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Research article

Existence and global exponential stability of compact almost automorphic solutions for Clifford-valued high-order Hopfield neutral neural networks with D operator

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Abstract: In this paper, a class of Clifford-valued higher-order Hopfield neural networks with D operator is studied by non-decomposition method. Except for time delays, all parameters, activation functions and external inputs of this class of neural networks are Clifford-valued functions. Based on Banach fixed point theorem and differential inequality technique, we obtain the existence, uniqueness and global exponential stability of compact almost automorphic solutions for this class of neural networks. Our results of this paper are new. In addition, two examples and their numerical simulations are given to illustrate our results.

Keywords: Clifford-valued neural network; high-order Hopfield neutral neural network with D operator; compact almost automorphic solution; global exponential stability
Mathematics Subject Classification: 34K14, 34K20, 34K40, 92B20

1. Introduction

In recent years, because higher-order Hopfield neural networks are superior to lower order Hopfield neural networks in many aspects, higher-order Hopfield neural networks have attracted more and more attention, and their dynamics have been widely studied [1–6].

Because time delay is inevitable, the neural network system described by functional differential equation is closer to reality than that described by ordinary differential equation. At the same time, due to the complexity of the interaction between neurons, the state change rate of neurons may be related to the state change rate in the history of neurons. Therefore, many neural networks are described by neutral functional differential equations. Neutral functional differential equations can be divided into D operator type and non operator type. As an important type of neutral functional differential equations, neutral functional differential equations of D operator type not only have relatively complete basic theory, but also have important applications in many fields [7]. As pointed out in [8–16] and references therein, in many practical applications of neural networks, neural networks with D operator have more practical significance than neural networks non-operator-based ones.

On the one hand, Clifford-valued neural networks are a generalization of real-valued, complex-valued and quaternion-valued neural networks. At the same time, it has more advantages than real-valued and complex-valued neural networks in dealing with multi-layer data and problems involving spatial transformation. Therefore, Clifford-valued neural network has attracted more and more attention [17–21]. However, the multiplication of Clifford numbers does not satisfy the commutative law. At present, it is rare to study the dynamics of Clifford-valued neural networks by non decomposition method [22–24]. Meanwhile, almost all papers on the dynamics of Clifford-valued neural networks assume that their self feedback coefficients are real numbers.

On the other hand, it is well known that the existence of almost periodic solutions is an important dynamics of differential equations and neural networks, and almost automorphic function is a natural generalization of almost periodic function. Therefore, it is of great theoretical and practical significance to study the existence of almost automorphic solutions of differential equations and neural networks. However, due to the fact that $f, \sigma : \mathbb{R} \to \mathbb{R}$ are almost automorphic does not imply that $f(t - \sigma(t))$ is almost automorphic. As a result, generally speaking, we cannot study the existence of almost automorphic solutions for differential equations and neural networks with time-varying delays. But, fortunately, if $f, \sigma : \mathbb{R} \to \mathbb{R}$ are compact almost automorphic, then $f(t - \sigma(t))$ is also compact almost automorphic. In other words, we can study the compact almost automorphic oscillation of neural networks with time-varying delays. Nevertheless, up to now, the results on the compact almost automorphic oscillation of neural networks are still very rare.

Inspired by the above discussion, the main purpose of this paper is to study the existence and global exponential stability of compact almost automorphic solutions of Clifford-valued higher-order Hopfield neural networks with D operator and whose self feedback coefficient is also Clifford number. As far as we know, this is the first paper to study the compact almost automorphic solution of Clifford-valued neural networks with D operator by direct method. Therefore, our research is innovative and meaningful.

This paper is organized as follows. In Section 2, we introduce some basic definitions and lemmas and give a model description. In Section 3, we study the existence of compact almost automorphic solutions for Clifford-valued high-order Hopfield neutral neural networks with D operator. In Section 4, we investigate the global exponential stability of compact almost automorphic solutions. In Section 5, we give two examples to show the feasibility of the results. In Section 6, we have a brief conclusion.

2. Model description and preliminaries

The real Clifford algebra over $\mathbb{R}^m$ is defined as

$$\mathcal{A} = \Big\{ \sum_{A \in \Omega} x^A e_A, x_A \in \mathbb{R} \Big\},$$

where $\Omega = \{\phi, 1, 2, \cdots, A, \cdots, 12, \cdots, m\}$, $e_A = e_{h_1} e_{h_2} \cdots e_{h_v}$, with $A = h_1 h_2 \cdots h_v$, $1 \le h_1 < h_2 < \cdots < h_v \le m$. Moreover, $e_\phi = e_0 = 1$, and e_h, $h = 1, 2, \cdots, m$ are said to be Clifford generators and satisfy $e_p = 1$, $p = 0, 1, 2, \cdots, s$, $e_p^2 = -1$, $p = s+1, s+2, \cdots, m$, where $s < m$, and $e_p e_q + e_q e_p = 0, p \neq q, p, q = 1, 2, \ldots, m$.

For $x = \sum_A x^A e_A$, we define $\|x\|_{\mathcal{A}} = \max\limits_A \left\{\left|x^A\right|\right\}$ and for $y = (y_1, y_2, \cdots, y_n)^T \in \mathcal{A}^n$, let $\|y\|_{\mathcal{A}^n} = \max\limits_{1\le i\le n}\{\|y_i\|_{\mathcal{A}}\}$, then $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{A}^n, \|\cdot\|_{\mathcal{A}^n})$ are Banach spaces.

We define the derivative of $u(t) = \sum_A u^A(t)e_A$ is $\dot{u}_i(t) = \sum_A \dot{u}^A(t)e_A$. For $x = \sum_A x^A e_A$, we define $x^c = \sum\limits_{A\neq\emptyset} x^A e_A$ and $x^\emptyset = x - x^c$. For readers interested in Clifford algebra, see [25].

In this paper, we are concerned with the following Clifford-valued high-order Hopfield neutral neural networks involving D operator:

$$\begin{aligned}
&[x_i(t) - a_i(t)x_i(t-\tau_i(t))]' \\
&= -b_i(t)x_i(t) + \sum_{j=1}^{n}\mu_{ij}(t)J_j(x_j(t)) + \sum_{j=1}^{n}v_{ij}(t)F_j(x_j(t-\sigma_{ij}(t))) \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}(t)R_j(x_j(t-\eta_{ijl}(t)))R_l(x_l(t-\gamma_{ijl}(t))) + I_i(t),
\end{aligned} \tag{2.1}$$

where $i = 1, 2, \cdots, n$, $x_i(t) \in \mathcal{A}$ corresponds to the state of the ith unit at time t; $\mathcal{A}$ is a real Clifford-algebra; $b_i(t) \in \mathcal{A}$ represents the self feedback coefficient at time t; $\mu_{ij} \in \mathcal{A}$ denotes the strength of the jth unit at time t; $\theta_{ijl}(t) \in \mathcal{A}$ is the second-order synaptic weight of the neural network; $a_i(t)$, $\mu_{ij}(t)$, $v_{ij}(t)$, $\theta_{ijl}(t) \in \mathcal{A}$ are the connection weights; J_j, F_j, $R_j : \mathcal{A} \to \mathcal{A}$ are the activation functions; $I_i(t) \in \mathcal{A}$ is the external input to the ith unit; $\tau_i(t), \sigma_{ij}(t), \eta_{ijl}(t), \gamma_{ijl}(t) \in \mathbb{R}^+$ denote the transmission delays.

For convenience, we introduce the following notations:

$$\bar{b}_i^\phi = \sup_{t\in\mathbb{R}} b_i^\phi(t), \quad \underline{b}_i^\phi = \inf_{t\in\mathbb{R}} b_i^\phi(t), \quad \bar{b}_i^c = \sup_{t\in\mathbb{R}} \|b_i^c(t)\|_{\mathcal{A}}, \quad a_i^+ = \sup_{t\in\mathbb{R}} \|a_i(t)\|_{\mathcal{A}},$$

$$\mu_{ij}^+ = \sup_{t\in\mathbb{R}} \|\mu_{ij}(t)\|_{\mathcal{A}}, \quad v_{ij}^+ = \sup_{t\in\mathbb{R}} \|v_{ij}(t)\|_{\mathcal{A}}, \quad \theta_{ijl}^+ = \sup_{t\in\mathbb{R}} \|\theta_{ijl}(t)\|_{\mathcal{A}}, \quad {\tau_i}^+ = \sup_{t\in\mathbb{R}} \tau_i(t),$$

$${\sigma_{ij}}^+ = \sup_{t\in\mathbb{R}} \sigma_{ij}(t), \quad {\eta_{ijl}}^+ = \sup_{t\in\mathbb{R}} \eta_{ijl}(t), \quad {\gamma_{ijl}}^+ = \sup_{t\in\mathbb{R}} \gamma_{ijl}(t), \quad \rho = \max_{1\le i,j,l\le n}\{\tau_i^+, \sigma_{ij}^+, \eta_{ijl}^+, \gamma_{ijl}^+\}.$$

The initial values of system (2.1) are given by

$$x_i(s) = \varphi_i(s) \in \mathcal{A}, \ \ s \in [-\rho, 0],$$

where $\varphi_i \in BC([-\rho, 0], \mathcal{A})$.

Throughout the rest of this paper, $BC(\mathbb{R}, \mathcal{A}^n)$ denotes the set of all bounded continuous functions from $\mathbb{R}$ to $\mathcal{A}^n$. Note that $(BC(\mathbb{R}, \mathcal{A}^n), \|\cdot\|_0)$ is a Banach space with the norm:

$$\|f\|_0 = \max_{1\le i\le n}\left\{\sup_{t\in\mathbb{R}} \|f_i(t)\|_{\mathcal{A}}\right\},$$

where $f = (f_1, f_2, \ldots f_i)^T \in BC(\mathbb{R}, \mathcal{A}^n)$.

Definition 2.1. *[26] A function $f \in BC(\mathbb{R}, \mathcal{A}^n)$ is said to be almost periodic if and only if for every sequence of real numbers $(\alpha'_n)_{n\in\mathbb{N}}$ there exist a subsequence $(\alpha_n)_{n\in\mathbb{N}}$ and a function f^* such that*

$$f(t + \alpha_n) \to f^*(t)$$

uniformly on $\mathbb{R}$ as $n \to \infty$. The collection of all such functions will be denoted by $AP(\mathbb{R}, \mathcal{A}^n)$.

Definition 2.2. *[27, 28] A function $f \in BC(\mathbb{R}, \mathcal{A}^n)$ is said to be almost automorphic, if for every sequence of real numbers $(\alpha'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(\alpha_n)_{n\in\mathbb{N}}$ such that*

$$f^*(t) = \lim_{n\to\infty} f(t + \alpha_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} f^*(t - \alpha_n) = f(t)$$

for each $t \in \mathbb{R}$.

