Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 6234–6258.
DOI:10.3934/math.2022347
Received: 30 November 2021
Revised: 03 January 2022
Accepted: 12 January 2022
Published: 18 January 2022

Research article

Optimal decay rates of higher–order derivatives of solutions for the compressible nematic liquid crystal flows in $\mathbb{R}^3$

Zhengyan Luo, Lintao Ma*and Yinghui Zhang

School of Mathematics and Statistics, Guangxi Normal University, Guilin, Guangxi 541004, China

* **Correspondence:** Email: mlt@mailbox.gxnu.edu.cn; Tel: +86-817-833-0282.

Abstract: In this paper, we are concerned with optimal decay rates of higher–order derivatives of the smooth solutions to the $3D$ compressible nematic liquid crystal flows. The main novelty of this paper is three–fold: First, under the assumptions that the initial perturbation is small in H^N–norm ($N \geq 3$) and bounded in L^1–norm, we show that the highest–order spatial derivatives of density and velocity converge to zero at the L^2–rates is $(1+t)^{-\frac{3}{4}-\frac{N}{2}}$, which are the same as ones of the heat equation, and particularly faster than the L^2–rate $(1+t)^{-\frac{1}{4}-\frac{N}{2}}$ in [J.C. Gao, et al., J. Differential Equations, 261: 2334–2383,2016]. Second, if the initial data satisfies some additional low frequency assumption, we also establish the lower optimal decay rates of solution as well as its all–order spatial derivatives. Therefore, our decay rates are optimal in this sense. Third, we prove that the lower bound of the time derivatives of density, velocity and macroscopic average converge to zero at the L^2–rate is $(1+t)^{-\frac{5}{4}}$. Our method is based on low-frequency and high-frequency decomposition and energy methods.

Keywords: compressible nematic liquid crystal flows; higher-order derivatives; optimal decay rates
Mathematics Subject Classification: 35Q35, 35B40, 76A15

1. Introduction

In this paper, we investigate the upper and lower bounds of decay rates for global solution to compressible nematic liquid crystal flows in three–dimensional whole space:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu\Delta u - (\mu+\nu)\nabla \operatorname{div} u + \nabla P(\rho) = -\gamma \nabla d \cdot \Delta d, \\ d_t + u\cdot\nabla d = \theta(\Delta d + |\nabla d|^2 d), \end{cases} \tag{1.1}$$

where $t \geq 0$ is time and $x \in \mathbb{R}^3$ is the spatial coordinate. Here, the symbol $\otimes$ is the Kronecker tensor product. We denote the unknown functions $\rho(x,t)$ denotes the fluid density, $u = (u_1, u_2, u_3)$ is the fluid velocity, d is the macroscopic average of the nematic liquid crystal orientation field. The pressure $P = P(\rho) = a\rho^{\gamma}(a > 0, \gamma \geq 1)$ is a smooth function in a neighborhood of 1 with $P'(1) = 1$. μ and ν are shear viscosity and the bulk viscosity coefficients of the fluid, respectively, which satisfy the physical assumptions:

$$\mu > 0, \quad 2\mu + 3\nu \geq 0.$$

The positive constants γ and θ represent the competition between the kinetic energy and the potential energy, and the microscopic elastic relaxation time for the molecular orientation field, respectively. For the sake of simplicity, we set the constants γ and θ to be 1. We consider the Cauchy problem of the system (1.1) subject to the initial conditions:

$$(\rho, u, d)(x, t) \mid_{t=0} = (\rho_0, u_0, d_0)(x) \to (1, 0, \omega_0), \quad as \quad |x| \to \infty, \tag{1.2}$$

where ω_0 is a unit constant vector.

1.1. History of the problem

Liquid crystal has important physical and chemical properties such as photoelectric effect, thermal effect, photochemical effect and so on. After nearly a century of research, liquid crystals have been widely used in production, life, and scientific research. Particularly, liquid crystal displays have been widely used in LED technology, airplanes, medical equipment, bioengineering, machinery manufacturing, and other fields. Next, let us present some explanations about the above model. The nematic liquid crystal flows are a coupling between the compressible Navier–Stokes equations and the transported flow harmonic maps. Ericksen [4] and Leslie [20] first established the continuum theory of liquid crystals in the 1960s. Since then, due to the physical importance and mathematical challenges, the study of the full Ericksen–Leslie model has attracted many physicists and mathematicians. Considering the compressible liquid crystal flows, Ding–Lin–Wang [2] gained both existence and uniqueness of global strong solution in one–dimensional space. For the case of multi-dimensional space, Jiang–Jiang–Wang in [16, 17] proved the global existence of weak solutions to the initial–boundary problem with large initial energy. Huang–Wang–Wen in [13] obtained the blow up criterion of strong solutions. Hu–Wu [11] showed the existence and uniqueness of global strong solution in critical Besov spaces. Under the assumption that the initial energy is suitably small, Wu–Tan in [32] proved the global existence of small energy weak solution. Li–Xu–Zhang in [22] established the global existence of classical solution with smooth initial data which are of small energy but possibly large oscillations. Recently, Gao–Tao–Yao [8] obtained the global well–posedness of classical solution under the condition that the initial data is a small perturbation of the constant equilibrium state in the $H^N(\mathbb{R}^3)(N \geq 3)$–framework. Furthermore, if the initial perturbation data belongs to L^1 additionally, they obtained the optimal decay rate of k–th ($k \leq N - 1$) order spatial derivative of solution in [8] as follow:

$$\|\nabla^k(\rho - 1)(t)\|_{H^{N-k}} + \|\nabla^k u(t)\|_{H^{N-k}} \leq C(1 + t)^{-\frac{3}{4}-\frac{k}{2}}, \ \ k \in [0, N - 1], \tag{1.3}$$

$$\|\nabla^l(d - \omega_0)(t)\|_{L^2} \leq C(1 + t)^{-\frac{3}{4}-\frac{l}{2}}, \ \ l \in [0, N + 1]. \tag{1.4}$$

Here, the positive constant C is independent of time.

When the director is a constant vector, then the compressible nematic liquid crystal flow (1.1) becomes the compressible Navier–Stokes equation. There are a lot of basic results about the global existence, unique and time decay rates of the solutions to the compressible Navier–Stokes equations, cf. [3,6,7,10,14,23,24,28,29] and references cited therein. We also mention that the problem of time-asymptotic behavior for the solutions of hydrodynamic equations is a hot topic, see [5,12,15,18,30,31] and references cited therein.

By observing the results of [8], decay rate (1.3) shows that the highest–order spatial derivative of density and velocity converge to zero at L^2–rate $(1+t)^{-\frac{1}{4}-\frac{N}{2}}$, which are slower than the L^2–rate $(1+t)^{-\frac{3}{4}-\frac{N}{2}}$ for the heat equation, and thus are not optimal in this sense. Thus, we are caught up in the upper and lower optimal decay rates of higher–order spatial derivatives of the solutions to the 3D compressible liquid crystal flows (1.1). More precisely, we focus on the following three problems:

(i) Can we show that the highest order spatial derivative of the density and velocity converge to zero at the same L^2–rate $(1+t)^{-\frac{3}{4}-\frac{N}{2}}$ as that of the heat equation?

(ii) Can we provide some lower bounds of decay rate for the solution as well as its all–order spatial derivatives?

(iii) Can we provide some information on the upper and lower bound of decay rate for the time derivatives of solution?

The main purpose of this article is to give a clear answer to the above three problems.

1.2. Main results

In this article, we use $H^k(\mathbb{R}^3)$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^k}$. Generally, we use L^p, $1 \le p \le \infty$ to denote the usual $L^p(\mathbb{R}^3)$ spaces with norm $\|\cdot\|_{L^p}$. The notation $a \lesssim b$ means that $a \le Cb$ for a universal positive constant which is independent of time t. Let Λ^s be the pseudo–differential operator defined by

$$\Lambda^s f = \mathfrak{F}^{-1}(|\xi|^s \widehat{f}) \quad \text{for } s \in \mathbb{R},$$

where $\widehat{f}$ and $\mathfrak{F}$ are the Fourier transform of f. For a radial function $\phi \in C_0^\infty(\mathbb{R}^3_\xi)$ such that $\phi(\xi) = 1$ when $|\xi| \le 1$ and $\phi(\xi) = 0$ when $|\xi| \ge 2$, we define the low–frequency part and the high–frequency part of f as follows

$$f^l = \mathfrak{F}^{-1}[\phi(\xi)\widehat{f}], \text{ and } f^h = \mathfrak{F}^{-1}[(1-\phi(\xi))\widehat{f}].$$

Before stating our main results, let us recall the result of [8] in the following.

Theorem 1.1.(See [8]) Assume that the initial data $(\rho_0 - 1, u_0, \nabla d_0) \in H^N(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ for any integer $N \ge 3$, $d_0(x) = 1$ in $\mathbb{R}^3$ and there exists a small constant $\delta_0 > 0$ such that

$$\|(\rho_0 - 1, u_0, \nabla d_0)\|_{H^3} \le \delta_0, \tag{1.5}$$

then the Cauchy problem (1.1)–(1.2) admits a unique globally classical solution (ρ, u, d) such that for any $t \in [0, \infty)$,

$$\|(\rho - 1, u, \nabla d)(t)\|_{H^N}^2 + \int_0^t (\|\nabla\rho\|_{H^{N-1}}^2 + \|(\nabla u, \nabla^2 d)\|_{H^N}^2) \mathrm{d}\tau \le C\|(\rho_0 - 1, u_0, \nabla d_0)\|_{H^N}^2. \tag{1.6}$$

If the initial data $\|d_0 - \omega_0\|_{L^2}$ and $\|(\rho_0 - 1, u_0, \nabla d_0)\|_{L^1}$ are finite additionally, the global solution (ρ, u, d) of problem (1.1)–(1.2) satisfies for all $t \ge 0$

$$\begin{aligned} \|\nabla^k(\rho - 1)(t)\|_{H^{N-k}} + \|\nabla^k u(t)\|_{H^{N-k}} &\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad k \in [0, N-1], \\ \|\nabla^l(d - \omega_0)(t)\|_{L^2} &\le C(1+t)^{-\frac{3}{4}-\frac{l}{2}}, \quad l \in [0, N+1]. \end{aligned} \tag{1.7}$$

Here, the positive constant C is independent of time.

Now, we are in a position to state our main results, which are stated in the following four theorems. First, we show that the highest–order spatial derivative of the density and velocity converge to zero at the same L^2–rate as ones of the heat equation.

Theorem 1.2. Under the assumptions of Theorem 1.1, for all $t \geq 0$, it holds that

$$\|\nabla^N(\rho-1)(t)\|_{L^2} + \|\nabla^N u(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{N}{2}}. \tag{1.8}$$

Here, the positive constant C is independent of time.

Next, we address the lower bound of decay rate for the solution as well as its all–order spatial derivatives.

Theorem 1.3. Assume that all the hypotheses of Theorem 1.1 are in force, denote $m_0 = \rho_0 u_0$, $b_0 = \Lambda d_0$, and the Fourier transform $(\widehat{\rho_0-1}, \widehat{m_0}, \widehat{b_0})$ satisfies

$$|\widehat{\rho_0-1}(\xi)| \geq c_0, \quad |\widehat{m_0}(\xi)| = 0, \quad |\widehat{b_0}(\xi)| \geq c_0, \quad for\ \ 0 \leq |\xi| \ll 1, \tag{1.9}$$

where c_0 is a positive constant. Then, the global solution (ρ, u, d) has the decay rates for all $t \geq t_*$

$$\begin{aligned} \min\{\|\nabla^k(\rho-1)(t)\|_{L^2}, \|\nabla^k u(t)\|_{L^2}\} &\geq c_1(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \ \ k \in [0, N], \\ \|\nabla^{l+1}(d-\omega_0)(t)\|_{L^2} &\geq c_2(1+t)^{-\frac{3}{4}-\frac{l}{2}}, \ \ l \in [0, N]. \end{aligned} \tag{1.10}$$

Here, t_* is a positive large time, the two positive constants c_1 and c_2 are independent of time.

Next, we will establish the upper bounds of decay rates for the time derivatives of the solution to the 3D compressible liquid crystal flows (1.1).

Theorem 1.4. Under the assumptions of Theorem 1.1 and Theorem 1.2, the global solution (ρ, u, d) of Cauchy problem (1.1)–(1.2) satisfies for all $t \geq 0$

$$\begin{aligned} \|\nabla^k \partial_t u(t)\|_{L^2} &\leq C(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \ \ k \in [0, N-2], \\ \|\nabla^l \partial_t \rho(t)\|_{L^2} &\leq C(1+t)^{-\frac{5}{4}-\frac{l}{2}}, \\ \|\nabla^l \partial_t (d-\omega_0)(t)\|_{L^2} &\leq C(1+t)^{-\frac{7}{4}-\frac{l}{2}}, \ \ l \in [0, N-1]. \end{aligned} \tag{1.11}$$

Here, the positive constant C is independent of time.

