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Research article

Inequalities for unified integral operators of generalized refined convex functions

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Abstract: In this article, the bounds of unified integral operators are studied by using a new notion called refined $(\alpha, h-m)-p$-convex function. The upper and lower bounds in the form of Hadamard inequality are established. From the results of this paper, refinements of well-known inequalities can be obtained by imposing additional conditions.

Keywords: refined $(\alpha, h-m)$-convex function; integral operators; fractional integrals; unified integral operators; bounds
Mathematics Subject Classification: 26D10, 31A10, 26A33

1. Introduction and preliminaries

Convex functions and their applications to construct new definitions are studied extensively to obtain generalized and new inequalities. The aim of this paper is to utilize a new notion of convexity so called refined $(\alpha, h-m)-p$-convex function for the establishment of new bounds of unified integral operators. These bounds will provide the refinements of inequalities already exist in literature which have been obtained for different kinds of fractional integral operators of several types of convexities. Some recent results related to findings of this paper we refer the readers to [1–6]. Next, we give definitions of functions and integral operators which will be useful in the formation of results of this paper.

All the functions are considered to be real valued until specified.

Definition 1.1. [7] Let $h : J \to \mathbb{R}$ be a function with $h \not\equiv 0$ and $(0,1) \subseteq J$. A positive function Ω is called refined $(\alpha, h-m)$-convex function, if $h \geq 0$ and for each $u, v \in [0, b] \subseteq \mathbb{R}$, we have

$$\Omega(tu + m(1-t)v) \leq h(t^{\alpha})h(1-t^{\alpha})\left(\Omega(u) + m\Omega(v)\right), \tag{1.1}$$

where $(\alpha, m) \in (0, 1]^2$ and $t \in (0, 1)$.

Definition 1.2. [8] Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Let $I \subset (0, \infty)$ be an interval and $p \in \mathbb{R} \setminus \{0\}$. A positive function Ω is said to be refined $(\alpha, h-m) - p$-convex, if

$$\Omega((tu^p + m(1-t)v^p))^{\frac{1}{p}}) \leq h(t^{\alpha})h(1-t^{\alpha})\left(\Omega(u) + m\Omega(v)\right) \tag{1.2}$$

holds provided $(tu^p + m(1-t)v^p))^{\frac{1}{p}} \in I$ for $t \in [0, 1]$ and $(\alpha, m) \in [0, 1]^2$.

It gives refinements of several types of convexities when $0 < h(t) < 1$, see [8]. Integral operators play a vital role in the theory of inequalities. In the recent era, integral operators are being used extensively to produce new literature results. For references see [9–14]. Unified integral operators are the well-known operators in the literature introduced in [15]. Also, these integrals are continuous and bounded.

In this paper, we are interested in giving the refined bounds of unified integral operators along with Hadamard-type inequalities for refined $(\alpha, h-m) - p$-convex functions.

Definition 1.3. [16] Let $\Omega \in L_1[u, v]$ and $x \in [u, v]$, also let $\sigma, \kappa, \alpha, \xi, \gamma, \iota \in \mathbb{C}$, $\Re(\kappa), \Re(\alpha), \Re(\xi) > 0$, $\Re(\iota) > \Re(\gamma) > 0$ with $p' \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\kappa)$. Then the generalized fractional integral operators $\epsilon^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,\sigma,u^+}\Omega$ and $\epsilon^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,\sigma,v^-}\Omega$ are defined by:

$$\left(\epsilon^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,u^+}\Omega\right)(x, \sigma; p') = \int_u^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}(\sigma(x-t)^{\kappa}; p')\Omega(t)dt, \tag{1.3}$$

$$\left(\epsilon^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,\sigma,v^-}\Omega\right)(x, \sigma; p') = \int_x^v (t-x)^{\alpha-1} E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}(\sigma(t-x)^{\kappa}; p')\Omega(t)dt, \tag{1.4}$$

where $E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}(t; p')$ is the extended generalized Mittag-Leffler function defined as:

$$E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}(t; p') = \sum_{n=0}^{\infty} \frac{\rho_{p'}(\gamma + nk, \iota - \gamma)}{\rho(\gamma, \iota - \gamma)} \frac{(\iota)_{nk}}{\Gamma(\kappa n + \alpha)} \frac{t^n}{(\xi)_{n\delta}}. \tag{1.5}$$

Definition 1.4. [17] Let Ω, Δ be real valued functions over $[u, v]$ with $0 < u < v$, where Ω is positive and integrable and Δ is differentiable and strictly increasing. Also let $\frac{\Upsilon}{x}$ be an increasing function on $[u, \infty)$ and $\alpha, \xi, \gamma, \iota \in \mathbb{C}$, $p', \kappa, \delta \geq 0$ and $0 < k \leq \delta + \kappa$. Then for $x \in [u, v]$ the left and right integral operators are defined by:

$$({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,u^+}\Omega)(x, \sigma; p') = \int_u^x J^y_x(E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}, \Delta; \Upsilon)\Delta'(y)\Omega(y)dy, \tag{1.6}$$

$$({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,v^-}\Omega)(x, \sigma; p') = \int_x^v J^x_y(E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}, \Delta; \Upsilon)\Delta'(y)\Omega(y)dy, \tag{1.7}$$

where

$$J^y_x(E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}, \Delta; \Upsilon) = \frac{\Upsilon(\Delta(x) - \Delta(y))}{\Delta(x) - \Delta(y)} E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}(\sigma(\Delta(x) - \Delta(y))^{\kappa}; p). \tag{1.8}$$

The kernel (1.8) involves an increasing function and the Mittag-Leffler functions from which one can deduce several fractional integral operators by choosing the particular parameters, see [17, Remarks 6 and 7].

The rest of the paper is organized as follows: In Section 2, the bounds of unified integral operators (1.6) and (1.7) are obtained using refined $(\alpha, h-m)-p$-convex functions. Their extensions are also obtained by imposing the condition $0 < h(t) < 1$. These extensions give refinements of already known results. In Section 3, some results for new deduced definitions are also presented.

