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Research article

Positive definite solution of non-linear matrix equations through fixed point technique

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Abstract: The goal of this study is to solve a non-linear matrix equation of the form $\mathcal{X} = \mathcal{Q} + \sum_{i=1}^{m} \mathcal{B}_i^* \mathcal{G}(\mathcal{X}) \mathcal{B}_i$, where $\mathcal{Q}$ is a Hermitian positive definite matrix, $\mathcal{B}_i^*$ stands for the conjugate transpose of an $n \times n$ matrix $\mathcal{B}_i$ and $\mathcal{G}$ an order-preserving continuous mapping from the set of all Hermitian matrices to the set of all positive definite matrices such that $\mathcal{G}(O) = O$. We explore the necessary and sufficient criteria for the existence of a unique positive definite solution to a particular matrix problem. For the said reason, we develop some fixed point results for $\mathcal{FG}$-contractive mappings on complete metric spaces equipped with any binary relation (not necessarily a partial order). We give adequate examples to confirm the fixed-point results and compare them to early studies, as well as four instances that show the convergence analysis of non-linear matrix equations using graphical representations.

Keywords: positive definite matrix; nonlinear matrix equation; fixed point; relational metric space
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1. Introductory notes

1.1. Nonlinear matrix equations

Nonlinear matrix equations (NME) were initially studied in the literature in relation to the algebraic Riccati problem. These equations appear in a wide range of problems in control theory, dynamical programming, ladder networks, stochastic filtering, queuing theory, statistics, and many other fields.

Let $\mathcal{H}(n)$ (resp. $\mathcal{K}(n)$, $\mathcal{P}(n)$) denote the set of all $n \times n$ Hermitian (resp. positive semi-definite, positive definite) matrices over $\mathbb{C}$ and $\mathcal{M}(n)$ the set of all $n \times n$ matrices over $\mathbb{C}$. In [20], Ran and Reurings discussed the existence of solutions of the following equation:

$$\mathcal{X} + \mathcal{B}^* F(\mathcal{X}) \mathcal{B} = \mathcal{Q} \tag{1.1}$$

in $\mathcal{K}(n)$, where $\mathcal{B} \in \mathcal{M}(n)$, Q is positive definite and F is a mapping from $\mathcal{K}(n)$ into $\mathcal{M}(n)$. Note that $\mathcal{X}$ is a solution of (1.1) if and only if it is a fixed point of the mapping $\mathcal{G}(\mathcal{X}) = Q - \mathcal{B}^* F(\mathcal{X})\mathcal{B}$. In [21], they used the notion of partial ordering and established a modification of Banach Contraction Principle, which they applied for solving a class of NMEs of the form $\mathcal{X} = Q + \sum_{i=1}^{m} \mathcal{B}_i^* F(\mathcal{X})\mathcal{B}_i$ using the Ky Fan norm in $\mathcal{M}(n)$.

Theorem 1.1. [21] *Let $F : \mathcal{H}(n) \to \mathcal{H}(n)$ be an order-preserving, continuous mapping which maps $\mathcal{P}(n)$ into itself and $Q \in \mathcal{P}(n)$. If $\mathcal{B}_i, \mathcal{B}_i^* \in \mathcal{P}(n)$ and $\sum_{i=1}^{m} \mathcal{B}_i\mathcal{B}_i^* < M \cdot \mathcal{I}_n$ for some $M > 0$ ($\mathcal{I}_n$ – the unit matrix in $\mathcal{M}(n)$) and if $|\operatorname{tr}(F(\mathcal{Y}) - F(\mathcal{X}))| \leq \frac{1}{M}|\operatorname{tr}(\mathcal{Y} - \mathcal{X})|$, for all $\mathcal{X}, \mathcal{Y} \in \mathcal{H}(n)$ with $\mathcal{X} \leq \mathcal{Y}$, then the equation $\mathcal{X} = Q + \sum_{i=1}^{m} \mathcal{B}_i^* F(\mathcal{X})\mathcal{B}_i$ has a unique positive definite solution (PDS).*

1.2. Fixed point results

On the other hand, many authors have obtained a large number of fixed point and common fixed point results over the course of the last few decades and have applied these results to obtain solutions of different kinds of equations arising in different situations in a wide range of mathematical problems.

Several mathematicians have recently established fixed point findings for contraction type mappings in partial order metric spaces. Turinici developed some early results in this technique in [23, 24]; it should be emphasised, however, that their starting points were amorphous contributions in the field due to Matkowski [15, 16]. These types of discoveries have been investigated by Ran and Reurings, and also Nieto and Rodriguez-López, who [17, 18]. Turinici's findings were broadened and enhanced in subsequent papers [17, 18]. Samet and Turinici refer to Bessem's new discovery of a fixed point theorem for nonlinear contraction when an arbitrary relation is symmetrically closed. Recently, Ahmadullah et al. [1, 2], and Alam and Imdad [4] established a relation-theoretic equivalent of the Banach Contraction Principle using an amorphous relation, which incorporates a number of well-known relevant order-theoretic fixed point theorems. In the paper [25], Wardowski created the term $\mathcal{F}$-contractions to describe a new type of contraction. He introduced the $\mathfrak{F}$ family of functions $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ that share certain properties. This concept has been used by many researchers in a variety of abstract situations.

1.3. Motivation

One of the most visually appealing uses of contraction mapping is found in the field of nonlinear matrix equation to solve it. The question now is whether the aforementioned $\mathcal{F}$-contraction can be enhanced and generalized. We explore a general class of contraction comprised of $\mathcal{F}\mathcal{G}$-contractions, extending certain fixed point findings from the conventional fixed point theory consisting of Banach contraction, $\mathcal{F}$-contraction, Geraghty-type-contraction, to provide an affirmative response. Additionally, two novel rational-type contraction are deduced. Two examples are offered to illustrate the topic.

1.4. Contribution

The following is the overview of the paper's structure. In Section 2, some notions related to relational metric spaces are discussed. In Section 3, we introduce a $\mathcal{F}\mathcal{G}$-contraction mapping on metric spaces equipped with an arbitrary binary relation (not necessarily partial order) and then show existence and uniqueness of fixed point findings under weaker conditions, and establish fixed point results. Section 4 provides two nontrivial instances to support the conclusion made here. In the final Section 5, we apply this conclusion to NMEs and examine their convergence behaviour with regard to three alternative initializations using graphical representations and solution by surface plot in MATLAB. Two randomly (real and complex) generated matrices of different orders are used to solve the nonlinear matrix equations.

2. Preliminaries

We fix, the notations $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ have their customary meanings, and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

A relational set is defined as $(\mathcal{W}, \mathcal{R})$ if (i) $\mathcal{W} \neq \emptyset$ is a set and (ii) $\mathcal{R}$ is a binary relation on $\mathcal{W}$.

In addition, if $(\mathcal{W}, d)$ is a metric space, we call $(\mathcal{W}, d, \mathcal{R})$ a relational metric space (RMS, for short).

The following are some commonly used terminology in relational set theory (see, for example, [4, 12–14, 22]).

Let $(\mathcal{W}, \mathcal{R})$ be a relational set, $(\mathcal{W}, d, \mathcal{R})$ be an RMS, and let $\mathfrak{I}$ be a self-mapping on $\mathcal{W}$. Then:

1) $u \in \mathcal{W}$ is $\mathcal{R}$-related to $v \in \mathcal{W}$ if and only if $(u, v) \in \mathcal{R}$.
   If for all $u, v \in \mathcal{W}$, $[u, v] \in \mathcal{R}$, where $[u, v] \in \mathcal{R}$ means either $(u, v) \in \mathcal{R}$ or $(v, u) \in \mathcal{R}$, the set $(\mathcal{W}, \mathcal{R})$ is said to be comparable.
2) The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$, that is, $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$. Indeed, $\mathcal{R}^s$ is the smallest symmetric relation on $\mathcal{W}$ containing $\mathcal{R}$.
3) A sequence (u_n) in $\mathcal{W}$ is said to be $\mathcal{R}$-preserving if $(u_n, u_{n+1}) \in \mathcal{R}$, $\forall n \in \mathbb{N} \cup \{0\}$.
4) $(\mathcal{W}, d, \mathcal{R})$ is said to be $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence converges in $\mathcal{W}$.
5) $\mathcal{R}$ is said to be $\mathfrak{I}$-closed if $(u, v) \in \mathcal{R} \Rightarrow (\mathfrak{I}u, \mathfrak{I}v) \in \mathcal{R}$. It is said to be weakly $\mathfrak{I}$-closed if $(u, v) \in \mathcal{R} \Rightarrow [\mathfrak{I}u, \mathfrak{I}v] \in \mathcal{R}$.
6) $\mathcal{R}$ is said to be d-self-closed if there is a subsequence (u_{n_k}) of (u_n) for every $\mathcal{R}$-preserving sequence with $u_n \to u$, such that $[u_{n_k}, u] \in \mathcal{R}$, for all $k \in \mathbb{N}^*$.
7) If for each $u, v \in \mathfrak{M}$, there exists $\mu \in \mathcal{W}$ such that $(u, \mu) \in \mathcal{R}$ and $(v, \mu) \in \mathcal{R}$, then the subset $\mathfrak{M}$ of $\mathcal{W}$ is termed $\mathcal{R}$-directed. If for any $u, v \in \mathfrak{M}$, there exists $\mu \in \mathcal{W}$ such that $(u, \mathfrak{I}\mu) \in \mathcal{R}$ and $(v, \mathfrak{I}\mu) \in \mathcal{R}$, it is said to be $(\mathfrak{I}, \mathcal{R})$-directed.
8) $\mathfrak{I}$ is said to be $\mathcal{R}$-continuous at u if we get $\mathfrak{I}(u_n) \to \mathfrak{I}(u)$ as $n \to \infty$ for every $\mathcal{R}$-preserving sequence (u_n) converging to u. Furthermore, $\mathfrak{I}$ is said to be $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at all points of $\mathcal{W}$.
9) For $u, v \in \mathcal{W}$, a path of length k (where k is a natural number) in $\mathcal{R}$ from u to v is a finite sequence $\{w_0, w_1, w_2, \ldots, w_k\} \subset \mathcal{W}$ satisfying the following conditions:
   (i) $z_0 = u$ and $\mu_k = v$,
   (ii) $(w_i, w_{i+1}) \in \mathcal{R}$ for each i $(0 \leq i \leq k-1)$,
   then this finite sequence is called a path of length k joining u to v in $\mathcal{R}$.
10) If for a pair of $u, v \in \mathcal{W}$, there is a finite sequence $\{w_0, w_1, w_2, \ldots, w_k\} \subset \mathcal{W}$ satisfying the following conditions:
   (i) $\mathfrak{I}w_0 = u$ and $\mathfrak{I}w_k = v$,
   (ii) $(\mathfrak{I}w_i, \mathfrak{I}w_{i+1}) \in \mathcal{R}$ for each i $(0 \leq i \leq k-1)$,
   then this finite sequence is called a $\mathfrak{I}$-path of length k joining u to v in $\mathcal{R}$.