If the above limits hold uniformly in compact subsets of $\mathbb{R}$, then f is said to be compact almost automorphic. We denote by $AA(\mathbb{R}, \mathcal{A}^n)$ the space of all almost automorphic functions and by $KAA(\mathbb{R}, \mathcal{A}^n)$ the space of all compact almost automorphic functions.

Lemma 2.1. *[29] A function f is compact almost automorphic if and only if it is almost automorphic and uniformly continuous.*

From the above definition, similar to the proofs of the corresponding results in [22], when $\mathbb{T} = \mathbb{R}$, one can easily prove the following four lemmas.

Lemma 2.2. *[22] If $\lambda \in \mathbb{R}$ and $f \in KAA(\mathbb{R}, \mathcal{A}^n)$, then $\lambda f \in KAA(\mathbb{R}, \mathcal{A}^n)$.*

Lemma 2.3. *[22] If $f, g \in KAA(\mathbb{R}, \mathcal{A}^n)$, then $f + g$, $fg \in KAA(\mathbb{R}, \mathcal{A}^n)$.*

Lemma 2.4. *[22] If $x \in KAA(\mathbb{R}, \mathcal{A})$, $\tau \in KAA(\mathbb{R}, \mathbb{R}^+)$, then $x(\cdot - \tau(\cdot)) \in KAA(\mathbb{R}, \mathcal{A})$.*

Lemma 2.5. *[22] Let $f \in C(\mathcal{A}, \mathcal{A})$ and satisfy the Lipschitz conditions. Through that $x \in KAA(\mathbb{R}, \mathcal{A})$, then $f(x(\cdot)) \in KAA(\mathbb{R}, \mathcal{A})$.*

The assumptions used in this paper are as follows:

(H_1) For $i, j, l = 1, 2, \cdots, n$, $b_i^\phi \in KAA(\mathbb{R}, \mathbb{R}^+)$ with $\underline{b}_i^\phi > 0$, $b_i^c, a_i, \mu_{ij}, \nu_{ij}, \theta_{ijl} \in KAA(\mathbb{R}, \mathcal{A})$.

(H_2) For $j = 1, 2, \cdots, n$, $J_j, F_j, R_j, I_j \in C(\mathcal{A}, \mathcal{A})$ and there exist positive constants L_j^J, L_j^F, L_j^R, L_j^I, M_l^R such that for any $x, y \in \mathcal{A}$,
$\|J_j(x) - J_j(y)\|_{\mathcal{A}} \le L_j^J \|x - y\|_{\mathcal{A}}$, $\|F_j(x) - F_j(y)\|_{\mathcal{A}} \le L_j^F \|x - y\|_{\mathcal{A}}$,
$\|R_j(x) - R_j(y)\|_{\mathcal{A}} \le L_j^R \|x - y\|_{\mathcal{A}}$, $\|I_j(x) - I_j(y)\|_{\mathcal{A}} \le L_j^I \|x - y\|_{\mathcal{A}}$,
$\|R_l(x)\|_{\mathcal{A}} \le M_l^R$ and $J_j(0) = F_j(0) = R_j(0) = I_j(0) = 0$.

(H_3) There are a positive function: $\tilde{b}_i^\phi \in BC(\mathbb{R}, \mathbb{R})$ and a constant $K_i > 0$ satisfying

$$e^{-\int_s^t b_i^\phi(u)du} \le K_i e^{-\int_s^t \tilde{b}_i^\phi(u)du}$$

for all $s, t \in \mathbb{R}$, $t - s \ge 0$.

(H_4) There exist constants $\beta_i > 0$ and $\wedge_i > 0$ such that

$$\sup_{t\in\mathbb{R}} \Big\{ \frac{1}{\tilde{b}_i^\phi(t)} K_i \Big[\bar{b}_i^\phi a_i^+ + \bar{b}_i^c + \beta_i^{-1} \sum_{j=1}^n \mu_{ij}^+ L_j^J \beta_j + \beta_i^{-1} \sum_{j=1}^n \nu_{ij}^+ L_j^F \beta_j + \beta_i^{-1} \sum_{j=1}^n \sum_{l=1}^n \theta_{ijl}^+ (L_j^R M_l^R \beta_j + L_l^R M_j^R \beta_l) \Big] \Big\} < \wedge_i, \tag{2.2}$$

$$\sup_{t\in\mathbb{R}}\Big\{-\tilde{b}_i^{\phi}(t)+K_i\Big[\bar{b}_i^{\phi}a_i^+\frac{1}{1-a_i^+}+\bar{b}_i^c\frac{1}{1-a_i^+}$$
$$+\beta_i^{-1}\sum_{j=1}^{n}\mu_{ij}^+L_j^J\beta_j\frac{1}{1-a_j^+}+\beta_i^{-1}\sum_{j=1}^{n}\nu_{ij}^+L_j^F\beta_j\frac{1}{1-a_j^+}$$
$$+\beta_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}^+\Big(L_j^RM_l^R\beta_j\frac{1}{1-a_j^+}+L_l^RM_j^R\beta_l\frac{1}{1-a_l^+}\Big)\Big]\Big\}<0 \tag{2.3}$$

and $r=\max_{1\le i\le n}\{a_i^+ + \wedge_i\}<1$, $i=1,2,\cdots,n$.

Remark 2.1. *In fact, condition* (H_1) *implies condition* (H_3). *For example, because* $e^{-\int_s^t b_i^{\phi}(u)du}\le e^{-\underline{b}_i^{\phi}(t-s)}$, *so, we can take* $K_i=1$ *and* $\tilde{b}_i^{\phi}(t)\equiv \underline{b}_i^{\phi}$. *But in order to make condition* (H_4) *less conservative, we still give condition* (H_3).

3. The existence of compact almost automorphic solutions

Let $X=\{\varphi\in BC(\mathbb{R},\mathcal{A}^n)\big|\varphi\in KAA(\mathbb{R},\mathcal{A}^n)\}$. For any $\varphi\in X$, we define the norm of φ as $\|\varphi\|_0=\max_{1\le i\le n}\Big\{\sup_{t\in\mathbb{R}}\|\varphi_i(t)\|_{\mathcal{A}}\Big\}$, then $(X,\|\cdot\|_0)$ is a Banach space.

Let $\varphi_0=\Big(\int_{-\infty}^t e^{-\int_s^t b_i^{\phi}(u)du}I_1(s)ds,\int_{-\infty}^t e^{-\int_s^t b_i^{\phi}(u)du}I_2(s)ds,\cdots,\int_{-\infty}^t e^{-\int_s^t b_i^{\phi}(u)du}I_n(s)ds\Big)^T$ and take a positive constant $W>\|\varphi_0\|_0$. Define

$$X_0=\Big\{\varphi\in X\Big|\|\varphi-\varphi_0\|_0\le\frac{rW}{1-r}\Big\}.$$

Then, for every $\varphi\in X_0$, we have

$$\|\varphi\|_0\le\|\varphi-\varphi_0\|+\|\varphi_0\|_0\le\frac{rW}{1-r}+W=\frac{W}{1-r}.$$

Lemma 3.1. *If* $G_i(t)=\int_{-\infty}^t e^{-\int_s^t b_i^{\phi}(u)du}H_i(s)ds$, *where* $b_i^{\phi}(t)\in KAA(\mathbb{R},\mathbb{R}^+)$ *with* $\underline{b}_i^{\phi}>0$ *and* $H_i(s)\in KAA(\mathbb{R},\mathcal{A})$, *then* $G_i(t)\in KAA(\mathbb{R},\mathcal{A})$, $i=1,2,\cdots,n$.

Proof. Let $(\alpha'_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, since $b_i^{\phi}\in KAA(\mathbb{R},\mathbb{R}^+)$ and $H_i(t)\in KAA(\mathbb{R},\mathcal{A})$, we can extract a subsequence $(\alpha_n)_{n\in\mathbb{N}}$ of $(\alpha'_n)_{n\in\mathbb{N}}$ such that for each compact subset Ω of $\mathbb{R}$

$$\lim_{n\to\infty}b_i^{\phi}(t+\alpha_n)=b_i^{\phi*}(t),\ \lim_{n\to\infty}b_i^{\phi*}(t-\alpha_n)=b_i^{\phi}(t),$$
$$\lim_{n\to\infty}H_i(t+\alpha_n)=H_i^*(t),\ \lim_{n\to\infty}H_i^*(t-\alpha_n)=H_i(t).$$

Set $G_i^*(t)=\int_{-\infty}^t e^{-\int_s^t b_i^{\phi*}(u)du}H_i^*(s)ds$, then we have

$$\begin{aligned}&\|G_i(t+\alpha_n)-G_i^*(t)\|_{\mathcal{A}}\\
=&\Big\|\int_{-\infty}^{t+\alpha_n}e^{-\int_s^{t+\alpha_n}b_i^{\phi}(u)du}H_i(s)ds-\int_{-\infty}^{t}e^{-\int_s^t b_i^{\phi*}(u)du}H_i^*(s)ds\Big\|_{\mathcal{A}}\\
=&\Big\|\int_{-\infty}^{t+\alpha_n}e^{-\int_{s-\alpha_n}^{t}b_i^{\phi}(u+\alpha_n)du}H_i(s)ds-\int_{-\infty}^{t}e^{-\int_s^t b_i^{\phi*}(u)du}H_i^*(s)ds\Big\|_{\mathcal{A}}\end{aligned}$$

$$
\begin{aligned}
=&\Big\|\int_{-\infty}^{t} e^{-\int_s^t b_i^{\phi}(u+\alpha_n)du} H_i(s+\alpha_n)ds - \int_{-\infty}^{t} e^{-\int_s^t b_i^{\phi^*}(u)du} H_i^*(s)ds\Big\|_{\mathcal{A}}\\
\leq&\Big\|\int_{-\infty}^{t} e^{-\int_s^t b_i^{\phi}(u+\alpha_n)du} H_i(s+\alpha_n)ds - \int_{-\infty}^{t} e^{-\int_s^t b_i^{\phi}(u+\alpha_n)du} H_i^*(s)ds\Big\|_{\mathcal{A}}\\
&+\Big\|\int_{-\infty}^{t} e^{-\int_s^t b_i^{\phi}(u+\alpha_n)du} H_i^*(s)ds - \int_{-\infty}^{t} e^{-\int_s^t b_i^{\phi^*}(u)du} H_i^*(s)ds\Big\|_{\mathcal{A}}\\
\leq&\int_{-\infty}^{t} e^{-\int_s^t b_i^{\phi}(u+\alpha_n)du} \big\|H_i(s+\alpha_n) - H_i^*(s)\big\|_{\mathcal{A}} ds\\
&+\Big\|\int_{-\infty}^{t} \Big(e^{-\int_s^t b_i^{\phi}(u+\alpha_n)du} - e^{-\int_s^t b_i^{\phi^*}(u)du}\Big) H_i^*(s)ds\Big\|_{\mathcal{A}}.
\end{aligned}
$$

By the Lebesgue dominated convergence theorem, we obtain that $\lim_{n\to\infty} G_i(t+\alpha_n) = G_i^*(t)$ for each $t \in \Omega$, $i = 1, 2, \cdots, n$. Similarly, one can prove that $\lim_{n\to\infty} G_i^*(t-\alpha_n) = G_i(t)$ for each $t \in \Omega$, $i = 1, 2, \cdots, n$. Hence, $G_i \in AA(\mathbb{R}, \mathcal{A})$, $i = 1, 2, \cdots, n$.