2. Main results

Throughout the paper we use the following notations:

$$\int_0^1 h(s)^\alpha h(1-s^\alpha)\Delta'(x-s(x-u))ds = H_x{}^u(s^\alpha; h, \Delta), \text{ and } \wedge(t) = t^{\frac{1}{p}}.$$

Theorem 2.1. *Let Ω be a positive, refined $(\alpha, h-m)-p$-convex and integrable function over $[u, v]$. Also, let $\frac{\Upsilon}{x}$ be an increasing function on $[u, v]$ and Δ be strictly increasing and differentiable function on (u, v). Then, for $\beta, \xi, \gamma, \iota \in \mathbb{R}$, $p', \kappa, \vartheta, \delta \geq 0$, $0 < k \leq \delta+\kappa$ and $0 < k \leq \delta+\vartheta$ following result holds:*

$$\begin{aligned}
&\left({}_\Delta\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^+}\Omega\circ\wedge\right)(x,\sigma;p') + \left({}_\Delta\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^-}\Omega\circ\wedge\right)(x,\sigma;p')\\
&\leq J^u_x(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((u)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(x-u)H_x{}^u(s^\alpha;h,\Delta)\\
&\quad + J^x_v(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(v-x)H^x_v(z^\alpha;h,\Delta),
\end{aligned} \tag{2.1}$$

where $p > 0$.

Proof. For the functions $\frac{\Upsilon}{x}$ and Δ, the following inequality holds

$$J^t_x(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\Delta'(t) \leq J^u_x(E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi},\Delta;\Upsilon)\Delta'(t). \tag{2.2}$$

Using refined $(\alpha, h-m)-p$-convexity of Ω, one can have

$$\Omega\left((t)^{\frac{1}{p}}\right) \leq h\left(\frac{x-t}{x-u}\right)^\alpha h\left(1-\left(\frac{x-t}{x-u}\right)^\alpha\right)\left(\Omega\left((u)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right). \tag{2.3}$$

From (2.2) and (2.3), the following integral inequality is obtained:

$$\begin{aligned}
\int_u^x J^t_x(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\Delta'(t)\Omega\left((t)^{\frac{1}{p}}\right)dt &\leq J^u_x(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((u)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)\\
&\quad\times\int_u^x h\left(\frac{x-t}{x-u}\right)^\alpha h\left(1-\left(\frac{x-t}{x-u}\right)^\alpha\right)\Delta'(t)dt.
\end{aligned} \tag{2.4}$$

Using (1.6) of Definition 1.4 on the left side of inequality (2.4) and making change of variable by setting $s = \frac{x-t}{x-u}$ on the right hand side of above inequality, we get

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^+}\Omega\circ\wedge\right)(x,\sigma;p')\leq J^u_x(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\,(x-u)\left(\Omega\left((u)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)$$
$$\times\int_0^1 h(s^\alpha)h(1-s^\alpha)\Delta'(x-s(x-u))ds. \tag{2.5}$$

The above inequality leads to the following inequality

$$\begin{aligned}&\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^+}\Omega\circ\wedge\right)(x,\sigma;p')\\&\leq J^u_x(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\,(x-u)\left(\Omega\left((u)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(x-u)H^u_x(s^\alpha;h,\Delta).\end{aligned} \tag{2.6}$$

Also, for $t\in(x,v]$, $x\in(u,v)$, we can write

$$J^x_t(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\Delta'(t)\leq J^x_v(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\Delta'(t). \tag{2.7}$$

By definition of refined $(\alpha,h-m)-p$-convex function, one can write

$$\Omega\left((t)^{\frac{1}{p}}\right)\leq h\left(\frac{t-x}{v-x}\right)^\alpha h\left(1-\left(\frac{t-x}{v-x}\right)^\alpha\right)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right). \tag{2.8}$$

From (2.7) and (2.8), the following integral inequality is obtained:

$$\begin{aligned}&\int_x^v J^x_t(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\Delta'(t)\Omega\left((t)^{\frac{1}{p}}\right)dt\\&\leq J^x_v(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\times\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)\int_x^v h\left(\frac{t-x}{v-x}\right)^\alpha h\left(1-\left(\frac{t-x}{v-x}\right)^\alpha\right)\Delta'(t)dt.\end{aligned}$$

Using (1.7) of Definition 1.4 on left hand side and making change of variable by setting $z=\frac{t-x}{v-x}$ on right hand side of above inequality, we get

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^-}\Omega\circ\wedge\right)(x,\sigma;p')\leq J^x_v(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)$$
$$\times(v-x)\int_0^1 h(z^\alpha)h(1-z^\alpha)\Delta'(x+z(v-x)dz. \tag{2.9}$$

The above inequality leads to the following inequality

$$\begin{aligned}&\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^-}\Omega\circ\wedge\right)(x,\sigma;p')\\&\leq J^x_v(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(v-x)H^x_v(z^\alpha;h,\Delta).\end{aligned} \tag{2.10}$$

Combining (2.6) and (2.10), the required inequality (2.1) is obtained. □

Next, we give the extension of Theorem 2.1 and refinement of [18, Theorem 1].

Theorem 2.2. *Under the assumptions of Theorem 2.1, if* $0 < h(t) < 1$*, then the following result holds:*

$$\begin{aligned}
&\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^{+}}\Omega\circ\wedge\right)(x,\sigma;p')+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^{-}}\Omega\circ\wedge\right)(x,\sigma;p')\\
&\leq J^{u}_{x}(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((u)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(x-u)H_{x}{}^{u}(s^{\alpha};h,\Delta)\\
&\quad+J^{x}_{v}(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(v-x)H^{x}_{v}(z^{\alpha};h,\Delta)\\
&\leq J^{u}_{x}(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)(x-u)\left(\Omega\left((u)^{\frac{1}{p}}\right)H^{u}_{x}(s^{\alpha};h,\Delta)\right.\\
&\quad\left.+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)H^{u}_{x}(1-s^{\alpha};h,\Delta)\right)+J^{x}_{v}(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)(v-x)\\
&\quad\times\left(\Omega\left((v)^{\frac{1}{p}}\right)H^{x}_{v}(z^{\alpha};h,\Delta)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)H^{x}_{v}(1-z^{\alpha};h,\Delta)\right).
\end{aligned}\tag{2.11}$$

