times\Omega \to [0,\infty)$ be given by

$$\begin{aligned} d_{\theta_\perp}(0,2) &= d_{\theta_\perp}(2,0) = 2, \\ d_{\theta_\perp}(1,1) &= d_{\theta_\perp}(2,2) = d_{\theta_\perp}(0,0) = 0, \\ d_{\theta_\perp}(0,1) &= d_{\theta_\perp}(1,0) = d_{\theta_\perp}(1,2) = d_{\theta_\perp}(2,1) = 1. \end{aligned}$$

Define a orthogonal binary relation as $\omega \perp \varpi$ iff $\omega > \varpi$ with $\varpi \neq 0$, and a mapping $\theta_\perp : \Omega\times\Omega \to \mathbb{R}^+$defined by $\theta_\perp(\omega,\varpi) = 1+\frac{\omega}{\varpi}$ for all $\omega,\varpi \in \Omega$,

$$d_{\theta_\perp}(0,2) = 2 \geq 1\,[1+1] = \theta_\perp(0,2)\left[d_{\theta_\perp}(0,1)+d_{\theta_\perp}(1,2)\right].$$

Hence, it is not an extended b-metric, but it is an O-extended b -metric. Indeed, we must take $\omega > \varpi$ for $\theta_\perp(\omega, \varpi)$. Therefore,

$$d_{\theta_\perp}(2,1) = 1 \leq 3[2+1] = \theta_\perp(2,1)\left[d_{\theta_\perp}(2,0) + d_{\theta_\perp}(0,1)\right].$$

Then $(\Omega, d_{\theta_\perp}, \perp)$ is an O-extended b-metric space.

Definition 3.4. Let $(\Omega, d_{\theta_\perp}, \perp)$ be an O-extended b-metric space with a Lipchitz constant $\lambda \in [0,1)$. Then mapping $Q : \Omega \to \Omega$ is called a $\theta_\perp$- extended contraction if

$$d_{\theta_\perp}(Q\omega, Q\varpi) \leq \lambda d_{\theta_\perp}(\omega, \varpi)$$

for all $\omega, \varpi \in \Omega$, with $\omega \perp \varpi$.

Definition 3.5. Let $\{\omega_n\}$ be an O-sequence in $(\Omega, d_{\theta_\perp}, \perp)$, that is $\omega_n \perp \omega_{n+1}$ or $\omega_{n+1} \perp \omega_n$ for all $n \in \mathbb{N}$. Then

(i) $\{\omega_n\}$ is a convergent sequence to some $\omega \in \Omega$ if $\lim\limits_{n\to\infty} d_{\theta_\perp}(\omega_n, \omega) = 0$ and $\omega_n \perp \omega$ for all $n \in \mathbb{N}$.

(ii) $\{\omega_n\}$ is Cauchy if $\lim\limits_{n,m\to\infty} d_{\theta_\perp}(\omega_n, \omega_m)$ exists and is finite.

Definition 3.6. Let $(\Omega, d_{\theta_\perp}, \perp)$ be an orthogonal extended metric space. Then Ω is said to be complete orthogonal extended b-metric space (O-complete) if every O-Cauchy sequence is convergent in Ω, briefly $(\Omega, d_{\theta_\perp}, \perp)$ is an complete O-extended metric space.

Definition 3.7. Let $(\Omega, d_{\theta_\perp}, \perp)$ be an orthogonal extended metric space. A mapping $Q : \Omega \to \Omega$ is O-continuous at $\omega \in \Omega$ if for each O-sequence $\{\omega_n\}_{n\in\mathbb{N}}$ converges to ω implies $Q(\omega_n) \to \omega$ as $n \to \infty$. Also, Q is said to be O-continuous at Ω if Q is O-continuous at each $\omega \in \Omega$.

Lemma 3.8. *Let $(\Omega, d_{\theta_\perp}, \perp)$ be an O-extended b-metric space. If $d_{\theta_\perp}$ is O-continuous, then every -convergent sequence has a unique limit.*

Now, we are ready to prove the main theorem of this paper which can be consider as a real extension of Banach contraction principle.