Finally, we will establish the lower bounds of decay rates for the time derivatives of the solution to the 3D compressible liquid crystal flows (1.1).

Theorem 1.5. Under all the assumptions in Theorem 1.1, and the condition (1.9) holds, then the global solution (ρ, u, d) of Cauchy problem (1.1)–(1.2) satisfies for all $t \geq t_*$

$$\begin{aligned} \|\partial_t u(t)\|_{L^2} &\geq c_3(1+t)^{-\frac{5}{4}}, \\ \|\partial_t (d-\omega_0)(t)\|_{L^2} &\geq c_3(1+t)^{-\frac{5}{4}}. \end{aligned} \tag{1.12}$$

Furthermore, if there exists a small constant δ_1 such that $\|u_0\|_{L^1} \le \delta_1$, it holds that for all $t \ge t_*$

$$\min\{\|\partial_t \varrho(t)\|_{L^2}, \|\mathrm{div} u(t)\|_{L^2}\} \ge c_3(1+t)^{-\frac{5}{4}}. \tag{1.13}$$

Here, t_* is a positive large time, and the positive constant c_3 is independent of time.

Remark 1.6. Compared to Theorem 1.1 of [8], the main innovation of Theorem 1.2–1.5 lies in the following three aspects: First, by observing the decay rates in (1.8), we prove that the highest–order spatial derivative of the density and velocity converge to their corresponding equilibrium states at the L^2–rate $(1+t)^{-\frac{3}{4}-\frac{N}{2}}$, which is the same as one of the heat equation and particularly faster than the rates $(1+t)^{-\frac{1}{4}-\frac{N}{2}}$ in [8]. Second, for well–chosen initial data, Theorem 1.3 also gives the lower bounds on solution as well as its all–order spatial derivatives. Thus, our time decay rates are really optimal in this sense. Third, we also gives the lower bound of decay rates for the time derivatives of density, velocity and macroscopic average for the $3D$ compressible liquid crystal flows, which converge to zero at the L^2–rates $(1+t)^{-\frac{5}{4}}$, $(1+t)^{-\frac{5}{4}}$ and $(1+t)^{-\frac{5}{4}}$, respectively.

Now, let us sketch the main strategy of proving Theorem 1.2–Theorem 1.5 and explain some main difficulties and techniques involved in the process. Roughly speaking, we will make full use of the benefit of the low–frequency and high–frequency decomposition $f = f^l + f^h$, where f^l and f^h stand for the low–frequency part and high–frequency part of f, respectively.

For the proof of Theorem 1.2, motivated by the work in [27], we will establish the highest–order spatial derivatives of the density and velocity to compressible nematic liquid crystal flows (1.1). There are four steps to achieve this. First, we rewrite the Cauchy problem (1.1)–(1.2) into the system (2.1). We notice that the low-frequency part $\nabla^k(\varrho, u)(0 \le k \le N)$ of the corresponding linear system to (2.1) has been obtained by [21]. Then, by employing Duhamel's principle, the key linear decay estimates in Lemma 3.1 and nonlinear energy estimates, we can get the optimal decay rate of $\|\nabla^N(\varrho^l, u^l)(t)\|_{L^2}$ (see the proof of Lemma 3.3 for details). Second, we want to obtain the optimal decay rate of $\|\nabla^N(\varrho^h, u^h)(t)\|_{L^2}$. Through high–frequency and low–frequency decomposition and precise energy estimates, we establish the energy inequality as follows:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |\nabla^N \varrho^h|^2 + |\nabla^N u^h|^2 \mathrm{d}x + C_1\|\nabla^{N+1}u^h\|_{L^2}^2 &\lesssim (1+t)^{-\frac{3}{2}-N} \\ &+(\delta + (1+t)^{-\frac{3}{2}})\|\nabla^{N+1}u\|_{L^2}^2 + \delta\|\nabla^N(\varrho, u)\|_{L^2}^2.\end{aligned} \tag{1.14}$$

Third, note that the energy equality (1.14) only gives the dissipation estimate for u^h. In order to explore the dissipation estimates for ϱ^h, we will construct the new interactive energy functionals between u^h and ϱ^h. Therefore, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla^{N-1}u^h \nabla^N \varrho^h \mathrm{d}x + C_2\|\nabla^N \varrho^h\|_{L^2}^2 &\lesssim (1+t)^{-\frac{3}{2}-N} + (\delta + (1+t)^{-\frac{3}{2}})\|\nabla^N \varrho\|_{L^2}^2 \\ &+(1 + (1+t)^{-\frac{3}{2}})\|\nabla^N u\|_{L^2}^2 + (1+\delta)\|\nabla^{N+1}u\|_{L^2}^2.\end{aligned} \tag{1.15}$$

Fourth, we choose two sufficiently large positive constant D_0 and T_0, then define the temporal energy functional as

$$\mathbf{E}(t) = D_0\|\nabla^N(\varrho^h, u^h)\|_{L^2}^2 + \int_{\mathbb{R}^3} \nabla^{N-1}u^h \nabla^N \varrho^h \mathrm{d}x. \tag{1.16}$$

Notice that $\mathbf{E}(t)$ is equivalent to $\|\nabla^N(\varrho^h, u^h)\|_{L^2}^2$. Multiplying (1.14) by D_0, adding the resulting inequality with (1.15), and then for all $t \geq T_0$, we obtain the Lyapunov-type energy inequality as follow:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(t) + C_3\mathbf{E}(t) \lesssim (1+t)^{-\frac{3}{2}-N} + \|\nabla^N \varrho^l\|_{L^2}^2 + \|\nabla^N u^l\|_{H^1}^2. \tag{1.17}$$

By virtue of Lemma 3.3 and Gronwall's inequality, we obtain the optimal L^2 time decay rates of $\|\nabla^N(\varrho^h, u^h)\|_{L^2}^2$. In addition, together with the N–order low–frequency decay rates of (3.7), we obtain the decay rates of (1.8) immediately. Thus, we have completed the proof of Theorem 1.2.

For Theorem 1.3, we show the lower bounds on the decay rates of solutions. Compared to the proof of Theorem 1.2, the new difficulty we encounter is that the system (2.1) is not a conservative form, which implies that it seems impossible to obtain the lower optimal decay rates as in (1.10). To this end, let's break it down into two parts. For the part of Navier–stokes system, the key idea here is that, instead of using the system $(2.1)_1$–$(2.1)_2$, we will employ the system (4.1) of (ϱ, m) with $m = (\varrho+1)u$, which can be rewritten in the conservative form. As a result, one can shift the derivative onto the solution semigroup to obtain the desired lower optimal decay rates (See the proof of (4.4) for details). For the part of macroscopic average, we notice that the linearized system of (4.6) is mere heat equation on ∇n. When the initial data satisfy (1.9), we will employ Plancherel theorem and careful analysis on the solution semigroup to obtain an optimal lower bound estimate for the linear part. We find a similar structure between (4.6) and the system (4.1), so we can prove the lower bound on the convergence rates in L^2–norm for the macroscopic average(see the proof of Lemma 4.2 for details). Thus, we complete the proof of Theorem 1.3.

For Theorem 1.4 and Theorem 1.5, we establish the upper and lower bound of decay rate for the time derivative of solution in L^2–norm. It is worth mentioning that the lower decay rate estimate for the time derivative of solution can be obtained which is inspired by the work of Gao–Lyu–yao in [9]. However, the lower bound of decay estimates in [9] are established for the compressible fluid model of Korteweg type.

2. Reformulation

In this section, we will reformulate the problem firstly. Set $\varrho = \rho - 1$ and $n = d - \omega_0$, the Cauchy problem (1.1)–(1.2) can be reformulated into:

$$\begin{cases} \varrho_t + \mathrm{div}(u) = S_1, \\ u_t - \mu\Delta u - (\mu+\nu)\nabla\mathrm{div}u + \nabla\varrho = S_2, \\ n_t - \Delta n = S_3, \\ (\varrho, u, n)(x,t)\mid_{t=0} = (\varrho_0, u_0, n_0)(x) \to (0,0,0), \quad as\ |x| \to \infty. \end{cases} \tag{2.1}$$

Here $S_i (i = 1, 2, 3)$ are defined as

$$\begin{aligned} S_1 &:= -\varrho\mathrm{div}u - u\cdot\nabla\varrho, \\ S_2 &:= -u\cdot\nabla u - h(\varrho)[\mu\Delta u + (\mu+\nu)\nabla\mathrm{div}u] - f(\varrho)\nabla\varrho - g(\varrho)\nabla n\cdot\Delta n, \\ S_3 &:= -u\cdot\nabla n + |\nabla n|^2(n+\omega_0), \end{aligned} \tag{2.2}$$

where the three nonlinear functions of ϱ are defined by

$$h(\varrho) := \frac{\varrho}{\varrho+1}, \quad f(\varrho) := \frac{P'(\varrho+1)}{\varrho+1} - 1, \quad and \quad g(\varrho) := \frac{1}{\varrho+1}. \tag{2.3}$$

Assume there exists a small positive constant δ satisfying following estimate

$$\|(\varrho, u, \nabla n)(t)\|_{H^3} \leq \delta, \tag{2.4}$$

for all $t \in [0, T]$. By virtue of (2.4) and Sobolev inequality, it is easy to get

$$\frac{1}{2} \leq \varrho + 1 \leq \frac{3}{2}.$$

Hence, we immediately have

$$|h(\varrho)|, |f(\varrho)| \leq C|\varrho| \quad and \quad |g^{k-1}(\varrho)|, |h^k(\varrho)|, |f^k(\varrho)| \leq C \quad \forall k \geq 1. \tag{2.5}$$

And, for the nonlinear terms of the model (2.1), employing the Hölder's inequality, we obtain

$$\begin{aligned} \|(S_1, S_2)\|_{L^1} \leq& (\|\varrho\|_{L^2} + \|u\|_{L^2} + \|\nabla n\|_{L^2})(\|\nabla\varrho\|_{L^2} + \|\nabla u\|_{H^1} + \|\nabla n\|_{H^1}) \\ \lesssim& \delta(\|\nabla\varrho\|_{L^2} + \|\nabla u\|_{H^1} + \|\nabla n\|_{H^1}). \end{aligned} \tag{2.6}$$

3. Proof of Theorem 1.2

Let us to consider the following linearized compressible nematic liquid crystal system:

$$\begin{cases} \varrho_t + \mathrm{div}(u) = 0, \\ u_t - \mu\Delta u - (\mu+\nu)\nabla \mathrm{div} u + \nabla\varrho = 0, \\ \nabla n_t - \Delta\nabla n = 0, \end{cases} \tag{3.1}$$

with the initial conditions:

$$(\varrho, u, \nabla n)(x,t)\mid_{t=0} = (\varrho_0, u_0, \nabla n_0)(x) \to (0,0,0), \quad as \ |x| \to \infty.$$

Since the system (3.1) is a decoupled system of the classical linearized Navier–Stokes equations and heat equations. If we set

$$U(t) = (\bar{\varrho}(t), \bar{u}(t))^t, \qquad U(0) = (\bar{\varrho}(0), \bar{u}(0))^t.$$

Then the solution to $(3.1)_1$–$(3.1)_2$ can be written as

$$U(t) = e^{-t\mathcal{A}} U(0), \tag{3.2}$$

where $\mathcal{A}$ is a matrix–valued differential operators of the form

$$\mathcal{A} = \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & -\mu\Delta - (\mu+\nu)\nabla\mathrm{div} \end{pmatrix}.$$

The solution semigroup $e^{-\mathcal{A}t}$ has the following lemma on the decay in time.

Lemma 3.1. *Let $k \geq -\frac{3}{2}$ and $2 \leq r \leq \infty$, then for any $t \geq 0$, it holds that*

$$\|\nabla^k e^{-t\mathcal{A}} U^l(0)\|_{L^r} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \|U(0)\|_{L^1}. \tag{3.3}$$

Moreover, if initial data satisfy (1.9)*, then there exists a positive large time t_* such that for all $t \geq t_*$, we have*

$$min\{\|\nabla^k \bar{\varrho}^l(t)\|_{L^2}, \|\nabla^k \bar{m}^l(t)\|_{L^2}\} \geq c_4(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{3.4}$$

for $k = 0, 1$. Here, the positive constant c_4 is independent of time.

Proof. The proof can be seen in [21]. □

To treat the macroscopic average, we notice that the Eq $(3.1)_3$ is a mere heat equation on ∇n. We state the large-time behavior of solutions to the heat equation as the following lemma which can be obtained by direct calculation or more can refer to [26, 33].

Lemma 3.2. *Let $k \geq -\frac{3}{2}$ and $2 \leq r \leq \infty$, then for any $t \geq 0$, it holds that*

$$\|\nabla^k (\nabla \bar{n})^l(t)\|_{L^r} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \|\nabla \bar{n}(0)\|_{L^1}. \tag{3.5}$$

Moreover, if initial data satisfy (1.9)*, then there exists a positive large time t_*, such that for all $t \geq t_*$, we have*

$$\|\nabla \bar{n}^l(t)\|_{L^2} \geq c_4(1+t)^{-\frac{3}{4}}. \tag{3.6}$$

Here, the positive constant c_4 is independent of time.