Proof. From (2.3) and (2.8) one can see that for $0 < h(t) < 1$

$$\begin{aligned}
\Omega\left((t)^{\frac{1}{p}}\right)&\leq h\left(\frac{x-t}{x-u}\right)^{\alpha}h\left(1-\left(\frac{x-t}{x-u}\right)^{\alpha}\right)\left(\Omega\left((u)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)\\
&\leq h\left(\frac{x-t}{x-u}\right)^{\alpha}\Omega\left((u)^{\frac{1}{p}}\right)+mh\left(1-\left(\frac{x-t}{x-u}\right)^{\alpha}\right)\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)
\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}
\Omega\left((t)^{\frac{1}{p}}\right)&\leq h\left(\frac{t-x}{v-x}\right)^{\alpha}h\left(1-\left(\frac{t-x}{v-x}\right)^{\alpha}\right)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)\\
&\leq h\left(\frac{t-x}{v-x}\right)^{\alpha}\Omega\left((v)^{\frac{1}{p}}\right)+mh\left(1-\left(\frac{t-x}{v-x}\right)^{\alpha}\right)\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right).
\end{aligned}\tag{2.13}$$

Hence, by following the proof of Theorem 2.1, one can get (2.11). □

Corollary 1. *For* $\kappa=\vartheta$*,* (2.1) *gives the following result:*

$$\begin{aligned}
&\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,u^{+}}\Omega\circ\wedge\right)(x,\sigma;p')+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,v^{-}}\Omega\circ\wedge\right)(x,\sigma;p')\\
&\leq J^{u}_{x}(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)(x-u)\left(\Omega\left((u)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)H_{x}{}^{u}(s^{\alpha};h,\Delta)\\
&\quad+J^{x}_{v}(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)(v-x)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)H^{x}_{v}(z^{\alpha};h,\Delta).
\end{aligned}\tag{2.14}$$

The following corollary presents the result of Theorem 2.1 for refined $(h-m)-p$-convex function.

Corollary 2. *Under the assumptions of Theorem 2.1, if* $0 < h(t) < 1$ *then the following result holds:*

$$\begin{aligned}
&\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^{+}}\Omega\circ\wedge\right)(x,\sigma;p')+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^{-}}\Omega\circ\wedge\right)(x,\sigma;p')\\
&\leq\left(J^{u}_{x}(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((u)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(\Delta(x)-\Delta(u))\right.\\
&\quad\left.+J^{x}_{v}(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(\Delta(v)-\Delta(x))\right)\|h\|^{2}_{\infty}.
\end{aligned}\tag{2.15}$$

Proof. Using $\alpha = 1$ and $h \in L_\infty[0,1]$ in (2.1), we get inequality (2.15). □

Corollary 3. *For $p' = \sigma = 0$ inequality* (2.1) *gives the following inequality for the fractional integral operator defined in [15].*

$$\frac{\left({}_{\Delta}\mathbb{F}^{\Upsilon,\Delta}_{u^+}\Omega \circ \wedge\right)(x)}{\Gamma(\kappa)} + \frac{\left({}_{\Delta}\mathbb{F}^{\Upsilon,\Delta}_{v^-}\Omega \circ \wedge\right)(x)}{\Gamma(\vartheta)}$$
$$\leq \frac{\Upsilon(\Delta(x)-\Delta(u))}{\Gamma(\kappa)(\Delta(x)-\Delta(u))}\left(\Omega\left((u)^{\frac{1}{p}}\right) + m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(x-u)H_x{}^u(s^\alpha; h, \Delta)$$
$$+ \frac{\Upsilon(\Delta(v)-\Delta(x))}{\Gamma(\vartheta)(\Delta(v)-\Delta(x))}\left(\Omega\left((v)^{\frac{1}{p}}\right) + m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(v-x)H_v^x(z^\alpha; h, \Delta). \tag{2.16}$$

Remark 1. (i) For $p = 1$, the inequality (2.1) coincides with [19, Theorem 1].

(ii) For $p = 1$, the inequality (2.11) coincides with [19, Theorem 2].

(iii) For $p = 1$, the inequality (2.14) coincides with [19, Corollary 1].

(iv) For $0 < h(t) < 1$, the inequality (2.1) coincides with [18, Theorem 1].

(v) For $0 < h(t) < 1$, the inequality (2.16) coincides with [18, Corollary 1].

(vi) For $\Upsilon(x) = \frac{x^{\frac{\alpha'}{k}}\Gamma(\alpha')}{k\Gamma_k(\alpha')}$, $\alpha' > k > 0$ and $p' = \sigma = 0$ along with the condition of (i), the inequality (2.1) coincides with [20, Theorem 10].

(vii) For $k = 1$ along with the conditions of (vi), the inequality (2.14) coincides with [20, Theorem 6].

(viii) For Δ as identity function along with the conditions of (vi), the inequality (2.14) coincides with [7, Theorem 5].

(ix) For Δ as identity function and $k = 1$ along with the conditions of (vi), the inequality (2.14) coincides with [7, Theorem 1].

(x) For $h(t) = t$ and $m = 1 = \alpha$ along with the condition of (i), the inequality (2.1) coincides with [21, Theorem 4].

(xi) For $h(t) = t$ and $m = 1 = \alpha$ along with the condition of (i), the inequality (2.14) coincides with [21, Corollary 1].

(xii) For $h(t) = t$ and $m = 1 = \alpha$ along with the conditions of (vi), the inequality (2.14) coincides with [22, Theorem 3.1].

(xiii) For $h(t) \leq \frac{1}{\sqrt{2}}$ along with the conditions of (ix), the inequality (2.14) coincides with [7, Theorem 2].

(xiv) For $h(t) = t$ and $m = 1 = \alpha$ along with the conditions of (viii), the inequality (2.14) coincides with [7, Corollary 8].

(xv) For $\alpha = 1$ and $h(t) = t$ a along with the conditions of (viii), the inequality (2.14) coincides with [7, Corollary 14].

(xvi) For $h(t) = t^s$ and $\alpha = 1$ along with the conditions of (viii), the inequality (2.14) coincides with [7, Corollary 15].