On the other hand, noting that

$$
\begin{aligned}
\sup_{t\in\mathbb{R}} \|G_i'(t)\|_{\mathcal{A}} =& \sup_{t\in\mathbb{R}} \Big\|H_i(t) - b_i^{\phi}(t)\int_{-\infty}^{t} e^{-\int_s^t b_i^{\phi}(u)du} H_i(s)ds\Big\|_{\mathcal{A}}\\
\leq& \sup_{t\in\mathbb{R}} \|H_i(t)\|_{\mathcal{A}} + \bar{b}_i^{\phi}\int_{-\infty}^{t} e^{-\underline{b}_i^{\phi}(t-s)} \sup_{s\in\mathbb{R}} \|H_i(s)\|_{\mathcal{A}} ds\\
\leq& \sup_{t\in\mathbb{R}} \|H_i(t)\|_{\mathcal{A}} + \frac{\bar{b}_i^{\phi}}{\underline{b}_i^{\phi}} \sup_{t\in\mathbb{R}} \|H_i(t)\|_{\mathcal{A}} < \infty,
\end{aligned}
$$

we have $G_i(t)$ is uniformly continuous on $\mathbb{R}$. Consequently, by Lemma 2.1, we have $G_i \in KAA(\mathbb{R}, \mathcal{A})$, $i = 1, 2, \cdots, n$. The proof is complete. □

Theorem 3.1. *Assume that* (H_1)-(H_4) *hold. Then system* (2.1) *has a unique compact almost automorphic solution in* X_0.

Proof. From (H_4) we can choose a constant $\lambda \in \Big(0, \min_{1\leq i\leq n}\Big\{\inf_{t\in\mathbb{R}} \|\tilde{b}_i^{\phi}(t)\|_{\mathcal{A}}\Big\}\Big)$ such that $a_i^+ e^{\lambda\tau_i^+} < 1$,

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}}\Big\{\frac{e^{\lambda}}{\tilde{b}_i^{\phi}(t)} K_i\Big[\bar{b}_i^{\phi} a_i^+ + \bar{b}_i^{c} + \beta_i^{-1}\sum_{j=1}^{n} \mu_{ij}^+ L_j^J \beta_j + \beta_i^{-1}\sum_{j=1}^{n} \nu_{ij}^+ L_j^F \beta_j\\
&\quad + \beta_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n} \theta_{ijl}^+ (L_j^R M_l^R \beta_j + L_l^R M_j^R \beta_l)\Big]\Big\} < \wedge_i. \qquad (3.1)
\end{aligned}
$$

Let $y_i(t) = \beta_i^{-1} x_i(t)$, $Y_i(t) = y_i(t) - a_i(t) y_i(t-\tau_i(t))$, $i = 1, 2, \cdots, n$, then, system (2.1) turns into

$$
\begin{aligned}
Y_i'(t) =& -b_i^{\phi}(t) Y_i(t) - b_i^{\phi}(t) a_i(t) y_i(t-\tau_i(t)) - b_i^{c}(t) y_i(t)\\
&+ \beta_i^{-1}\sum_{j=1}^{n} \mu_{ij}(t) J_j(\beta_j y_j(t)) + \beta_i^{-1}\sum_{j=1}^{n} \nu_{ij}(t) F_j(\beta_j y_j(t-\sigma_{ij}(t)))\\
&+ \beta_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n} \theta_{ijl}(t) R_j(\beta_j y_j(t-\eta_{ijl}(t))) R_l(\beta_l y_l(t-\gamma_{ijl}(t)))
\end{aligned}
$$

$$+\beta_i^{-1}I_i(t), \ i=1,2,\cdots,n. \tag{3.2}$$

Multiplying both sides of (3.2) by $e^{\int_0^s b_i^{\phi}(u)du}$, and integrating it on $(-\infty, t]$, we get

$$\begin{aligned} y_i(t) =& a_i(t)y_i(t-\tau_i(t)) + \int_{-\infty}^t e^{-\int_s^t b_i^{\phi}(u)du}\Big[-b_i^{\phi}(s)a_i(s)y_i(s-\tau_i(s)) \\ & - b_i^c(s)y_i(s) + \beta_i^{-1}\sum_{j=1}^n \mu_{ij}(s)J_j(\beta_j y_j(s)) + \beta_i^{-1}\sum_{j=1}^n \nu_{ij}(s)F_j(\beta_j y_j(s-\sigma_{ij}(s))) \\ & + \beta_i^{-1}\sum_{j=1}^n\sum_{l=1}^n \theta_{ijl}(s)R_j(\beta_j y_j(s-\eta_{ijl}(s)))R_l(\beta_l y_l(s-\gamma_{ijl}(s))) + \beta_i^{-1}I_i(s)\Big]ds. \end{aligned}$$

It is easy to see that if $y^* = (y_1^*(t), y_2^*(t), \ldots, y_n^*(t))$ is a solution of (3.2), then $x^* = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t)) = (\beta_1 y_1^*(t), \beta_2 y_2^*(t), \ldots, \beta_n y_n^*(t))$ is a solution of system (2.1).

Define an operator $T : X \to BC(\mathbb{R}, \mathcal{A}^n)$ by $T\varphi = (T_1\varphi, T_2\varphi, \cdots T_n\varphi)^T$, where

$$(T_i\varphi)(t) = a_i(t)\varphi_i(t-\tau_i(t)) + \int_{-\infty}^t e^{-\int_s^t b_i^{\phi}(u)du}H_i(s)ds$$

and

$$\begin{aligned} H_i(s) =& -b_i^{\phi}(s)a_i(s)\varphi_i(s-\tau_i(s)) - b_i^c(s)\varphi_i(s) \\ & + \beta_i^{-1}\sum_{j=1}^n \mu_{ij}(s)J_j(\beta_j\varphi_j(s)) + \beta_i^{-1}\sum_{j=1}^n \nu_{ij}(s)F_j(\beta_j\varphi_j(s-\sigma_{ij}(s))) \\ & + \beta_i^{-1}\sum_{j=1}^n\sum_{l=1}^n \theta_{ijl}(s)R_j(\beta_j\varphi_j(s-\eta_{ijl}(s)))R_l(\beta_l\varphi_l(s-\gamma_{ijl}(s))) + \beta_i^{-1}I_i(s). \end{aligned}$$

Then by Lemmas 2.2–2.4, we have $H_i \in KAA(\mathbb{R}, \mathcal{A})$, $i = 1, 2, \cdots, n$. Therefore by Lemma 3.1, for $\varphi \in KAA(\mathbb{R}, \mathcal{A}^n)$, we get $T\varphi \in KAA(\mathbb{R}, \mathcal{A}^n)$. In order to finish the proof of this theorem, we will divide the rest of the proof into the following two steps.

Step 1. We will prove that the mapping T is a self-mapping from X_0 to X_0. In fact, for each $\varphi \in X_0$, we have

$$\begin{aligned} & \|T\varphi - \varphi_0\|_0 \\ \le& \max_{1\le i\le n}\Big\{\sup_{t\in\mathbb{R}}\Big[\|a_i(t)\varphi_i(t-\tau_i(t))\|_{\mathcal{A}} + \int_{-\infty}^t e^{-\int_s^t b_i^{\phi}(u)du}\Big(\big\|-b_i^{\phi}(s)a_i(s)\varphi_i(s-\tau_i(s))\big\|_{\mathcal{A}} \\ & + \big\|-b_i^c(s)\varphi_i(s)\big\|_{\mathcal{A}} + \Big\|\beta_i^{-1}\sum_{j=1}^n \mu_{ij}(s)J_j(\beta_j\varphi_j(s))\Big\|_{\mathcal{A}} \\ & + \Big\|\beta_i^{-1}\sum_{j=1}^n \nu_{ij}(s)J_j(\beta_j\varphi_j(s-\sigma_{ij}(s)))\Big\|_{\mathcal{A}} \\ & + \Big\|\beta_i^{-1}\sum_{j=1}^n\sum_{l=1}^n \theta_{ijl}(s)R_j(\beta_j\varphi_j(s-\eta_{ijl}(s)))R_l(\beta_l\varphi_l(s-\gamma_{ijl}(s)))\Big\|_{\mathcal{A}}\Big)ds\Big]\Big\} \end{aligned}$$

$$
\begin{aligned}
&\le \max_{1\le i\le n}\Big\{a_i^+ + \int_{-\infty}^{t} e^{-\int_s^t \tilde{b}_i^{\phi}(u)du} K_i\Big[\bar{b}_i^{\phi} a_i^+ + \bar{b}_i^{c} + \beta_i^{-1}\sum_{j=1}^{n}\mu_{j\,i}^{+} L_j^J \beta_j \\
&\quad + \beta_i^{-1}\sum_{j=1}^{n} v_{ij}^{+} L_j^F \beta_j + \beta_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}^{+} L_j^R M_l^R \beta_j\Big]\|\varphi\|_0 ds\Big\} \\
&\le \max_{1\le i\le n}\Big\{a_i^+ + \wedge_i \int_{-\infty}^{t} e^{-\int_s^t \tilde{b}_i^{\phi}(u)du}\frac{1}{e^{\lambda}}\tilde{b}_i^{\phi}(s)ds\Big\}\|\varphi\|_0 \\
&\le \max_{1\le i\le n}\Big\{a_i^+ + \frac{1}{e^{\lambda}}\wedge_i\Big\}\|\varphi\|_0 \\
&< \frac{rW}{1-r},
\end{aligned} \tag{3.3}
$$

which implies that $\varphi(X_0) \subset X_0$.