Second, we state the L^2–time decay rates on the low–frequency part of the solution in the nonlinear system (2.1).

Lemma 3.3. *Under the assumptions of Theorem 1.1, then solution (ϱ, u) of the nonlinear system* (2.1) *satisfies the following decay rates:*

$$\|\nabla^N(\varrho^l, u^l)(t)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{N}{2}}. \tag{3.7}$$

Proof. We write the nonlinear terms $S := (\,S_1,\ S_2)^T$. By virtue of the Eq $(2.1)_1$–$(2.1)_2$, Lemma 3.1, Duhamel's principle, Plancherel theorem and Hausdorff–Young's inequality, we have

$$\begin{aligned}\|\nabla^N(\varrho^l, u^l)(t)\|_{L^2} \lesssim\ & (1+t)^{-\frac{3}{4}-\frac{N}{2}} \|(\varrho, u)(0)\|_{L^1} + \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{N}{2}} \|\mathcal{S}(\tau)\|_{L^1} \mathrm{d}\tau \\ & + \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{5}{4}} \left\| |\xi|^{N-1} \widehat{\mathcal{S}^l}(\tau) \right\|_{L^\infty} \mathrm{d}\tau.\end{aligned} \tag{3.8}$$

On the other hand, by employing decay rate (1.7), Lemma 7.1, Lemma 7.3 and Hölder's inequality, we can bound the third term on the right–hand of (3.8) as follows

$$
\begin{aligned}
\left\||\xi|^{N-1}\widehat{S^l}(\tau)\right\|_{L^\infty} \lesssim& \|\nabla^{N-2}S(t)^l\|_{L^1}\\
\lesssim& \|\nabla^{N-2}(\varrho\nabla\cdot u)\|_{L^1}+\|\nabla^{N-2}(\nabla\varrho\cdot u)\|_{L^1}+\|\nabla^{N-2}(u\cdot\nabla u)\|_{L^1}\\
&+\|\nabla^{N-2}(h(\varrho)\Delta u)\|_{L^1}+\|\nabla^{N-2}(h(\varrho)\nabla\mathrm{div}u)\|_{L^1}+\|\nabla^{N-2}(f(\varrho)\nabla\varrho)\|_{L^1}\\
&+\|\nabla^{N-2}(g(\varrho)\nabla n\cdot\Delta n)\|_{L^1}\\
\lesssim& \|\nabla(\varrho,u)\|_{L^2}\|\nabla^{N-2}(\varrho,u)\|_{L^2}+\|(\varrho,u)\|_{L^2}\|\nabla^{N-1}(\varrho,u)\|_{L^2}\\
&+\|\varrho\|_{L^2}\|\nabla^N u\|_{L^2}+\|\nabla^2 u\|_{L^2}\|\nabla^{N-2}\varrho\|_{L^2}\\
&+\|\varrho\|_{L^2}\|\nabla^{N-2}(\nabla n\cdot\Delta n)\|_{L^2}+\|\nabla n\cdot\Delta n\|_{L^2}\|\nabla^{N-2}\varrho\|_{L^2}\\
\lesssim& (1+t)^{-\frac{5}{4}}(1+t)^{-\frac{2N-1}{4}}+(1+t)^{-\frac{3}{4}}(1+t)^{-\frac{2N+1}{4}}+(1+t)^{-\frac{3}{4}}(1+t)^{-\frac{2N+3}{4}}\\
&+(1+t)^{-\frac{7}{4}}(1+t)^{-\frac{2N-1}{4}}+(1+t)^{-\frac{3}{4}}(1+t)^{-\frac{5+N}{2}}+(1+t)^{-\frac{7}{2}}(1+t)^{-\frac{2N-1}{4}}\\
\lesssim& (1+t)^{-1-\frac{N}{2}}.
\end{aligned}
\tag{3.9}
$$

Here, for the term $\|\nabla n\cdot\Delta n\|_{L^2}$, we have

$$
\begin{aligned}
\|\nabla n\cdot\Delta n\|_{L^2} &\lesssim \|\nabla n\|_{L^3}\|\Delta n\|_{L^6}\lesssim\|\nabla n\|_{H^1}\|\nabla^3 n\|_{L^2}\\
&\lesssim (1+t)^{-\frac{5}{4}}(1+t)^{-\frac{9}{4}}\lesssim(1+t)^{-\frac{7}{2}}.
\end{aligned}
\tag{3.10}
$$

For the term $\|\nabla^{N-2}(\nabla n\cdot\Delta n)\|_{L^2}$, we get

$$
\begin{aligned}
\|\nabla^{N-2}(\nabla n\cdot\Delta n)\|_{L^2} &\lesssim \|\nabla n\|_{L^\infty}\|\nabla^N n\|_{L^2}+\|\Delta n\|_{L^3}\|\nabla^{N-1}n\|_{L^6}\\
&\lesssim \|\nabla^2 n\|_{H^1}\|\nabla^N n\|_{L^2}\\
&\lesssim (1+t)^{-\frac{7}{4}}(1+t)^{-\frac{3+2N}{4}}\\
&\lesssim (1+t)^{-\frac{5}{2}-\frac{N}{2}}.
\end{aligned}
\tag{3.11}
$$

Substituting (3.9) and (2.6) into (3.8), we can get the follow estimates

$$
\begin{aligned}
\|\nabla^N(\varrho^l,u^l)(t)\|_{L^2} &\lesssim C(1+t)^{-\frac{3}{4}-\frac{N}{2}}+\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{5}{4}}(1+t)^{-1-\frac{N}{2}}\mathrm{d}\tau.\\
&\lesssim (1+t)^{-\frac{3}{4}-\frac{N}{2}}.
\end{aligned}
\tag{3.12}
$$

Hence, we complete the proof of this lemma. □

Third, we will give the energy estimates which contains the dissipation of $\nabla^N u^h$.

Lemma 3.4. *Under the assumptions of Theorem 1.1, then we have*

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla^N\varrho^h|^2+|\nabla^N u^h|^2\mathrm{d}x+C_1\|\nabla^{N+1}u^h\|_{L^2}^2 &\lesssim (1+t)^{-\frac{3}{2}-N}\\
&+(\delta+(1+t)^{-\frac{3}{2}})\|\nabla^{N+1}u\|_{L^2}^2+\delta\|\nabla^N(\varrho,u)\|_{L^2}^2,
\end{aligned}
\tag{3.13}
$$

for any $t\in[0,\infty)$.

Proof. Taking

$$\langle \mathfrak{F}^{-1}(1-\phi(\xi))\nabla^N(2.1)_1, \nabla^N\varrho^h\rangle + \langle \mathfrak{F}^{-1}(1-\phi(\xi))\nabla^N(2.1)_2, \nabla^N u^h\rangle, \tag{3.14}$$

and using integration by parts, we can obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla^N\varrho^h|^2+|\nabla^N u^h|^2\mathrm{d}x+(2\mu+\nu)\|\nabla^{N+1}u^h\|_{L^2}^2\\ &=\langle\nabla^N S_1^h,\nabla^N\varrho^h\rangle+\langle\nabla^N S_2^h,\nabla^N u^h\rangle=I_1+I_2.\end{aligned} \tag{3.15}$$

The right-hand side of (3.15) can be estimated one by one. For the term I_1, it holds that

$$\begin{aligned}I_1 &= \langle\nabla^N S_1^h,\nabla^N\varrho^h\rangle\\ &= -\langle\nabla^N(u\cdot\nabla\varrho)^h,\nabla^N\varrho^h\rangle-\langle\nabla^N(\varrho\nabla\cdot u)^h,\nabla^N\varrho^h\rangle\\ &= I_{11}+I_{12}.\end{aligned} \tag{3.16}$$

The first term I_{11} can be rewritten as follows

$$\begin{aligned}I_{11} &= -\langle\nabla^N(u\cdot\nabla\varrho)^h,\nabla^N\varrho^h\rangle\\ &= -\langle\nabla^N(u\cdot\nabla\varrho)-\nabla^N(u\nabla\varrho)^l,\nabla^N\varrho^h\rangle\\ &= -\langle\nabla^N(u\cdot\nabla\varrho^h)+\nabla^N(u\cdot\nabla\varrho^l)-\nabla^N(u\cdot\nabla\varrho)^l,\nabla^N\varrho^h\rangle\\ &= I_{111}+I_{112}+I_{113}.\end{aligned} \tag{3.17}$$

For the term I_{111}, due to Lemma 7.4, Hölder's inequality, Young's inequality and Sobolev interpolation theorem, we arrive at

$$\begin{aligned}|I_{111}| &\le |\langle u\nabla^{N+1}\varrho^h,\nabla^N\varrho^h\rangle|+|\langle[\nabla^N,u]\nabla\varrho^h,\nabla^N\varrho^h\rangle|\\ &\lesssim \frac{1}{2}\langle \mathrm{div}u,(\nabla^N\varrho^h)^2\rangle+\|[\nabla^N,u]\nabla\varrho^h\|_{L^2}\|\nabla^N\varrho^h\|_{L^2}\\ &\lesssim \|\nabla u\|_{L^\infty}\|\nabla^N\varrho\|_{L^2}^2+(\|\nabla u\|_{L^\infty}\|\nabla^N\varrho^h\|_{L^2}+\|\nabla^N u\|_{L^2}\|\nabla\varrho^h\|_{L^\infty})\|\nabla^N\varrho\|_{L^2}^2\\ &\lesssim \|\nabla^2 u\|_{H^1}\|\nabla^N\varrho\|_{L^2}^2+(\|\nabla^2 u\|_{H^1}\|\nabla^N\varrho^h\|_{L^2}+\|\nabla^2\varrho\|_{H^1}\|\nabla^N u\|_{L^2})\|\nabla^N\varrho\|_{L^2}^2\\ &\lesssim \delta\|\nabla^N(\varrho,u)\|_{L^2}^2.\end{aligned} \tag{3.18}$$

Here we have defined the commutator:

$$[\nabla^N,u]\nabla\varrho^h=\nabla^N(u\cdot\nabla\varrho^h)-u\cdot\nabla^{N+1}\varrho^h.$$

For the term I_{112}, we can obtain

$$\begin{aligned}|I_{112}| &= |\langle\nabla^N(u\cdot\nabla\varrho^l),\nabla^N\varrho^h\rangle|\\ &\lesssim \|\nabla^N(u\cdot\nabla\varrho^l)\|_{L^2}\|\nabla^N\varrho^h\|_{L^2}\\ &\lesssim (\|u\|_{L^\infty}\|\nabla^{N+1}\varrho^l\|_{L^2}+\|\nabla\varrho^l\|_{L^3}\|\nabla^N u\|_{L^6})\|\nabla^N\varrho^h\|_{L^2}\\ &\lesssim (\|u\|_{H^2}\|\nabla^N\varrho\|_{L^2}+\|\nabla\varrho\|_{H^1}\|\nabla^{N+1}u\|_{L^2})\|\nabla^N\varrho^h\|_{L^2}\\ &\lesssim \delta\|\nabla^{N+1}u\|_{L^2}^2+\delta\|\nabla^N\varrho\|_{L^2}^2.\end{aligned} \tag{3.19}$$

Similarly, it is easy to see that

$$
\begin{aligned}
|I_{113}| =&|\langle\nabla^N(u\cdot\nabla\varrho)^l,\nabla^N\varrho^h\rangle|\\
\lesssim&\|\nabla^{N-1}(u\cdot\nabla\varrho)\|_{L^2}\|\nabla^N\varrho^h\|_{L^2}\\
\lesssim&(\|u\|_{L^\infty}\|\nabla^N\varrho\|_{L^2}+\|\nabla\varrho\|_{L^3}\|\nabla^{N-1}u\|_{L^6})\|\nabla^N\varrho^h\|_{L^2}\\
\lesssim&(\|u\|_{H^2}\|\nabla^N\varrho\|_{L^2}+\|\nabla\varrho\|_{H^1}\|\nabla^N u\|_{L^2})\|\nabla^N\varrho^h\|_{L^2}\\
\lesssim&\ \delta\|\nabla^N(\varrho,u)\|_{L^2}^2.
\end{aligned}
\tag{3.20}
$$

Substituting (3.18)–(3.20) into (3.17), we can conclude that

$$
|I_{11}|\lesssim\delta\|\nabla^N u\|_{H^1}^2+\delta\|\nabla^N\varrho\|_{L^2}^2. \tag{3.21}
$$