(xvii) For $h(t) = t$ and $\alpha = 1$ along with the conditions of (viii), the inequality (2.14) coincides with [7, Corollary 16].

(xviii) For $h(t) = t$ and $m = 1 = \alpha$ along with the conditions of (ix), the inequality (2.14) coincides with [7, Corollary 1].

(xix) For $\alpha = 1$ and $h(t) = t$ a along with the conditions of (ix), the inequality (2.14) coincides with [7, Corollary 2].

(xx) For $h(t) = t^s$ and $\alpha = 1$ along with the conditions of (ix), the inequality (2.14) coincides with [7, Corollary 4].

(xxi) For $h(t) = t$ and $\alpha = 1$ along with the conditions of (ix), the inequality (2.14) coincides with [7, Corollary 5].

By using $0 < h(t) < 1$ and making different choices of functions h and Δ and other parameters in (2.1) one can get the refinements for many known classes of convex functions and integral operators which are given in [18, Remark 2].

Next we give a lemma which we will use in upcoming Theorem 2.4.

Lemma 2.3. *Let $\Omega : [0,\infty) \to \mathbb{R}$ be refined $(\alpha, h-m)-p$-convex function. If $\Omega(x) = \Omega\left(\frac{u^p+v^p-x^p}{m}\right)^{\frac{1}{p}}$, $x \in [u, v]$, then the following inequality holds:*

$$\Omega\left(\frac{u^p+v^p}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(m+1)\Omega(x). \tag{2.17}$$

Proof. Since Ω is refined $(\alpha, h-m)-p$ convex, then following inequality holds:

$$\begin{aligned}
&\Omega\left(\frac{u^p+v^p}{2}\right)\\
\leq& h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left[\Omega\left(\left(\frac{x^p-u^p}{v^p-u^p}v^p+\frac{v^p-x^p}{v^p-u^p}u^p\right)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{\frac{x^p-u^p}{v^p-u^p}u^p+\frac{v^p-x^p}{v^p-u^p}v^p}{m}\right)^{\frac{1}{p}}\right)\right]\\
\leq& h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(\Omega(x)+m\Omega\left(\frac{u^p+v^p-x^p}{m}\right)^{\frac{1}{p}}\right).
\end{aligned} \tag{2.18}$$

Using $\Omega(x) = \Omega\left(\frac{u^p+v^p-x^p}{m}\right)^{\frac{1}{p}}$ in above inequality, we get (2.17). □

Remark 2. (i) For $p = 1$, the inequality (2.17) coincides with [19, Lemma 1].

(ii) For $p = 1, h(t) = t$ and $m = \alpha = 1$, the inequality (2.17) coincides with [21, Lemma 1].

(iii) For $0 < h(t) < 1$, the inequality (2.17) gives refinement of [18, Lemma 1].

Theorem 2.4. *Under the assumptions of Theorem 2.1, the following result holds for $\Omega(x) = \Omega\left(\frac{u^p+v^p-x^p}{m}\right)^{\frac{1}{p}}$:*

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(m+1)}\Omega\left(\frac{u^p+v^p}{2}\right)\left(\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^-}1\right)(u,\sigma;p')+\left({}_{\Delta}F^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^+}1\right)(v,\sigma;p')\right)\\
&\leq \left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,v^-}\Omega\circ\wedge\right)(u,\sigma;p')+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,u^+}\Omega\circ\wedge\right)(v,\sigma;p')\\
&\leq (v-u)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{u}{m}\right)^{\frac{1}{p}}\right)\right)H^u_v(z^\alpha;h,\Delta)\left[J^u_v(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)+J^u_v(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\right].
\end{aligned} \tag{2.19}$$

Proof. For the kernel defined in (1.8) and function Δ, the following inequality holds:

$$J^u_x(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\Delta'(x) \leq J^u_v(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\Delta'(x),\ x \in (u,v). \tag{2.20}$$

Using refined $(\alpha, h-m)$-convexity of Ω, we have

$$\Omega\left((x)^{\frac{1}{p}}\right) \leq h\left(\frac{x-u}{v-u}\right)^{\alpha} h\left(1-\left(\frac{x-u}{v-u}\right)^{\alpha}\right)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{u}{m}\right)^{\frac{1}{p}}\right)\right). \tag{2.21}$$

From (2.20) and (2.21), the following integral inequality is obtained:

$$\begin{aligned}\int_u^v J_x^u(E_{\vartheta,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\Omega\left((x)^{\frac{1}{p}}\right)\Delta'(x)dx \leq J_v^u(E_{\vartheta,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{u}{m}\right)^{\frac{1}{p}}\right)\right)\\ \times\int_{x_1}^v h\left(\frac{x-u}{v-u}\right)^{\alpha} h\left(1-\left(\frac{x-u}{v-u}\right)^{\alpha}\right)\Delta'(x)dx.\end{aligned}$$

Using (1.7) of Definition 1.4 on right hand side, making change of variable by setting $z=\frac{x-u}{v-u}$ on right hand side of above inequality, we get

$$\left({}_{\Delta}\mathbb{F}_{\vartheta,\beta,\xi,v^-}^{\Upsilon,\gamma,\delta,k,\iota}\Omega\circ\wedge\right)(u,\sigma;p') \leq J_v^u(E_{\vartheta,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)(v-u)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{u}{m}\right)^{\frac{1}{p}}\right)\right)H_v^u(z^{\alpha};h,\Delta). \tag{2.22}$$

The following inequality also holds true for $x\in(u,v)$:

$$J_v^x(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\Delta'(x) \leq J_v^u(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\Delta'(x). \tag{2.23}$$

From (2.21) and (2.23), the following integral inequality is obtained:

$$\begin{aligned}\int_u^v J_v^x(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\Delta'(x)\Omega\left((x)^{\frac{1}{p}}\right)dx \leq J_v^u(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{u}{m}\right)^{\frac{1}{p}}\right)\right)\\ \times\int_u^v h\left(\frac{x-u}{v-u}\right)^{\alpha} h\left(1-\left(\frac{x-u}{v-u}\right)^{\alpha}\right)\Delta'(x)dx.\end{aligned}$$