Step 2. We will prove that T is a contracting mapping. Indeed, for any $\varphi, \psi \in X_0$, we have that

$$
\begin{aligned}
&\|T\varphi - T\psi\|_0 \\
&\le \max_{1\le i\le n}\Big\{\sup_{t\in\mathbb{R}}\Big[\Big\|a_i(t)(\varphi_i(t-\tau_i(t)) - \psi_i(t-\tau_i(t)))\Big\|_{\mathcal{A}} + \int_{-\infty}^{t} e^{-\int_s^t b_i^{\phi}(u)du} \\
&\quad \times\Big(\Big\|-b_i^{\phi}(s)a_i(s)(\varphi_i(s-\tau_i(s)) - \psi_i(s-\tau_i(s)))\Big\|_{\mathcal{A}} + \Big\|-b_i^{c}(s)(\varphi_i(s) - \psi_i(s))\Big\|_{\mathcal{A}} \\
&\quad + \Big\|\beta_i^{-1}\sum_{j=1}^{n}\mu_{ij}(s)(J_j(\beta_j\varphi_j(s)) - J_j(\beta_j\psi_j(s)))\Big\|_{\mathcal{A}} \\
&\quad + \Big\|\beta_i^{-1}\sum_{j=1}^{n} v_{ij}(s)(F_j(\beta_j\varphi_j(s-\sigma_{ij}(s))) - F_j(\beta_j\psi_j(s-\sigma_{ij}(s))))\Big\|_{\mathcal{A}} \\
&\quad + \Big\|\beta_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}(s)[R_j(\beta_j\varphi_j(s-\eta_{ijl}(s)))R_l(\beta_l\varphi_l(s-\gamma_{ijl}(s))) \\
&\quad - R_j(\beta_j\psi_j(s-\gamma_{ijl}(s)))R_l(\beta_l\psi_l(s-\gamma_{ijl}(s)))]\Big\|_{\mathcal{A}}\Big)ds\Big]\Big\} \\
&\le \max_{1\le i\le n}\Big\{a_i^+ + \int_{-\infty}^{t} e^{-\int_s^t \tilde{b}_i^{\phi}(u)du} K_i\Big(\bar{b}_i^{\phi} a_i^+ + \bar{b}_i^{c} + \beta_i^{-1}\sum_{j=1}^{n}\mu_{ij}^{+} L_j^J \beta_j \\
&\quad + \beta_i^{-1}\sum_{j=1}^{n} v_{ij}^{+} L_j^F \beta_j + \beta_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}^{+}(L_j^R M_l^R \beta_j + L_l^R M_j^R \beta_l)\Big)ds\|\varphi - \psi\|_0\Big\} \\
&\le \max_{1\le i\le n}\Big\{a_i^+ + \wedge_i \int_{-\infty}^{t} e^{-\int_s^t \tilde{b}_i^{\phi}(u)du}\frac{1}{e^{\lambda}}\tilde{b}_i^{\phi}(s)ds\Big\}\|\varphi - \psi\|_0 \\
&\le \max_{1\le i\le n}\Big\{a_i^+ + \frac{1}{e^{\lambda}}\wedge_i\Big\}\|\varphi - \psi\|_0 \\
&< r\|\varphi - \psi\|_0,
\end{aligned}
$$

which combined with the fact that $r = \max_{1\le i\le n}\{a_i^+ + \wedge_i\} < 1$ implies that the mapping T possesses a unique fixed point $y^* = \{y_i^*(t)\} \in KAA(\mathbb{R}, \mathcal{A}^n)$. Consequently, $x^* = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t)) = (\beta_1 y_1^*(t), \beta_2 y_2^*(t), \ldots, \beta_n y_n^*(t))$ is a compact almost automorphic solution of system (2.1). The proof is complete. □

4. Global exponential stability

Theorem 4.1. *Assume that* (H_1)–(H_4) *hold. Then system* (2.1) *has a unique compact almost automorphic solution that is gllobally exponentially stable.*

Proof. From (H_4) we can choose a constant $\lambda \in \left(0, \min\limits_{1\le i\le n}\left\{\inf\limits_{t\in\mathbb{R}} \|\tilde{b}_i^{\phi}(t)\|_{\mathcal{A}}\right\}\right)$ such that $a_i^+ e^{\lambda\tau_i^+} < 1$,

$$\begin{aligned}\sup_{t\in\mathbb{R}}\Bigg\{&\lambda - \tilde{b}_i^{\phi}(t) + K_i\Bigg[\bar{b}_i^{\phi} a_i^+ \frac{e^{\lambda\tau_i^+}}{1-a_i^+e^{\lambda\tau_i^+}} + \bar{b}_i^c \frac{1}{1-a_i^+e^{\lambda\tau_i^+}}\\ &+ \beta_i^{-1}\sum_{j=1}^n \mu_{ij}^+ L_j^J \beta_j \frac{1}{1-a_j^+e^{\lambda\tau_j^+}} + \beta_i^{-1}\sum_{j=1}^n \nu_{ij}^+ L_j^F \beta_j \frac{e^{\lambda\sigma_{ij}^+}}{1-a_j^+e^{\lambda\tau_j^+}}\\ &+ \beta_i^{-1}\sum_{j=1}^n\sum_{l=1}^n \theta_{ijl}^+\Bigg(L_j^R M_l^R \beta_j \frac{e^{\lambda\eta_{ijl}^+}}{1-a_j^+e^{\lambda\tau_j^+}} + \frac{e^{\lambda\gamma_{ijl}^+}}{1-a_l^+e^{\lambda\tau_l^+}} L_l^R M_j^R \beta_l\Bigg)\Bigg]\Bigg\} < 0. \end{aligned} \tag{4.1}$$

Let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ be the compact almost automorphic solution of system (2.1) with initial value $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ and $x = (x_1, x_2, \ldots, x_n)$ be an arbitrary solution of system (2.1) with initial value $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$.

Denote $y_i(t) = \beta_i^{-1} x_i(t), y_i^*(t) = \beta_i^{-1} x_i^*(t)$, $z_i(t) = y_i(t) - y_i^*(t)$, $Z_i(t) = z_i(t) - a_i(t) z_i(t - \tau_i(t))$, $i = 1, 2, \cdots, n$. Then we have

$$\begin{aligned} Z_i'(t) = &- b_i^{\phi}(t) Z_i(t) - b_i^{\phi}(t) a_i(t) z_i(t-\tau_i(t)) - b_i^c(t) z_i(t)\\ &+ \beta_i^{-1}\sum_{j=1}^n \mu_{ij}(t)(J_j(\beta_j y_j(t)) - J_j(\beta_j y_j^*(t)))\\ &+ \beta_i^{-1}\sum_{j=1}^n \nu_{ij}(t)(F_j(\beta_j y_j(t-\sigma_{ij}(t))) - F_j(\beta_j y_j^*(t-\sigma_{ij}(t))))\\ &+ \beta_i^{-1}\sum_{j=1}^n\sum_{l=1}^n \theta_{ijl}(t)(R_j(\beta_j y_j(t-\eta_{ijl}(t))) - R_j(\beta_j y_j^*(t-\eta_{ijl}(t))))\\ &\times (R_l(\beta_l y_l(t-\gamma_{ijl}(t))) - R_l(\beta_l y_l^*(t-\gamma_{ijl}(t)))). \end{aligned} \tag{4.2}$$

Without loss of generality, let

$$\begin{aligned} &\|\varphi - \phi\|_{\xi}\\ =& \max_{1\le i\le n}\left\{\sup_{t\in(-\rho,0]} \beta_i^{-1}\Big\|\varphi_i(t) - a_i(t)\varphi_i(t-\tau_i(t)) - [\phi_i(t) - a_i(t)\phi_i(t-\tau_i(t))]\Big\|_{\mathcal{A}}\right\} > 0 \end{aligned}$$

and M be a constant such that

$$M > \sum_{i=1}^n K_i + 1, \ i = 1, 2, \cdots, n. \tag{4.3}$$

Consequently, for any $\varepsilon > 0$, it is obvious that

$$\|Z(t)\|_{\mathcal{A}^n} < M(\|\varphi - \phi\|_{\xi} + \varepsilon)e^{-\lambda t}, \ t \in (-\rho, 0].$$

We claim that

$$\|Z(t)\|_{\mathcal{A}^n} < M(\|\varphi - \phi\|_\xi + \varepsilon)e^{-\lambda t}, \text{ for all } t > 0. \tag{4.4}$$

In the contrary case, there must exist a certain $t_1 > 0$ such that

$$\|Z(t_1)\|_{\mathcal{A}^n} = M(\|\varphi - \phi\|_\xi + \varepsilon)e^{-\lambda t_1}, \tag{4.5}$$

$$\|Z(t)\|_{\mathcal{A}^n} < M(\|\varphi - \phi\|_\xi + \varepsilon)e^{-\lambda t}, \text{ for all } t \in (-\rho, t_1). \tag{4.6}$$

Therefore, we obtain

$$\begin{aligned}
&e^{\lambda v}\|z_j(v)\|_{\mathcal{A}} \\
\leq& e^{\lambda v}\|z_j(v) - a_j(v)z_j(v - \tau_j(v))\|_{\mathcal{A}} + e^{\lambda v}\|a_j(v)z_j(v - \tau_j(v))\|_{\mathcal{A}} \\
\leq& e^{\lambda v}\|Z_j(v)\|_{\mathcal{A}} + a_j^+ e^{\lambda \tau_j^+} e^{\lambda(v - \tau_j(v))}\|z_j(v - \tau_j(v))\|_{\mathcal{A}} \\
\leq& M(\|\varphi - \phi\|_\xi + \varepsilon) + a_j^+ e^{\lambda \tau_j^+} \sup_{s \in (-\rho, t]} e^{\lambda s}\|z_j(s)\|_{\mathcal{A}},
\end{aligned} \tag{4.7}$$

for all $v \in (-\rho, t]$, $t \in (-\rho, t_1)$, $j = 1, 2, \cdots, n$, which implies that

$$e^{\lambda t}\|z_j(t)\|_{\mathcal{A}} \leq \sup_{s \in (-\rho, t]} e^{\lambda s}\|z_j(s)\|_{\mathcal{A}} \leq \frac{M(\|\varphi - \phi\|_\xi + \varepsilon)}{1 - a_j^+ e^{\lambda \tau_j^+}}. \tag{4.8}$$

Multiplying both sides of (4.2) by $e^{\int_0^s b_i^\phi(u)du}$ and integrating it over the interval $[0, t]$, we get

$$\begin{aligned}
Z_i(t) =& Z_i(0)e^{-\int_0^t b_i^\phi(u)du} + \int_0^t e^{-\int_s^t b_i^\phi(u)du}\Big[-b_i^\phi(s)a_i(s)z_i(s - \tau_i(s)) \\
&- b_i^c(s)z_i(s) + \beta_i^{-1}\sum_{j=1}^n \mu_{ij}(s)(J_j(\beta_j y_j(s)) - J_j(\beta_j {y_j}^*(s))) \\
&+ \beta_i^{-1}\sum_{j=1}^n v_{ij}(s)(F_j(\beta_j y_j(s - \sigma_{ij}(s))) - F_j(\beta_j {y_j}^*(s - \sigma_{ij}(s)))) \\
&+ \beta_i^{-1}\sum_{j=1}^n\sum_{l=1}^n \theta_{ijl}(s)(R_j(\beta_j y_j(t - \eta_{ijl}(s))) - R_j(\beta_j y_j^*(s - \eta_{ijl}(s)))) \\
&\times (R_l(\beta_l y_l(s - \gamma_{ijl}(s))) - R_l(\beta_l y_l^*(s - \gamma_{ijl}(s))))\Big]ds, \ t \in [0, t_1], \ i = 1, 2, \ldots, n.
\end{aligned}$$

