Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 6153–6172.
DOI: 10.3934/math.2022342
Received: 22 August 2021
Revised: 09 December 2021
Accepted: 23 December 2021
Published: 17 January 2022

*Research article*

# A two-grid $P_0^2$-$P_1$ mixed finite element scheme for semilinear elliptic optimal control problems

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**Abstract:** This paper aims to construct a two-grid mixed finite element scheme for distributed optimal control governed by semilinear elliptic equations. The state and co-state are approximated by the $P_0^2$-$P_1$ pair and the control variable is approximated by the piecewise constant functions. First, a superclose result for the control variable and a priori error estimates for all variables are obtained. Second, a two-grid $P_0^2$-$P_1$ mixed finite element algorithm is presented and the corresponding error is analyzed. In the two-grid scheme, the solution of the semilinear elliptic optimal control problem on a fine grid is reduced to the solution of the semilinear elliptic optimal control problem on a much coarser grid and the solution of a linear decoupled algebraic system on the fine grid and the resulting solution still maintains an asymptotically optimal accuracy. We find that the two-grid method achieves the same convergence property as the $P_0^2$-$P_1$ mixed finite element method if the two mesh sizes satisfy $h = H^2$. Finally, a numerical example demonstrating our theoretical results is presented.

**Keywords:** semilinear elliptic equations; distributed optimal control problems; two-grid; superconvergence; $P_0^2$-$P_1$ mixed finite element
**Mathematics Subject Classification:** 49J20, 65N30

## 1. Introduction

In this paper, we consider the following semilinear optimal control problems for the state variables $\boldsymbol{p}$, $y$, and the control $u$:

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\} \tag{1.1}$$

subject to the state equation

$$-\mathrm{div}(A(x)\nabla y)+\phi(y)=f+u,\quad x\in\Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{p} + \phi(y) = f + u, \quad \boldsymbol{p} = -A(x)\nabla y, \quad x \in \Omega, \tag{1.3}$$

and the boundary condition

$$y = 0, \quad x \in \partial\Omega, \tag{1.4}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$. $U_{ad}$ is the admissible set, defined by

$$U_{ad} = \left\{ u \in L^\infty(\Omega) : \int_\Omega u dx \geq 0 \right\}. \tag{1.5}$$

We assume that the function $\phi(\cdot) \in W^{2,\infty}(-R, R) \cap H^3(-R, R)$ for any $R > 0$, $\phi'(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, and $\phi' \geq 0$. Moreover, we assume that $y_d \in W^{1,\infty}(\Omega)$ and $\boldsymbol{p}_d \in (H^2(\Omega))^2$. $\nu$ is a fixed positive number. The coefficient $A(x) = (a_{ij}(x))$ is a symmetric matrix function with $a_{ij}(x) \in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition

$$c_*|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \leq c^*|\xi|^2, \quad \forall\ (\xi, x) \in \mathbb{R}^2 \times \bar{\Omega}, \quad c^* > c_* > 0.$$

It is well known that finite element approximation plays an important role in the numerical treatment of optimal control problems ( [6, 8, 9, 13, 16, 18, 19, 23–25]). In particular, finite element approximation of semilinear elliptic optimal control problems has been widely studied in the literature, see, for example, finite element method [1, 12, 30], mixed finite element method [14, 20, 29]. Arada et al. [1] studied the numerical approximation of distributed nonlinear optimal control problems governed by semilinear elliptic partial differential equations, they derived the maximum-norm error estimates of finite element approximation for optimal controls. Chen et al. investigated the superconvergence of the finite element approximation for quadratic optimal control problem governed by semilinear elliptic equations in [12]. Chen et al. [14] and Hou [20] obtained the superconvergence and the $L^\infty$-error estimates of semilinear elliptic optimal control problems with two different admissible sets by using the Raviart-Thomas mixed finite element method, respectively. Recently, Hou et al. [21] applied $P_0^2$-$P_1$ mixed finite element to solve the elliptic optimal control problems and derived a priori error estimates for all variables. Compared with the standard mixed finite element method, the velocity of $P_0^2$-$P_1$ mixed finite element method not only belongs to the square integrable space instead of the classical $H(\mathrm{div}; \Omega)$ space but also needs the less regularity and less degrees of freedom. Moreover, the central processing unit (CPU) time of $P_0^2$-$P_1$ mixed finite element method is less than that of the lowest order Raviart-Thomas mixed finite element method because of less degrees of freedom.