Theorem 3.9. *Let $(\Omega, d_{\theta_\perp}, \perp)$ be a complete O-extended b-metric space. Suppose that $Q : \Omega \to \Omega$ is O-continuous, O-preserving and $\theta_\perp$- extended contraction with a Lipchitz constant $\lambda \in [0,1)$ such that for each $\omega \in \Omega$, $\lim\limits_{n,m\to\infty} \theta(\omega_n, \omega_m) < \frac{1}{\lambda}$ where $\omega_n = Q^n\omega$ for all $n \geq 1$. Then $\omega^* \in \Omega$ has a unique fixed point.*

Proof. By the defnition of orthogonality, there exists $\omega_0 \in Q$ such that

$$(\forall\ \omega \in Q,\ \omega_0 \perp \omega) \text{ or } (\forall\ \omega \in Q,\ \omega \perp \omega_0)$$

It follows that $\omega_0 \perp Q(\omega_0)$ or $Q(\omega_0) \perp \omega_0$. Let $\omega_1 = Q\omega_0$, $\omega_2 = Q\omega_1 = Q^2\omega_0$, ..., $\omega_{n+1} = Q\omega_n = Q^{n+1}\omega_0$ for all $n \in \mathbb{N}$. Since Q is O-preserving, $\{\omega_n\}$ is an $\perp$-sequence. Also, Q is an O-contraction so that,

$$\begin{aligned} d_{\theta_\perp}(\omega_{n+1}, \omega_n) &= d_{\theta_\perp}\left(Q^{n+1}\omega_0, Q^n\omega_0\right) \\ &\leq \lambda d_{\theta_\perp}(\omega_n, \omega_{n-1}). \end{aligned}$$

Consequently,

$$d_{\theta_\perp}(\omega_{n+1},\omega_n) \leq \lambda^n d_{\theta_\perp}(\omega_1,\omega_0) \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

By triangular inequality and equation (3.1) for all $m, n \in \mathbb{N}$, such that $m > n$,

$$\begin{aligned}
& d_{\theta_\perp}(\omega_n,\omega_m) \\
\leq\ & \theta(\omega_n,\omega_m)\lambda^n d_{\theta_\perp}(\omega_0,\omega_1) + \theta(\omega_n,\omega_m)\theta(\omega_{n+1},\omega_m)\lambda^{n+1} d_{\theta_\perp}(\omega_0,\omega_1) \\
& + \cdots \\
& + \theta(\omega_n,\omega_m)\theta(\omega_{n+1},\omega_m)\theta(\omega_{n+2},\omega_m)\cdots\theta(\omega_{m-2},\omega_m)\theta(\omega_{m-1},\omega_m)\lambda^{m-1} d_{\theta_\perp}(\omega_0,\omega_1) \\
\leq\ & d_{\theta_\perp}(\omega_0,\omega_1)[\theta(\omega_1,\omega_m)\theta(\omega_2,\omega_m)\cdots\theta(\omega_{n-1},\omega_m)\theta(\omega_n,\omega_m)\lambda^n + \\
& \theta(\omega_1,\omega_m)\theta(\omega_2,\omega_m)\cdots\theta(\omega_n,\omega_m)\theta(\omega_{n+1},\omega_m)\lambda^{n+1} + \cdots + \\
& \theta(\omega_1,\omega_m)\theta(\omega_2,\omega_m)\cdots\theta(\omega_n,\omega_m)\theta(\omega_{n+1},\omega_m)\cdots\theta(\omega_{m-2},\omega_m)\theta(\omega_{m-1},\omega_m)\lambda^{m-1}].
\end{aligned}$$

Since, $\lim_{n,m\to\infty} \theta(\omega_n,\omega_m) < \frac{1}{\lambda}$, so that the series $\sum_{n=1}^{\infty} \lambda^n \prod_{i=1}^{n} \theta(\omega_i,\omega_m)$ converges by the ratio test for each $m \in \mathbb{N}$. Let

$$P = \sum_{n=1}^{\infty} \lambda^n \prod_{i=1}^{n} \theta(\omega_i,\omega_m), \quad P_n = \sum_{j=1}^{\infty} \lambda^j \prod_{i=1}^{j} \theta(\omega_i,\omega_m).$$