This, with the help of (4.1), (4.3), (4.5) and (4.8), for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
\|Z_i(t_1)\|_{\mathcal{A}} \leq& \|Z_i(0)\|_{\mathcal{A}} e^{-\int_0^{t_1} b_i^\phi(u)du} \\
&+ \int_0^{t_1} e^{-\int_s^{t_1} b_i^\phi(u)du}\Big[\|-b_i^\phi(s)a_i(s)z_i(s - \tau_i(s)) - b_i^c(s)z_i(s)\|_{\mathcal{A}} \\
&+ \Big\|\beta_i^{-1}\sum_{j=1}^n \mu_{ij}(s)(J_j(\beta_j y_j(s)) - J_j(\beta_j y_j^*(s)))\Big\|_{\mathcal{A}}
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\beta_i^{-1}\sum_{j=1}^{n}v_{ij}(s)(F_j(\beta_j y_j(s-\sigma_{ij}(s)))-F_j(\beta_j y_j^*(s-\sigma_{ij}(s))))\right\|_{\mathcal{A}}\\
&+\left\|\beta_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}(s)(R_j(\beta_j y_j(s-\eta_{ijl}(s)))R_l(\beta_l y_l(s-\gamma_{ijl}(s))))\right.\\
&\left.\times(R_j(\beta_j y_j^*(s-\eta_{ijl}(s)))R_l(\beta_l y_l^*(s-\gamma_{ijl}(s))))\right\|_{\mathcal{A}}\Big]ds\\
\le&(\|\varphi-\phi\|_{\xi}+\varepsilon)K_i e^{-\int_0^{t_1}\tilde{b}_i^{\phi}(u)du}\\
&+\int_0^{t_1}e^{-\int_s^{t_1}\tilde{b}_i^{\phi}(u)du}K_i\Big[\bar{b}_i^{\phi}a_i^+\|z_i(s-\tau_i(s))\|_{\mathcal{A}}+\bar{b}_i^c\|z_i(s)\|_{\mathcal{A}}\\
&+\beta_i^{-1}\sum_{j=1}^{n}\mu_{ij}^+L_j^J\beta_j\|z_j(s)\|_{\mathcal{A}}+\beta_i^{-1}\sum_{j=1}^{n}v_{ij}^+L_j^F\beta_j\|z_j(s-\sigma_{ij}(s))\|_{\mathcal{A}}\\
&+\beta_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}^+(L_j^RM_l^R\beta_j\|z_j(s-\eta_{ijl}(s))\|_{\mathcal{A}}+L_l^RM_j^R\beta_l\|z_j(s-\gamma_{ijl}(s))\|_{\mathcal{A}})\Big]ds\\
\le&(\|\varphi-\phi\|_{\xi}+\varepsilon)e^{-\lambda t_1}K_i e^{-\int_0^{t_1}[\tilde{b}_i^{\phi}(u)-\lambda]du}\\
&+\int_0^{t_1}e^{-\int_s^{t_1}\tilde{b}_i^{\phi}(u)du}K_i\Big[\bar{b}_i^{\phi}a_i^+\frac{M(\|\varphi-\phi\|_{\xi}+\varepsilon)}{1-a_i^+e^{\lambda\tau_i^+}}e^{-\lambda(s-\tau_i(s))}\\
&+\bar{b}_i^c\frac{M(\|\varphi-\phi\|_{\xi}+\varepsilon)}{1-a_i^+e^{\lambda\tau_i^+}}e^{-\lambda s}+\beta_i^{-1}\sum_{j=1}^{n}\mu_{ij}^+L_j^J\beta_j\frac{M(\|\varphi-\phi\|_{\xi}+\varepsilon)}{1-a_j^+e^{\lambda\tau_j^+}}e^{-\lambda s}\\
&+\beta_i^{-1}\sum_{j=1}^{n}v_{ij}^+L_j^F\beta_j\frac{M(\|\varphi-\phi\|_{\xi}+\varepsilon)}{1-a_j^+e^{\lambda\tau_j^+}}e^{-\lambda(s-\sigma_{ij}(s))}\\
&+\beta_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}^+\Big(L_j^RM_l^R\beta_j\frac{M(\|\varphi-\phi\|_{\xi}+\varepsilon)}{1-a_j^+e^{\lambda\tau_j^+}}e^{-\lambda(s-\eta_{ijl}(s))}\\
&+L_l^RM_j^R\beta_l\frac{M(\|\varphi-\phi\|_{\xi}+\varepsilon)}{1-a_l^+e^{\lambda\tau_l^+}}e^{-\lambda(s-\eta_{ijl}(s))}\Big)\Big]ds\\
\le&(\|\varphi-\phi\|_{\xi}+\varepsilon)e^{-\lambda t_1}K_i e^{-\int_0^{t_1}[\tilde{b}_i^{\phi}(u)-\lambda]du}\\
&+M(\|\varphi-\phi\|_{\xi}+\varepsilon)\int_0^{t_1}e^{-\int_s^{t_1}\tilde{b}_i^{\phi}(u)du}K_i e^{-\lambda s}\Big[\bar{b}_i^{\phi}a_i^+\frac{e^{\lambda\tau_i^+}}{1-a_i^+e^{\lambda\tau_i^+}}+\bar{b}_i^c\frac{1}{1-a_i^+e^{\lambda\tau_i^+}}\\
&+\beta_i^{-1}\sum_{j=1}^{n}\mu_{ij}^+L_j^J\beta_j\frac{1}{1-a_j^+e^{\lambda\tau_j^+}}+\beta_i^{-1}\sum_{j=1}^{n}v_{ij}^+L_j^F\beta_j\frac{e^{\lambda\sigma_{ij}^+}}{1-a_j^+e^{\lambda\tau_j^+}}\\
&+\beta_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}^+\Big(L_j^RM_l^R\beta_j\frac{e^{\lambda\eta_{ijl}^+}}{1-a_j^+e^{\lambda\tau_j^+}}+\frac{e^{\lambda\gamma_{ijl}^+}}{1-a_l^+e^{\lambda\tau_l^+}}L_l^RM_j^R\beta_l\Big)\Big]ds\\
\le&(\left\|\varphi-\phi\right\|_{\xi}+\varepsilon)e^{-\lambda t_1}K_i e^{-\int_0^{t_1}[\tilde{b}_i^{\phi}(u)-\lambda]du}\\
&+M(\|\varphi-\phi\|_{\xi}+\varepsilon)e^{-\lambda t_1}\int_0^{t_1}e^{-\int_s^{t_1}[\tilde{b}_i^{\phi}(u)-\lambda]du}K_i\Big[\bar{b}_i^{\phi}a_i^+\frac{e^{\lambda\tau_i^+}}{1-a_i^+e^{\lambda\tau_i^+}}+\bar{b}_i^c\frac{1}{1-a_i^+e^{\lambda\tau_i^+}}
\end{aligned}
$$

$$
\begin{aligned}
&+\beta_i^{-1}\sum_{j=1}^{n}\mu_{ij}^{+}L_j^{J}\beta_j\frac{1}{1-a_j^{+}e^{\lambda\tau_j^{+}}}+\beta_i^{-1}\sum_{j=1}^{n}\nu_{ij}^{+}L_j^{F}\beta_j\frac{e^{\lambda\sigma_{ij}^{+}}}{1-a_j^{+}e^{\lambda\tau_j^{+}}}\\
&+\beta_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}^{+}\Big(L_j^{R}M_l^{R}\beta_j\frac{e^{\lambda\eta_{ijl}^{+}}}{1-a_j^{+}e^{\lambda\tau_j^{+}}}+\frac{e^{\lambda\gamma_{ijl}^{+}}}{1-a_l^{+}e^{\lambda\tau_l^{+}}}L_l^{R}M_j^{R}\beta_l\Big)\Big]ds\\
\leq&M(\big\|\varphi-\phi\big\|_{\xi}+\varepsilon)e^{-\lambda t_1}\Big(\frac{K_i}{M}e^{-\int_0^{t_1}[\tilde{b}_i^{\phi}(u)-\lambda]du}+\int_0^{t_1}e^{-\int_s^{t_1}[\tilde{b}_i^{\phi}(u)-\lambda]du}\Big[\tilde{b}_i^{\phi}(s)-\lambda\Big]ds\Big)\\
\leq&M(\|\varphi-\phi\|_{\xi}+\varepsilon)e^{-\lambda t_1}\Big(\frac{K_i}{M}e^{-\int_0^{t_1}[\tilde{b}_i^{\phi}(u)-\lambda]du}+1-e^{-\int_0^{t_1}[\tilde{b}_i^{\phi}(u)-\lambda]du}\Big)\\
\leq&M(\|\varphi-\phi\|_{\xi}+\varepsilon)e^{-\lambda t_1}\Big[1-\Big(1-\frac{K_i}{M}\Big)e^{-\int_0^{t_1}[\tilde{b}_i^{\phi}(u)-\lambda]du}\Big]\\
<&M(\|\varphi-\phi\|_{\xi}+\varepsilon)e^{-\lambda t_1},
\end{aligned}
$$

thus,

$$\|Z(t_1)\|_{\mathcal{A}^n}<M(\|\varphi-\phi\|_{\xi}+\varepsilon)e^{-\lambda t_1}.$$

This, according to (4.5), is a contradiction. Hence, (4.4) holds. Letting $\varepsilon\to 0^+$, we obtain

$$\|Z(t)\|_{\mathcal{A}^n}\leq M\|\varphi-\phi\|_{\xi}e^{-\lambda t},\ \text{ for all } t>0. \tag{4.9}$$

Then, using a similar derivation in the proofs of (4.7) and (4.8), with the help of (4.9), we can see that

$$e^{\lambda t}\|z_j(t)\|_{\mathcal{A}}\leq\sup_{s\in(-\rho,t]}e^{\lambda s}\|z_j(s)\|_{\mathcal{A}}\leq\frac{M\|\varphi-\phi\|_{\xi}}{1-a_j^{+}e^{\lambda\tau_j^{+}}},\ i=1,2,\ldots,n.$$

Consequently, we arrive at

$$\|z(t)\|_{\mathcal{A}^n}\leq M_0\|\varphi-\phi\|_{\xi}e^{-\lambda t},\ \text{ for all } t>0,$$

where $M_0=\max\limits_{1\leq i\leq n}\Big\{\frac{M}{1-a_j^{+}e^{\lambda\tau_j^{+}}}\Big\}$, which means that the solution x^* is globally exponentially stable. The proof is completed. □