It is worth noting that a path of length k contains $k+1$ components of $\mathcal{W}$, which are not necessarily distinct.

For a relational space $(\mathcal{W}, \mathcal{R})$, a self-mapping $\mathfrak{I}$ on $\mathcal{W}$ and an $\mathcal{R}$-directed subset $\mathfrak{D}$ of $\mathcal{W}$, we use the following notation:

(i) $Fix(\mathfrak{I})$:= the set of all fixed points of $\mathfrak{I}$,
(ii) $\mathcal{N}(\mathfrak{I}, \mathcal{R}) := \{u \in \mathcal{W} : (u, \mathfrak{I}u) \in \mathcal{R}\}$,
(iii) $\Lambda(u, v, \mathcal{R})$:= the class of all paths in $\mathcal{R}$ from u to v in $\mathcal{R}$, where $u, v \in \mathcal{W}$.

3. Main results

3.1. $\mathcal{FG}$-contractive mappings

Wardowski [25] introduced the family $\mathfrak{F}$ of functions $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ with the following properties:

(F1) $\mathcal{F}$ is strictly increasing;
(F2) for each sequence $\{\xi_n\}$ of positive numbers,

$$\lim_{n\to\infty} \xi_n = 0 \text{ if and only if } \lim_{n\to\infty} \mathcal{F}(\xi_n) = -\infty.$$

(F3) There exists $k \in (0, 1)$ such that $\lim_{t\to 0^+} \xi^k \mathcal{F}(\xi) = 0$.

Parvaneh et al. [19] used following set of slightly modified family of functions.

Definition 3.1. [19] The collection of all functions $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}$ satisfying:

($\mathbb{F}_1$) $\mathcal{F}$ is continuous and strictly increasing;
($\mathbb{F}_2$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\lim_{n\to\infty} \xi_n = 0$ iff $\lim_{n\to\infty} \mathcal{F}(\xi_n) = -\infty$,

will be denoted by $\mathbb{F}$.

The collection of all pairs of mappings $(\mathcal{G}, \beta)$, where $\mathcal{G} : \mathbb{R}_+ \to \mathbb{R}$, $\beta : \mathbb{R}_+ \to [0, 1)$, satisfying:

($\mathbb{F}_3$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\limsup_{n\to\infty} \mathcal{G}(\xi_n) \geq 0$ iff $\limsup_{n\to\infty} \xi_n \geq 1$;
($\mathbb{F}_4$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\limsup_{n\to\infty} \beta(\xi_n) = 1$ implies $\lim_{n\to\infty} \xi_n = 0$;
($\mathbb{F}_5$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\sum_{n=1}^{\infty} \mathcal{G}(\beta(\xi_n)) = -\infty$,

will be denoted by $\mathbb{G}_\beta$.

Definition 3.2. Let $(\mathcal{W}, d, \mathcal{R})$ be an *RMS* and $\mathfrak{I} : \mathcal{W} \to \mathcal{W}$ be a given mapping. A mapping $\mathfrak{I}$ is said to be a $\mathcal{FG}$-contractive mapping, if there exist $\mathcal{F} \in \mathbb{F}$ and $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$, such that for $(u, v) \in \mathcal{W}^2$ with $(u, v) \in \mathcal{R}^*$,

$$\mathcal{F}(d(\mathfrak{I}u, \mathfrak{I}v)) \leq \mathcal{F}(\Delta(u, v)) + \mathcal{G}(\beta(\Delta(u, v))), \tag{3.1}$$

where

$$\Delta(u, v) = \max\left\{d(u, v), d(u, \mathfrak{I}u), d(v, \mathfrak{I}v), \frac{d(u, \mathfrak{I}v) + d(v, \mathfrak{I}u)}{2}\right\}, \tag{3.2}$$

$$\mathcal{R}^* = \{(u, v) \in \mathcal{R} \mid \mathfrak{I}u \neq \mathfrak{I}v\}.$$

We denote by $(\mathcal{FG})_{\mathcal{R}}$ the collection of all $\mathcal{FG}$-contractive mappings on $(\mathcal{W}, d, \mathcal{R})$.

3.2. Fixed point results on $(\mathcal{F}\mathcal{G})_{\mathcal{R}}$-contractive mappings

We present and verify our conclusions on $(\mathcal{F}\mathcal{G})_{\mathcal{R}}$-contractive mappings described in Sub-Section 3.1. The following is the first main outcome.

Theorem 3.3. *Let $(\mathcal{W}, d, \mathcal{R})$ be an RMS and $\mathfrak{I}: \mathcal{W} \to \mathcal{W}$. Suppose that the following conditions hold:*

(C_1) $\mathcal{N}(\mathfrak{I}, \mathcal{R}) \neq \emptyset$;
(C_2) $\mathcal{R}$ is $\mathfrak{I}$-closed and $\mathfrak{I}$-transitive;
(C_3) $\mathcal{W}$ is $\mathfrak{I}$-$\mathcal{R}$-complete;
(C_4) $\mathfrak{I} \in (\mathcal{F}\mathcal{G})_{\mathcal{R}}$;
(C_5) $\mathfrak{I}$ is $\mathcal{R}$-continuous or
(C_5') $\mathcal{R}$ is d-self-closed.

Then there exists a point $u^* \in Fix(\mathfrak{I})$.

Proof. Starting with $u_0 \in \mathcal{W}$ given by (C_1), we construct a sequence $\{u_n\}$ of Picard iterates $u_{n+1} = \mathfrak{I}^n(u_0)$ for all $n \in \mathbb{N}^*$.

Using (C_1) and (C_2), we have that $(\mathfrak{I}u_0, \mathfrak{I}^2 u_0) \in \mathcal{R}$. Continuing this process inductively, we obtain

$$(\mathfrak{I}^n u_0, \mathfrak{I}^{n+1} u_0) \in \mathcal{R} \tag{3.3}$$

for any $n \in \mathbb{N}^*$. Hence, $\{u_n\}$ is an $\mathcal{R}$-preserving sequence.

Now, if there exists some $n_0 \in \mathbb{N}^*$ such that $d(u_{n_0}, \mathfrak{I}u_{n_0}) = 0$, then the result follows immediately. Otherwise, for all $n \in \mathbb{N}^*$, $d(u_n, \mathfrak{I}u_n) > 0$ so that $\mathfrak{I}u_n \neq \mathfrak{I}u_{n+1}$ which implies that $(u_n, u_{n+1}) \in \mathcal{R}^*$. Therefore, using (C_4) for $u = u_n$, $v = u_{n+1}$, we have

$$\mathcal{F}(d(\mathfrak{I}u_n, \mathfrak{I}u_{n+1})) \leq \mathcal{F}(\Delta(u_n, u_{n+1})) + \mathcal{G}(\beta(u_n, u_{n+1}))),$$

where

$$\begin{aligned}
\Delta(u_n, u_{n+1})) &= \max\left\{ \begin{array}{c} d(u_n, u_{n+1}), d(u_n, \mathfrak{I}u_n), d(u_{n+1}, \mathfrak{I}u_{n+1}), \\ \frac{d(u_n, \mathfrak{I}u_{n+1}) + d(u_{n+1}, \mathfrak{I}u_n)}{2} \end{array} \right\} \\
&= \max\left\{ \begin{array}{c} d(u_n, u_{n+1}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), \\ \frac{d(u_n, u_{n+2})}{2} \end{array} \right\} \\
&\leq \max\left\{ \begin{array}{c} d(u_n, u_{n+1}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), \\ \frac{d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2})}{2} \end{array} \right\} \\
&= \max\left\{ d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}) \right\}.
\end{aligned}$$

If $\Delta(u_n, u_{n+1})) = d(u_{n+1}, u_{n+2})$, then

$$\mathcal{F}(d(u_{n+1}, u_{n+2})) \leq \mathcal{F}(d(u_{n+1}, u_{n+2})) + \mathcal{G}(\beta(d(u_{n+1}, u_{n+2})))$$

which implies $\mathcal{G}(\beta(d(u_{n+1}, u_{n+2}))) \geq 0$ i.e. $\beta(d(u_{n+1}, u_{n+2})) \geq 1$, a contradiction. Therefore

$$d(u_{n+1}, u_{n+2}) \leq d(u_n, u_{n+1}) \text{ for all } n \in \mathbb{N} \tag{3.4}$$

and so

$$\mathcal{F}(d(u_{n+1}, u_{n+2})) \leq \mathcal{F}(d(u_n, u_{n+1})) + \mathcal{G}(\beta(d(u_n, u_{n+1})))$$

for all $n \in \mathbb{N}$. Consequently

$$\begin{aligned}\mathcal{F}(d(u_n, u_{n+1})) &\leq \mathcal{F}(d(u_{n-1}, u_n)) + \mathcal{G}(\beta(d(u_{n-1}, u_n))) \\ &\leq \ldots \\ &\leq \mathcal{F}(d(u_0, u_1)) + \sum_{i=1}^{i=n} \mathcal{G}(\beta(d(u_i, u_{i-1}))).\end{aligned} \tag{3.5}$$

Letting $n \to \infty$ gives $\lim_{n\to\infty} \mathcal{F}(d(u_n, u_{n+1})) = -\infty$ and $\mathcal{F} \in \mathbb{F}$ gives