So for $m > n$ the above inequality become

$$\lim_{n,m\to\infty} d_{\theta_\perp}(\omega_n,\omega_m) \leq d_{\theta_\perp}(\omega_0,\omega_1)[P_{m-1} - P_n].$$

Taking $\lim_{n\to\infty}$ we conculde that $\{\omega_n\}$ is an O-Cauchy sequence. Since Ω is O-complete, there exists $\omega^* \in \Omega$ such that $\lim_{n\to\infty} d_{\theta_\perp}(\omega_n,\omega^*) = 0$. On the other hand Q is $\perp$ −continuous and then and $Q(\omega_n) \to Q(\omega^*)$ and $Q(\omega^*) = Q(\lim_{n\to\infty} Q(\omega_n)) = \lim_{n\to\infty} \omega_{n+1} = \omega^*$. Hence ω^* is fixed point of Q.

To prove the unique fixed point, let ω^{**} be another fixed point for Q. Then we have $Q^n\omega^* = \omega^*$ and $Q^n\omega^{**} = \omega^{**}$ for all $n \in \mathbb{N}$. By the choice of ω_0, we obtain

$$[\omega_0 \perp \omega^* \text{ and } \omega_0 \perp \omega^{**}] \quad \text{or} \quad [\omega^* \perp \omega_0 \text{ and } \omega^{**} \perp \omega_0].$$

Since Q is O-preserving, we have

$$[Q^n(\omega_0) \perp Q^n(\omega^*) \text{ and } Q^n(\omega_0) \perp Q^n(\omega^{**})]$$

or

$$[Q^n(\omega^*) \perp Q^n(\omega_0) \text{ and } Q^n(\omega^{**}) \perp Q^n(\omega_0)]$$

for all $n \in \mathbb{N}$. Therefore, by the triangular inequality, we get

$$\begin{aligned}
d_{\theta_\perp}(\omega^*,\omega^{**}) &= d_{\theta_\perp}(Q^n\omega^*, Q^n\omega^{**}) \\
&\leq \theta(Q^n\omega^*, Q^n\omega^{**})[d_{\theta_\perp}(Q^n\omega^*, Q^n\omega) + d_{\theta_\perp}(Q^n\omega, Q^n\omega^{**})] \\
&\leq \theta(Q^n\omega^*, Q^n\omega^{**}) d_{\theta_\perp}(Q^n\omega^*, Q^n\omega) + \theta(Q^n\omega^*, Q^n\omega^{**}) d_{\theta_\perp}(Q^n\omega, Q^n\omega^{**}) \\
&\leq \theta(Q^n\omega^*, Q^n\omega^{**})\lambda^n d_{\theta_\perp}(\omega^*,\omega) + \theta(Q^n\omega^*, Q^n\omega^{**})\lambda^n d(\omega,\omega^{**}).
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality,

$$d_{\theta_\perp}(\omega^*, \omega^{**}) = 0$$

which implies that $\omega^* = \omega^{**}$.Hence ω^* is unique fixed point. Let $\omega \in \Omega$ be an arbitrary point. Then

$$d_{\theta_\perp}(\omega^*, Q^n\omega) \leq \lambda^n d_{\theta_\perp}(\omega_*, \omega).$$

Taking the limit as $n \to \infty$ in the above inequality, we have $d_{\theta_\perp}(\omega^*, Q^n\omega) = 0$. So,

$$\lim_{n\to\infty} Q^n\omega = \omega^* \text{ for all } \omega \in \Omega.$$

□

Example 3.10. Let $\Omega = [0, 12]$ and $d_{\theta_\perp} : \Omega \times \Omega \to [0, \infty)$ be given by

$$d_{\theta_\perp}(\omega, \varpi) = |\omega - \varpi|^2$$

for all $\omega, \varpi \in \Omega$. Define a binary relation $\perp$on Ω by $\omega \perp \varpi$ if $\omega\varpi \leq \omega$ or $\omega\varpi \leq \varpi$. Then $(\Omega, d_{\theta_\perp}, \perp)$ is an O-extended b-metric space with $\theta = \omega + \varpi + 2$. Define a mapping $Q : \Omega \to \Omega$ by

$$Q\omega = \begin{cases} \frac{\omega}{3}, & \text{if } 0 \leq \omega \leq 3, \\ 0, & \text{if } 3 \leq \omega \leq 12. \end{cases}$$