5. Numerical examples

Example 5.1. *In system* (2.1)*, let* $n=m=2$, $s=0$, *and for* $i,j,l=1,2$, *take*

$$
\begin{aligned}
x_i(t)=&e_0x_i^0(t)+e_1x_i^1(t)+e_2x_i^2(t)+e_{12}x_i^{12}(t),\\
J_j(x_j)=&\frac{1}{100}e_0\sin(x_j^0)+\frac{1}{70}e_1\sin(x_j^1)+\frac{1}{20}e_2\tanh(x_j^2)+\frac{1}{40}e_{12}\sin(x_j^{12}),\\
F_j(x_j)=&\frac{1}{20}e_0\sin(x_j^0)+\frac{1}{25}e_1\sin(x_j^1)+\frac{1}{50}e_2\sin(x_j^2)+\frac{1}{40}e_{12}\sin(x_j^{12}),\\
R_j(x_j)=&\frac{1}{50}e_0\sin(x_j^0)+\frac{1}{60}e_1\sin(x_j^1)+\frac{1}{70}e_2\tanh(x_j^2)+\frac{1}{90}e_{12}\sin(x_j^{12}),\\
I_1(t)=&I_2(t)=0.49e_0\sin t+0.51e_1\sin\sqrt{2}t+0.56e_2\sin t+0.49e_{12}\sin\sqrt{2}t,
\end{aligned}
$$

$$\begin{aligned}
\tau_1(t) =& 1 - \sin t,\ \ \tau_2(t) = 1.4 + 0.2\cos t,\\
a_1(t) =& (0.01 + 0.004\cos t)e_0 + (0.01 + 0.001\sin\sqrt{6}t)e_1,\\
&+ (0.01 + 0.001\cos\sqrt{3}t)e_2 + (0.01 + 0.002\cos\sqrt{5}t)e_{12},\\
a_2(t) =& (0.01 + 0.002\sin t)e_0 + (0.01 + 0.001\sin\sqrt{3}t)e_1\\
&+ (0.01 + 0.001\cos\sqrt{2}t)e_2 + (0.01 + 0.001\cos\sqrt{7}t)e_{12},\\
b_1(t) =& (0.5 + 0.07\sin t)e_0 + (0.3 + 0.08\cos\sqrt{2}t)e_1\\
&+ (0.3 + 0.01\sin t)e_2 + (0.3 + 0.01\sin\sqrt{3}t)e_{12},\\
b_2(t) =& (0.5 + 0.01\cos\sqrt{3}t)e_0 + (0.3 + 0.01\sin t)e_1\\
&+ (0.3 + 0.01\cos t)e_2 + (0.3 + 0.01\cos\sqrt{5}t)e_{12},
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} \mu_{11}(t) & \mu_{12}(t)\\ \mu_{21}(t) & \mu_{22}(t)\end{pmatrix} &= \begin{pmatrix} \frac{1}{100}e_0\sin\sqrt{2}t + \frac{1}{50}e_1\sin t & \frac{3}{100}e_2\sin\sqrt{3}t + \frac{1}{20}e_{12}\cos t\\ \frac{1}{20}e_1\sin\sqrt{3}t + \frac{2}{25}e_2\cos t & \frac{7}{100}e_0\sin t + \frac{2}{25}e_{12}\cos\sqrt{5}t\end{pmatrix},\\
\begin{pmatrix} \nu_{11}(t) & \nu_{12}(t)\\ \nu_{21}(t) & \nu_{22}(t)\end{pmatrix} &= \begin{pmatrix} \frac{1}{100}e_0\cos t + \frac{1}{100}e_1\sin t & \frac{1}{25}e_2\sin t + \frac{3}{100}e_{12}\sin t\\ \frac{1}{25}e_1\cos t + \frac{1}{50}e_2\sin\sqrt{2}t & \frac{1}{100}e_0\sin t + \frac{1}{25}e_{12}\cos\sqrt{5}t\end{pmatrix},\\
\begin{pmatrix} \theta_{111}(t) & \theta_{112}(t)\\ \theta_{121}(t) & \theta_{122}(t)\end{pmatrix} &= \begin{pmatrix} \frac{1}{100}e_0\sin t + \frac{3}{100}e_1\cos\sqrt{2}t & \frac{1}{20}e_1\sin\sqrt{2}t + \frac{3}{100}e_2\\ \frac{7}{100}e_2\sin t + \frac{9}{100}e_{12}\cos t & \frac{2}{25}e_0\cos t + \frac{1}{25}e_{12}\sin t\end{pmatrix},\\
\begin{pmatrix} \theta_{211}(t) & \theta_{212}(t)\\ \theta_{221}(t) & \theta_{222}(t)\end{pmatrix} &= \begin{pmatrix} \frac{1}{100}e_0\sin\sqrt{2}t + \frac{1}{50}e_1\sin t & \frac{7}{100}e_2\sin\sqrt{7}t + \frac{9}{100}e_{12}\cos\sqrt{5}t\\ \frac{1}{50}e_1\cos\sqrt{3}t + \frac{3}{100}e_2\sin t & \frac{1}{25}e_0\sin t + \frac{1}{100}e_{12}\cos\sqrt{7}t\end{pmatrix},\\
\begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t)\\ \sigma_{21}(t) & \sigma_{22}(t)\end{pmatrix} &= \begin{pmatrix} 1 - 0.5\sin t & 0.5 + 0.2\sin 3t\\ 1.1 + 0.2\cos\sqrt{5}t & 1.2 + 0.7\sin t\end{pmatrix},\\
\begin{pmatrix} \eta_{111}(t) & \eta_{112}(t)\\ \eta_{121}(t) & \eta_{122}(t)\end{pmatrix} &= \begin{pmatrix} 1 - 0.2\sin t & 0.01\sin\sqrt{6}t + 0.07\\ 1 - 0.08\sin\sqrt{5}t & 1.5 - 0.3\cos t\end{pmatrix},\\
\begin{pmatrix} \eta_{211}(t) & \eta_{212}(t)\\ \eta_{221}(t) & \eta_{222}(t)\end{pmatrix} &= \begin{pmatrix} 1.3 - 0.2\sin t & 0.11\sin\sqrt{6}t + 0.7\\ 1 - 0.08\sin\sqrt{5}t & 1.2 - 0.8\cos t\end{pmatrix},\\
\begin{pmatrix} \gamma_{111}(t) & \gamma_{112}(t)\\ \gamma_{121}(t) & \gamma_{122}(t)\end{pmatrix} &= \begin{pmatrix} 0.13 + 0.02\cos t & 0.7 + 0.5\cos\sqrt{2}t\\ 0.5 - 0.07\sin\sqrt{5}t & 0.9 - 0.8\sin t\end{pmatrix},\\
\begin{pmatrix} \gamma_{211}(t) & \gamma_{212}(t)\\ \gamma_{221}(t) & \gamma_{222}(t)\end{pmatrix} &= \begin{pmatrix} 0.4 + 0.02\sin t & 0.5 + 0.2\cos\sqrt{3}t\\ 0.7 - 0.07\sin\sqrt{7}t & 0.9 - 0.6\cos t\end{pmatrix}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&L_j^J = 0.05,\ L_j^F = 0.05,\ L_j^R = 0.02,\ M_l^R = 0.02,\ a_1^+ = 0.014,\ a_2^+ = 0.012,\ \bar{b}_1^{\phi} = 0.57,\\
&\bar{b}_2^{\phi} = 0.51,\ \bar{b}_1^c = 0.38,\ \bar{b}_2^c = 0.31,\ \mu_{11}^+ = 0.02,\ \mu_{12}^+ = 0.05,\ \mu_{21}^+ = 0.08,\ \mu_{22}^+ = 0.08,\\
&\nu_{11}^+ = 0.01,\ \nu_{12}^+ = 0.04,\ \nu_{21}^+ = 0.04,\ \nu_{22}^+ = 0.04,\ \theta_{111}^+ = 0.03,\ \theta_{112}^+ = 0.05,\ \theta_{121}^+ = 0.09,\\
&\theta_{122}^+ = 0.08,\ \theta_{211}^+ = 0.02,\ \theta_{212}^+ = 0.09,\ \theta_{221}^+ = 0.03,\ \theta_{222}^+ = 0.04,\ \beta_1 = \beta_2 = 1,\\
&K_1 = 0.95,\ K_2 = 1,\ \tilde{b}_1^{\phi}(t) = 0.5,\ \tilde{b}_2^{\phi}(t) = 0.5,\ r \approx 0.762942 < 1,
\end{aligned}$$

$$\sup_{t\in\mathbb{R}}\Big\{-\tilde{b}_i^{\phi}(t)+K_i\Big[\bar{b}_i^{\phi}a_i^+\frac{1}{1-a_i^+}+\bar{b}_i^c\frac{1}{1-a_i^+} + \beta_i^{-1}\sum_{j=1}^{n}\mu_{ij}^+L_j^J\beta_j\frac{1}{1-a_j^+}+\beta_i^{-1}\sum_{j=1}^{n}\nu_{ij}^+L_j^F\beta_j\frac{1}{1-a_j^+} + \beta_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}^+\Big(L_j^RM_l^R\beta_j\frac{1}{1-a_j^+}+L_l^RM_j^R\beta_l\frac{1}{1-a_l^+}\Big)\Big]\Big\}<-0.121.$$

So (H_1), (H_2), (H_3) *and* (H_4) *are satisfied. Hence, by Theorem 4.1, we see that system* (2.1) *has a unique compact almost automorphic solution that is globally exponentially stable (see Figures 1–4).*

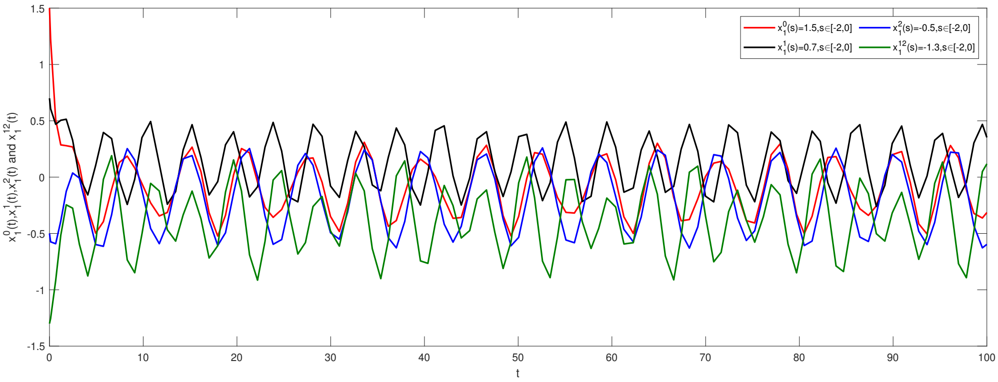

Figure 1. States $x_1^0, x_1^1, x_1^2, x_1^{12}$ of (2.1) with different initial values.