For the term I_{12}, by using the Lemma 7.3, Hölder's inequality, Young's inequality, Sobolev interpolation theorem, we have

$$
\begin{aligned}
|I_{12}| \lesssim&\|\nabla^N(\varrho\nabla\cdot u)^h\|_{L^2}\|\nabla^N\varrho^h\|_{L^2}\\
\lesssim&\|\nabla^N(\varrho\nabla\cdot u)\|_{L^2}\|\nabla^N\varrho^h\|_{L^2}\\
\lesssim&(\|\varrho\|_{L^\infty}\|\nabla^N\nabla\cdot u\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla^N\varrho\|_{L^2})\|\nabla^N\varrho^h\|_{L^2}\\
\lesssim&(\|\varrho\|_{H^2}\|\nabla^{N+1}u\|_{L^2}+\|\nabla u\|_{H^2}\|\nabla^N\varrho\|_{L^2})\|\nabla^N\varrho^h\|_{L^2}\\
\lesssim&\delta\|\nabla^N\varrho\|_{L^2}^2+\delta\|\nabla^{N+1}u\|_{L^2}^2.
\end{aligned}
\tag{3.22}
$$

For the term I_2, it holds that

$$
\begin{aligned}
I_2 =&\langle\nabla^N(u\cdot\nabla u)^h,\nabla^N u^h\rangle+\langle\nabla^N(h(\varrho)\Delta u)^h,\nabla^N u^h\rangle+\langle\nabla^N(h(\varrho)\nabla(\mathrm{div}u))^h,\nabla^N u^h\rangle\\
&+\langle\nabla^N(f(\varrho)\nabla\varrho)^h,\nabla^N u^h\rangle+\langle\nabla^N(g(\varrho)\nabla n\cdot\Delta n)^h,\nabla^N u^h\rangle=\sum_{i=1}^5 I_{2i}.
\end{aligned}
\tag{3.23}
$$

For the term I_{21}, making use of integration by parts, we have

$$
\begin{aligned}
|I_{21}| =&|\langle\nabla^{N-1}(u\cdot\nabla u)^h,\nabla^N\nabla\cdot u^h\rangle|\\
\lesssim&\|\nabla^{N-1}(u\cdot\nabla u)\|_{L^2}\|\nabla^N\nabla\cdot u^h\|_{L^2}\\
\lesssim&(\|u\|_{L^\infty}\|\nabla^N u\|_{L^2}+\|\nabla u\|_{L^3}\|\nabla^{N-1}u\|_{L^6})\|\nabla^N\nabla\cdot u^h\|_{L^2}\\
\lesssim&\|\nabla u\|_{H^1}\|\nabla^N u\|_{L^2}\|\nabla^N\nabla\cdot u\|_{L^2}\\
\lesssim&(1+t)^{-\frac54}(1+t)^{-\frac14-\frac N2}\|\nabla^{N+1}u\|_{L^2}\\
\lesssim&(1+t)^{-\frac32-N}+(1+t)^{-\frac32}\|\nabla^{N+1}u\|_{L^2}^2.
\end{aligned}
\tag{3.24}
$$

For the term I_{22}, we obtain

$$
\begin{aligned}
|I_{22}| =&|\langle\nabla^{N-1}(h(\varrho)\Delta u)^h,\nabla^N\nabla\cdot u^h\rangle|\\
\lesssim&\|\nabla^{N-1}(h(\varrho)\Delta u)\|_{L^2}\|\nabla^N\nabla\cdot u^h\|_{L^2}\\
\lesssim&(\|\rho\|_{L^\infty}\|\nabla^{N+1}u\|_{L^2}+\|\nabla^2 u\|_{L^3}\|\nabla^{N-1}\varrho\|_{L^6})\|\nabla^N\nabla\cdot u^h\|_{L^2}\\
\lesssim&(\|\nabla\varrho\|_{H^1}\|\nabla^{N+1}u\|_{L^2}+\|\nabla^2 u\|_{H^1}\|\nabla^N\varrho\|_{L^2})\|\nabla^N\nabla\cdot u^h\|_{L^2}\\
\lesssim&\delta\|\nabla^{N+1}u\|_{L^2}^2+(1+t)^{-\frac74}(1+t)^{-\frac14-\frac N2}\|\nabla^{N+1}u\|_{L^2}\\
\lesssim&(1+t)^{-\frac32-N}+(\delta+(1+t)^{-\frac52})\|\nabla^{N+1}u\|_{L^2}^2.
\end{aligned}
\tag{3.25}
$$

Similarly, it is easy to see that

$$
\begin{aligned}
|I_{23}| =&|\langle\nabla^{N-1}(h(\varrho)\nabla(\mathrm{div}u))^h, \nabla^N\nabla\cdot u^h\rangle| \\
\lesssim&\|\nabla^{N-1}(h(\varrho)\nabla(\mathrm{div}u))\|_{L^2}\|\nabla^N\nabla\cdot u^h\|_{L^2} \\
\lesssim&(\|\rho\|_{L^\infty}\|\nabla^{N+1}u\|_{L^2} + \|\nabla^2 u\|_{L^3}\|\nabla^{N-1}\varrho\|_{L^6})\|\nabla^N\nabla\cdot u^h\|_{L^2} \\
\lesssim&(\|\nabla\varrho\|_{H^1}\|\nabla^{N+1}u\|_{L^2} + \|\nabla^2 u\|_{H^1}\|\nabla^N\varrho\|_{L^2})\|\nabla^N\nabla\cdot u^h\|_{L^2} \\
\lesssim&\delta\|\nabla^{N+1}u\|_{L^2}^2 + (1+t)^{-\frac{7}{4}}(1+t)^{-\frac{1}{4}-\frac{N}{2}}\|\nabla^{N+1}u\|_{L^2} \\
\lesssim&(1+t)^{-\frac{3}{2}-N} + (\delta + (1+t)^{-\frac{5}{2}})\|\nabla^{N+1}u\|_{L^2}^2.
\end{aligned}
\tag{3.26}
$$

For the term I_{24}, similar to the proof of (3.24), we have

$$
\begin{aligned}
|I_{24}| =&|\langle\nabla^{N-1}(f(\varrho)\nabla\varrho)^h, \nabla^N\nabla\cdot u^h\rangle| \\
\lesssim&\|\nabla^{N-1}(f(\varrho)\nabla\varrho)\|_{L^2}\|\nabla^N\nabla\cdot u^h\|_{L^2} \\
\lesssim&(\|\varrho\|_{L^\infty}\|\nabla^N\varrho\|_{L^2} + \|\nabla\varrho\|_{L^3}\|\nabla^{N-1}\varrho\|_{L^6})\|\nabla^N\nabla\cdot u^h\|_{L^2} \\
\lesssim&\|\nabla\varrho\|_{H^1}\|\nabla^N\varrho\|_{L^2}\|\nabla^N\nabla\cdot u\|_{L^2} \\
\lesssim&(1+t)^{-\frac{5}{4}}(1+t)^{-\frac{1}{4}-\frac{N}{2}}\|\nabla^{N+1}u\|_{L^2} \\
\lesssim&(1+t)^{-\frac{3}{2}-N} + (1+t)^{-\frac{3}{2}}\|\nabla^{N+1}u\|_{L^2}^2.
\end{aligned}
\tag{3.27}
$$

For the term I_{25}, by virtue of Lemma 7.3, we have

$$
\begin{aligned}
|I_{25}| =&|\langle\nabla^{N-1}(g(\varrho)\nabla n\cdot\Delta n)^h, \nabla^N\nabla\cdot u^h\rangle| \\
\lesssim&\|\nabla^{N-1}(g(\varrho)\nabla n\cdot\Delta n)\|_{L^2}\|\nabla^N\nabla\cdot u^h\|_{L^2} \\
\lesssim&(\|g(\varrho)\|_{L^\infty}\|\nabla^{N-1}(\nabla n\cdot\Delta n)\|_{L^2} + \|\nabla n\cdot\Delta n\|_{L^3}\|\nabla^{N-1}g(\varrho)\|_{L^6})\|\nabla^N\nabla\cdot u^h\|_{L^2} \\
\lesssim&(\|\nabla\varrho\|_{H^1}\|\nabla^{N-1}(\nabla n\cdot\Delta n)\|_{L^2} + \|\nabla n\cdot\Delta n\|_{L^3}\|\nabla^N\varrho\|_{L^2})\|\nabla^N\nabla\cdot u\|_{L^2} \\
\lesssim&[\|\nabla\varrho\|_{H^1}(\|\nabla n\|_{L^\infty}\|\nabla^{N+1}n\|_{L^2} + \|\Delta n\|_{L^3}\|\nabla^N n\|_{L^6}) + (\|\nabla n\|_{L^6}\|\Delta n\|_{L^6})\|\nabla^N\varrho\|_{L^2}]\|\nabla^N\nabla\cdot u\|_{L^2} \\
\lesssim&\|\nabla\varrho\|_{H^1}\|\nabla^2 n\|_{H^1}\|\nabla^{N+1}n\|_{L^2}\|\nabla^N\nabla\cdot u\|_{L^2} + \|\nabla^2 n\|_{L^2}\|\nabla^3 n\|_{L^2}\|\nabla^N\varrho\|_{L^2}\|\nabla^N\nabla\cdot u\|_{L^2} \\
\lesssim&(1+t)^{-\frac{5}{4}}(1+t)^{-\frac{7}{4}}(1+t)^{-\frac{5}{4}-\frac{N}{2}}\|\nabla^N\nabla\cdot u\|_{L^2} + \delta\|\nabla^N\varrho\|_{L^2}\|\nabla^N\nabla\cdot u\|_{L^2} \\
\lesssim&(1+t)^{-\frac{3}{2}-N} + (\delta + (1+t)^{-7})\|\nabla^{N+1}u\|_{L^2}^2 + \delta\|\nabla^N\varrho\|_{L^2}^2.
\end{aligned}
\tag{3.28}
$$

Substituting (3.24)–(3.28) into (3.23), we have

$$
|I_2| \lesssim (1+t)^{-\frac{3}{2}-N} + (\delta + (1+t)^{-\frac{3}{2}})\|\nabla^{N+1}u\|_{L^2}^2 + \delta\|\nabla^N\varrho\|_{L^2}^2. \tag{3.29}
$$

Substituting (3.21)–(3.22) and (3.29) into (3.15), we obtain the estimate (3.13). □

Next, we will establish the dissipation estimates for $\nabla^N\varrho^h$.

Lemma 3.5. *Suppose that the assumptions of Theorem 1.1 are in force, then we have*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^{N-1}u^h\nabla^N\varrho^h\mathrm{d}x + C_2\|\nabla^N\varrho^h\|_{L^2}^2 \lesssim& (1+t)^{-\frac{3}{2}-N} + (\delta + (1+t)^{-\frac{3}{2}})\|\nabla^N\varrho\|_{L^2}^2 \\
&+(1 + (1+t)^{-\frac{3}{2}})\|\nabla^N u\|_{L^2}^2 + (1+\delta)\|\nabla^{N+1}u\|_{L^2}^2,
\end{aligned}
\tag{3.30}
$$

for any $t\in[0,\infty)$.

Proof. Taking $\langle \mathfrak{F}^{-1}(1-\phi(\xi))\nabla^{N-1}(2.1)_2, \nabla^N \varrho^h\rangle$, and making use of the integration by parts, it hold that

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla^{N-1}u_t^h \nabla^N \varrho^h \mathrm{d}x + P'(1)\|\nabla^N \varrho^h\|_{L^2}^2 =&\mu\langle \nabla^{N-1}\Delta u^h, \nabla^N \varrho^h\rangle + (\mu+\nu)\langle \nabla^N \mathrm{div} u^h, \nabla^N \varrho^h\rangle \\ &+ \langle \nabla^{N-1} S_2^h, \nabla^N \varrho^h\rangle.\end{aligned} \tag{3.31}$$

In order to deal with the term $\int_{\mathbb{R}^3} \nabla^{N-1}u_t^h \nabla^N \varrho^h \mathrm{d}x$, we apply the transport equation $(2.1)_1$. More precisely, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla^{N-1}u_t^h \nabla^N \varrho^h \mathrm{d}x =&\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3} \nabla^{N-1}u^h \nabla^N \varrho^h \mathrm{d}x - \int_{\mathbb{R}^3} \nabla^{N-1}u^h \nabla^N \varrho_t^h \mathrm{d}x \\ =&\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3} \nabla^{N-1}u^h \nabla^N \varrho^h \mathrm{d}x + \int_{\mathbb{R}^3} \nabla^{N-1}\mathrm{div} u^h \nabla^{N-1} \varrho_t^h \mathrm{d}x \\ =&\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3} \nabla^{N-1}u^h \nabla^N \varrho^h \mathrm{d}x + \int_{\mathbb{R}^3} \nabla^{N-1}\mathrm{div} u^h \nabla^{N-1} (\mathrm{div} u - S_1)^h \mathrm{d}x.\end{aligned} \tag{3.32}$$

Substituting (3.33) into (3.32), and employing Young's inequality, it is easy to deduce