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0. \tag{3.6}$$

We will now show that the sequence $\{u_n\}$ is a $\mathcal{R}$ preserving Cauchy sequence in $(\mathcal{W}, d)$. On the contrary, we suppose that there exist $\zeta > 0$ and two subsequences $\{u_{n(j)}\}$ and $\{u_{m(j)}\}$ of $\{u_n\}$ such that $m(j)$ is the smallest index for which $m(j) > n(j) > j$ and

$$d(u_{m(j)}, u_{n(j)}) \geq \zeta. \tag{3.7}$$

This means that $m(j) > n(j) > j$ and

$$d(u_{m(j)-1}, u_{n(j)}) < \zeta. \tag{3.8}$$

On the other hand

$$\zeta \leq d(u_{m(j)}, u_{n(j)}) \leq d(u_{m(j)}, u_{m(j)-1}) + d(u_{m(j)-1}, u_{nj)}) \leq d(u_{m(j)}, u_{m(j)-1}) + \zeta.$$

Taking $j \to \infty$ and using (3.6), we get

$$\lim_{j\to\infty} d(u_{m(j)}, u_{n(j)}) = \zeta, \tag{3.9}$$

and hence

$$\lim_{j\to\infty} d(u_{m(j)+1}, u_{n(j)+1}) = \zeta. \tag{3.10}$$

As the sequence $\{u_n\}$ is $\mathcal{R}$-preserving and $\mathcal{R}$ is $\mathfrak{I}$-transitive, therefore $(u_{m(j)}, u_{n(j)}) \in \mathcal{R}^*$ and we get

$$\begin{aligned}&\mathcal{F}(\limsup_{j\to\infty} d(u_{m(j)+1}, u_{n(j)+1})) \\ &\leq \mathcal{F}(\limsup_{j\to\infty} \Delta(u_{m(j)}, u_{n(j)})) + \limsup_{j\to\infty} \mathcal{G}(\beta(\Delta(u_{m(j)}, u_{n(j)})))\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}&\Delta(u_{m(j)}, u_{n(j)}) \\ &= \max\left\{\begin{array}{c} d(u_{m(j)}, u_{n(j)}), d(u_{m(j)}, \mathfrak{I}u_{m(j)}), d(u_{n(j)}, \mathfrak{I}u_{n(j)}), \\ \frac{d(u_{n(j)}, \mathfrak{I}u_{m(j)}) + d(u_{m(j)}, \mathfrak{I}u_{n(j)})}{2} \end{array}\right\}\end{aligned}$$

$$= \max\left\{ \begin{array}{c} d(u_{m(j)}, u_{n(j)}), d(u_{m(j)}, u_{m(j)+1}), d(u_{n(j)}, u_{n(j)+1}), \\ \frac{d(u_{n(j)}, u_{m(j)+1}) + d(u_{m(j)}, u_{n(j)+1})}{2} \end{array} \right\}$$

$$\leq \max\left\{ \begin{array}{c} d(u_{m(j)}, u_{n(j)}), d(u_{m(j)}, u_{m(j)+1}), d(u_{n(j)}, u_{n(j)+1}), \\ \frac{d(u_{n(j)}, u_{m(j)}) + d(u_{m(j)}, u_{m(j)+1}) + d(u_{m(j)}, u_{n(j)}) + d(u_{n(j)}, u_{n(j)+1})}{2} \end{array} \right\}.$$

Taking upper limit as $j \to \infty$ and making use of (3.6), (3.9) and (3.10), we get

$$\limsup_{j\to\infty} \Delta(u_{m(j)}, u_{n(j)}) = \limsup_{j\to\infty} d(u_{m(j)}, u_{n(j)}). \tag{3.12}$$

Therefore, from (3.11), (3.10) and (3.12), we have

$$\begin{aligned} \mathcal{F}(\zeta) &= \mathcal{F}(\limsup_{j\to\infty} d(u_{m(j)+1}, u_{n(j)+1})) \\ &\leq \mathcal{F}(\limsup_{j\to\infty} d(u_{m(j)}, u_{n(j)})) + \limsup_{j\to\infty} \mathcal{G}(\beta(d(u_{m(j)}, u_{n(j)}))) \\ &= \mathcal{F}(\zeta) + \limsup_{j\to\infty} \mathcal{G}(\beta(d(u_{m(j)}, u_{n(j)})), \end{aligned}$$

which implies that $\limsup_{j\to\infty} \mathcal{G}(\beta(d(u_{m(j)}, u_{n(j)}))) \geq 0$, which gives $\limsup_{j\to\infty} \beta(d(u_{m(j)}, u_{n(j)})) \geq 1$, and taking in account that $\beta(\xi) < 1$ for all $\xi \geq 0$, we have $\limsup_{j\to\infty} \beta(d(u_{m(j)}, u_{n(j)})) = 1$. Therefore, $\limsup_{j\to\infty} d(u_{m(j)}, u_{n(j)}) = 0$, a contradiction. Hence, $\{u_n\}$ is $\mathcal{R}$ preserving Cauchy sequence in $\mathcal{W}$.

The $\mathcal{R}$-completeness of $\mathcal{W}$ implies that there exists $u^* \in \mathcal{W}$ such that $\lim_{n\to\infty} u_n = u^*$. Now first by (C5), we have

$$u^* = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} \Im u_n = \Im u^* \tag{3.13}$$

and hence u^* is a fixed point of $\Im$.

Alternatively, suppose that $\mathcal{R}$ is d-self-closed. Then, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u^*] \in \mathcal{R}$ for all $k \in \mathbb{N}^*$.

Now, we distinguish two cases for $\Gamma = \{k \in \mathbb{N} : \Im u_{n_k} = \Im u^*\}$. If Γ is finite, then there exists $k_0 \in \mathbb{N}$ such that $\Im u_{n_k} \neq \Im u^*$ for all $k > k_0$. It follows from (3.1), (for all $k > k_0$) that

$$\mathcal{F}(d(\Im u_{n_k}, \Im u^*)) \leq \mathcal{F}(\Delta(u_{n_k}, u^*)) + \mathcal{G}(\beta(\Delta(u_{n_k}, u^*)))$$

where

$$\Delta(u_{n_k}, u^*) = \max\left\{ \begin{array}{c} d(u_{n_k}, u^*), d(u_{n_k}, \Im u_{n_k}), d(u^*, \Im u^*), \\ \frac{d(u_{n_k}, \Im u^*) + d(u^*, \Im u_{n_k})}{2} \end{array} \right\}.$$

Applying limit as $n \to \infty$, we get $\lim_{n\to\infty} \Delta(u_{n_k}, u^*) = d(u^*, \Im u^*)$ which implies that $\limsup_{n\to\infty} \mathcal{G}(\beta(\Delta(u_{n_k}, u^*)) \geq 0$, which gives $\limsup_{n\to\infty} \beta(\Delta(u_{n_k}, u^*)) \geq 1$, and taking in account that $\beta(\xi) < 1$ for all $\xi \geq 0$, we have $\limsup_{n\to\infty} \beta(\Delta(u_{n_k}, u^*)) = 1$. Therefore, $\limsup_{n\to\infty} \Delta(u_{n_k}, u^*) = 0$. Hence, $d(u^*, \Im u^*) = 0$, we get $u^* = \Im u^*$.

Otherwise, if Γ is not finite, a subsequence exists. $\{u_{n(k(\varsigma))}\}$ of $\{u_{n_k}\}$ such that

$$u_{n(k(\varsigma))+1} = \Im u_{n(k(\varsigma))} = \Im u^*, \ \forall \varsigma \in \mathbb{N}.$$

As $u_{n_k} \to^d u^*$, therefore $\Im u^* = u^*$. □

Theorem 3.4. *In addition to the assumptions of Theorem 3.3, let* $\Lambda(u, v; \mathcal{R}|_{\Im(\mathcal{W})}) \neq \emptyset$ *for all* $u, v \in \Im(\mathcal{W})$*. Then* $\Im$ *has a unique fixed point.*

Proof. In view of Theorem 3.3, $Fix(\Im) \neq \emptyset$. If $Fix(\Im)$ is a singleton, then we concluded the proof. Otherwise, let $u^* \neq \varpi \in Fix(\Im)$. Then $\Im u^* = u^* \neq \varpi = \Im\varpi$. Since $\Lambda(u, v; \mathcal{R}|_{\Im(\mathcal{W})}) \neq \emptyset$ for all $v,\ u \in \Im(\mathcal{W})$, there exists a path $\{\Im z_0,\ \Im z_1,\ \cdots,\ \Im z_k\}$ of some length k in $\mathcal{R}|_{\Im(\mathcal{W})}$ such that $\Im z_0 = u^*$, $\Im z_k = \varpi$ and $(\Im z_j,\ \Im z_{j+1}) \in \mathcal{R}|_{\Im(\mathcal{W})}$ for each $j = 0, 1, 2, \cdots, k-1$. Since $\mathcal{R}$ is $\Im$-transitive, we have

$$(u^*,\ \Im z_1) \in \mathcal{R},\ (\Im z_1,\ \Im z_2) \in \mathcal{R},\ \cdots,\ (\Im z_{k-1},\ \varpi) \in \mathcal{R} \Rightarrow (u^*,\ \varpi) \in \mathcal{R}.$$

Therefore from $(u^*,\ \varpi) \in \mathcal{R}$ and $\Im u^* \neq \Im\varpi$, we have $(u^*,\ \varpi) \in \Re^*$. Using (3.1) for $(u^*,\ \varpi) \in \Re^*$, we have

$$\mathcal{F}(d(\Im u^*, \Im\varpi)) \leq \mathcal{F}(\Delta(\varpi, u^*) + \mathcal{G}(\beta(\Delta(\varpi, u^*))), \tag{3.14}$$

where

$$\begin{aligned} \Delta(\varpi, u^*) &= \max\left\{d(u^*, \varpi), d(u^*, \Im u^*), d(\varpi, \Im\varpi), \frac{d(u^*, \Im\varpi) + d(\varpi, \Im u^*)}{2}\right\} \\ &= d(u^*, \varpi) \end{aligned}$$

which on substituting in (3.14) gives

$$\mathcal{F}(d(u^*, \varpi)) \leq \mathcal{F}(d(u^*, \varpi)) + \mathcal{G}(\beta(d(u^*, \varpi))),$$

which gives $\mathcal{G}(\beta(d(u^*, \varpi)) \geq 0$ implies $\beta(d(u^*, \varpi) \geq 1$, a contradiction. Thus, $d(u^*, \varpi) = 0$. □

Theorem 3.5. *In addition to the hypotheses of Theorem 3.3, if any of the following conditions is fulfilled:*

(I) for all $u, v \in \mathcal{W}$*, there exists a* $z \in \mathcal{W}$ *such that*

$$\{(z, \Im z), (z, u), (z, v)\} \subseteq \mathcal{R}; \tag{3.15}$$

(II) the set $\Im(\mathcal{W})$ *is* $\mathcal{R}$*-directed;*
(III) $\mathcal{R}|_{\Im(\mathcal{E})}$ *is complete;*
(IV) $\Lambda(u, v, Fix(\Im), \mathcal{R}^s)$ *is nonempty, for each* $u, v \in Fix(\Im)$*,*

then $\Im$ *has a unique fixed point.*

Proof. In view of Theorem 3.3, $Fix(\Im) \neq \emptyset$.