Using (1.6) of Definition 1.4 on left hand side and making change of variable on right hand side of above inequality, we get

$$\left({}_{\Delta}\mathbb{F}_{\kappa,\beta,\xi,u^+}^{\Upsilon,\gamma,\delta,k,\iota}\Omega\circ\wedge\right)(v,\sigma;p') \leq J_v^u(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)(v-u)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{u}{m}\right)^{\frac{1}{p}}\right)\right)H_v^u(z^{\alpha};h,\Delta). \tag{2.24}$$

Now, using Lemma 2.3, we can write

$$\begin{aligned}&\int_u^v \Omega\left(\frac{u^p+v^p}{2}\right)J_x^u(E_{\vartheta,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\Delta'(x)dx\\ &\leq h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)\int_u^v J_x^u(E_{\vartheta,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\Delta'(x)\Omega(x)dx,\end{aligned}$$

which by using (1.7) of Definition 1.4 gives the following integral inequality:

$$\frac{1}{h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)}\Omega\left(\frac{u^p+v^p}{2}\right)\left({}_{\Delta}\mathbb{F}_{\vartheta,\beta,\xi,v^-}^{\Upsilon,\gamma,\delta,k,\iota}1\right)(u,\sigma;p') \leq \left({}_{\Delta}\mathbb{F}_{\vartheta,\beta,\xi,v^-}^{\Upsilon,\gamma,\delta,k,\iota}\Omega\right)(u,\sigma;p'). \tag{2.25}$$

Again, using Lemma 2.3, we can write

$$\Omega\left(\frac{u^p+v^p}{2}\right) J_v^x(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\Delta'(x)dx \tag{2.26}$$
$$\leq h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(m+1)\int_u^v J_v^x(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\Delta'(x)\Omega(x)dx,$$

which by using (1.6) of Definition 1.4 gives the following fractional integral inequality:

$$\frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(m+1)}\Omega\left(\frac{u^p+v^p}{2}\right)\left({}_\Delta\mathbb{F}_{\kappa,\beta,\xi,u^+}^{\Upsilon,\gamma,\delta,k,\iota}1\right)(v,\sigma;p') \leq \left({}_\Delta\mathbb{F}_{\kappa,\beta,\xi,u^+}^{\Upsilon,\gamma,\delta,k,\iota}\Omega\right)(v,\sigma;p'). \tag{2.27}$$

By Combining (2.22), (2.24), (2.25) and (2.27), the required inequality (2.19) is obtained. □

The following theorem is the extension of Theorem 2.4 and refinement of [18, Theorem 2].

Theorem 2.5. *Under the assumptions of Theorem 2.4, the following result holds:*

$$\frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(m+1)}\Omega\left(\frac{u^p+v^p}{2}\right)\left(\left({}_\Delta\mathbb{F}_{\vartheta,\beta,\xi,v^-}^{\Upsilon,\gamma,\delta,k,\iota}1\right)(u,\sigma;p')+\left({}_\Delta F_{\kappa,\beta,\xi,u^+}^{\Upsilon,\gamma,\delta,k,\iota}1\right)(v,\sigma;p')\right)$$
$$\leq \frac{1}{h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)}\Omega\left(\frac{u^p+v^p}{2}\right)\left(\left({}_\Delta\mathbb{F}_{\vartheta,\beta,\xi,v^-}^{\Upsilon,\gamma,\delta,k,\iota}1\right)(u,\sigma;p')+\left({}_\Delta F_{\kappa,\beta,\xi,u^+}^{\Upsilon,\gamma,\delta,k,\iota}1\right)(v,\sigma;p)\right)$$
$$\leq \left({}_\Delta\mathbb{F}_{\kappa,\beta,\xi,v^-}^{\Upsilon,\gamma,\delta,k,\iota}\Omega\right)(u,\sigma;p')+\left({}_\Delta\mathbb{F}_{\vartheta,\beta,\xi,u^+}^{\Upsilon,\gamma,\delta,k,\iota}\Omega\right)(v,\sigma;p')$$
$$\leq (v-u)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{u}{m}\right)^{\frac{1}{p}}\right)\right)H_v^u(z^\alpha;h,\Delta)\left[J_v^u(E_{\vartheta,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)+J_v^u(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\right]$$
$$\leq (v-u)\left(J_v^u(E_{\vartheta,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\Omega\left((v)^{\frac{1}{p}}\right)H_v^u(z^\alpha;h,\Delta)\right.$$
$$\left.+mJ_v^u(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)\Omega\left(\left(\frac{u}{m}\right)^{\frac{1}{p}}\right)H_v^u(1-z^\alpha;h,\Delta)\right). \tag{2.28}$$

Proof. From (2.21) one can see that for $0<h(t)<1$

$$\Omega\left((x)^{\frac{1}{p}}\right)\leq h\left(\frac{x-u}{v-u}\right)^\alpha h\left(1-\left(\frac{x-u}{v-u}\right)^\alpha\right)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{u}{m}\right)^{\frac{1}{p}}\right)\right)$$
$$\leq h\left(\frac{x-u}{v-u}\right)^\alpha\Omega\left((v)^{\frac{1}{p}}\right)+mh\left(1-\left(\frac{x-u}{v-u}\right)\right)^\alpha\Omega\left(\left(\frac{u}{m}\right)^{\frac{1}{p}}\right). \tag{2.29}$$

Hence by following the proof of Theorem 2.4, one can get (2.28). □

Corollary 4. *For $\kappa=\vartheta$,* (2.19) *gives the following result:*

$$\frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(m+1)}\Omega\left(\frac{u^p+v^p}{2}\right)\left(\left({}_\Delta\mathbb{F}_{\kappa,\beta,\xi,v^-}^{\Upsilon,\gamma,\delta,k,\iota}1\right)(u,\sigma;p')+\left({}_\Delta F_{\kappa,\beta,\xi,u^+}^{\Upsilon,\gamma,\delta,k,\iota}1\right)(v,\sigma;p')\right)$$
$$\leq \left({}_\Delta\mathbb{F}_{\kappa,\beta,\xi,v^-}^{\Upsilon,\gamma,\delta,k,\iota}\Omega\circ\wedge\right)(u,\sigma;p')+\left({}_\Delta\mathbb{F}_{\kappa,\beta,\xi,u^+}^{\Upsilon,\gamma,\delta,k,\iota}\Omega\circ\wedge\right)(v,\sigma;p')$$
$$\leq 2(v-u)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{u}{m}\right)^{\frac{1}{p}}\right)\right)J_v^u(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon)H_v^u(z^\alpha;h,\Delta). \tag{2.30}$$

The following corollary presents the result of Theorem 2.4 for refined $(h-m)-p$-convex function.