(i) If $\omega = 0$ and $0 \leq \varpi \leq 3$, then $Q\omega = 0$ and $Q\varpi = \frac{\varpi}{3}$.
(ii) If $\omega = 0$ and $3 \leq \varpi \leq 12$, then $Q\omega = 0$ and $Q\varpi = 0$.
(iii) If $0 \leq \varpi \leq 3$ and $0 \leq \omega \leq 3$, then $Q\omega = \frac{\omega}{3}$ and $Q\varpi = \frac{\varpi}{3}$.
(iv) If $0 \leq \varpi \leq 3$ and $3 \leq \omega \leq 12$, then $Q\varpi = \frac{\varpi}{3}$ and $Q\omega = 0$.
From (i) – (iv),

$$|Q\omega - Q\varpi| \leq \frac{1}{3}|\omega - \varpi|,$$

i.e.,

$$|Q\omega - Q\varpi|^2 \leq \frac{1}{9}|\omega - \varpi|^2,$$

that is,

$$d_{\theta_\perp}(Q\omega, Q\varpi) \leq \frac{1}{9} d_{\theta_\perp}(\omega, \varpi).$$

So, Q is an $\perp$-contraction with $k = \frac{1}{9}$. Note that for each $\omega \in \Omega$, $Q^n\omega = \frac{\omega^n}{3}$. Thus, we obtain that

$$\lim_{n,m\to\infty} d_{\theta_\perp}(Q^m\omega, Q^n\omega) < 9.$$

Therefore, all the conditions of Theorem 3.9 are satisfied. Hence, Q has a unique fixed point.

Theorem 3.11. *Let $(\Omega, d_{\theta_\perp}, \perp)$ be a complete O-extended b-metric space. Suppose that $Q : \Omega \to \Omega$ is an O-continuous and O-preserving mapping such that*

$$d_{\theta_\perp}(Q\omega, Q\varpi) \leq \lambda_1 d_{\theta_\perp}(\omega, \varpi) + \lambda_2[d_{\theta_\perp}(\omega, Q\omega) + d_{\theta_\perp}(\varpi, Q\varpi)] \tag{3.2}$$

where $\lambda_i \geq 0$, for $i = 1, 2$ and $\lim_{n,m\to\infty} \frac{\lambda_1 + \lambda_2\theta_\perp(\omega_{n-1}, \omega_{n+1})}{1 - \lambda_2\theta_\perp(\omega_{n-1}, \omega_{n+1})}\theta_\perp(\omega_n, \omega_m) < 1$ with $\omega_n = Q^n\omega_0$ for $\omega_0 \in \Omega$. Then Q has a unique fixed point $\omega^* \in \Omega$.

Proof. By the defnition of orthogonality, there exists $\omega_0 \in Q$ such that

$$(\forall\ \omega \in Q,\ \omega_0 \perp \omega) \text{ or } (\forall\ \omega \in Q,\ \omega \perp \omega_0)$$

It follows that $\omega_0 \perp Q(\omega_0)$ or $Q(\omega_0) \perp \omega_0$. Let $\omega_1 = Q\omega_0$, $\omega_2 = Q\omega_1 = Q^2\omega_0$, ..., $\omega_{n+1} = Q\omega_n = Q^{n+1}\omega_0$ for all $n \in \mathbb{N}$.Then ω_n is a fixed point of Q, We suppose that $\omega_n \neq \omega_{n-1}$, for all $n \geq 1$. From (3.2), we have

$$d_{\theta_\perp}(Q\omega_n, Q\omega_{n-1}) \leq \lambda_1 d_{\theta_\perp}(\omega_n, \omega_{n-1}) + \lambda_2[d_{\theta_\perp}(\omega_n, Q\omega_n) + d_{\theta_\perp}(\omega_{n-1}, Q\omega_{n-1})].$$