- **Assume (I)**. Suppose $u, v \in \mathcal{W}$ are the two distinct fixed points of $\Im$, that is, $\Im u = u \neq v = \Im v$. We will consider the following two cases.
  Case (A): We have $(u, v) \in \mathcal{R}$, then $\Im^n u = u$ and $\Im^n v = v$ such that $(\Im^n u, \Im^n v) \in \mathcal{R}^*$ for $n = 0, 1, \ldots$. Therefore, using condition (3.1),

$$\mathcal{F}(d(\Im^{n+1} u, \Im^{n+1} v) \leq \mathcal{F}(\Delta(\Im^n u, \Im^n v)) + \mathcal{G}(\beta(\Delta(\Im^n u, \Im^n v)))$$

where

$$\Delta(\Im^n u, \Im^n v)) \quad = \quad \max \left\{ \begin{array}{c} d(\Im^n u, \Im^n v), d(\Im^n u, \Im^{n+1} u), d(\Im^n v, \Im^{n+1} v), \\ \frac{d(\Im^n u, \Im^{n+1} v) + d(\Im^n v, \Im^{n+1} u)}{2} \end{array} \right\}.$$

Since u and v are fixed points of $\Im$, we have

$$\Delta(\Im^n u, \Im^n v) = d(u, v)$$

and we get

$$\mathcal{F}(d(u, v) \leq \mathcal{F}(d(u, v) + \mathcal{G}(\beta(d(u, v))$$

which gives $\mathcal{G}(\beta(d(u, v)) \geq 0$ and so $\beta(d(u, v) \geq 1$, a contradiction. Thus, the fixed point is unique.
Case (B): By the assumption (I), there exists a distinct element z in $\mathcal{W}$ such that $\Im u = u \neq z \neq v = \Im v$, satisfying condition (3.15), otherwise proof follows from **Case (A)**. Next due to $\Im$-closedness of $\mathcal{R}$, we get

$$(\Im^{n-1} z, u) \in \mathcal{R}, \quad (\Im^{n-1} z, v) \in \mathcal{R}.$$

Also distinctness of z from u and v, we have

$$(\Im^{n-1} z, \Im^{n-1} u) = (\Im^{n-1} z, u) \in \mathcal{R}^*, \quad (\Im^{n-1} z, \Im^{n-1} v) = (\Im^{n-1} z, v) \in \mathcal{R}^*.$$

Therefore using condition (3.1) for $(\Im^{n-1} z, u) \in \mathcal{R}^*$, we have

$$\mathcal{F}(d(\Im^n z, u)) \leq \mathcal{F}(\Delta(\Im^{n-1} z, u)) + \mathcal{G}(\beta(\Delta(\Im^{n-1} z, u))), \tag{3.16}$$

where

$$\begin{aligned}
&\Delta(\Im^{n-1} z, u) \\
&= \max \left\{ d(\Im^{n-1} z, u), d(\Im^{n-1} z, \Im^n z), d(u, \Im u), \frac{d(\Im^{n-1} z, \Im u) + d(u, \Im^n z)}{2} \right\} \\
&\leq \max \left\{ d(\Im^{n-1} z, u), d(\Im^{n-1} z, \Im^n z), d(u, \Im u), \frac{2d(\Im^{n-1} z, u) + d(\Im^{n-1} z, \Im^n z)}{2} \right\} \\
&\leq \max\{d(\Im^{n-1} z, u), d(\Im^{n-1} z, \Im^n z), d(u, \Im u)\}.
\end{aligned}$$

Using $(z, \Im z) \in \mathcal{R}$, similarly as in the proof of Theorem 3.3, it can be shown that $d(\Im^{n-1} z, \Im^n z) \to 0$ as $n \to \infty$. Therefore, for n sufficiently large,

$$\max\{d(\Im^{n-1} z, u), d(\Im^{n-1} z, \Im^n z), d(u, \Im u)\} = d(\Im^{n-1} z, u)$$

and, from (3.16), we have

$$\mathcal{F}(d(\Im^n z, u)) \leq \mathcal{F}(d(\Im^{n-1} z, u)) + \mathcal{G}(\beta(d(\Im^{n-1} z, u))).$$

As in the proof of Theorem 3.3, it can be shown that $d(\Im^n z, u) \leq d(\Im^{n-1} z, u)$. It follows that the sequence $\{d(\Im^n z, u)\}$ is nonincreasing. As earlier, we have

$$\lim_{n\to\infty} d(\Im^n z, u) = 0.$$

Also, since $(z, v) \in \mathcal{R}$, proceeding as earlier, we can prove that

$$\lim_{n\to\infty} d(\Im^n z, v) = 0,$$

and by using limit uniqueness, we infer that $u = v$; i.e., the fixed point of $\Im$ is unique.

- **Assume (II)**. For any two fixed points u, v of $\Im$, there must be an element $z \in \Im(\mathcal{W})$, such that

$$(z, u) \in \mathcal{R} \text{ and } (z, v) \in \mathcal{R}.$$

As $\mathcal{R}$ is $\Im$-closed, so for all $n \in \mathbb{N} \cup \{0\}$,

$$(\Im^n z, u) \in \mathcal{R} \text{ and } (\Im^n z, v) \in \mathcal{R}.$$

In the line of proof of Case(B) (I), we obtain $u = v$, i.e., $\Im$ has a unique fixed point.

- **Assume (III)**. Suppose u, v are two fixed points of $\Im$. Then, we must have $(u, v) \in \mathcal{R}$ and since $u \neq \Im v$, we have $(v, u) \in \mathcal{R}^*$. Therefore, using condition (3.1),

$$\mathcal{F}(d(\Im u, \Im v) \leq \mathcal{F}(\Delta(u, v) + \mathcal{G}(\beta(\Delta(u, v)))$$

where

$$\begin{aligned}\Delta(u, v) &= \max\left\{d(u, v), d(u, \Im u), d(v, \Im v), \frac{d(u, \Im v) + d(v, \Im u)}{2}\right\} \\ &= d(u, v)\end{aligned}$$

which gives $\mathcal{G}(\beta(d(u, v)) \geq 0$ and so $\beta(d(u, v) \geq 1$, a contradiction. Thus, the fixed point is unique. In a similar way, if $(v, u) \in \mathcal{R}$, we have $u = v$.

- **Assume (IV)**. Suppose u, v are two fixed points of $\Im$. Let $\{z_0, z_1, \ldots, z_k\}$ be an $\mathcal{R}^s$-path in $Fix(\Im)$ connecting u and v. As in Case (I,A), it must be $z_{i-1} = z_i$ for each $i = 1, 2, \ldots, k$, and it follows that $u = v$.

□

If we take $\mathcal{R} = \{(u,\ u) \in \mathcal{W} \times \mathcal{W} \mid u \preceq u\}$, then we have more new results as consequences of Theorem 3.3.

Corollary 3.6. *Let $(\mathcal{W}, d, \preceq)$ be an ordered complete metric space. Let $\Im : \mathcal{W} \to \mathcal{W}$ be increasing and $(\mathcal{F}\mathcal{G})_{\mathcal{R}}$ on $\mathcal{W}_{\preceq}$. Suppose there exists $u_0 \in \mathcal{W}$ such that $u_0 \preceq \Im u_0$. If $\Im$ is $\mathcal{W}_{\preceq}$-continuous or $\mathcal{W}_{\preceq}$ is d-self-closed, then $u^* \in Fix(\Im)$. Moreover, for each $u_0 \in \mathcal{W}$ with $u_0 \preceq \Im u_0$, the Picard sequence $\Im^n(u_0)$ for all $n \in \mathbb{N}$ converges to a $u^* \in Fix(\Im)$.*

Considering a range of concrete functions $\mathcal{F} \in \mathbb{F}$ and $(\mathcal{G},\beta) \in \mathbb{G}_\beta$ in the condition (1.1) of Theorems 3.3–3.5 and Corollary 3.6, we can get various classes of $(\mathcal{F}\mathcal{G})_\mathcal{R}$-contractive conditions in an *RMS*. We state just a few examples (recall that $\Delta(u, v)$ is defined in (3.14)).

Corollary 3.7. *Let all the hypothesis of Theorem 3.3 hold except (C_4) is replaced by*

(C_4') $\mathfrak{I}$ *satisfy Wardowski-type* [25] *condition, that is, for* $(u, v) \in \mathcal{W}$ *with* $(u, v) \in \mathcal{R}^*$, $\tau > 0$

$$\tau + \mathcal{F}(d(\mathfrak{I}u, \mathfrak{I}v)) \leq \mathcal{F}(\Delta(u, v)).$$

Then there exists a point $u^* \in Fix(\mathfrak{I})$.

Proof. If we take $\beta(t) = e^{-\tau}$, and $\mathcal{G}(t) = \ln t$ $(t > 0)$ in the Eq (3.1), then the result follows from Theorem 3.3. □

Corollary 3.8. *Let all the hypothesis of Theorem 3.3 hold except (C_4) is replaced by*

(C_4') $\mathfrak{I}$ *satisfy Geraghty-type [9, 10] condition, that is, for* $(u, v) \in \mathcal{W}$ *with* $(u, v) \in \mathcal{R}^*$,

$$d(\mathfrak{I}u, \mathfrak{I}v) \leq \beta(\Delta(u, v))\Delta(u, v))).$$

Then there exists a point $u^* \in Fix(\mathfrak{I})$.

Proof. If we take $\mathcal{F}(t) = \mathcal{G}(t) = \ln t$ $(t > 0)$ in the Eq (3.1), then the result follows from Theorem 3.3. □

Corollary 3.9. *Let all the hypothesis of Theorem 3.3 hold except (C_4) is replaced by*

(C_4') $\mathfrak{I}$ *satisfy mixed type-I condition, that is, for* $(u, v) \in \mathcal{W}$ *with* $(u, v) \in \mathcal{R}^*$,

$$d(\mathfrak{I}u, \mathfrak{I}v) \leq \frac{\Delta(u, v)}{[1 - \sqrt{\Delta(u, v)}\ln(\beta(\Delta(u, v)))]^2}.$$

Then there exists a point $u^* \in Fix(\mathfrak{I})$.