Corollary 5. *Under the assumptions of Theorem 2.4, the following result holds:*

$$\frac{1}{h^2\left(\frac{1}{2}\right)(m+1)}\Omega\left(\frac{u^p+v^p}{2}\right)\left(\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,v^-}1\right)(u,\sigma;p')+\left({}_{\Delta}F^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^+}1\right)(v,\sigma;p')\right)$$
$$\leq\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,v^-}\Omega\circ\wedge\right)(u,\sigma;p')+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^+}\Omega\circ\wedge\right)(v,\sigma;p')$$
$$\leq 2\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{u}{m}\right)^{\frac{1}{p}}\right)\right)J^u_v(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\,(\Delta(v)-\Delta(u))\,\|h\|^2_{\infty}. \tag{2.31}$$

Proof. For $h\in L_\infty[0,1]$ and $\alpha=1$ in (2.19), one can get (2.31). □

Corollary 6. *For $p'=\sigma=0$, inequality* (2.19) *gives the following inequality:*

$$\frac{\Omega\left(\frac{u^p+v^p}{2}\right)}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(m+1)}\left[\frac{\left({}_{\Delta}\mathbb{F}^{\Upsilon,\Delta}_{u^+}1\right)(x)}{\Gamma(\kappa)}+\frac{\left({}_{\Delta}\mathbb{F}^{\Upsilon,\Delta}_{v^-}1\right)(x)}{\Gamma(\vartheta)}\right]$$
$$\leq\frac{\left({}_{\Delta}\mathbb{F}^{\Upsilon,\Delta}_{u^+}\Omega\circ\wedge\right)(x)}{\Gamma(\kappa)}+\frac{\left({}_{\Delta}\mathbb{F}^{\Upsilon,\Delta}_{v^-}\Omega\circ\wedge\right)(x)}{\Gamma(\vartheta)}$$
$$\leq\frac{\Upsilon(\Delta(v)-\Delta(u))(v-u)H^u_v(z^\alpha;h,\Delta)}{\Delta(v)-\Delta(u)}\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)\left(\frac{1}{\Gamma(\vartheta)}+\frac{1}{\Gamma(\kappa)}\right). \tag{2.32}$$

Remark 3. (i) For $p=1$, the inequality (2.19) coincides with [19, Theorem 3].
(ii) For $p=1$, the inequality (2.30) coincides with [19, Corollary 3].
(iii) For $h(t)=t$ and $m=p=1=\alpha$, the inequality (2.19) coincides with [21, Theorem 5].
(iv) For $h(t)=t$ and $m=p=1=\alpha$, the inequality (2.30) coincides with [21, Corollary 2].

For $0<h(t)<1$ in (2.19), we can get refinements for different classes of convex functions and integral operators given in [18, Remark 3].

Theorem 2.6. *Let Ω,Δ be differentiable functions such that $|\Omega'|$ is refined $(\alpha,h-m)-p$-convex and Δ be strictly increasing over $[u,v]$ and differentiable over $[u,v]$. Also, $\frac{\Upsilon}{x}$ be an increasing function on $[u,v]$ and $\beta,\xi,\gamma,\iota\in\mathbb{C}$, $p',\kappa,\vartheta,\delta\geq 0$ and $0<k\leq\delta+\kappa$ and $0<k\leq\delta+\vartheta$. Then for $x\in(u,v)$, we have*

$$\left|\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^+}(\Omega*\Delta)\circ\wedge\right)(x,\sigma;p')+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^-}(\Omega*\Delta)\circ\wedge\right)(x,\sigma;p')\right|$$
$$\leq J^u_x(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)(x-u)\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right|+m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)H_x{}^u(s^\alpha;h,\Delta)$$
$$+J^x_v(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)(v-x)\left(\left|\Omega'\left((v)^{\frac{1}{p}}\right)\right|+m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)H^x_v(z^\alpha;h,\Delta), \tag{2.33}$$

where

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^+}\Omega*\Delta\right)(x,\sigma;p')=\int_u^x J^t_x(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\Delta'(t)\Omega'(t)dt \tag{2.34}$$

and

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^-}\Omega*\Delta\right)(x,\sigma;p')=\int_x^v J^x_t(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\Delta'(t)\Omega'(t)dt, \tag{2.35}$$

where $p>0$.

Proof. Using refined $(\alpha, h-m)-p$-convexity of $|\Omega'|$ over $[u,v]$ implies

$$\left|\Omega'\left((t)^{\frac{1}{p}}\right)\right| \leq h\left(\frac{x-t}{x-u}\right)^{\alpha} h\left(1-\left(\frac{x-t}{x-u}\right)^{\alpha}\right)\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right| + m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right). \tag{2.36}$$

Absolute value property implies to the following relation:

$$\begin{aligned} &-h\left(\frac{x-t}{x-u}\right)^{\alpha} h\left(1-\left(\frac{x-t}{x-u}\right)^{\alpha}\right)\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right| + m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right) \\ &\leq \Omega'\left((t)^{\frac{1}{p}}\right) \\ &\leq h\left(\frac{x-t}{x-u}\right)^{\alpha} h\left(1-\left(\frac{x-t}{x-u}\right)^{\alpha}\right)\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right| + m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right). \end{aligned} \tag{2.37}$$