The two-grid method was first introduced by Xu [33–35] as a discretization method for nonsymmetric, indefinite and nonlinear partial differential equations. For nonlinear equations, the main idea of two-grid method is to use a coarse-grid space to produce a rough approximation of the solution of nonlinear problems, and then use it as the initial guess for one Newton-like iteration on the fine grid. In recent years, the two-grid method combined with various numerical methods was further investigated by many authors, such as finite difference method [17, 31], mixed finite element method [10], finite volume element method [4, 7], discontinuous Galerkin method [5, 36]. In order to solve the nonlinear fully discrete system more efficiently, Qiu et al. [31] proposed a time two-grid algorithm based on finite difference scheme and obtained the stability and $L^2$-norm error estimate. Yang et al. [36] proposed a two-grid algorithm of discontinuous Galerkin approximation for a kind of nonlinear parabolic problems and discussed the convergence in $H^1$-norm. At present, some scholars have applied the two-grid method to compute optimal control problems. Liu et al. [27] firstly attempted to apply and analyze the two-grid finite element method for the elliptic optimal control problems. Hou et al. [22] discussed the two-grid method of $P_0^2$-$P_1$ mixed finite element approximation for general elliptic optimal control problems with low regularity. As far as we know, there is no convergence result on $P_0^2$-$P_1$ mixed finite element approximation combined with two-grid method for semilinear elliptic optimal control problems.

This paper is motivated by the ideas of the works [20, 21]. We design a two-grid scheme for semilinear elliptic optimal control problems discretized by $P_0^2$-$P_1$ mixed finite element [11]. In the proposed two-grid scheme, we first solve a nonlinear system on the coarse-grid space, then we use the coarse grid solution to extrapolate the solution on the fine grid. On the fine grid we need to solve a decoupled system of linear equations.

The paper is organized as follows. We construct $P_0^2$-$P_1$ mixed finite element approximation scheme for optimal control problem (1.1)–(1.4) in Section 2. The superclose and error analysis is carried out in Section 3. We propose a two-grid algorithm and discuss its convergence in Section 4. We presented a numerical example to demonstrate our theoretical results in Section 5. In the last section, we briefly summarize the results obtained and some possible future extensions.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on $\Omega$ with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. In addition $C$ denotes a general positive constant independent of $h$, where $h$ is the spatial mesh-size for the control and state discretization.

## 2. $P_0^2$-$P_1$ mixed finite element approximation

In this section, we shall construct mixed finite element approximation scheme of the control problem (1.1)–(1.4). For sake of simplicity, we assume that the domain $\Omega$ is a convex polygon. Now, we introduce the co-state elliptic equation

$$-\mathrm{div}(A(x)(\nabla z + \boldsymbol{p} - \boldsymbol{p}_d)) + \phi'(y)z = y - y_d, \quad x \in \Omega, \tag{2.1}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{q} + \phi'(y)z = y - y_d, \quad \boldsymbol{q} = -A(x)(\nabla z + \boldsymbol{p} - \boldsymbol{p}_d), \quad x \in \Omega, \tag{2.2}$$

and the boundary condition

$$z = 0, \quad x \in \partial\Omega. \tag{2.3}$$

Let

$$\boldsymbol{V} = (L^2(\Omega))^2, \quad W = H_0^1(\Omega). \tag{2.4}$$

So the weak formulation of the optimal control problem (1.1)–(1.4) can be restated as the following (OCP):

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\} \tag{2.5}$$

$$(A^{-1}\boldsymbol{p},\boldsymbol{v})+(\nabla y,\boldsymbol{v})=0,\ \forall\ \boldsymbol{v}\in\boldsymbol{V}, \tag{2.6}$$

$$-(\boldsymbol{p},\nabla w)+(\phi(y),w)=(f+u,w),\ \forall\ w\in W, \tag{2.7}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$ or $(L^2(\Omega))^2$.

Since the objective functional is convex, it follows from [26] that the optimal control problem (OCP) has a locally unique solution $(\boldsymbol{p},y,u)$, and that a triplet $(\boldsymbol{p},y,u)$ is the solution of (OCP) if there is a co-state $(\boldsymbol{q},z)\in\boldsymbol{V}\times W$ such that $(\boldsymbol{p},y,\boldsymbol{q},z,u)$ satisfies the following optimality conditions (OCP-OPT):

$$(A^{-1}\boldsymbol{p},\boldsymbol{v})+(\nabla y,\boldsymbol{v})=0,\ \forall\ \boldsymbol{v}\in\boldsymbol{V}, \tag{2.8}$$

$$-(\boldsymbol{p},\nabla w)+(\phi(y),w)=(f+u,w),\ \forall\ w\in W, \tag{2.9}$$

$$(A^{-1}\boldsymbol{q},\boldsymbol{v})+(\nabla z,\boldsymbol{v})=-(\boldsymbol{p}-\boldsymbol{p}_d,\boldsymbol{v}),\ \forall\ \boldsymbol{v}\in\boldsymbol{V}, \tag{2.10}$$

$$-(\boldsymbol{q},\nabla w)+(\phi'(y)z,w)=(y-y_d,w),\ \forall\ w\in W, \tag{2.11}$$

$$(\nu u+z,\tilde{u}-u)\geq 0,\ \forall\ \tilde{u}\in U_{ad}. \tag{2.12}$$

As in [15], the inequality (2.12) can be expressed as

$$u=\{\max\{0,\bar{z}\}-z\}/\nu, \tag{2.13}$$

where $\bar{z}=\frac{\int_\Omega z dx}{\int_\Omega dx}$ denotes the integral average on $\Omega$ of the function $z$.