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3} \nabla^{N-1}u^h \nabla^N \varrho^h \mathrm{d}x + C_2\|\nabla^N \varrho^h\|_{L^2}^2 \\ \lesssim&\|\nabla^N u^h\|_{L^2}^2 + \|\nabla^{N+1} u^h\|_{L^2}^2 + \langle \nabla^{N-1} S_2^h, \nabla^N \varrho^h\rangle - \int_{\mathbb{R}^3} \nabla^{N-1}\mathrm{div} u^h \nabla^{N-1} S_1^h \mathrm{d}x.\end{aligned} \tag{3.33}$$

Here, we write

$$\Pi_1 + \Pi_2 = -\int_{\mathbb{R}^3} \nabla^{N-1}\mathrm{div} u^h \nabla^{N-1} S_1^h \mathrm{d}x + \langle \nabla^{N-1} S_2^h, \nabla^N \varrho^h\rangle.$$

For the term Π_1, with the help of Hölder's inequality and Lemma 7.3, it holds that

$$\begin{aligned}|\Pi_1| =&|\langle \nabla^N(\varrho u)^h, \nabla^N u^h\rangle| \\ \lesssim&\|\nabla^N(\varrho u)\|_{L^2}\|\nabla^N u^h\|_{L^2} \\ \lesssim&\|(\varrho,u)\|_{L^\infty}\|\nabla^N(\varrho,u)\|_{L^2}\|\nabla^N u^h\|_{L^2} \\ \lesssim&\|\nabla(\varrho,u)\|_{H^1}\|\nabla^N(\varrho,u)\|_{L^2}\|\nabla^N u^h\|_{L^2} \\ \lesssim&(1+t)^{-\frac{5}{4}}(1+t)^{-\frac{1}{4}-\frac{N}{2}}\|\nabla^N u^h\|_{L^2} \\ \lesssim&(1+t)^{-\frac{3}{2}-N} + (1+t)^{-\frac{3}{2}}\|\nabla^N u^h\|_{L^2}^2.\end{aligned} \tag{3.34}$$

For the term Π_2, similarly to the proof of (3.16), we have

$$\begin{aligned}\Pi_2 =&\langle \nabla^{N-1}(u\cdot\nabla u)^h, \nabla^N \varrho^h\rangle + \langle \nabla^{N-1}(h(\varrho)\Delta u)^h)^h, \nabla^N \varrho^h\rangle + \langle \nabla^{N-1}(h(\varrho)\nabla(\mathrm{div} u))^h, \nabla^N \varrho^h\rangle \\ &+ \langle \nabla^{N-1}(f(\varrho)\nabla\varrho)^h, \nabla^N \varrho^h\rangle + \langle \nabla^{N-1}(g(\varrho)\nabla n\cdot\Delta n)^h, \nabla^N \varrho^h\rangle := \sum_{i=1}^{5}\Pi_{2i}.\end{aligned} \tag{3.35}$$

For the term Π_{21}, we have

$$\begin{aligned}|\Pi_{21}| \lesssim&\|\nabla^{N-1}(u\cdot\nabla u)\|_{L^2}\|\nabla^N \varrho^h\|_{L^2} \\ \lesssim&(\|u\|_{L^\infty}\|\nabla^N u\|_{L^2} + \|\nabla u\|_{L^3}\|\nabla^{N-1} u\|_{L^6})\|\nabla^N \varrho^h\|_{L^2} \\ \lesssim&\|\nabla u|_{H^1}\|\nabla^N u\|_{L^2}\|\nabla^N \varrho^h\|_{L^2} \\ \lesssim&(1+t)^{-\frac{5}{4}}(1+t)^{-\frac{1}{4}-\frac{N}{2}}\|\nabla^N \varrho^h\|_{L^2} \\ \lesssim&(1+t)^{-\frac{3}{2}-N} + (1+t)^{-\frac{3}{2}}\|\nabla^N \varrho^h\|_{L^2}.\end{aligned} \tag{3.36}$$

For the term Π_{22}, it easy to deduce

$$\begin{aligned}|\Pi_{22}| \lesssim&\|\nabla^{N-1}(h(\varrho)\Delta u)\|_{L^2}\|\nabla^{k+1}\varrho^h\|_{L^2}\\ \lesssim&(\|\varrho\|_{L^\infty}\|\nabla^{N+1}u\|_{L^2}+\|\nabla^2 u\|_{L^3}\|\nabla^{N-1}\varrho\|_{L^6})\|\nabla^N\varrho^h\|_{L^2}\\ \lesssim&(\|\varrho\|_{H^2}\|\nabla^{N+1}u\|_{L^2}+\|\nabla^2 u\|_{H^1}\|\nabla^N\varrho\|_{L^2})\|\nabla^N\varrho\|_{L^2}\\ \lesssim&\delta\|\nabla^{N+1}u\|_{L^2}^2+\delta\|\nabla^N\varrho\|_{L^2}^2.\end{aligned}\tag{3.37}$$

For the term Π_{23}, we can get

$$\begin{aligned}|\Pi_{23}| \lesssim&\|\nabla^{N-1}(h(\varrho)\nabla(\mathrm{div}u))\|_{L^2}\|\nabla^{k+1}\varrho^h\|_{L^2}\\ \lesssim&(\|\varrho\|_{L^\infty}\|\nabla^{N+1}u\|_{L^2}+\|\nabla^2 u\|_{L^3}\|\nabla^{N-1}\varrho\|_{L^6})\|\nabla^N\varrho^h\|_{L^2}\\ \lesssim&(\|\varrho\|_{H^2}\|\nabla^{N+1}u\|_{L^2}+\|\nabla^2 u\|_{H^1}\|\nabla^N\varrho\|_{L^2})\|\nabla^N\varrho\|_{L^2}\\ \lesssim&\delta\|\nabla^{N+1}u\|_{L^2}^2+\delta\|\nabla^N\varrho\|_{L^2}^2.\end{aligned}\tag{3.38}$$

For the term Π_{24}, similarly to the proof of (3.34), we have

$$\begin{aligned}|\Pi_{24}| \lesssim&\|\nabla^{N-1}(f(\varrho)\nabla\varrho)\|_{L^2}\|\nabla^N\varrho^h\|_{L^2}\\ \lesssim&(\|\varrho\|_{L^\infty}\|\nabla^N\varrho\|_{L^2}+\|\nabla\varrho\|_{L^3}\|\nabla^{N-1}\varrho\|_{L^6})\|\nabla^N\varrho^h\|_{L^2}\\ \lesssim&\|\nabla\varrho\|_{H^1}\|\nabla^N\varrho\|_{L^2}\|\nabla^N\varrho^h\|_{L^2}\\ \lesssim&(1+t)^{-\frac{5}{4}}(1+t)^{-\frac{1}{4}-\frac{N}{2}}\|\nabla^N\varrho^h\|_{L^2}\\ \lesssim&(1+t)^{-\frac{3}{2}-N}+(1+t)^{-\frac{3}{2}}\|\nabla^N\varrho^h\|_{L^2}.\end{aligned}\tag{3.39}$$

For the term Π_{25}, by virtue of Lemma 7.3, it is easy to obtain

$$\begin{aligned}|\Pi_{25}| \lesssim&\|\nabla^{N-1}(g(\varrho)\nabla n\cdot\Delta n)\|_{L^2}\|\nabla^N\varrho^h\|_{L^2}\\ \lesssim&(\|g(\varrho)\|_{L^\infty}\|\nabla^{N-1}(\nabla n\cdot\Delta n)\|_{L^2}+\|\nabla n\cdot\Delta n\|_{L^3}\|\nabla^{N-1}g(\varrho)\|_{L^6})\|\nabla^N\varrho^h\|_{L^2}\\ \lesssim&\|\varrho\|_{L^\infty}(\|\nabla n\|_{L^\infty}\|\nabla^{N+1}n\|_{L^2}+\|\Delta n\|_{L^3}\|\nabla^N n\|_{L^6})\|\nabla^N\varrho^h\|_{L^2}+\|\nabla n\|_{L^6}\|\Delta n\|_{L^6}\|\nabla^N\varrho\|_{L^2}^2\\ \lesssim&\|\nabla\varrho\|_{H^1}\|\nabla^2 n\|_{H^1}\|\nabla^{N+1}n\|_{L^2}\|\nabla^N\varrho\|_{L^2}+\|\nabla^2 n\|_{L^2}\|\nabla^3 n\|_{L^2}\|\nabla^N\varrho\|_{L^2}^2\\ \lesssim&(1+t)^{-\frac{5}{4}}(1+t)^{-\frac{7}{4}}(1+t)^{-\frac{5}{4}-\frac{N}{2}}\|\nabla^N\varrho\|_{L^2}+\delta\|\nabla^N\varrho\|_{L^2}^2\\ \lesssim&(1+t)^{-\frac{3}{2}-N}+(\delta+(1+t)^{-7})\|\nabla^N\varrho\|_{L^2}^2.\end{aligned}\tag{3.40}$$

Substituting (3.36)–(3.41) into (3.35), yields directly

$$|\Pi_2| \lesssim(1+t)^{-\frac{3}{2}-N}+(\delta+(1+t)^{-\frac{3}{2}})\|\nabla^N\varrho\|_{L^2}^2+\delta\|\nabla^{N+1}u\|_{L^2}^2.\tag{3.41}$$

Substituting (3.34) and (3.41) into (3.33), we complete the proof of the Lemma 3.5.

□

Finally, we will close the estimate by combining them which have been already proved along the proof of Lemma (3.3)–(3.5). To achieve this, we choose sufficiently large time T_0 and positive constant D_0, and then define the temporary energy functional

$$\mathbf{E}(t)=D_0\|\nabla^N(\varrho^h,u^h)\|_{L^2}^2+\int_{\mathbb{R}^3}\nabla^{N-1}u^h\nabla^N\varrho^h\mathrm{d}x,$$

for $t \geq T_0$, it is noticed that $\mathbf{E}(t)$ is equivalent to $\|\nabla^N(\varrho^h, u^h)\|_{L^2}^2$. Substituting (3.13) and (3.30) into

$$D_0 \times (3.13) + (3.30), \tag{3.42}$$

which together with the smallness of δ, for $t \geq T_0$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(t) + \|\nabla^N \varrho^h\|_{L^2}^2 + \|\nabla^{N+1} u^h\|_{L^2}^2 \lesssim (1+t)^{-\frac{3+2N}{2}} + \|\nabla^N \varrho^l\|_{L^2}^2 + \|\nabla^N u^l\|_{H^1}^2, \tag{3.43}$$

where we have used the fact that T_0 is large enough. On other hand, it is clear that

$$\|\nabla^N \varrho^h\|_{L^2}^2 + \|\nabla^{N+1} u^h\|_{L^2}^2 \geq C_3 \mathbf{E}(t).$$

Hence, by virtue of estimate (3.7) and Gronwall's inequality, we can arrive at

$$\|\nabla^N(\varrho^h, u^h)\|_{L^2}^2 \lesssim (1+t)^{-\frac{3}{2}-N}. \tag{3.44}$$

So, the proof of Theorem 1.2 has been completed.

4. Proof of Theorem 1.3

In this section, we will establish the lower optimal decay rates of the solution as well as its all–order spatial derivatives in Theorem 1.3. At first, we rewrite $(1.1)_1$–$(1.1)_2$ with $m = (\varrho + 1)u$ in the perturbation form as follow:

$$\begin{cases} \varrho_t + \mathrm{div} m = 0, \\ m_t + P'(1)\nabla\varrho - \mu\Delta m - (\mu+\nu)\nabla \mathrm{div} m = -\mathrm{div}\mathcal{F}, \end{cases} \tag{4.1}$$

where the functions $\mathcal{F} = \mathcal{F}(\varrho, u, n)$ is defined as

$$\begin{aligned} \mathcal{F} =& [P(\varrho+1) - P(1) - \varrho]\mathbb{I}_{3\times3} + \mu\nabla(\varrho u) + (\mu+\nu)\mathrm{div}(\varrho u)\mathbb{I}_{3\times3} \\ &+ (1+\varrho)(u \otimes u) + \frac{1}{2}(|\nabla n|^2)\mathbb{I}_{3\times3} - (\nabla n \odot \nabla n). \end{aligned}$$

Here $\nabla n \odot \nabla n = (\partial_i n, \partial_j n)_{1\leq i,j\leq 3}$. It is easy to check

$$\nabla n \cdot \Delta n = \mathrm{div}(\nabla n \odot \nabla n) - \frac{1}{2}\nabla(|\nabla n|^2), \tag{4.2}$$

with the initial conditions:

$$(\varrho, m)(x,t)\mid_{t=0} = (\varrho_0, m_0)(x) \to (0,0), \quad as \ |x| \to \infty.$$

Next, we will establish the lower bound decay rate of $\nabla^k(\varrho, u)$ to the system (2.1).

Lemma 4.1. *Under all the assumptions of Theorem 1.3, then the global solution (ϱ, u) has the decay rates for all $t \geq t_*$ and $0 \leq k \leq N$*

$$\min\{\|\nabla^k \varrho(t)\|_{L^2}, \|\nabla^k u(t)\|_{L^2}\} \geq c_1(1+t)^{-\frac{3}{4}-\frac{k}{2}}. \tag{4.3}$$

Here, t_* is a positive large time, the positive constant c_1 is independent of time.