From the triangle inequality, we get

$$\begin{aligned} d_{\theta_\perp}(Q\omega_n, Q\omega_{n-1}) &\leq \lambda_1 d_{\theta_\perp}(\omega_n, \omega_{n-1}) + \lambda_2\theta_\perp(\omega_{n-1}, \omega_{n+1})[d_{\theta_\perp}(\omega_{n-1}, \omega_n) \\ &\quad + d_{\theta_\perp}(\omega_n, \omega_{n+1})], \end{aligned}$$

$$\begin{aligned} d_{\theta_\perp}(Q\omega_n, Q\omega_{n-1}) &\leq \lambda_1 d_{\theta_\perp}(\omega_n, \omega_{n-1}) + \lambda_2\theta_\perp(\omega_{n-1}, \omega_{n+1}) d_{\theta_\perp}(\omega_{n-1}, \omega_n) \\ &\quad + \lambda_2\theta_\perp(\omega_{n-1}, \omega_{n+1}) d_{\theta_\perp}(\omega_n, \omega_{n+1})]. \end{aligned}$$

This implies that

$$\begin{aligned} d_{\theta_\perp}(\omega_{n+1}, \omega_n) &\leq (\lambda_1 + \lambda_2\theta_\perp(\omega_{n-1}, \omega_{n+1})) d_{\theta_\perp}(\omega_n, \omega_{n-1}) \\ &\quad + \lambda_2\theta_\perp(\omega_{n-1}, \omega_{n+1}) d_{\theta_\perp}(\omega_n, \omega_{n+1}), \end{aligned}$$

that is,

$$(1 - \lambda_2\theta_\perp(\omega_{n-1}, \omega_{n+1}))\, d_{\theta_\perp}(\omega_{n+1}, \omega_n) \leq (\lambda_1 + \lambda_2\theta_\perp(\omega_{n-1}, \omega_{n+1})) d_{\theta_\perp}(\omega_n, \omega_{n-1})$$

which yields that

$$d_{\theta_\perp}(\omega_{n+1}, \omega_n) \leq \zeta d_{\theta_\perp}(\omega_n, \omega_{n-1}),$$

where

$$\zeta = \frac{\lambda_1 + \lambda_2\theta_\perp(\omega_{n-1}, \omega_{n+1})}{1 - \lambda_2\theta_\perp(\omega_{n-1}, \omega_{n+1})}.$$

So,

$$\frac{d_{\theta_\perp}(\omega_{n+1}, \omega_n)}{d_{\theta_\perp}(\omega_n, \omega_{n-1})} \leq \zeta < 1.$$

By Theorem 3.9, $\{\omega_n\}$ is an O-Cauchy sequence. Since Ω is O-complete, therefore there exists $\omega \in \Omega$ such that $\lim\limits_{n\to\infty} \omega_n = \omega$ and $\omega_n \perp \omega^*$ for all $n \geq \lambda$.

Next, we will show that ω is a fixed point of Q. From the triangle inequality and using (3.1), we have

$$\begin{aligned} d_{\theta_\perp}(\omega, Q\omega) &\leq \theta_\perp(\omega, Q\omega)[d_{\theta_\perp}(\omega, \omega_{n+1}) + d_{\theta_\perp}(\omega_{n+1}, Q\omega)] \\ &\leq \theta_\perp(\omega, Q\omega)[d_{\theta_\perp}(\omega, \omega_{n+1}) + \lambda_1 d_{\theta_\perp}(\omega_n, \omega) \\ &\quad + \lambda_2[d_{\theta_\perp}(\omega_n, Q\omega) + d_{\theta_\perp}(\omega, Q\omega_n)]], \end{aligned}$$

hence,

$$d_{\theta_\perp}(\omega, Q\omega) \leq \limsup \theta_\perp(\omega, Q\omega)[d_{\theta_\perp}(\omega, \omega_{n+1}) + \lambda_1 d_{\theta_\perp}(\omega_n, \omega)$$

$$+ \lambda_2[d_{\theta_\perp}(\omega_n, Q\omega) + d_{\theta_\perp}(\omega, Q\omega_n)]] \leq \lambda_2[\theta_\perp(\omega, Q\omega)]^2 d_{\theta_\perp}(\omega, Q\omega) \text{ as } n \to \infty.$$