Proof. If we take $\mathcal{F}(t) = -\frac{1}{\sqrt{t}}$, $\mathcal{G}(t) = \ln t$ $(t > 0)$ in Eq (3.1), then the result follows from Theorem 3.3. □

Corollary 3.10. *Let all the hypothesis of Theorem 3.3 hold except (C_4) is replaced by*

(C_4') $\mathfrak{I}$ *satisfy mixed type-II condition, that is, for* $(u, v) \in \mathcal{W}$ *with* $(u, v) \in \mathcal{R}^*$, $\tau > 0$

$$d(\mathfrak{I}u, \mathfrak{I}v) \leq \frac{\Delta(u, v)}{[1 + \tau\sqrt{\Delta(u, v)}]^2}.$$

Then there exists a point $u^* \in Fix(\mathfrak{I})$.

Proof. If we take $\beta(t) = e^{-\tau}$, $\mathcal{G}(t) = \ln t$ and $\mathcal{F}(t) = -1/\sqrt{t}$ $(t > 0)$ in the Eq (3.1), then the result follows from Theorem 3.3. □

Remark 3.11. If we take $\mathcal{R} = \{(u,\ u) \in \mathcal{W} \times \mathcal{W} \mid u \leq u\}$ in the Corollarys 3.7–3.10, then it belong to [4, 9, 21].

Remark 3.12. If we replace (C_3) by relatively weaker notions, namely, $\mathcal{R}$-precompleteness [3] of $\mathfrak{I}(\mathcal{W})$, our results will be true.

4. Illustrations

Example 4.1. Let $\mathcal{W} = [0,9)$ be equipped with usual metric d. Consider the binary relation on $\mathcal{W}$ as follows:

$$\mathcal{R} = \{(0,2),\ (3,2),\ (3,3),\ (3,6),\ (4,2),\ (4,3),\ (4,4),\ (4,6),\ (6,2),\ (6,3),\ (6,6)\}.$$

Define a mapping $\Im : \mathcal{W} \to \mathcal{W}$ by

$$\Im u = \begin{cases} 2, & 0 \le u < 1; \\ 4, & u = 1; \\ 6, & 1 < u < 8. \end{cases}$$

Then $\Im$ is not continuous while $\Im$ is $\mathcal{R}$-continuous, $\mathcal{R}$ is $\Im$-closed, and $\Im$-transitive; $\mathcal{W}$ is $\Im$-$\mathcal{R}$-complete. Also $\mathcal{R}^* = \{(0,2),\ (6,2)\}$ and $\mathcal{N}(\Im;\mathcal{R}) \neq \emptyset$ as $(6,\ \Im 6) = (6,6) \in \mathcal{R}$.

Now we take $\mathcal{F}(\xi) = -\frac{1}{\sqrt{\xi}}$, $\mathcal{G}(\xi) = \ln \xi$ $(\xi > 0)$ and $\beta(\xi) = \lambda \in (0,1)$, $\tau = -\ln \lambda > 0$, then (3.1) converted to

$$d(\Im u, \Im v) \le \frac{\Delta(u,v)}{(1 + \tau\sqrt{\Delta(u,v)})^2} \tag{4.1}$$

where $\Delta(u,v)$ given in (3.14).

Consider $(u,v) = (6,2) \in \mathcal{R}^*$. Then $d(\Im u, \Im v) = 2$ and $\Delta(u,v) = 4$. Therefore, the condition (4.1) reduces to $2 \le \frac{4}{(1+\tau\sqrt{4})^2}$, which is true for $\tau = 0.1$. Thus, all the conditions of Theorem 3.3 are satisfied, hence $\Im$ has a fixed point. Moreover, $\mathcal{R}|_{\Im(\mathcal{W}?)}$ is transitive while $\mathcal{R}$ is not and for all $u,\ v \in \Im(\mathcal{W})$, we have $(u,\ v) \in \mathcal{R}$, so $\Lambda(u,v,\mathcal{R})|_{\Im(\mathcal{W})})$ is nonempty for all $u,\ v \in \Im(\mathcal{W})$. Following Theorem 3.4, $\Im$ has a unique fixed point which is $u^* = 5$.

Now for $(0,2) \in \mathcal{R}$,

$$d(\Im u, \Im v) = 2 \nleq 2k = k\ \max\left\{d(u,v), d(u,\Im u), d(v,\Im v), \frac{1}{2}[d(u,\Im v) + d(v,\Im u)]\right\}$$

which is not true for any $k \in (0,1)$, and hence $\Im$ is not contraction mapping (Ćirić type contraction) on $(\mathcal{W}, d, \mathcal{R})$. Hence Ćirić et al. [8] cannot be applied to the present example.

Also, as $2,0 \in \mathcal{W}$, $(2,0) \notin \mathcal{R}$ with $\Im 2 = 4 \neq 2 = \Im 0$ such that $d(\Im 2,\ \Im 0) \nleq \frac{d(2,0)}{(1+\tau\sqrt{d(2,0)})^2}$ and $d(\Im u, \Im v) \nleq \frac{\Delta(u,v)}{(1+\tau\sqrt{\Delta(u,v)})^2}$. Also

$$d(\Im u, \Im v) = 2 \nleq 2k = k\ \max\left\{d(u,v), d(u,\Im u), d(v,\Im v), \frac{1}{2}[d(u,\Im v) + d(v,\Im u)]\right\},$$

which shows that $\Im$ is neither contraction nor generalized contraction for any $k \in [0,1)$. Hence results of Wardowski [25] and Ćirić [7] cannot be applied to the present example, while our Theorems 3.3 and 3.4 are applicable. This shows that our results are genuine improvements over the corresponding results contained in Wardowski [25], Ćirić [7] and Ćirić et al. [8].

Example 4.2. Consider the set $\mathcal{W} = [\frac{1}{5}, 1]$ with the usual metric d. Define a binary relation $\mathcal{R}$ by

$$\mathcal{R} = \left\{\left(\frac{1}{5}, \frac{1}{5}\right), \left(\frac{1}{5}, 1\right)\left(\frac{1}{4}, 1\right), \left(\frac{1}{4}, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{4}\right)\right\}.$$

Consider the $\Im$ self-mapping on $\mathcal{W}$ defined by

$$\Im(v) = \begin{cases} \frac{1}{5}, & \frac{1}{5} \leq v \leq \frac{1}{4} \\ \frac{1}{4}, & \frac{3}{4} < v \leq 1. \end{cases}$$

It is obvious that $\mathcal{W}$ is $\Im-\mathcal{R}$ is $\Im$-closed and $\mathcal{R}$ is $\Im$-complete. Also $\mathcal{R}^* = \{(\frac{1}{5}, 1), (\frac{1}{4}, 1)\}$ and $\mathcal{N}(\Im;\mathcal{R}) \neq \emptyset$ as $(\frac{1}{5}, \Im\frac{1}{5}) = (\frac{1}{5}, \frac{1}{5}) \in \mathcal{R}$.

We consider (4.1) of previous Example 4.1 to verify $\Im \in (\mathcal{FG})_{\mathcal{R}}$.

- Let $(v, u) = (\frac{1}{5}, 1)$. Then $d(\Im v, \Im u) = \frac{1}{20}$ and $\Delta(u, v) = \frac{4}{5}$. Therefore, the condition (4.1) reduces to $\frac{1}{20} \leq \frac{4/5}{(1+2\tau\sqrt{1/5})^2}$.
- Let $(v, u) = (\frac{1}{4}, 1)$. Then $d(\Im v, \Im u) = \frac{1}{20}$ and $\Delta(u, v) = 4/5$. Therefore, the condition (4.1) reduces to $\frac{1}{20} \leq \frac{4/5}{(1+2\tau\sqrt{1/5})^2}$.

It is reasonable to verify that the aforementioned instances hold true for $\tau > 0$ (especially $\tau = 0.1$). Thus, $\Im \in (\mathcal{FG})_{\mathcal{R}}$.

Let (v_n) be a sequence that preserves $\mathcal{R}$ and converges to v as $n \to \infty$. Then we'll need

$$(v_n, v_{n+1}) \in \left\{\left(\frac{1}{4}, \frac{1}{5}\right), \left(\frac{1}{4}, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{4}\right)\right\}$$

implies that

$$v_n \in \left\{\frac{1}{5}, \frac{1}{4}\right\}.$$

This means that either $v_n \to 0$ or $v_n \to \frac{1}{5}$ is $n \to \infty$, and we have $[v_n, v] \in \mathcal{R}$ for all $n \in \mathbb{N}$, where $v = \frac{1}{5}$ and $\frac{1}{4}$. This shows that $\mathcal{R}$ is d-self-closed. Thus, all the conditions of Theorem 3.3 are satisfied, hence $\Im$ has a fixed point ($u^* = 1/5$).

Remark 4.3.

- Take note that the binary relation $\mathcal{R}$ used in Examples 4.1 and 4.2 is not one of the more well-known conventional binary relations such as reflexive, irreflexive, symmetric, antisymmetric, complete, or weakly complete.
- It is worth noting that the comparable theorems in [6, 17, 18, 20–24] cannot be used in the context of the preceding examples (i.e., Example 4.1 and 4.2), which illustrate the superiority of Theorem 3.3 over many other conclusions. As a result, all of the traditional discoveries have been extended to an arbitrary binary connection.

5. Application to nonlinear matrix equations

For a matrix $\mathcal{B} \in \mathcal{H}(n)$, we will denote by $s(\mathcal{B})$ any of its singular values and by $s^+(\mathcal{B})$ the sum of all of its singular values, that is, the trace norm $\|\mathcal{B}\|_{tr} = s^+(\mathcal{B})$. For $C, \mathcal{D} \in \mathcal{H}(n)$, $C \geq \mathcal{D}$ (resp. $C > \mathcal{D}$) will mean that the matrix $C - \mathcal{D}$ is positive semi-definite (resp. positive definite).

In [21], Ran and Raurings derived the Lemma 3.1 to get positive solution of linear and nonlinear matrix equations. We state in the following which is needed in the subsequent discussion.