Proof. By virtue of Duhamel's principle, system (4.1), Lemma 3.1, Theorem 1.1 for $k = 0, 1$, we have

$$\begin{aligned}
&min\{\|\nabla^k \varrho(t)\|_{L^2}, \|\nabla^k u(t)\|_{L^2}\} \\
&\geq min\{\|\nabla^k \varrho(t)\|_{L^2}, \|\nabla^k m(t)\|_{L^2}\} \\
&\geq min\{\|\nabla^k \varrho^l(t)\|_{L^2}, \|\nabla^k m^l(t)\|_{L^2}\} \\
&\geq c_4(1+t)^{-\frac{3}{4}-\frac{k}{2}} - \int_0^t (1+t-\tau)^{-\frac{5}{4}-\frac{k}{2}} \|\mathcal{F}\|_{L^1} d\tau \\
&\geq c_4(1+t)^{-\frac{3}{4}-\frac{k}{2}} - \int_0^t (1+t-\tau)^{-\frac{5}{4}-\frac{k}{2}} (1+\tau)^{-\frac{3}{2}} d\tau \\
&\geq c_4(1+t)^{-\frac{3}{4}-\frac{k}{2}} - C(1+t)^{-\frac{5}{4}-\frac{k}{2}} \\
&\geq c_1(1+t)^{-\frac{3}{4}-\frac{k}{2}}.
\end{aligned} \tag{4.4}$$

Then, for $2 \leq k \leq N$, by applying (4.3) and Lemma 7.2, we have from Sobolev interpolation that

$$\|\nabla^k(\varrho, u)(t)\|_{L^2} \geq C\|\nabla(\varrho, u)(t)\|_{L^2}^k \|(\varrho, u)(t)\|_{L^2}^{-(k-1)} \geq c_1(1+t)^{-\frac{3}{4}-\frac{k}{2}}. \tag{4.5}$$

Therefore, we have completed the proof of this lemma.

□

In order to obtain the decay rate of $\|\nabla^{k+1} n\|_{L^2}$, employing ∇ operator to the equation $(2.3)_3$, we have

$$\nabla n_t - \Delta \nabla n = \nabla S_3. \tag{4.6}$$

Then, we can deduce the lower bound decay rate of $\nabla^{k+1} n$ which is given in the following lemma.

Lemma 4.2. *Under all the assumptions of Theorem 1.3, then there is a positive constant c_2 independent of time, such that for all $t \geq t_*$ and $0 \leq k \leq N$,*

$$\|\nabla^{k+1} n(t)\|_{L^2} \geq c_2(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{4.7}$$

where t_* is a positive large time.

Proof. If t_* is large enough, by using Lemma 3.2 and Lemma 7.5, for $k = 0, 1$, we have

$$\begin{aligned}
\|\nabla^{k+1} n(t)\|_{L^2} \geq& \|\nabla^{k+1} n^l(t)\|_{L^2} \\
\geq& c_4(1+t)^{-\frac{3}{4}-\frac{k}{2}} - C\int_0^t (1+t-\tau)^{-\frac{5}{4}-\frac{k}{2}} \|\mathcal{S}_3\|_{L^1} d\tau \\
\geq& c_4(1+t)^{-\frac{3}{4}-\frac{k}{2}} - C\int_0^t (1+t-\tau)^{-\frac{5}{4}-\frac{k}{2}} (1+\tau)^{-2} d\tau \\
\geq& c_4(1+t)^{-\frac{3}{4}-\frac{k}{2}} - C(1+t)^{-\frac{5}{4}-\frac{k}{2}} \\
\geq& c_2(1+t)^{-\frac{3}{4}-\frac{k}{2}}.
\end{aligned} \tag{4.8}$$

Next, we employ interpolation inequality to establish the lower bound of decay rate for the higher–order spatial derivative of solution. The decay rates (4.8) together with the interpolations ($k \geq 2$)

$$\|\nabla^k f\|_{L^2} \geq C\|\nabla f\|_{L^2}^k \|f\|_{L^2}^{-(k-1)} \tag{4.9}$$

we have

$$\|\nabla^{k+1} n(t)\|_{L^2} \geq c_2(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{4.10}$$

for large time t_*. Thus the proof of Lemma 4.2 is completed.

Therefore, we have completed the proof of Theorem 1.3. □

5. Proof of Theorem 1.4

In this section, we will establish the decay rate for the mixed space–time derivatives of solution to the Cauchy problem (1.1)–(1.2).

Lemma 5.1. *Suppose that the assumptions of Theorem 1.1 and Theorem 1.2 are in force, the global solution* (ϱ, u, n) *has the time decay rate:*

$$\begin{aligned}
&\|\nabla^k \partial_t u(t)\|_{L^2} \le C(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad k \in [0, N-2],\\
&\|\nabla^l \partial_t \varrho(t)\|_{L^2} \le C(1+t)^{-\frac{5}{4}-\frac{l}{2}},\\
&\|\nabla^l \partial_t n(t)\|_{L^2} \le C(1+t)^{-\frac{7}{4}-\frac{l}{2}}, \quad l \in [0, N-1].
\end{aligned} \tag{5.1}$$

Here, the positive constant C is independent of time.

Proof. First of all, we shall estimate $\|\nabla^k \partial_t \varrho\|_{L^2}$. For $l = 0, 1, 2, ..., N-1$, taking l–th spatial derivative to $(2.1)_1$, then multiplying the resulting identities by $\nabla^l \partial_t \varrho$ and integrating over $\mathbb{R}^3$, we have

$$\|\nabla^l \partial_t \varrho\|_{L^2}^2 = -\int_{\mathbb{R}^3} \nabla^l(\varrho \mathrm{div} u + u \cdot \nabla \varrho + \mathrm{div} u) \cdot \nabla^l \partial_t \varrho \mathrm{d}x := K_1 + K_2 + K_3. \tag{5.2}$$

We will estimate the right hand side of (5.2) term by term. First, for the term K_1, by virtue of Lemma 7.3, and Young's inequality, we obtain

$$\begin{aligned}
K_1 &\lesssim (\|\nabla^l \varrho\|_{L^6}\|\nabla u\|_{L^3} + \|\varrho\|_{L^\infty}\|\nabla^{l+1} u\|_{L^2})\|\nabla^l \partial_t \varrho\|_{L^2}\\
&\le \delta\|\nabla^l \partial_t \varrho\|_{L^2}^2 + C\|\nabla^{l+1}\varrho\|_{L^2}^2\|\nabla u\|_{L^3}^2 + C\|\varrho\|_{L^\infty}^2\|\nabla^{l+1} u\|_{L^2}^2\\
&\le \delta\|\nabla^l \partial_t \varrho\|_{L^2}^2 + C(1+t)^{-\frac{5}{2}-l}(1+t)^{-3} + C(1+t)^{-3}(1+t)^{-\frac{5}{2}-l}\\
&\le \delta\|\nabla^l \partial_t \varrho\|_{L^2}^2 + C(1+t)^{-\frac{11}{2}-l},
\end{aligned} \tag{5.3}$$

where δ is small enough. Similarly, we can get

$$\begin{aligned}
K_2 &\lesssim (\|\nabla^l u\|_{L^6}\|\nabla \varrho\|_{L^3} + \|u\|_{L^\infty}\|\nabla^{l+1} \varrho\|_{L^2})\|\nabla^l \partial_t \varrho\|_{L^2}\\
&\le \delta\|\nabla^l \partial_t \varrho\|_{L^2}^2 + C\|\nabla^{l+1}u\|_{L^2}^2\|\nabla \varrho\|_{L^3}^2 + C\|u\|_{L^\infty}^2\|\nabla^{l+1} \varrho\|_{L^2}^2\\
&\le \delta\|\nabla^l \partial_t \varrho\|_{L^2}^2 + C(1+t)^{-\frac{5}{2}-l}(1+t)^{-3} + C(1+t)^{-3}(1+t)^{-\frac{5}{2}-l}\\
&\le \delta\|\nabla^l \partial_t \varrho\|_{L^2}^2 + C(1+t)^{-\frac{11}{2}-l}.
\end{aligned} \tag{5.4}$$

For the term K_3, using Young's inequality, it is easy to see that

$$\begin{aligned}
K_3 &\le \delta\|\nabla^l \partial_t \varrho\|_{L^2}^2 + C\|\nabla^{l+1} u\|_{L^2}^2\\
&\le \delta\|\nabla^l \partial_t \varrho\|_{L^2}^2 + C(1+t)^{-\frac{5}{2}-l}.
\end{aligned} \tag{5.5}$$

Combining (5.3), (5.4) with (5.5), we obtain

$$\|\nabla^l \partial_t \varrho\|_{L^2}^2 \lesssim (1+t)^{-\frac{5}{2}-l}. \tag{5.6}$$

Then we shall estimate $\|\nabla^k\partial_t u\|_{L^2}$. For $k = 0, 1, 2, ..., N-2$, taking k–th spatial derivative to $(2.1)_2$, multiplying the resulting identities by $\nabla^k\partial_t u$ and integrating over $\mathbb{R}^3$, we have

$$
\begin{aligned}
\|\nabla^k\partial_t u\|_{L^2}^2 =& \int_{\mathbb{R}^3} \nabla^k[\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u - \nabla P(\rho) - u\cdot\nabla u \\
& - h(\varrho)[\mu\Delta u + (\mu+\nu)\nabla\mathrm{div}u] - f(\varrho)\nabla\varrho - g(\varrho)\nabla n\cdot\Delta n]\cdot\nabla^k\partial_t u \mathrm{d}x \\
\lesssim& \delta\|\nabla^k\partial_t u\|_{L^2}^2 + \|\nabla^k[\nabla^2 u - \nabla P(\rho)]\|_{L^2}^2 + \|\nabla^k[u\cdot\nabla u + f(\varrho)\nabla\varrho]\|_{L^2}^2 \\
& + \|\nabla^k[h(\varrho)\nabla^2 u]\|_{L^2}^2 + \|\nabla^k[g(\varrho)\nabla n\cdot\Delta n]\|_{L^2}^2 \\
=& \delta\|\nabla^k\partial_t u\|_{L^2}^2 + L_1 + L_2 + L_3 + L_4.
\end{aligned}
\tag{5.7}
$$

By applying Lemma 7.3 again, we estimate the second term in the right hand of (5.7).

$$
\begin{aligned}
L_1 \lesssim& \|\nabla^{k+2}u\|_{L^2}^2 + \|\nabla^{k+1}\varrho\|_{L^2}^2 \\
\lesssim& (1+t)^{-\frac{7}{2}-k} + (1+t)^{-\frac{5}{2}-k} \\
\lesssim& (1+t)^{-\frac{5}{2}-k}.
\end{aligned}
\tag{5.8}
$$

Similarly, for the term L_2, we can bound

$$
\begin{aligned}
L_2 \lesssim& \|\nabla^k(\varrho,u)\|_{L^6}^2\|\nabla(\varrho,u)\|_{L^3}^2 + \|\nabla^{k+1}(\varrho,u)\|_{L^2}^2\|(\varrho,u)\|_{L^\infty}^2 \\
\lesssim& (1+t)^{-\frac{5}{2}-k}(1+t)^{-3} \\
\lesssim& (1+t)^{-\frac{11}{2}-k}.
\end{aligned}
\tag{5.9}
$$

For the term L_3, it is easy to get

$$
\begin{aligned}
L_3 \lesssim& \|\nabla^k\varrho\|_{L^6}^2\|\nabla^2 u\|_{L^3}^2 + \|\nabla^{k+2}u\|_{L^2}^2\|\varrho\|_{L^\infty}^2 \\
\lesssim& (1+t)^{-\frac{5}{2}-k}(1+t)^{-4} + (1+t)^{-\frac{7}{2}-k}(1+t)^{-3} \\
\lesssim& (1+t)^{-\frac{13}{2}-k}.
\end{aligned}
\tag{5.10}
$$

By employing the Leibniz formula and Lemma 7.3, we estimate the last term of (5.7)

$$
\begin{aligned}
L_4 \lesssim& \|\nabla^k[g(\varrho)\nabla n\cdot\Delta n]\|_{L^2}^2 \\
\lesssim& \|g(\varrho)\|_{L^\infty}^2\|\nabla^k(\nabla n\cdot\Delta n)\|_{L^2}^2 + \|\nabla n\cdot\Delta n\|_{L^3}^2\|\nabla^k g(\varrho)\|_{L^6}^2 \\
\lesssim& \|\varrho\|_{L^\infty}^2\|\nabla^k(\nabla n\cdot\Delta n)\|_{L^2}^2 + \|\nabla n\cdot\Delta n\|_{L^3}^2\|\nabla^{k+1}\varrho\|_{L^2}^2 \\
\lesssim& \|\varrho\|_{L^\infty}^2(\|\nabla n\|_{L^\infty}^2\|\nabla^{k+2}n\|_{L^2}^2 + \|\Delta n\|_{L^3}^2\|\nabla^{k+1}n\|_{L^6}^2) + \|\nabla n\|_{L^6}^2\|\Delta n\|_{L^6}^2\|\nabla^{k+1}\varrho\|_{L^2}^2 \\
\lesssim& (1+t)^{-3}(1+t)^{-4}(1+t)^{-\frac{7}{2}-k} + (1+t)^{-\frac{7}{2}}(1+t)^{-\frac{9}{2}}(1+t)^{-\frac{5}{2}-k} \\
\lesssim& (1+t)^{-\frac{5}{2}-k}.
\end{aligned}
\tag{5.11}
$$