From (2.2) and second inequality of (2.37) the following inequality is obtained:

$$\begin{aligned} \int_{x_1}^{x} J_x^t(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota}, \Delta; \Upsilon)\Delta'(t)\Omega'\left((t)^{\frac{1}{p}}\right)dt \leq J_x^u(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota}, \Delta; \Upsilon)\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right| + m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right) \\ \times \int_u^x h\left(\frac{x-t}{x-u}\right)^{\alpha} h\left(1-\left(\frac{x-t}{x-u}\right)^{\alpha}\right)\Delta'(t)dt, \end{aligned} \tag{2.38}$$

which leads to the following fractional integral inequality:

$$\begin{aligned} &\left({}_{\Delta}\mathbb{F}_{\kappa,\beta,\xi,u^+}^{\Upsilon,\gamma,\delta,k,\iota}(\Omega * \Delta) \circ \wedge\right)(x,\sigma;p') \\ &\leq J_x^u(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota}, \Delta; \Upsilon)(x-u)\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right| + m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)H_x{}^u(s^{\alpha};h,\Delta). \end{aligned} \tag{2.39}$$

Also, inequality (2.2) and first inequality of (2.37) gives the following fractional integral inequality:

$$\begin{aligned} &\left({}_{\Delta}\mathbb{F}_{\kappa,\beta,\xi,u^+}^{\Upsilon,\gamma,\delta,k,\iota}(\Omega * \Delta) \circ \wedge\right)(x,\sigma;p') \\ &\geq -J_x^u(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota}, \Delta; \Upsilon)(x-u)\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right| + m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)H_x{}^u(s^{\alpha};h,\Delta). \end{aligned} \tag{2.40}$$

Again, using refined $(\alpha, h-m)-p$-convexity of $|\Omega'|$ over $[u,v]$, we can write

$$|\Omega'(t)^{\frac{1}{p}}| \leq h\left(\frac{t-x}{v-x}\right)^{\alpha} h\left(1-\left(\frac{t-x}{v-x}\right)^{\alpha}\right)(m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right| + \left|\Omega'\left((v)^{\frac{1}{p}}\right)\right|) \tag{2.41}$$

and

$$\begin{aligned} &-h\left(\frac{t-x}{v-x}\right)^{\alpha} h\left(1-\left(\frac{t-x}{v-x}\right)^{\alpha}\right)\left(m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right| + \left|\Omega'\left((v)^{\frac{1}{p}}\right)\right|\right) \\ &\leq \Omega'\left((t)^{\frac{1}{p}}\right) \\ &\leq h\left(\frac{t-x}{v-x}\right)^{\alpha} h\left(1-\left(\frac{t-x}{v-x}\right)^{\alpha}\right)\left(m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right| + \left|\Omega'\left((v)^{\frac{1}{p}}\right)\right|\right). \end{aligned} \tag{2.42}$$

From (2.7) and second inequality of (2.42) following fractional integral inequality is obtained:

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^-}(\Omega*\Delta)\circ\wedge\right)(x,\sigma;p')$$
$$\leq J_v^x(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)(v-x)\left(\left|\Omega'\left((v)^{\frac{1}{p}}\right)\right|+m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)H_v^x(z^\alpha;h,\Delta). \tag{2.43}$$

The inequality (2.7) and the first inequality of (2.42) give the following fractional integral inequality:

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^-}(\Omega*\Delta)\circ\wedge\right)(x,\sigma;p')$$
$$\geq -J_v^x(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)(v-x)\left(\left|\Omega'\left((v)^{\frac{1}{p}}\right)\right|+m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)H_v^x(z^\alpha;h,\Delta). \tag{2.44}$$

By combining (2.39), (2.40), (2.43) and (2.44), the inequality (2.33) is obtained. □

Next, we give extension of Theorem 2.6 and refinement of [18, Theorem 3].

Theorem 2.7. *Under the assumptions of Theorem 2.6, the following inequality holds:*

$$\left|\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^+}(\Omega*\Delta)\circ\wedge\right)(x,\sigma;p')+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^-}(\Omega*\Delta)\circ\wedge\right)(x,\sigma;p')\right|$$
$$\leq J_x^u(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)(x-u)\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right|+m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)H_x{}^u(s^\alpha;h,\Delta)$$
$$+J_v^x(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)(v-x)\left(\left|\Omega'\left((v)^{\frac{1}{p}}\right)\right|+m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)H_v^x(z^\alpha;h,\Delta)$$
$$\leq J_x^u(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)(x-u)\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right|H_x{}^u(s^\alpha;h,\Delta)\right.$$
$$\left.+m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|H_x{}^u(1-s^\alpha;h,\Delta)\right)+J_v^x(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)(v-x)$$
$$\times\left(\left|\Omega'\left((v)^{\frac{1}{p}}\right)\right|H_v^x(z^\alpha;h,\Delta)+m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|H_v^x(1-z^\alpha;h,\Delta)\right). \tag{2.45}$$

Proof. From (2.36), one can see that for $0<h(t)<1$

$$\left|\Omega'\left((t)^{\frac{1}{p}}\right)\right|\leq h\left(\frac{x-t}{x-u}\right)^\alpha h\left(1-\left(\frac{x-t}{x-u}\right)^\alpha\right)\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right|+m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)$$
$$\leq h\left(\frac{x-t}{x-u}\right)^\alpha\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right|+mh\left(1-\left(\frac{x-t}{x-u}\right)^\alpha\right)\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|. \tag{2.46}$$

Hence by following the proof of Theorem 2.6, one can get (2.45). □

Corollary 7. *For $\kappa=\vartheta$,* (2.33) *gives the following result:*

$$\left|\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^+}(\Omega*\Delta)\circ\wedge\right)(x,\sigma;p')+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,v^-}(\Omega*\Delta)\circ\wedge\right)(x,\sigma;p')\right|$$
$$\leq J_x^u(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)(x-u)\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right|+m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)H_x{}^u(s^\alpha;h,\Delta)$$
$$+J_v^x(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)(v-x)\left(\left|\Omega'\left((v)^{\frac{1}{p}}\right)\right|+m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)H_v^x(z^\alpha;h,\Delta). \tag{2.47}$$

Following corollary presents the result of Theorem 2.6 for refined $(h - m) - p$-convex function.