Lemma 5.1. *[21, Lemma 3.1] If $\mathcal{A} \geq O$ and $\mathcal{B} \geq O$ are $n \times n$ matrices, then*

$$0 \leq tr(\mathcal{AB}) \leq \|\mathcal{A}\| tr(\mathcal{B}).$$

Lemma 5.2. *[5] If $\mathcal{A} \in \mathcal{H}(n)$ such that $\mathcal{A} \prec I_n$, then $\|\mathcal{A}\| < 1$.*

Here we present an example that satisfying the above lemmas.

Example 5.3. Consider the matrices

$$\mathcal{A} = \begin{bmatrix} 0.1444 & 0.1089 & 0.0766 \\ 0.1089 & 0.2697 & 0.2064 \\ 0.0766 & 0.2064 & 0.2039 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0.1864 & 0.1352 & 0.0923 \\ 0.1352 & 0.1292 & 0.0585 \\ 0.0923 & 0.0585 & 0.1563 \end{bmatrix}$$

Both the matrices are positive definite having the minimum eigenvalues as 0.0259 and 0.0180 respectively. Also, satisfying Lemma 5.1, since

$$0 \leq tr(\mathcal{A}\mathcal{B}) = 0.1614 \leq \|\mathcal{A}\| tr(\mathcal{B}) = 0.2917.$$

In addition, here $\mathcal{A} \prec I_n$, and $\|\mathcal{A}\| = 0.6181 < 1$ which validate the Lemma 5.2.

In the following, we demonstrate that the solution to the nonlinear matrix problem exists and is unique

$$\mathcal{X} = \mathcal{Q} + \sum_{i=1}^{m} \mathcal{B}_i^* \mathcal{G}(\mathcal{X}) \mathcal{B}_i, \tag{5.1}$$

where $\mathcal{Q}$ is a Hermitian positive definite matrix, $\mathcal{B}_i^*$ stands for the conjugate transpose of an $n \times n$ matrix $\mathcal{B}_i$ and $\mathcal{G}$ an order-preserving continuous mapping from the set of all Hermitian matrices to the set of all positive definite matrices such that $\mathcal{G}(O) = O$.

Theorem 5.4. *Consider NME (5.1). Assume that there exists a positive real number η such that*

(H_1) *there exists $\mathcal{Q} \in \mathcal{P}(n)$ such that* $\sum_{i=1}^{m} \mathcal{B}_i^* \mathcal{G}(\mathcal{Q}) \mathcal{B}_i \succ 0$;

(H_2) $\sum_{i=1}^{m} \mathcal{B}_i \mathcal{B}_i^* \prec \eta I_n$.

(H_3) *for every $\mathcal{X}$, $\mathcal{Y} \in \mathcal{P}(n)$ such that $\mathcal{X} \preceq \mathcal{Y}$ with,*

$$\sum_{i=1}^{m} \mathcal{B}_i^* \mathcal{G}(\mathcal{X}) \mathcal{B}_i \neq \sum_{i=1}^{m} \mathcal{B}_i^* \mathcal{G}(\mathcal{Y}) \mathcal{B}_i$$

for $\tau > 0$, we have

$$\begin{aligned} &|s^+(\mathcal{G}(\mathcal{X}) - \mathcal{G}(\mathcal{Y}))| \\ &\leq \frac{1}{\eta} \times \max \left\{ \begin{array}{c} \frac{|s^+(\mathcal{X}-\mathcal{Y})|}{[1+\tau|s^+(\mathcal{X}-\mathcal{Y})|^{1/2}]^2}, \frac{|s^+(\mathcal{X}-\mathcal{Q}-\sum_{i=1}^{m}\mathcal{B}_i^*\mathcal{G}(\mathcal{X})\mathcal{B}_i)|}{[1+\tau|s^+(\mathcal{X}-\mathcal{Q}-\sum_{i=1}^{m}\mathcal{B}_i^*\mathcal{G}(\mathcal{X})\mathcal{B}_i)|^{1/2}]^2}, \\ \frac{|s^+(\mathcal{Y}-\mathcal{Q}-\sum_{i=1}^{m}\mathcal{B}_i^*\mathcal{G}(\mathcal{Y})\mathcal{B}_i)|}{[1+\tau|s^+(\mathcal{Y}-\mathcal{Q}-\sum_{i=1}^{m}\mathcal{B}_i^*\mathcal{G}(\mathcal{Y})\mathcal{B}_i)|^{1/2}]^2}, \\ \frac{1}{2}\left[\begin{array}{c} \frac{|s^+(\mathcal{X}-\mathcal{Q}-\sum_{i=1}^{m}\mathcal{B}_i^*\mathcal{G}(\mathcal{Y})\mathcal{B}_i)|}{[1+\tau|s^+(\mathcal{X}-\mathcal{Q}-\sum_{i=1}^{m}\mathcal{B}_i^*\mathcal{G}(\mathcal{Y})\mathcal{B}_i)|^{1/2}]^2} \\ +\frac{|s^+(\mathcal{Y}-\mathcal{Q}-\sum_{i=1}^{m}\mathcal{B}_i^*\mathcal{G}(\mathcal{X})\mathcal{B}_i)|}{[1+\tau|s^+(\mathcal{Y}-\mathcal{Q}-\sum_{i=1}^{m}\mathcal{B}_i^*\mathcal{G}(\mathcal{X})\mathcal{B}_i)|^{1/2}]^2} \end{array} \right] \end{array} \right\}. \end{aligned}$$

Then the NME (5.1) has a unique solution. Moreover, the iteration

$$\mathcal{X}_n = \mathcal{Q} + \sum_{i=1}^{m} \mathcal{B}_i^* \mathcal{G}(\mathcal{X}_{n-1}) \mathcal{B}_i \tag{5.2}$$

where $\mathcal{X}_0 \in \mathcal{P}(n)$ satisfies

$$\mathcal{X}_0 \preceq Q + \sum_{i=1}^{m} \mathcal{B}_i^* \mathcal{G}(\mathcal{X}_0)\mathcal{B}_i,$$

converges in the sense of trace norm $\|.\|_{tr}$ to the solution of the matrix equation (5.1).

Proof. Define a mapping $\Im : \mathcal{P}(n) \to \mathcal{P}(n)$ by

$$\Im(\mathcal{X}) = Q + \sum_{i=1}^{m} \mathcal{B}_i^* \mathcal{G}(\mathcal{X})\mathcal{B}_i, \text{ for all } \mathcal{X} \in \mathcal{P}(n),$$

and a binary relation

$$\mathcal{R} = \{(\mathcal{X},\ \mathcal{Y}) \in \mathcal{P}(n) \times \mathcal{P}(n) : \mathcal{X} \preceq \mathcal{Y}\}.$$

Then a solution of the matrix Eq (5.1) is a fixed point of the mapping $\Im$. It's worth noting that $\Im$ is well defined, $\mathcal{R}$ is $\Im$-closed, and $\mathcal{R}$ is $\mathcal{R}$-continuous.

$$\sum_{i=1}^{m} \mathcal{B}_i^* \mathcal{G}(Q)\mathcal{B}_i \succ 0,$$

for some $Q \in \mathcal{P}(n)$, we have $(Q, \Im(\mathcal{K})) \in \mathcal{R}$ and hence $\mathcal{P}(n)(\Im; \mathcal{R}) \neq \emptyset$.

Now, let $(\mathcal{X},\ \mathcal{Y}) \in \mathcal{R}^* = \{(\mathcal{X},\ \mathcal{Y}) \in \mathcal{R} : \Im(\mathcal{X}) \neq \Im(\mathcal{Y})\}$. Then

$$\begin{aligned}
&\|\Im(\mathcal{X}) - \Im(\mathcal{Y})\|_{tr} \\
&= s^+(\Im(\mathcal{X}) - \Im(\mathcal{Y})) \\
&= s^+(\sum_{i=1}^{m} \mathcal{B}_i^*(\mathcal{G}(\mathcal{X}) - \mathcal{G}(\mathcal{Y}))\mathcal{B}_i) \\
&= \sum_{i=1}^{m} s^+(\mathcal{B}_i^*(\mathcal{G}(\mathcal{X}) - \mathcal{G}(\mathcal{Y}))\mathcal{B}_i) \\
&= \sum_{i=1}^{m} s^+(\mathcal{B}_i\mathcal{B}_i^*(\mathcal{G}(\mathcal{X}) - \mathcal{G}(\mathcal{Y}))) \\
&= s^+(\sum_{i=1}^{m} \mathcal{B}_i\mathcal{B}_i^*)s^+(\mathcal{G}(\mathcal{X}) - \mathcal{G}(\mathcal{Y})) \\
&\leq \frac{\|\sum_{i=1}^{m} \mathcal{B}_i\mathcal{B}_i^*\|}{\eta} \times \max\left\{ \begin{array}{c} \frac{\|\mathcal{X}-\mathcal{Y}\|_{tr}}{[1+\tau\|\mathcal{X}-\mathcal{Y}\|_{tr}^{1/2}]^2}, \frac{\|\mathcal{X}-\Im\mathcal{X}\|_{tr}}{[1+\tau\|\mathcal{X}-\Im\mathcal{X}\|_{tr}^{1/2}]^2}, \frac{\|\mathcal{Y}-\Im\mathcal{Y}\|_{tr}}{[1+\tau\|\mathcal{Y}-\Im\mathcal{Y}\|_{tr}^{1/2}]^2}, \\ \frac{1}{2}\left[\frac{\|\mathcal{X}-\Im\mathcal{Y}\|_{tr}}{[1+\tau\|\mathcal{X}-\Im\mathcal{Y}\|_{tr}^{1/2}]^2} + \frac{\|\mathcal{Y}-\Im\mathcal{X}\|_{tr}}{[1+\tau\|\mathcal{Y}-\Im\mathcal{X}\|_{tr}^{1/2}]^2}\right] \end{array} \right\} \\
&\leq \frac{\Theta(\mathcal{U}, \mathcal{V})}{[1 + \tau(\Theta(\mathcal{U}, \mathcal{V}))^{1/2}]^2}
\end{aligned} \tag{5.3}$$

where

$$\Theta(\mathcal{U}, \mathcal{V}) = \max\left\{ \begin{array}{c} \frac{\|\mathcal{X}-\mathcal{Y}\|_{tr}}{[1+\tau\|\mathcal{X}-\mathcal{Y}\|_{tr}^{1/2}]^2}, \frac{\|\mathcal{X}-\Im\mathcal{X}\|_{tr}}{[1+\tau\|\mathcal{X}-\Im\mathcal{X}\|_{tr}^{1/2}]^2}, \frac{\|\mathcal{Y}-\Im\mathcal{Y}\|_{tr}}{[1+\tau\|\mathcal{Y}-\Im\mathcal{Y}\|_{tr}^{1/2}]^2}, \\ \frac{1}{2}\left[\frac{\|\mathcal{X}-\Im\mathcal{Y}\|_{tr}}{[1+\tau\|\mathcal{X}-\Im\mathcal{Y}\|_{tr}^{1/2}]^2} + \frac{\|\mathcal{Y}-\Im\mathcal{X}\|_{tr}}{[1+\tau\|\mathcal{Y}-\Im\mathcal{X}\|_{tr}^{1/2}]^2}\right] \end{array} \right\}. \tag{5.4}$$