Substituting estimates (5.8)–(5.11) into (5.7), we have

$$
\|\nabla^k\partial_t u\|_{L^2}^2 \lesssim (1+t)^{-\frac{5}{2}-k}. \tag{5.12}
$$

Similar to the estimate of the term $\|\nabla^l\partial_t\varrho\|_{L^2}$. For $l = 0, 1, 2, ..., N-1$, taking l–th spatial derivative to $(2.1)_3$, then multiplying the resulting identities by $\nabla^l\partial_t n$ and integrating over $\mathbb{R}^3$, we have

$$
\begin{aligned}
\|\nabla^l\partial_t n\|_{L^2}^2 =& \int_{\mathbb{R}^3}\nabla^l[\Delta n - u\cdot\nabla n + |\nabla n|^2(n+\omega_0)]\cdot\nabla^l\partial_t n\mathrm{d}x \\
\lesssim& \delta\|\nabla^l\partial_t n\|_{L^2}^2 + \|\nabla^l[\Delta n + u\cdot\nabla n]\|_{L^2}^2 + \|\nabla^l[|\nabla n|^2(n+\omega_0)]\|_{L^2}^2 \\
=& \delta\|\nabla^l\partial_t n\|_{L^2}^2 + X_1 + X_2.
\end{aligned}
\tag{5.13}
$$

By using Lemma 7.3 again, we estimate the second term of (5.13)

$$\begin{aligned} X_1 \lesssim & \|\nabla^{l+2} n\|_{L^2}^2 + \|\nabla^l u\|_{L^6}^2 \|\nabla n\|_{L^3}^2 + \|\nabla^{l+1} n\|_{L^2}^2 \|u\|_{L^\infty}^2 \\ \lesssim & (1+t)^{-\frac{7}{2}-l} + (1+t)^{-\frac{5}{2}-l}(1+t)^{-3} \\ \lesssim & (1+t)^{-\frac{7}{2}-l}. \end{aligned} \tag{5.14}$$

By employing the Leibniz formula and Lemma 7.3, we estimate the last term of (5.13)

$$\begin{aligned} X_2 = & \|\nabla^l [|\nabla n|^2 (n+\omega_0)]\|_{L^2}^2 \\ \lesssim & \|\nabla^l (|\nabla n|^2) \cdot (n+\omega_0)\|_{L^2}^2 + \sum_{m=1}^{l} C_k^m \|\nabla^m (n+\omega_0) \nabla^{l-m} (|\nabla n|^2)\|_{L^2}^2 \\ \lesssim & \|\nabla n\|_{L^3}^2 \|\nabla^{l+1} n\|_{L^6}^2 + \|\nabla^m (n+\omega_0)\|_{L^6}^2 \|\nabla^{l-m} (|\nabla n|^2)\|_{L^3}^2 \\ \lesssim & \|\nabla n\|_{L^3}^2 \|\nabla^{l+2} n\|_{L^2}^2 + \|\nabla^{m+1} n\|_{L^2}^2 \|\nabla n\|_{L^6}^2 \|\nabla^{l+1-m} n\|_{L^6}^2 \\ \lesssim & (1+t)^{-3}(1+t)^{-\frac{7}{2}-l} + (1+t)^{-\frac{5}{2}-m}(1+t)^{-\frac{7}{2}}(1+t)^{-\frac{7}{2}-(l-m)} \\ \lesssim & (1+t)^{-\frac{7}{2}-l}. \end{aligned} \tag{5.15}$$

Here, we have the basic fact that $|n(x,t)+\omega_0| = |d(x,t)| = 1$, and

$$\nabla^l (|\nabla n|^2 (n+\omega_0)) = \nabla^l (|\nabla n|^2) \cdot (n+\omega_0) + \sum_{m=1}^{l} C_k^m \nabla^m (n+\omega_0) \nabla^{l-m} (|\nabla n|^2). \tag{5.16}$$

Combining (5.14) with (5.15), we arrive at

$$\|\nabla^l \partial_t n\|_{L^2}^2 \lesssim (1+t)^{-\frac{7}{2}-l}. \tag{5.17}$$

Combining estimates (5.6), (5.12) with (5.17), then we complete the proof of this lemma. □

6. Proof of Theorem 1.5

In this section, we will establish the lower bound of decay rate for the time derivative of solution of the system (2.1).

Lemma 6.1. *Under all the assumptions of Theorem 1.3, then the global solution (ϱ, u, n) has the time decay rate for all $t \geq t_*$*

$$\begin{aligned} min\{\|\partial_t \varrho(t)\|_{L^2}, \|\partial_t u(t)\|_{L^2}, \|\mathrm{div} u(t)\|_{L^2}\} \geq c_3 (1+t)^{-\frac{5}{4}}, \\ \|\partial_t n(t)\|_{L^2} \geq c_3 (1+t)^{-\frac{5}{4}}. \end{aligned} \tag{6.1}$$

Here, t_* is a positive large constant, the positive constant c_3 is independent of time.

Proof. At first, we establish the lower bound time decay rate for $\partial_t n$ in the L^2–norm. With the help of the Eq $(2.1)_3$, we have

$$\|\partial_t n\|_{L^2} \geq \|\Delta n\|_{L^2} - \|S_3\|_{L^2} \geq c_2 (1+t)^{-\frac{5}{4}} - \|S_3\|_{L^2}. \tag{6.2}$$

By applying the Sobolev inequality and decay rate (1.7), it is easy to get

$$\begin{aligned}\|S_3\|_{L^2} \le&\|u\cdot\nabla n\|_{L^2}+\||\nabla n|^2(n+\omega_0)\|_{L^2}\\ \le&C\|u\|_{L^\infty}\|\nabla n\|_{L^2}+C\|\nabla n\|_{L^3}\|\nabla n\|_{L^6}\\ \le&C\|\nabla u\|_{H^1}\|\nabla n\|_{L^2}+C\|\nabla n\|_{H^1}\|\nabla^2 n\|_{L^2}\\ \le&C(1+t)^{-\frac{5}{2}}.\end{aligned}\tag{6.3}$$

And hence, we have

$$\|\partial_t n\|_{L^2}\ge c_2(1+t)^{-\frac{5}{4}}-C(1+t)^{-\frac{5}{2}}\ge c_3(1+t)^{-\frac{5}{4}},\tag{6.4}$$

for all some large time t_*.

Next, we establish lower bound time decay rate for $\partial_t u$ in the L^2–norm. Using the Eq $(2.1)_2$, we can obtain

$$\|\nabla\varrho\|_{L^2}\le\|\partial_t u\|_{L^2}+\|\nabla^2 u\|_{L^2}+\|S_2\|_{L^2}.\tag{6.5}$$

And hence, we have

$$\|\partial_t u\|_{L^2}\ge\|\nabla\varrho\|_{L^2}-\|\nabla^2 u\|_{L^2}-\|S_2\|_{L^2}\ge c_1(1+t)^{-\frac{5}{4}}-C(1+t)^{-\frac{7}{4}}-\|S_2\|_{L^2}.\tag{6.6}$$

By virtue of the Sobolev inequality and time decay rate (1.7), it is easy to deduce

$$\begin{aligned}\|S_2\|_{L^2}\le&\|u\cdot\nabla u\|_{L^2}+\|h(\varrho)\nabla^2 u\|_{L^2}+\|f(\varrho)\nabla\varrho\|_{L^2}+\|g(\varrho)\nabla n\cdot\Delta n\|_{L^2}\\ \le&C\|(\varrho,u)\|_{L^\infty}\|\nabla(\varrho,u)\|_{L^2}+C\|\varrho\|_{L^\infty}\|\nabla^2 u\|_{L^2}+C\|\varrho\|_{L^\infty}\|\nabla n\|_{L^3}\|\nabla^2 n\|_{L^6}\\ \le&C\|\nabla(\varrho,u)\|_{H^1}\|\nabla(\varrho,u)\|_{L^2}+C\|\nabla\varrho\|_{H^1}\|\nabla^2 u\|_{L^2}+C\|\nabla\varrho\|_{H^1}\|\nabla n\|_{H^1}\|\nabla^3 n\|_{L^2}\\ \le&C(1+t)^{-\frac{5}{2}}.\end{aligned}\tag{6.7}$$

Together with estimates (6.6) and (6.7), yields directly

$$\|\partial_t u\|_{L^2}\ge c_1(1+t)^{-\frac{5}{4}}-C(1+t)^{-\frac{7}{4}}-C(1+t)^{-\frac{5}{2}}\ge c_3(1+t)^{-\frac{5}{4}}.\tag{6.8}$$

Finally, we establish lower bound time decay rate for $\partial_t\varrho$ in the L^2–norm. To accomplish this objective, we use the Eq $(2.1)_1$ to obtain

$$\|\mathrm{div}u\|_{L^2}\le\|\partial_t\varrho\|_{L^2}+\|S_1\|_{L^2}.\tag{6.9}$$

By virtue of the Sobolev inequality and decay rate (1.7), we can get

$$\begin{aligned}\|S_1\|_{L^2}\le&\|\varrho\mathrm{div}u\|_{L^2}+\|u\cdot\nabla\varrho\|_{L^2}\\ \le&C\|(\varrho,u)\|_{L^\infty}\|\nabla(\varrho,u)\|_{L^2}\\ \le&C\|\nabla(\varrho,u)\|_{H^1}\|\nabla(\varrho,u)\|_{L^2}\\ \le&C(1+t)^{-\frac{5}{2}},\end{aligned}\tag{6.10}$$

and hence, we arrive at

$$\|\partial_t\varrho\|_{L^2}\ge\|\mathrm{div}u\|_{L^2}-C(1+t)^{-\frac{5}{2}}.\tag{6.11}$$

Now, we need to establish the lower bound decay rate for $\|\mathrm{div}u\|_{L^2}$. Notice the relation differential relation $\Delta = \nabla\mathrm{div} - \nabla\times\nabla\times$, we get

$$\|\nabla u\|_{L^2}^2 = \|\mathrm{div}u\|_{L^2}^2 + \|\nabla\times u\|_{L^2}^2. \tag{6.12}$$

And hence, we have

$$\|\mathrm{div}u\|_{L^2} \geq C\|\nabla u\|_{L^2} - C\|\nabla\times u\|_{L^2} \geq c_1 C(1+t)^{-\frac{5}{4}} - C\|\nabla\times u\|_{L^2}, \tag{6.13}$$

which implies that we need to establish upper bound decay rate for $\|\nabla\times u\|_{L^2}$. To this end, we take the $\nabla\times$ operator the the velocity equation $(2.1)_2$ to get

$$\partial_t(\nabla\times u) - \mu\Delta(\nabla\times u) = \nabla\times G_2. \tag{6.14}$$

With the help of the Sobolev inequality, uniform bound (1.6), decay rate (1.7) and calculus identity (4.2), we have

$$\begin{aligned}
&\|S_2\|_{L^1} + \|S_2\|_{L^2}\\
&\leq C\|u\cdot\nabla u\|_{L^1} + C\|h(\varrho)\nabla^2 u\|_{L^1} + C\|f(\varrho)\nabla\varrho\|_{L^1} + C\|g(\varrho)\nabla n\cdot\Delta n\|_{L^1}\\
&\quad + C\|u\cdot\nabla u\|_{L^2} + C\|h(\varrho)\nabla^2 u\|_{L^2} + C\|f(\varrho)\nabla\varrho\|_{L^2} + C\|g(\varrho)\nabla n\cdot\Delta n\|_{L^2}\\
&\leq C(\|(\varrho,u,\nabla n)\|_{L^2}\|\nabla(\varrho,u,\nabla n)\|_{H^1} + \|\nabla(\varrho,u,\nabla n)\|_{H^1}\|\nabla(\varrho,u,\nabla n)\|_{H^2})\\
&\leq C\delta_0(1+t)^{-\frac{5}{4}}.
\end{aligned} \tag{6.15}$$