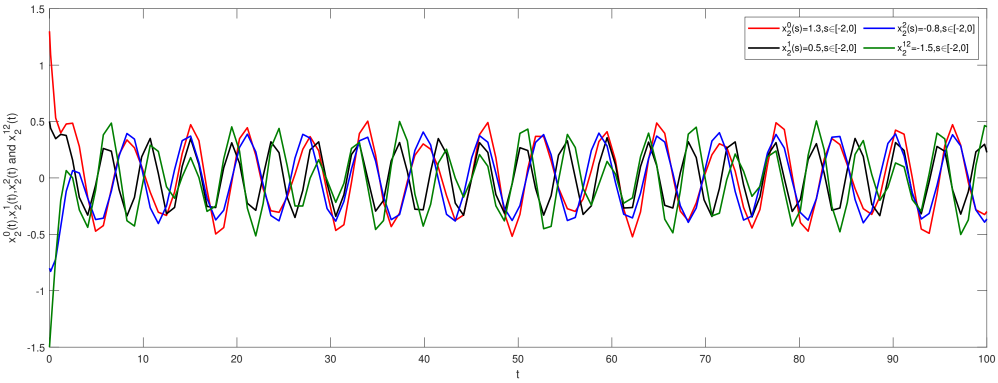

Figure 2. States $x_2^0, x_2^1, x_2^2, x_2^{12}$ of (2.1) with different initial values.

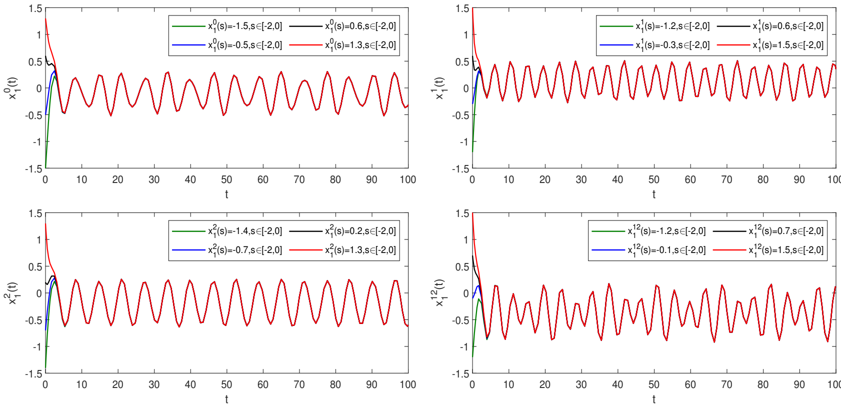

Figure 3. Global exponential stability of states $x_1^0, x_1^1, x_1^2, x_1^{12}$ of (2.1) with different initial values.

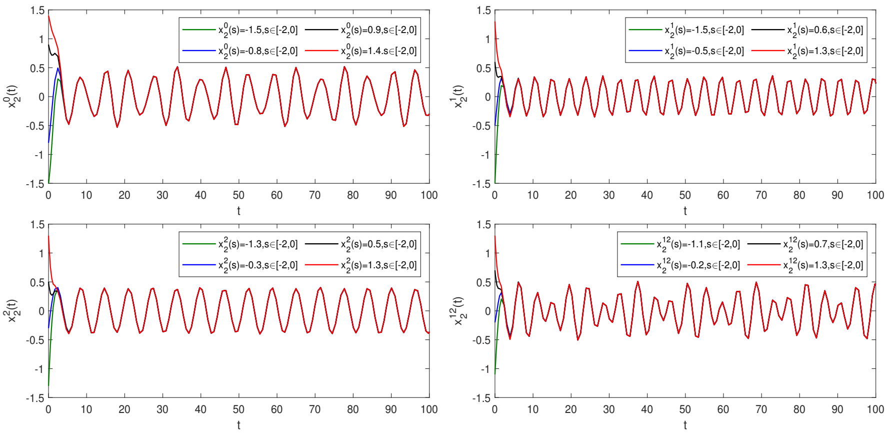

Figure 4. Global exponential stability of states $x_2^0, x_2^1, x_2^2, x_2^{12}$ of (2.1) with different initial values.

Example 5.2. *In system* (2.1)*, let* $m = 3, n = 2, s = 0$*, and for* $i,\ \ j,\ \ l = 1, 2$*, take*

$$
\begin{aligned}
x_i(t) =& e_0 x_i^0(t) + e_1 x_i^1(t) + e_2 x_i^2(t) + +e_3 x_i^3(t) + e_{12} x_i^{12}(t) + e_{13} x_i^{13}(t) + e_{23} x_i^{23}(t) + e_{123} x_i^{123}(t),\\
J_j(x_j) =& \frac{1}{100} e_0 \sin(x_j^0) + \frac{1}{150} e_1 \sin(x_j^1) + \frac{1}{125} e_2 \tanh(x_j^2) + \frac{1}{175} e_3 \sin(x_j^3)\\
&+ \frac{1}{140} e_{12} \sin(x_j^{12}) + \frac{1}{170} e_{13} \sin(x_j^{13}) + \frac{1}{120} e_{23} \sin(x_j^{23}) + \frac{1}{160} e_{123} \sin(x_j^{123}),\\
F_j(x_j) =& \frac{1}{100} e_0 \sin(x_j^0) + \frac{1}{200} e_1 \sin(x_j^1) + \frac{1}{150} e_2 \sin(x_j^2) + \frac{1}{120} e_3 \sin(x_j^3)\\
&+ \frac{1}{135} e_{12} \sin(x_j^{12}) + \frac{1}{150} e_{13} \sin(x_j^{13}) + \frac{1}{120} e_{23} \sin(x_j^{23}) + \frac{1}{140} e_{123} \sin(x_j^{123}),\\
R_j(x_j) =& \frac{1}{120} e_0 \sin(x_j^0) + \frac{1}{110} e_1 \sin(x_j^1) + \frac{1}{100} e_2 \tanh(x_j^2) + \frac{1}{130} e_3 \sin(x_j^3)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{160}e_{12}\sin(x_j^{12})+\frac{1}{170}e_{13}\sin(x_j^{13})+\frac{1}{150}e_{23}\sin(x_j^{23})+\frac{1}{180}e_{123}\sin(x_j^{123}),\\
I_1(t)=&I_2(t)=0.49e_0\sin t+0.51e_1\sin\sqrt{2}t+0.50e_2\sin t+0.49e_3\sin\sqrt{2}t\\
&+0.49e_{12}\sin\sqrt{2}t+0.49e_{13}\sin\sqrt{2}t+0.49e_{23}\sin\sqrt{2}t+0.49e_{123}\sin\sqrt{2}t,\\
\tau_1(t)=&1-\sin t,\ \tau_2(t)=1.4+0.2\cos t,\\
a_1(t)=&(0.01+0.004\sin t)e_0+(0.01+0.001\sin\sqrt{6}t)e_1\\
&+(0.01+0.001\cos\sqrt{3}t)e_2+(0.01+0.002\sin\sqrt{5}t)e_3\\
&+(0.01+0.003\sin t)e_{12}+(0.01+0.002\cos t)e_{13}\\
&+(0.01+0.002\cos\sqrt{2}t)e_{23}+(0.01+0.002\sin\sqrt{6}t)e_{123},\\
a_2(t)=&(0.01+0.002\sin t)e_0+(0.01+0.002\cos 3t)e_1\\
&+(0.01+0.001\sin\sqrt{2}t)e_2+(0.01+0.001\sin\sqrt{7}t)e_3\\
&+(0.01+0.001\sin\sqrt{5}t)e_{12}+(0.01+0.002\cos t)e_{13}\\
&+(0.01+0.002\cos\sqrt{5}t)e_{23}+(0.01+0.001\sin\sqrt{3}t)e_{123},\\
b_1(t)=&(0.5+0.05\sin t)e_0+(0.3+0.01\cos\sqrt{2}t)e_1\\
&+(0.3+0.02\sin t)e_2+(0.3+0.07\sin\sqrt{3}t)e_3\\
&+(0.3+0.06\sin 3t)e_{12}+(0.3+0.05\sin 2t)e_{13}\\
&+(0.3+0.01\sin t)e_{23}+(0.3+0.01\cos\sqrt{3}t)e_{123},\\
b_2(t)=&(0.5+0.01\cos\sqrt{3}t)e_0+(0.3+0.02\sin t)e_1\\
&+(0.3+0.07\cos t)e_2+(0.3+0.05\cos\sqrt{5}t)e_3,\\
&+(0.3+0.01\sin\sqrt{5}t)e_{12}++(0.3+0.02\cos\sqrt{5}t)e_{13}\\
&+(0.3+0.01\sin t)e_{23}++(0.3+0.01\sin 7t)e_{123},
\end{aligned}
$$

$$
\begin{aligned}
\mu_{11}(t)=&\frac{1}{50}e_0\sin\sqrt{2}t+\frac{1}{70}e_1\sin t+\frac{1}{80}e_3\sin\sqrt{3}t,\\
\mu_{12}(t)=&\frac{1}{20}e_0\cos\sqrt{2}t+\frac{1}{40}e_{13}\sin t+\frac{1}{60}e_{123}\sin\sqrt{5}t,\\
\mu_{21}(t)=&\frac{2}{25}e_0\cos\sqrt{2}t+\frac{7}{100}e_{12}\sin t-\frac{1}{100}e_{23}\sin t,\\
\mu_{22}(t)=&\frac{2}{25}e_0\sin 2t+\frac{3}{100}e_2\cos t+\frac{7}{100}e_{23}\sin\sqrt{2}t,\\
\nu_{11}(t)=&\frac{1}{100}e_0\cos\sqrt{2}t+\frac{1}{120}e_1\cos 2t+\frac{1}{110}e_3\sin\sqrt{6}t,\\
\nu_{12}(t)=&\frac{1}{25}e_0\sin\sqrt{2}t+\frac{3}{100}e_{13}\sin t+\frac{1}{50}e_{123}\sin\sqrt{6}t,\\
\nu_{21}(t)=&\frac{1}{25}e_0\cos\sqrt{7}t+\frac{3}{100}e_{12}\sin 3t-\frac{1}{25}e_{23}\sin\sqrt{6}t,\\
\nu_{22}(t)=&\frac{1}{25}e_0\sin 2t+\frac{1}{50}e_2\sin 3t+\frac{1}{75}e_{23}\cos\sqrt{7}t,
\end{aligned}
$$

$$\begin{aligned}
\theta_{111}(t) =&\frac{3}{100}e_0\sin\sqrt{2}t+\frac{1}{50}e_1\cos t+\frac{1}{90}e_3\sin 5t,\\
\theta_{112}(t) =&\frac{1}{20}e_0\sin\sqrt{2}t+\frac{1}{25}e_{13}\cos t+\frac{1}{85}e_{123}\cos\sqrt{3}t,\\
\theta_{121}(t) =&\frac{7}{100}e_0\cos\sqrt{2}t+\frac{3}{50}e_{12}\sin t-\frac{1}{25}e_{23}\cos t,\\
\theta_{122}(t) =&\frac{2}{25}e_0\sin 2t+\frac{7}{100}e_2\sin\sqrt{7}t+\frac{3}{100}e_{23}\sin t,\\
\theta_{211}(t) =&\frac{1}{50}e_0\sin t+\frac{1}{60}e_1\sin t+\frac{1}{80}e_3\cos\sqrt{3}t,\\
\theta_{212}(t) =&\frac{9}{100}e_0\sin\sqrt{2}t+\frac{7}{100}e_{13}\cos t+\frac{1}{50}e_{123}\cos\sqrt{3}t,\\
\theta_{221}(t) =&\frac{3}{100}e_0\cos\sqrt{2}t+\frac{1}{50}e_{12}\sin\sqrt{3}t-\frac{1}{75}e_{23}\sin\sqrt{3}t,\\
\theta_{222}(t) =&\frac{1}{25}e_0\sin\sqrt{2}t+\frac{1}{50}e_2\sin t+\frac{1}{100}e_{23}\sin\sqrt{7}t,\\
\sigma_{ij} =&1.1+0.1\sin\sqrt{7}t, \eta_{ijl}=1.3-0.9\sin t, \gamma_{ijl}=1.2-0.2\cos t
\end{aligned}$$