Corollary 8.

$$\begin{aligned}&\left|\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^+}(\Omega * \Delta) \circ \wedge\right)(x,\sigma;p') + \left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^-}(\Omega * \Delta) \circ \wedge\right)(x,\sigma;p')\right|\\&\leq \left[J^u_x(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)(\Delta(x)-\Delta(u))\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right| + m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)\right.\\&\quad\left.+J^x_v(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)(\Delta(v)-\Delta(x))\left(\left|\Omega'\left((v)^{\frac{1}{p}}\right)\right| + m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)\right]\|h\|^2_\infty.\end{aligned}\tag{2.48}$$

Proof. For $h \in L_\infty[0,1]$ and $\alpha = 1$ in (2.33), we get (2.48). □

Corollary 9. *For $p' = \sigma = 0$, inequality* (2.33) *gives the following result is obtained for the fractional integral operator defined in [15]:*

$$\begin{aligned}&\left|\frac{\left({}_{\Delta}\mathbb{F}^{\Upsilon,\Delta}_{u^+}(\Omega * \Delta) \circ \wedge\right)(x)}{\Gamma(\kappa)} + \frac{\left({}_{\Delta}\mathbb{F}^{\Upsilon,\Delta}_{v^-}(\Omega * \Delta) \circ \wedge\right)(x)}{\Gamma(\vartheta)}\right|\\&\leq \frac{\Upsilon(\Delta(x)-\Delta(u))}{\Gamma(\kappa)(\Delta(x)-\Delta(u))}(x-u)\left(\left|\Omega'\left((u)^{\frac{1}{p}}\right)\right| + m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)H_x{}^u(s^\alpha;h,\Delta)\\&\quad+\frac{\Upsilon(\Delta(v)-\Delta(x))}{\Gamma(\vartheta)(\Delta(v)-\Delta(x))}(v-x)\left(\left|\Omega'\left((v)^{\frac{1}{p}}\right)\right| + m\left|\Omega'\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right|\right)H^x_v(z^\alpha;h,\Delta).\end{aligned}\tag{2.49}$$

Remark 4. (i) For $p = 1$, the inequality (2.33) coincide with [19, Theorem 5].
(ii) For $p = 1$, the inequality (2.47) coincide with [19, Corollary 5].
(iii) For $h(t) = t$ and $m = p = 1 = \alpha$, the inequality (2.33) coincide with [21, Theorem 6].
(iv) For $h(t) = t$ and $m = p = 1 = \alpha$, the inequality (2.47) coincide with [21, Corollary 3].

For $0 < h(t) < 1$ in (2.33), one can get refinements for different classes of convex functions and integral operators given in [18, Remark 4].

3. Deductions of main results

In this section, we present the bounds for refined $(h - m) - p$-convex function, refined $(\alpha, m) - p$-convex function, refined $(\alpha, h) - p$-convex function, which will be deduced from the results of Section 2.

Theorem 3.1. *Under the assumptions of Theorem 2.1, the following result for refined $(h-m)-p$-convex function holds:*

$$\begin{aligned}&\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^+}\Omega \circ \wedge\right)(x,\sigma;p') + \left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^-}\Omega \circ \wedge\right)(x,\sigma;p')\\&\leq J^u_x(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((u)^{\frac{1}{p}}\right) + m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(x-u)\int_0^1 h(s)h(1-s)\Delta'(x-s(x-u))ds\\&\quad+J^x_v(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((v)^{\frac{1}{p}}\right) + m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(v-x)\int_0^1 h(z)h(1-z)\Delta'(x+z(v-x)dz.\end{aligned}\tag{3.1}$$

Proof. For $\alpha = 1$ in the proof of Theorem 2.1, we get (3.1). □

Theorem 3.2. *Under the assumptions of Theorem 2.1, the following inequality refined $(\alpha, h)-p$-convex function holds:*

$$\begin{aligned}&\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^{+}}\Omega\circ\wedge\right)(x,\sigma;p')+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^{-}}\Omega\circ\wedge\right)(x,\sigma;p')\\&\leq J^{u}_{x}(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((u)^{\frac{1}{p}}\right)+\Omega\left((x)^{\frac{1}{p}}\right)\right)(x-u)H_{x}{}^{u}(s^{\alpha};h,\Delta)\\&\quad+J^{x}_{v}(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((v)^{\frac{1}{p}}\right)+\Omega\left((x)^{\frac{1}{p}}\right)\right)(v-x)H^{x}_{v}(z^{\alpha};h,\Delta).\end{aligned}\tag{3.2}$$

Proof. Using $m = 1$ in the proof of Theorem 2.1, we get (3.2). □

Theorem 3.3. *Under the assumptions of Theorem 2.1, the following inequality refined $(\alpha, m)-p$-convex function holds:*

$$\begin{aligned}&\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,u^{+}}\Omega\circ\wedge\right)(x,\sigma;p')+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,v^{-}}\Omega\circ\wedge\right)(x,\sigma;p')\\&\leq J^{u}_{x}(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((u)^{\frac{1}{p}}\right)+\Omega\left((x)^{\frac{1}{p}}\right)\right)(x-u)\int_{0}^{1}s(1-s)\Delta'(x-s(x-u))ds\\&\quad+J^{x}_{v}(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon)\left(\Omega\left((v)^{\frac{1}{p}}\right)+m\Omega\left(\left(\frac{x}{m}\right)^{\frac{1}{p}}\right)\right)(v-x)\int_{0}^{1}z(1-z)\Delta'(x+z(v-x)dz.\end{aligned}\tag{3.3}$$

Proof. Using $h(t) = t$ in the proof of Theorem 2.1, we get (3.3). □

The readers can obtain the similar results for Theorem 2.4 and Theorem 2.6, which are left as an exercise.

4. Conclusions

This work determines inequalities for unified integral operators for refined convex functions of different kinds. The results of this paper at once implies inequalities for unified integral operators of refined $(h-m)-p$-convex, refined $(\alpha, m)-p$-convex, refined $(\alpha, h)-p$-convex, refined $(h-p)$-convex, refined $(\alpha-p)$-convex, refined $(m-p)$-convex, refined $(s, m)-p$-convex, refined (s, p)-convex, refined $(s, m)-p-$Godunova-Levin function and refined $(\alpha, h-m)-HA$-convex functions. The reader can deduce similar inequalities for fractional integral operators associated with unified integral operators.

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Conflict of interest

The authors declare that they have no competing interests.

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