By virtue of the Duhamel principle formula, we obtain

$$\begin{aligned}
\|\nabla\times u\|_{L^2} \leq& C(1+t)^{-\frac{5}{4}}(\|\Lambda^{-1}\mathfrak{F}(\nabla\times u_0)\|_{L^\infty} + \|\Lambda^{-1}\mathfrak{F}(\nabla\times u_0)\|_{L^2})\\
&+ C\int_0^t (1+t-\tau)^{-\frac{5}{4}}(\|\Lambda^{-1}\mathfrak{F}(\nabla\times S_2)\|_{L^\infty} + \|\Lambda^{-1}\mathfrak{F}(\nabla\times S_2)\|_{L^2})\mathrm{d}x\\
\leq& C(\delta_0+\delta_1)(1+t)^{-\frac{5}{4}} + C\delta_0\int_0^t (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{5}{4}}\mathrm{d}x\\
\leq& C(\delta_0+\delta_1)(1+t)^{-\frac{5}{4}},
\end{aligned} \tag{6.16}$$

which, together with estimate (6.13) and smallness of $\delta_i (i = 0, 1)$, yields directly

$$\|\mathrm{div}u\|_{L^2} \geq c_1 C(1+t)^{-\frac{5}{4}} - C(\delta_0+\delta_1)(1+t)^{-\frac{5}{4}} \geq c_3(1+t)^{-\frac{5}{4}}. \tag{6.17}$$

This and the estimate (6.11) yields

$$\|\partial_t\varrho\|_{L^2} \geq c_3(1+t)^{-\frac{5}{4}} - C(1+t)^{-\frac{5}{2}}, \tag{6.18}$$

which implies directly

$$\|\partial_t\varrho\|_{L^2} \geq c_3(1+t)^{-\frac{5}{4}}, \tag{6.19}$$

for some large time t_*. Therefore, we complete the proof of this lemma. □

7. Analytic tools.

For ease of use and clear reference, some Sobolev inequalities are listed as follows:

Lemma 7.1. *(i) If $u(x) \in H^1(\mathbb{R}^3)$, then the following inequalities hold:*

$$\|u\|_{L^6} \leq C\|\nabla u\|,$$
$$\|u\|_{L^3} \leq C(\|u\| + \|u\|_{L^6}) \leq C\|u\|_{H^1}.$$

(ii) Assume $u(x) \in H^2(\mathbb{R}^3)$, then

$$\|u\|_{L^\infty} \leq C\|\nabla u\|_{H^1}.$$

Proof. One can found them in [1]. □

Now, we will introduce the Gagliardo-Nirenberg inequality that is frequently used in this paper.

Lemma 7.2. *Let $0 \leq i, j \leq k$; then we have*

$$\|\nabla^i f\|_{L^p} \lesssim \|\nabla^j f\|_{L^q}^{1-\theta} \|\nabla^k f\|_{L^r}^{\theta},$$

where θ satisfies

$$\frac{i}{3} - \frac{1}{p} = \left(\frac{j}{3} - \frac{1}{q}\right)(1-\theta) + \left(\frac{k}{3} - \frac{1}{r}\right)\theta.$$

Especially, while $p = q = r = 2$, we have

$$\|\nabla^i f\|_{L^2} \lesssim \|\nabla^j f\|_{L^2}^{\frac{k-i}{k-j}} \|\nabla^k f\|_{L^2}^{\frac{i-j}{k-j}}.$$

Proof. The proof can be seen in [25]. □

Next, to estimate the L^2-norm of the spatial derivatives of the product of two functions, we shall use the following formula:

Lemma 7.3. *Let $k \geq 1$ be an integer; then one have*

$$\|\nabla^k (fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|\nabla^k g\|_{L^{p_2}} + \|\nabla^k f\|_{L^{p_3}} \|g\|_{L^{p_4}},$$

where $p, p_2, p_3 \in [1, +\infty]$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Proof. For $p = p_2 = p_3 = 2$, it can be proved by using Lemma 7.2. For the general case, one may to refer to [19]. □

From Lemma 7.3, we can deduce the following commutator estimate:

Lemma 7.4. *Let f and g be smooth functions belonging to $H^k \cap L^\infty$ for any integer $k \geq 1$ and define the commutator*

$$g = \nabla^k (fg) - f\nabla^k g.$$

Then we have

$$\|[\nabla^k, f]g\|_{L^p} \lesssim \|\nabla f\|_{L^{p_1}} \|\nabla^{k-1} g\|_{L^{p_2}} + \|\nabla^k f\|_{L^{p_3}} \|g\|_{L^{p_4}}.$$

Here $p_i (i = 1, 2, 3, 4)$ are defined in Lemma 7.3.

Finally, we introduce the estimate for the low–frequency part and the high–frequency part of f.

Lemma 7.5. *If* $2 \le p \le \infty$, $f \in L^p(\mathbb{R}^3)$*, then we have*

$$\|f^l\|_{L^p} + \|f^h\|_{L^p} \lesssim \|f\|_{L^p}.$$

Proof. For $2 \le p \le \infty$, by Young's inequality for convolutions, for the low-frequency, we have

$$\|f^l\|_{L^p} \lesssim \|\mathcal{F}^{-1}\phi\|_{L^1}\|f\|_{L^p} \lesssim \|f\|_{L^p},$$

and hence

$$\|f^h\|_{L^p} \lesssim \|f\|_{L^p} + \|f^l\|_{L^p} \lesssim \|f\|_{L^p}.$$

□

Acknowledgments

This work is partially supported by Guangxi Natural Science Foundation #2019JJG110003, #2019AC20214, and #2019JJA110071.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. R. Adams, J. F. Fournier, *Sobolev spaces*, 2 Eds., New York: Academic Press, 2003.
2. S. J. Ding, J. L. Lin, C. Y. Wang, H. Y. Wen, Compressible hydrodynamic flow of liquid crystals in 1–D, *Discrete Contin. Dyn. Syst.*, **32** (2012), 539–563. http://dx.doi.org/10.3934/dcds.2012.32.539
3. R. J. Duan, H. X. Liu, S. Ukai, T. Yang, Optimal $L^p - L^q$ convergence rate for the compressible Navier-Stokes quations with potential force, *J. Differential Equations.*, **238** (2007), 220–223. https://doi.org/10.1016/j.jde.2007.03.008
4. J. L. Ericksen, Hydrostatic theory of liquid crystals, *Arch. Ration. Mech. Anal.*, **9** (1962), 371–378. https://doi.org/10.1007/BF00253358
5. J. S. Fan, F. Jiang, Large-time behavior of liquid crystal flows with a trigonometric condition in two dimensions, *Commun. Pure Appl. Anal.*, **15** (2016), 73–90. http://dx.doi.org/10.3934/cpaa.2016.15.73
6. E. Feireisl, A. Novotný, H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, *J. Math. Fluid Mech.*, **3** (2001), 358–392. https://doi.org/10.1007/PL00000976
7. Y. Guo, Y. J. Wang, Decay of dissipative equations and negative Sobolev spaces, *Comm. Partial Differential Equations*, **37** (2012), 2165–2208. https://doi.org/10.1080/03605302.2012.696296
8. J. Gao, Q. Tao, Z. A. Yao, Long-time behavior of solution for the compressible nematic liquid crystal flows in $\mathbb{R}^3$, *J. Differential Equations.*, **261** (2016), 2334–2383. https://doi.org/10.1016/j.jde.2016.04.033.

9. J. Gao, Z. Y. Lyu, Z. A. Yao, Lower bound of decay rate for higher–order derivatives of solution to the compressible fluid models of Korteweg type, *Z. Angew. Math. Phys.,* **71** (2020), 1–19. https://doi.org/10.1007/s00033-020-01330-8

10. D. Hoff, Global solutions of the Navier–Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differential Equations.,* **200** (1995), 215–254. https://doi.org/10.1006/jdeq.1995.1111.

11. X. P. Hu, H. Wu, Global solution to the three-dimensional compressible flow of liquid crystals, *SIAM J. Math. Anal.,* **45** (2013), 2678–2699. https://doi.org/10.1137/120898814

12. W. T. Huang, X. Y. Lin, W. W. Wang, Decay-in-time of the highest-order derivatives of solutions for the compressible isentropic MHD equations, *J. Math. Anal. Appl.,* **502** (2021), 125273. https://doi.org/10.1016/j.jmaa.2021.125273.

13. T. Huang, C. Y. Wang, H. Y. Wen, Blow up criterion for compressible nematic liquid crystal flows in dimension three, *Arch. Ration. Mech. Anal.,* **204** (2012), 285–311. https://doi.org/10.1007/s00205-011-0476-1

14. X. D. Huang, J. Li, Z. P. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations, *Comm. Pure Appl. Math.,* **65** (2012), 549–585. https://doi.org/10.1002/cpa.21382

15. F. Jiang, S. Jiang, Asymptotic behaviors of global solutions to the two-dimensional non-resistive MHD equations with large initial perturbations, *Adv. Math.,* **393** (2021), 108084. https://doi.org/10.1016/j.aim.2021.108084

16. F. Jiang, S. Jiang, D. H. Wang, On multi-dimensional compressible flows of nematic liquid crystals with large initial energy in a bounded domain, *J. Funct. Anal.,* **265** (2013), 3369–3397. https://doi.org/10.1016/j.jfa.2013.07.026

17. F. Jiang, S. Jiang, D. H. Wang, Global weak solutions to the equations of compressible flow of nematic liquid crystals in two dimensions, *Arch. Ration. Mech. Anal.,* **214** (2014), 403–451. https://doi.org/10.1007/s00205-014-0768-3

18. F. Jiang, G. C. Wu, X. Zhong, On exponential stability of gravity driven viscoelastic flows, *J. Differential Equations.,* **260** (2016), 7498–7534. https://doi.org/10.1016/j.jde.2016.01.030.

19. N. Ju, Existence and uniqueness of the solution to the dissipative $2D$ quasi-geostrophic equations in the Sobolev space, *Comm. Math. Phys.,* **251** (2004), 365–376. https://doi.org/10.1007/s00220-004-1062-2

20. F. M. Leslie, Some constitutive equations for liquid crystals, *Arch. Ration. Mech. Anal.,* **28** (1968), 265–283. https://doi.org/10.1007/BF00251810

21. H. Li, T. Zhang, Large time behavior of isentropic compressible Navier–Stokes system in $\mathbb{R}^3$, *Math. Meth. Appl. Sci.,* **34** (2011), 670–682. https://doi.org/10.1002/mma.1391

22. J. Li, Z. Xu, J. Zhang, Global existence of classical solutions with large oscillations and vacuum to the three–dimensional compressible nematic liquid crystal flows, *J. Math. Fluid Mech.,* **20** (2018), 2105–2145. https://doi.org/10.1007/s00021-018-0400-7

23. A. Matsumura, T. Nishida, The initial value problems for the equations of motion of viscous and heat-conductive fluids, *Proc. Jpn. Acad. Ser. A,* **55** (1979), 337–342. https://doi.org/10.3792/pjaa.55.337

24. A. Matsumura, T. Nishida, The initial value problems for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.,* **20** (1980), 67–104. https://doi.org/10.1215/kjm/1250522322

25. L. Nirenberg, On elliptic partial diferential equations, *Annali della Scuola Normale Superiore di Pisa,* **13** (1959), 115–162. https://doi.org/10.1007/978-3-642-10926-3_1

26. M. E. Schonbek, Large time behaviour of solutions to the Navier–Stokes equations, *Comm. Partial Differential Equations,* **11** (1986), 733–763. https://doi.org/10.1080/03605308608820443

27. J. Wang, C. G. Xiao, Y. H. Zhang, Optimal large time behavior of the compressible Navier–Stokes–Korteweg system in $\mathbb{R}^3$, *Appl. Math. Lett.,* **120** (2021), 107274. https://doi.org/10.1016/j.aml.2021.107274.

28. Y. J. Wang, Z. Tan, Global existence and optimal decay rate for the strong solutions in H^2 to the compressible Navier-Stokes equations, *Appl. Math. Lett.,* **24** (2011), 1778–1784. https://doi.org/10.1016/j.aml.2011.04.028.

29. X. Y. Wang, W. W. Wang, On global behavior of weak solutions to the Navier–Stokes equations of compressible fluid for $\gamma = 5/3$, *Bound. Value Probl.,* **2015** (2015), 1–13. https://doi.org/10.1186/s13661-015-0443-0

30. W. W. Wang, On global behavior of weak solutions of compressible flows of nematic liquid crystals, *Acta Math. Sci. Ser. B (Engl. Ed.),* **35** (2015), 650–672. https://doi.org/10.1016/S0252-9602(15)30011-4

31. W. W. Wang, Y. Y. Zhao, Time-decay solutions of the initial-boundary value problem of rotating magnetohydrodynamic fluids, *Bound. Value Probl.,* **2017** (2017), 1–31. https://doi.org/10.1186/s13661-017-0845-2

32. G. C. Wu, Z. Tan, Global low–energy weak solution and large–time behavior for the compressible flow of liquid crystals, *J. Differential Equations,* **264** (2018), 6603–6632. https://doi.org/10.1016/j.jde.2018.01.045

33. E. Zuazua, *Time asymptotics for heat and dissipative wave equations*, Preprint Boling Guo Institute of Applied Physics and Computational Mathematics PO Box, 8009, 2003. Available from: `http://www.uam.es/enrique.zuazua`.

AIMS Press