Then we have

$$\begin{aligned}
&L_j^J=0.01,\ L_j^F=0.01,\ L_j^R=0.01,\ M_l^R=0.01,\ a_1^+=0.014,\ a_2^+=0.012,\ \bar{b}_1^\phi=0.55,\\
&\bar{b}_2^\phi=0.51,\ \bar{b}_1^c=0.37,\ \bar{b}_2^c=0.37,\ \mu_{11}^+=0.02,\ \mu_{12}^+=0.05,\ \mu_{21}^+=0.08,\ \mu_{22}^+=0.08,\\
&v_{11}^+=0.01,\ v_{12}^+=0.04,\ v_{21}^+=0.04,\ v_{22}^+=0.04,\ \theta_{111}^+=0.03,\ \theta_{112}^+=0.05,\ \theta_{121}^+=0.09,\\
&\theta_{122}^+=0.08,\ \theta_{211}^+=0.02,\ \theta_{212}^+=0.09,\ \theta_{221}^+=0.03,\ \theta_{222}^+=0.04,\ \beta_1=\beta_2=1,\\
&K_1=1,\ K_2=1,\ \tilde{b}_1^\phi(t)=0.5,\ \tilde{b}_2^\phi(t)=0.5,\ r\approx 0.7719<1,
\end{aligned}$$

$$\begin{aligned}
\sup_{t\in\mathbb{R}}\Big\{&-\tilde{b}_i^\phi(t)+K_i\Big[\bar{b}_i^\phi a_i^+\frac{1}{1-a_i^+}+\bar{b}_i^c\frac{1}{1-a_i^+}\\
&+\beta_i^{-1}\sum_{j=1}^n\mu_{ij}^+L_j^J\beta_j\frac{1}{1-a_j^+}+\beta_i^{-1}\sum_{j=1}^n v_{ij}^+L_j^F\beta_j\frac{1}{1-a_j^+}\\
&+\beta_i^{-1}\sum_{j=1}^n\sum_{l=1}^n\theta_{ijl}^+\Big(L_j^RM_l^R\beta_j\frac{1}{1-a_j^+}+L_l^RM_j^R\beta_l\frac{1}{1-a_l^+}\Big)\Big]\Big\}<-0.11575.
\end{aligned}$$

So (H_1), (H_2), (H_3) *and* (H_4) *are satisfied. Hence, by Theorem 4.1, we see that system* (2.1) *has a unique compact almost automorphic solution that is globally exponentially stable (see Figures 5–10).*

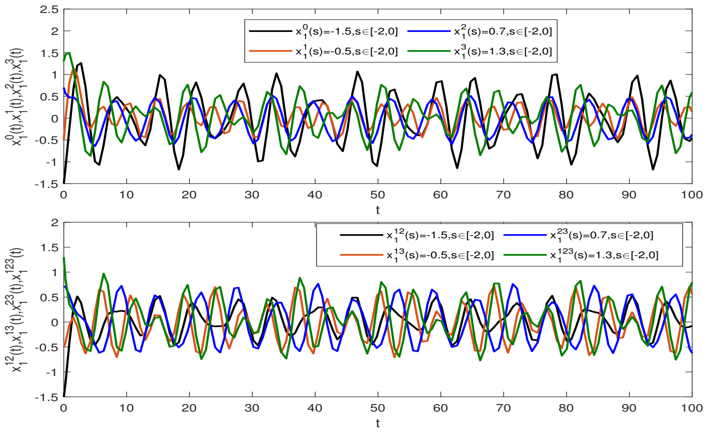

Figure 5. States $x_1^0, x_1^1, x_1^2, x_1^3, x_1^{12}, x_1^{13}, x_1^{23}, x_1^{123}$ of (2.1) with different initial values.

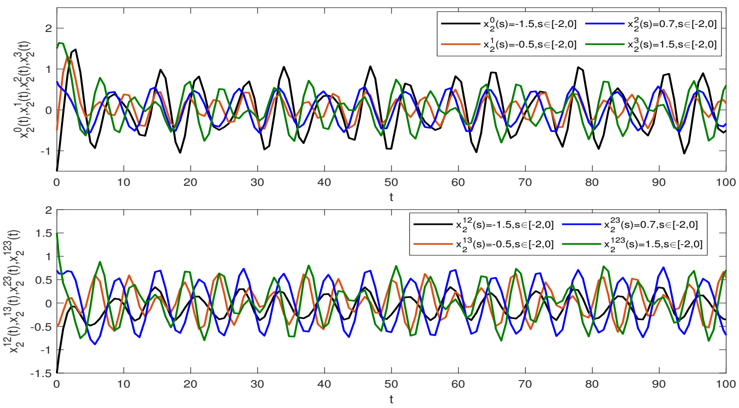

Figure 6. States $x_2^0, x_2^1, x_2^2, x_2^3, x_2^{12}, x_2^{13}, x_2^{23}, x_2^{123}$ of (2.1) with different initial values.

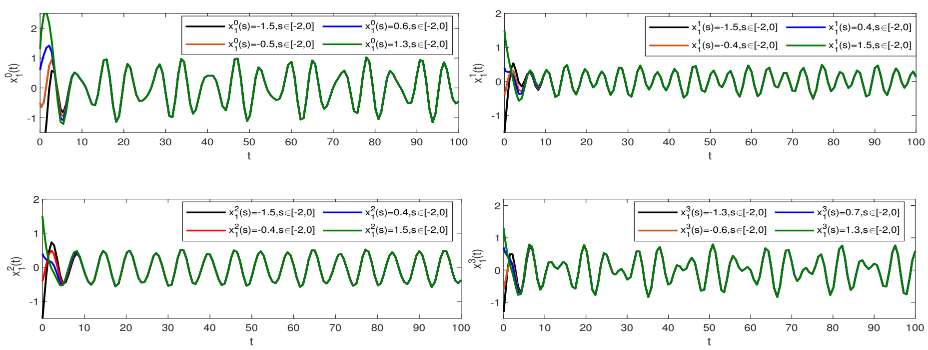

Figure 7. Global exponential stability of states $x_1^0, x_1^1, x_1^2, x_1^3$ of (2.1) with different initial values.

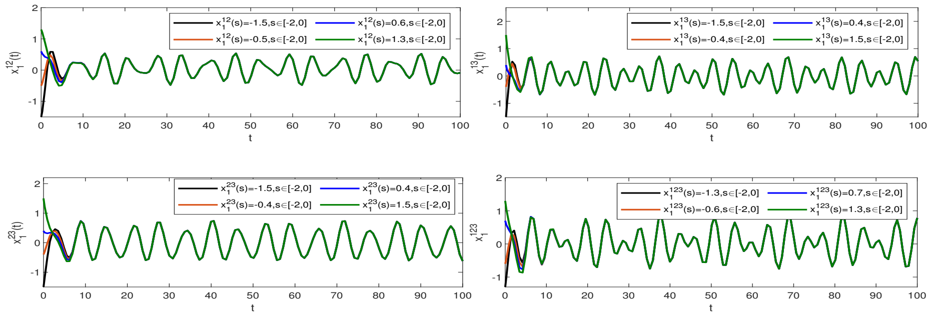

Figure 8. Global exponential stability of states $x_1^{12}, x_1^{13}, x_1^{23}, x_1^{123}$ of (2.1) with different initial values.

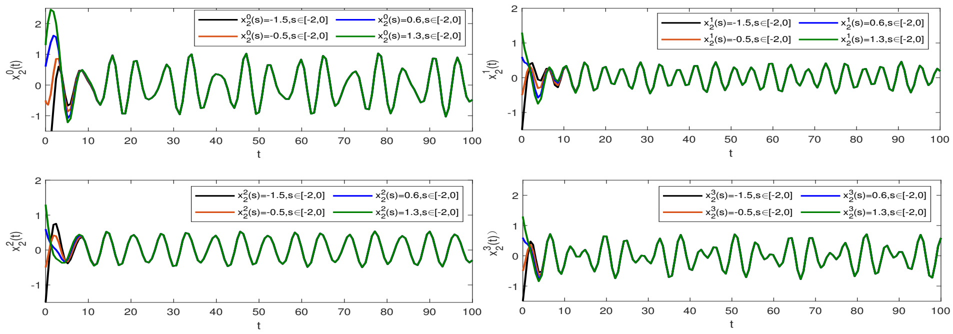

Figure 9. Global exponential stability of states $x_2^0, x_2^1, x_2^2, x_2^3$ of (2.1) with different initial values.

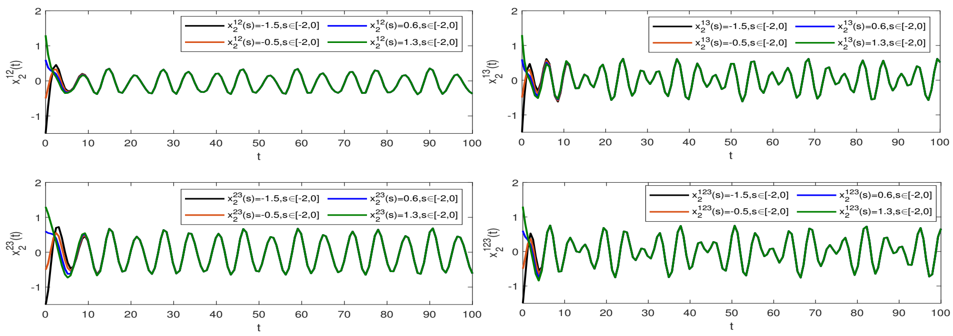

Figure 10. Global exponential stability of states $x_2^{12}, x_2^{13}, x_2^{23}, x_2^{123}$ of (2.1) with different initial values.

Remark 5.1. *In Examples 5.1 and 5.2, since $a_i \not\equiv 0$ and b_i is not a real number, no known results can be used to draw the conclusions of Examples 5.1 and 5.2.*

6. Conclusions

In this paper, the existence and global exponential stability of compact almost automorphic solutions for a class of Clifford-valued higher-order Hopfield neural networks with D operator are established by direct method. It is worth mentioning that the self feedback coefficients of this class of neural networks are also Clifford numbers. The results of this paper are new. The method in this paper can be used to study the compact almost automorphic oscillation of other types of Clifford-valued neural networks with D operator.

Acknowledgments

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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