Consider $\mathcal{F}(t) = -\frac{1}{\sqrt{t}}$, $\mathcal{G}(t) = \ln t$ $(t > 0)$ and $\beta(t) = \lambda \in (0, 1)$, $\tau = -\ln \lambda > 0$, then (5.3) converted to

$$\mathcal{F}(\|\Im(\mathcal{X}) - \Im(\mathcal{Y})\|_{tr}) \leq \mathcal{F}(\Theta(\mathcal{U}, \mathcal{V})) + \mathcal{G}(\beta(\Theta(\mathcal{U}, \mathcal{V})))$$

where $\Theta(\mathcal{U}, \mathcal{V})$ given in (5.4). Thus, all the hypotheses of Theorem 3.3 are satisfied, therefore there exists $\hat{\mathcal{X}} \in \mathcal{P}(n)$ such that $\Im(\hat{\mathcal{X}}) = \hat{\mathcal{X}}$, and hence the matrix Eq (5.1) has a solution in $\mathcal{P}(n)$. Furthermore, we have $\Lambda(\mathcal{X}, \mathcal{Y}; \mathcal{R}|_{\Im(\mathcal{P}(n))}) \neq \emptyset$ for every $\mathcal{X}, \mathcal{Y} \in \Im(\mathcal{P}(n))$ owing to the presence of least upper bound and largest lower bound for each $\mathcal{X}, \mathcal{Y} \in \Im(\mathcal{P}(n))$. As a result of using Theorem 3.4, we may infer that $\Im$ has a unique fixed point, and that the matrix Eq (5.1) has a unique solution in $\mathcal{P}(n)$.

□

Example 5.5. Consider the NME (5.1) for $m = 3$, $\eta = 4$, $n = 3$, with $\mathcal{G}(\mathcal{X}) = \mathcal{X}^{1/4}$, i.e.,

$$\mathcal{X} = \mathcal{Q} + \mathcal{B}_1^* \mathcal{X}^{1/4} \mathcal{B}_1 + \mathcal{B}_2^* \mathcal{X}^{1/4} \mathcal{B}_2 + \mathcal{B}_3^* \mathcal{X}^{1/4} \mathcal{B}_3, \tag{5.5}$$

where

$$\mathcal{Q} = \begin{bmatrix} 12.722272690000000 & 1.464788500000000 & 2.414163701250000 \\ 1.464788500000000 & 11.328605119375000 & 2.000862796281250 \\ 2.414163701250000 & 2.000862796281250 & 13.332179887689062 \end{bmatrix},$$

$$\mathcal{B}_1 = \begin{bmatrix} 0.070500000000000 & 0.094800000000000 & 0.187200000000000 \\ 0.076200000000000 & 0.046200000000000 & 0.191400000000000 \\ 0.196200000000000 & 0.077400000000000 & 0.036600000000000 \end{bmatrix},$$

$$\mathcal{B}_2 = \begin{bmatrix} 0.022400000000000 & 0.029000000000000 & 0.033000000000000 \\ 0.047000000000000 & 0.031400000000000 & 0.036800000000000 \\ 0.049000000000000 & 0.047800000000000 & 0.031800000000000 \end{bmatrix},$$

$$\mathcal{B}_3 = \begin{bmatrix} 0.859375000000000 & 1.343750000000000 & 0.421875000000000 \\ 0.718750000000000 & 0.375000000000000 & 0.812500000000000 \\ 1.500000000000000 & 0.562500000000000 & 0.875000000000000 \end{bmatrix}.$$

The conditions of Theorem 5.4 can be checked numerically, taking various special values for matrices involved. For example, they can be tested (and verified to be true) for

$$\mathcal{X} = \begin{bmatrix} 2.722167968750000 & 1.464355468750000 & 2.414062500000000 \\ 1.464355468750000 & 1.317382812500000 & 2.000000000000000 \\ 2.414062500000000 & 2.000000000000000 & 3.332031250000000 \end{bmatrix},$$

$$\mathcal{Y} = \begin{bmatrix} 10.000104721250000 & 0.000433031250000 & 0.000101201250000 \\ 0.000433031250000 & 10.011222306875000 & 0.000862796281250 \\ 0.000101201250000 & 0.000862796281250 & 10.000148637689062 \end{bmatrix}.$$

To see the convergence of the sequence $\{\mathcal{X}_n\}$ defined in (5.2), we start with three different initial values

$$\mathcal{X}_0 = \begin{bmatrix} 0.025970559290683 & 0.014219828729812 & 0.004760641350592 \\ 0.014219828729812 & 0.055823355744100 & 0.011986278815522 \\ 0.004760641350592 & 0.011986278815522 & 0.024342909184651 \end{bmatrix}$$

with $\|\mathcal{X}_0\| = 0.106136824219434$, $\mathcal{Y}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $\|\mathcal{Y}_0\| = 3$, $\mathcal{W}_0 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ with $\|\mathcal{W}_0\| = 12$.

We have the following approximation of the unique positive definite solution of the system (5.1) after 10 iterations:

$$\widehat{\mathcal{X}} \approx \mathcal{X}_{10} = \begin{bmatrix} 20.940543248755521 & 7.041955400915334 & 7.763256211115045 \\ 7.041955400915334 & 16.607981685744864 & 5.475739411366025 \\ 7.763256211115045 & 5.475739411366025 & 17.183592428336805 \end{bmatrix}$$

$$\widehat{\mathcal{Y}} \approx \mathcal{Y}_{10} = \begin{bmatrix} 20.940543256819559 & 7.041955407550640 & 7.763256216731211 \\ 7.041955407550640 & 16.607981691214334 & 5.475739415981650 \\ 7.763256216731211 & 5.475739415981650 & 17.183592432298781 \end{bmatrix}$$

$$\widehat{\mathcal{W}} \approx \mathcal{W}_{10} = \begin{bmatrix} 20.940543262336771 & 7.041955412090349 & 7.763256220573650 \\ 7.041955412090350 & 16.607981694956411 & 5.475739419139545 \\ 7.763256220573650 & 5.475739419139543 & 17.183592435009462 \end{bmatrix}.$$

Also, the elements of each sequence are order preserving. The Figure 1 represents the convergence analysis of sequence and Figure 2 represents surface plot of solution.

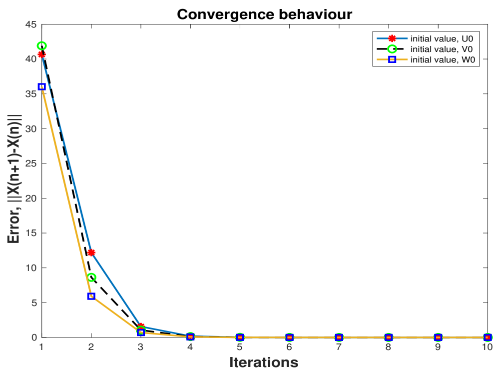

Figure 1. Convergence behavior.

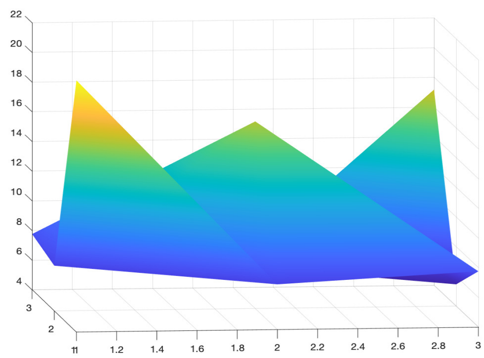

Figure 2. Solution graph.

Example 5.6. Consider the NME

$$\mathcal{X} = \mathcal{Q} + \mathcal{B}_1^* \mathcal{X}^{0.1} \mathcal{B}_1 + \mathcal{B}_2^* \mathcal{X}^{0.1} \mathcal{B}_2$$

where $\mathcal{Q}, \mathcal{B}_1, \mathcal{B}_2$ are uniformly distributed positive definite matrices.

$$\mathcal{B}_1 = \begin{bmatrix} 1.0000 & -0.3863 & 0.0275 & -0.1669 & 0.0528 \\ -0.3863 & 1.0000 & 0.4274 & -0.9587 & 0.6466 \\ 0.0275 & 0.4274 & 1.0000 & 0.7799 & -0.9775 \\ -0.1669 & -0.9587 & 0.7799 & 1.0000 & 0.7176 \\ 0.0528 & 0.6466 & -0.9775 & 0.7176 & 1.0000 \end{bmatrix},$$

$$\mathcal{B}_2 = \begin{bmatrix} 1.0000 & 0.2678 & -0.0068 & -0.2905 & 0.1674 \\ 0.2678 & 1.0000 & -0.6979 & 0.3640 & -0.1111 \\ -0.0068 & -0.6979 & 1.0000 & 0.6944 & 0.9336 \\ -0.2905 & 0.3640 & 0.6944 & 1.0000 & 0.8174 \\ 0.1674 & -0.1111 & 0.9336 & 0.8174 & 1.0000 \end{bmatrix},$$

$$\mathcal{Q} = \begin{bmatrix} 1.0000 & -0.1515 & 0.0282 & 0.2155 & 0.6334 \\ -0.1515 & 1.0000 & 0.0855 & -0.9102 & -0.5035 \\ 0.0282 & 0.0855 & 1.0000 & 0.1743 & -0.0878 \\ 0.2155 & -0.9102 & 0.1743 & 1.0000 & 0.4105 \\ 0.6334 & -0.5035 & -0.0878 & 0.4105 & 1.0000 \end{bmatrix}.$$