This implies that

$$(1 - \lambda_2[\theta_\perp(\omega, Q\omega)]^2) d_{\theta_\perp}(\omega, Q\omega) \leq 0.$$

But

$$(1 - \lambda_2[\theta_\perp(\omega, Q\omega)]^2) > 0,$$

so, we get

$$d_{\theta_\perp}(\omega, Q\omega) = 0 \Rightarrow Q\omega = \omega.$$

Hence, ω is a fixed point. Now, we will prove that ω is the unique fixed point, let ω^{**} be another fixed point for Q. Then we have $Q^n\omega = \omega$ and $Q^n\omega^{**} = \omega^{**}$ for all $n \in \mathbb{N}$. By the choice of ω_0, we obtain

$$[\omega_0 \perp \omega \text{ and } \omega_0 \perp \omega^{**}] \quad \text{or} \quad [\omega \perp \omega_0 \text{ and } \omega^{**} \perp \omega_0] \ .$$

Since Q is O-preserving, we have

$$[Q^n(\omega_0) \perp Q^n(\omega) \text{ and } Q^n(\omega_0) \perp Q^n(\omega^{**})]$$

or

$$[Q^n(\omega) \perp Q^n(\omega_0) \text{ and } Q^n(\omega^{**}) \perp Q^n(\omega_0)]$$

Therefore, by the triangular inequality we get

$$\begin{aligned} d_{\theta_\perp}(\omega, \omega^{**}) &= d_{\theta_\perp}(Q\omega, Q\omega^{**}) \\ &\leq \lambda_1 d_{\theta_\perp}(\omega, \omega^{**}) + \lambda_2[d_{\theta_\perp}(\omega, Q\omega^{**}) + d_{\theta_\perp}(\omega^{**}, Q\omega)], \end{aligned}$$

that is

$$(1 - \lambda_1 - 2\lambda_2)\, d_{\theta_\perp}(\omega, \omega^{**}) \leq 0.$$

As,

$$(1 - \lambda_1 - 2\lambda_2) > 0.$$

So, $d_{\theta_\perp}(\omega, \omega^{**}) = 0$ implies that $\omega = \omega^{**}$. Hence ω is unique fixed point. □

4. Application

We will apply Theorem 3.9 to achieve the existence of solution to the following system of boundary value problems:

$$-\frac{d^2\omega}{dt^2} = H(t, \omega(t)); \quad t \in J, \omega(0) = \omega(1) = 0 \tag{4.1}$$

$$-\frac{d^2\varpi}{dt^2} = k(t, \varpi(t)); \ t \in J, \varpi(0) = \varpi(1) = 0 \tag{4.2}$$

Where $J = [0, 1], C(J)$ represents the set of continuous functions defined on J. The functions $H, K : [0, 1] \times C(J) \to \mathbb{R}$ are continuous and non-decreasing according to ordinates. We define the binary relation

$$\omega \perp \varpi \text{ iff } \omega(t) \leq \varpi(t) \ \text{ for all } t \in J.$$

The associated Green's function $g : J \times J \to J$ to (4.1) and (4.2) can be defined as follow:

$$g(t,\ b) = \left\{ \begin{array}{l} t(1-b)\ \text{if}\ 0 \le t \le b \le 1 \\ b(1-t)\ \text{if}\ 0 \le b \le t \le 1 \end{array} \right\}$$

Let the mapping $d : C(J) \times C(J) \to [0,\ \infty)$ be defined by

$$d(\omega, \varpi) = \left\| (\omega - \varpi)^2 \right\| = \sup |\omega(t) - \varpi(t)|^2, \forall \omega, \varpi \in C(J) \text{ and } t \in J.$$

It is claimed that $(C(J), d, 2)$ is a O-complete extended b-metric space. By integration, we see that can be written as $\omega = Q(\omega)$,where $Q : \Omega \to \Omega$ are defined by:

$$Q(\omega(t)) = \int_0^1 g(t,b)H(b,\omega(b))db.$$

It is remarked that the solution to (4.1) and (4.2) is the fixed point of the operator Q.Suppose the following conditions:

(a). $\exists\, k > 0$ such that for $\omega(t) \neq \varpi(t)\ \forall\, t$ we have

$$|H(t,\omega(t)) - K(t,\varpi(t))|^2 \le 16e^{-k} |\omega(t) - \varpi(t)|^2 \ \ \forall\, t \in J.$$

(b). $\exists\, \omega, \varpi \in C(J)$ such that

$$\omega_0(t)) \le \int_0^1 g(t,b)H(b,\omega(b))db,$$

$$\varpi_0(t)) \le \int_0^1 g(t,b)H(b,\varpi(b))db.$$

The following theorem states the conditions under which the Eqs (4.1) and (4.2) have a common solution.

Theorem 4.1. *Let the functions $H, K : [0,1] \times C(J) \to \mathbb{R}$ satisfy the conditions (a) and (b). Then the Eqs* (4.1) *and* (4.2) *have a solution.*

Proof. We will apply Theorem to show the existence of the solution to Q. Since, the functions H, K are continuous, so $Q : \Omega \to \Omega$ defined above is continuous.

There exists ω such that $\omega \perp \varpi$.Since, it is given that H, K are non-decreasing. To show that the mappings Q form $\theta_\perp$- extended contraction, we proceed as follow:

$$|Q(\omega(t)) - Q(\varpi(t)|^2 = \left| \int_0^1 g(t,b)\left(H(b,\omega(b)) - K(b,\varpi(b))\right) db \right|^2$$

$$\le \left(\int_0^1 g(t,b) \left| H(b,\omega(b)) - K(b,\varpi(b)) \right| db \right)^2$$

$$\leq \left(\int_0^1 g(t,b) \sqrt{16e^{-k}|\omega(t) - \varpi(t)|^2 dt} \right)^2$$

since, $\left(\sup \int_0^1 g(t,b)\, db\right)^2 = \frac{1}{64}$, for all $t \in J$, thus, taking supremum on both sides of above inequlaity we have,

$$s^2 d_{\theta_\perp}(Q(\omega), Q(\varpi)) \leq e^{-k} d_{\theta_\perp}(\omega, \varpi)\ \forall\ \omega(\cdot), \varpi(\cdot) \in C(J).$$

Define the O-extended b-metric d on $C(J)$ by

$$d_{\theta_\perp}(\omega, \varpi) = \begin{cases} p_b(\omega, \varpi) \text{ if } \omega \neq \varpi \\ 0 \text{ if } \omega = \varpi \end{cases}$$

The inequlaity can be written as:

$$s^2 d_{\theta_\perp}(Q(\omega), Q(\varpi)) \leq e^{-k} d_{\theta_\perp}(\omega, \varpi)\ \forall\ \omega(\cdot), \varpi(\cdot) \in C(J).$$

Defining the function C, F and D by $C(t) = e^{-k}t, F(t) = \ln t$, and $D(t) = e^{F(t)}$ respectively, for all $t \in [0, \infty)$, we have,

$$\begin{aligned} k + F\left(s^2 d_{\theta_\perp}(Q(\omega), Q(\varpi))\right) &\leq F(p_b(\omega, \varpi)) \\ e^k . e^{F\left(s^2 d_{\theta_\perp}(Q(\omega), Q(\varpi))\right)} &\leq e^{F\left(d_{\theta_\perp}(\omega, \varpi)\right)} \\ e^{F\left(s^2 d_{\theta_\perp}(Q(\omega), Q(\varpi))\right)} &\leq e^{-\tau} e^{F\left(d_{\theta_\perp}(\omega, \varpi)\right)} \end{aligned}$$

We say that the boundary value problem have a solution in $C(J)$. □

5. Conclusions and future work

Orthogonal-extended b-metric space is a combination of Orthogonal relation and extended b-metric spaces. The presented theorems provide a general criterion for the existence of a unique fixed point of $\theta_\perp$- extended contractions in orthogonal extended b-metric spaces. This concept of orthogonal sets and binary relation can be applied to different generalized metric structure to investigate fixed point.

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Conflict of interest

The authors declare that they have no competing interests.

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