Using three approximations as a starting point

$$\mathcal{X}_0 = 5 \times \begin{bmatrix} 1.0000 & -0.1061 & -0.3330 & 0.3496 & 0.6669 \\ -0.1061 & 1.0000 & 0.6168 & 0.4456 & -0.0913 \\ -0.3330 & 0.6168 & 1.0000 & 0.2424 & -0.0166 \\ 0.3496 & 0.4456 & 0.2424 & 1.0000 & 0.1357 \\ 0.6669 & -0.0913 & -0.0166 & 0.1357 & 1.0000 \end{bmatrix},$$

$$\mathcal{Y}_0 = 2 \times \begin{bmatrix} 1.0000 & 0.5814 & -0.2994 & -0.3206 & -0.3171 \\ 0.5814 & 1.0000 & 0.4472 & 0.6869 & -0.0376 \\ -0.2994 & 0.4472 & 1.0000 & -0.2630 & -0.2886 \\ -0.3206 & 0.6869 & -0.2630 & 1.0000 & -0.3170 \\ -0.3171 & -0.0376 & -0.2886 & -0.3170 & 1.0000 \end{bmatrix}$$

$$\mathcal{Z}_0 = 4 \times \begin{bmatrix} 1.0000 & -0.8225 & 0.4813 & 0.7906 & 0.1445 \\ -0.8225 & 1.0000 & 0.7100 & 0.0642 & -0.0427 \\ 0.4813 & 0.7100 & 1.0000 & 0.4340 & 0.3744 \\ 0.7906 & 0.0642 & 0.4340 & 1.0000 & -0.5953 \\ 0.1445 & -0.0427 & 0.3744 & -0.5953 & 1.0000 \end{bmatrix}.$$

We take $\eta = \|\mathcal{B}_1^* \mathcal{B}_1 + \mathcal{B}_2^* \mathcal{B}_2\|_{tr} = 16.602$, and $\mathcal{G}(\mathcal{X}) = \mathcal{X}^{0.1}$ to test our algorithm. The numerical results are discussed in Table 1.

Table 1. Numerical Results of Example 5.6.

Initial Matrix	$\mathcal{G}(\mathcal{U})$	Iter no.	Error(1.e-10)	CPU
$\mathcal{X}_0$	$\mathcal{X}_0^{0.1}$	12	0.9491	0.030092
$\mathcal{Y}_0$	$\mathcal{Y}_0^{0.1}$	13	0.1432	0.036789
$\mathcal{Z}_0$	$\mathcal{W}_0^{0.1}$	13	0.1085	0.028577

The following positive-definite solution is obtained after 12 iterations.

$$\widehat{\mathcal{X}} = \begin{bmatrix} 3.7159 & -0.3242 & -0.6653 & -0.0887 & 0.3253 \\ -0.3242 & 6.1770 & -2.1357 & -2.2443 & -1.0107 \\ -0.6653 & -2.1357 & 7.9363 & 3.3075 & 1.8312 \\ -0.0887 & -2.2443 & 3.3075 & 7.9845 & 3.3266 \\ 0.3253 & -1.0107 & 1.8312 & 3.3266 & 7.8012 \end{bmatrix},$$

with min eigenvalue 3.3143. The Figure 3 represents the convergence analysis of sequence and Figure 4 represents surface plot of solution $\widehat{\mathcal{X}}$.

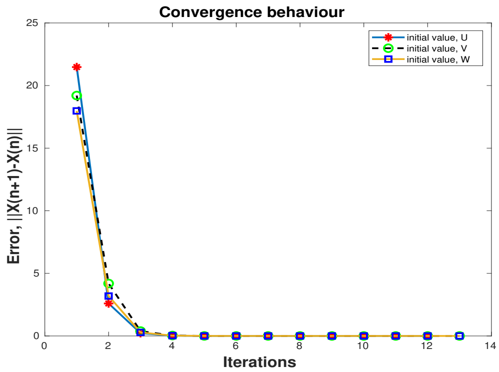

Figure 3. Convergence behavior.

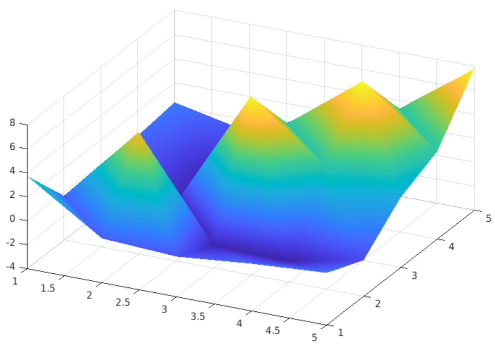

Figure 4. Solution graph.

Next, we introduce a new example consisting of randomly generated real coefficient matrices with various dimensions.

Example 5.7. Consider a randomly generated real coefficient matrices of the equation

$$\mathcal{X} = \mathcal{Q} + \mathcal{B}_1^* \mathcal{X}^{3/10} \mathcal{B}_1 + \mathcal{B}_2^* \mathcal{X}^{3/10} \mathcal{B}_2 + \mathcal{B}_3^* \mathcal{X}^{3/10} \mathcal{B}_3, \tag{5.6}$$

where the coefficients $\mathcal{B}_j$ $(j = 1, 2, 3)$ are chosen by $\mathcal{B}_j = rand(n)$, $\mathcal{Q} = \mathcal{B}_1 \mathcal{B}_1^*$, $\tau = 0.008$. All the experimental data such as iteration number, cpu time, error shown in the Table 2. The last column shows that the solution is positive definite. The Figure 5 represents the convergence analysis of sequence for various dimensions.

Table 2. Numerical analysis for different dimension in Example 5.7.

Dimension	No. of Iteration	Error	CPU Time	Min. Eigen Value
4	20	5.7919e-11	0.055063	0.7974
8	22	9.7623e-11	0.072524	0.5615
12	25	3.5907e-11	0.109079	1.2494
16	26	4.5219e-11	0.139981	1.4793
20	27	3.1658e-11	0.262055	1.7858
30	28	5.0170e-11	0.439142	2.872
64	30	8.4997e-11	5.319442	5.8014

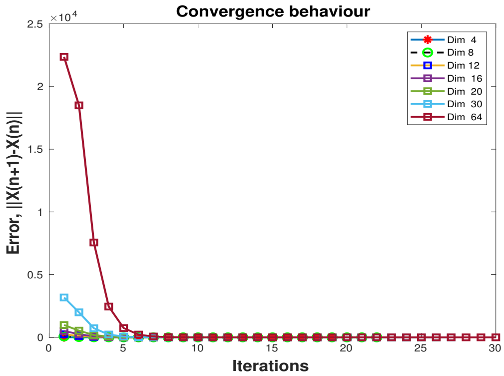

Figure 5. Iteration vs Error graph of the Example 5.7.

Definition 5.8. [11] For a complex matrix $\mathcal{X}$ to be positive definite if and only if the Hermitian portion $\mathcal{X}_H = \frac{1}{2}(\mathcal{X} + \mathcal{X}^*)$ be positive definite, where $\mathcal{X}^*$ denotes the conjugate transpose.

Using this above Definition 5.8, a new example is illustrated below:

Example 5.9. Consider a randomly generated complex coefficient matrices of the equation

$$\mathcal{X} = \mathcal{Q} + \mathcal{B}_1^* \mathcal{X}^{1/2} \mathcal{B}_1 + \mathcal{B}_2^* \mathcal{X}^{1/2} \mathcal{B}_2 + \mathcal{B}_3^* \mathcal{X}^{1/2} \mathcal{B}_3, \tag{5.7}$$

where the coefficients $\mathcal{B}_j$ $(j = 1, 2, 3)$ are chosen by $\mathcal{B}_j = rand(n) + i\ rand(n)$, $\mathcal{Q} = \mathcal{B}_1 \mathcal{B}_1^*$, $\tau = 0.01$. All the experimental data such as iteration number, cpu time, error shown are reported in the Table 3. The last column presents the minimum eigenvalue of the solution matrix to ensure the solution to be positive definite. The Figure 6 represents the convergence analysis of sequence for various dimensions.

Table 3. Numerical analysis for different dimension in Example 5.9.

Dimension	No. of Iteration	Error	CPU Time	Min. Eigen Value
5	43	8.6462e-11	0.116591	2.5393
8	47	7.8654e-11	0.213147	3.7129
10	43	5.9279e-11	0.303800	3.4470
16	52	7.8891e-11	0.664103	6.9888
20	54	6.7639e-11	1.213817	9.8417
22	128	9.8101e-11	3.262496	12.0391

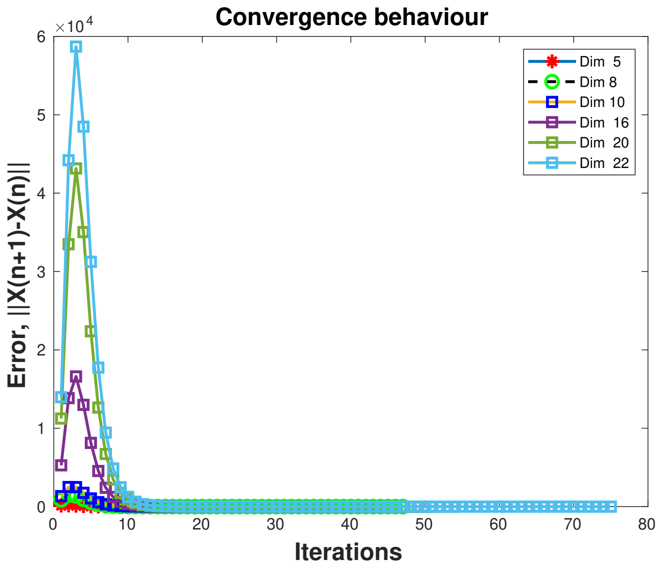

Figure 6. Iteration vs Error graph of the Example 5.9.

6. Conclusions

It is clear from the discussion in the Examples 5.5–5.7 and 5.9 that the solutions of NMEs are positive definite, as the lowest eigenvalues are positive for any starting matrices. Additionally, it is also clear from the solution's surface plot as it is pointing upward. Convergence analysis demonstrates that for any starting guess, it will convergence to a common value. Finally from Tables 2 and 3, it is obvious that result is true for real and complex coefficient matrices for any dimension.

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