Let $\mathcal{T}_h$ denotes a shape-regular triangulation of the polygonal domain $\Omega$, $h_T$ denotes the diameter of the element $T$ and $h=\max_{T\in\mathcal{T}_h}\{h_T\}$. Let $\boldsymbol{V}_h\times W_h\subset\boldsymbol{V}\times W$ be defined by the following finite element pair $P_0^2$-$P_1$ [11,32]:

$$\boldsymbol{V}_h=\{\boldsymbol{v}_h=(v_{1h},v_{2h})\in\boldsymbol{V}|v_{1h},v_{2h}\in P_0(T),\ \forall\ T\in\mathcal{T}_h\}, \tag{2.14}$$

$$W_h=\{w_h\in C^0(\Omega)\cap W|w_h\in P_1(T),\ \forall\ T\in\mathcal{T}_h\}. \tag{2.15}$$

Let

$$V_h:=\{v_h\in L^2(\Omega):\forall\ T\in\mathcal{T}_h,\ v_h|_T=\text{constant}\}, \tag{2.16}$$

and $U_h=V_h\cap U_{ad}$.

Next, we introduce three projection operators. Firstly, we define the standard elliptic projection [16] $R_h:\ W\to W_h$, which satisfies: for any $\phi\in W$

$$(\nabla(\phi-R_h\phi),\nabla w_h)=0,\ \ \forall\ w_h\in W_h, \tag{2.17}$$

$$\|\phi-R_h\phi\|_s\leq Ch^{2-s}\|\phi\|_2,\ s=0,1,\ \forall\ \phi\in H^2(\Omega). \tag{2.18}$$

Secondly, we define the standard $L^2$ projection [3] $\Pi_h:\ \boldsymbol{V}\to\boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q}\in\boldsymbol{V}$

$$(\boldsymbol{q}-\Pi_h\boldsymbol{q},\boldsymbol{v}_h)=0,\ \forall\ \boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.19}$$

$$\|\Pi_h \boldsymbol{q}\| \leq C\|\boldsymbol{q}\|, \tag{2.20}$$
$$\|\boldsymbol{q} - \Pi_h \boldsymbol{q}\| \leq Ch\|\boldsymbol{q}\|_1, \ \forall \ \boldsymbol{q} \in (H^1(\Omega))^2. \tag{2.21}$$

At last, we define the standard $L^2$-orthogonal projection $P_h : L^2(\Omega) \to V_h$ which satisfies: for any $\varphi \in L^2(\Omega)$

$$(\varphi - P_h\varphi, v_h) = 0, \ \forall \ v_h \in V_h. \tag{2.22}$$

We have the approximation property:

$$\|\varphi - P_h\varphi\|_{-s,r} \leq Ch^{1+s}|\varphi|_{1,r}, \ s = 0, 1, \ \forall \ \varphi \in W^{1,r}(\Omega). \tag{2.23}$$

Then the mixed finite element discretization of (OCP) is the control problem $(\text{OCP})_h$: find $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times U_h$ such that

$$\min_{u_h\in U_h} \left\{\frac{1}{2}\|\boldsymbol{p}_h - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y_h - y_d\|^2 + \frac{\nu}{2}\|u_h\|^2\right\} \tag{2.24}$$
$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) + (\nabla y_h, \boldsymbol{v}_h) = 0, \ \forall \ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.25}$$
$$-(\boldsymbol{p}_h, \nabla w_h) + (\phi(y_h), w_h) = (f + u_h, w_h), \ \forall \ w_h \in W_h. \tag{2.26}$$

The above problem has a locally unique solution $(\boldsymbol{p}_h, y_h, u_h)$, and that a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of $(\text{OCP})_h$ if there is a co-state $(\boldsymbol{q}_h, z_h) \in \boldsymbol{V}_h \times W_h$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions $(\text{OCP-OPT})_h$:

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) + (\nabla y_h, \boldsymbol{v}_h) = 0, \ \forall \ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.27}$$
$$-(\boldsymbol{p}_h, \nabla w_h) + (\phi(y_h), w_h) = (f + u_h, w_h), \ \forall \ w_h \in W_h, \tag{2.28}$$
$$(A^{-1}\boldsymbol{q}_h, \boldsymbol{v}_h) + (\nabla z_h, \boldsymbol{v}_h) = -(\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{v}_h), \ \forall \ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.29}$$
$$-(\boldsymbol{q}_h, \nabla w_h) + (\phi'(y_h)z_h, w_h) = (y_h - y_d, w_h), \ \forall \ w_h \in W_h, \tag{2.30}$$
$$(\nu u_h + z_h, \tilde{u}_h - u_h) \geq 0, \ \forall \ \tilde{u}_h \in U_h. \tag{2.31}$$

Similar to (2.13), the control inequality (2.31) can be expressed as

$$u_h = \{\max\{0, \overline{z_h}\} - P_h z_h\} / \nu, \tag{2.32}$$

where $\overline{z_h} = \frac{\int_\Omega z_h dx}{\int_\Omega dx}$ denotes the integral average on $\Omega$ of the function $z_h$.

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u} \in U_{ad}$, we first define the state solution $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u})) \in (\boldsymbol{V} \times W)^2$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}(\tilde{u}), \boldsymbol{v}) + (\nabla y(\tilde{u}), \boldsymbol{v}) = 0, \ \ \forall \ \boldsymbol{v} \in \boldsymbol{V}, \tag{2.33}$$
$$-(\boldsymbol{p}(\tilde{u}), \nabla w) + (\phi(y(\tilde{u})), w) = (f + \tilde{u}, w), \ \ \forall \ w \in W, \tag{2.34}$$
$$(A^{-1}\boldsymbol{q}(\tilde{u}), \boldsymbol{v}) + (\nabla z(\tilde{u}), \boldsymbol{v}) = -(\boldsymbol{p}(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}), \ \ \forall \ \boldsymbol{v} \in \boldsymbol{V}, \tag{2.35}$$
$$-(\boldsymbol{q}(\tilde{u}), \nabla w) + (\phi'(y(\tilde{u}))z(\tilde{u}), w) = (y(\tilde{u}) - y_d, w), \ \ \forall \ w \in W. \tag{2.36}$$

Then, we define the discrete state solution $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u})) \in (\boldsymbol{V}_h \times W_h)^2$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}_h(\tilde{u}), \boldsymbol{v}_h) + (\nabla y_h(\tilde{u}), \boldsymbol{v}_h) = 0, \ \ \forall \ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.37}$$

$$-(\boldsymbol{p}_h(\tilde{u}), \nabla w_h) + (\phi(y_h(\tilde{u})), w_h) = (f + \tilde{u}, w_h), \quad \forall\, w_h \in W_h, \tag{2.38}$$
$$(A^{-1}\boldsymbol{q}_h(\tilde{u}), \boldsymbol{v}_h) + (\nabla z_h(\tilde{u}), \boldsymbol{v}_h) = -(\boldsymbol{p}_h(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.39}$$
$$-(\boldsymbol{q}_h(\tilde{u}), \nabla w_h) + (\phi'(y_h(\tilde{u}))z_h(\tilde{u}), w_h) = (y_h(\tilde{u}) - y_d, w_h), \quad \forall\, w_h \in W_h. \tag{2.40}$$

Thus, as we defined, the exact solution and its approximation can be written in the following way:

$$(\boldsymbol{p}, y, \boldsymbol{q}, z) = (\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)),$$
$$(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) = (\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)).$$

## 3. Superconvergence analysis

In this section, we will give a detailed superclose analysis. In order to derive the main results, we need the following lemmas.

**Lemma 3.1.** *Let* $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h)) \in (\boldsymbol{V} \times W)^2$ *and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) \in (\boldsymbol{V}_h \times W_h)^2$ *be the solutions of (2.33)–(2.36) and (2.37)–(2.40) with* $\tilde{u} = u_h$ *respectively. Assume that*

$$\boldsymbol{p}(u_h),\ \boldsymbol{q}(u_h) \in (H^1(\Omega))^2 \text{ and } y(u_h),\ z(u_h) \in W^{1,\infty}(\Omega),$$

*then we have*

$$\|\nabla(y(u_h) - y_h)\| + \|\boldsymbol{p}(u_h) - \boldsymbol{p}_h\| \le Ch, \tag{3.1}$$
$$\|\nabla(z(u_h) - z_h)\| + \|\boldsymbol{q}(u_h) - \boldsymbol{q}_h\| \le Ch. \tag{3.2}$$

*Proof.* From Eqs (2.33)–(2.36) and (2.37)–(2.40), we can easily obtain the following error equations

$$(A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \boldsymbol{v}_h) + (\nabla(y(u_h) - y_h), \boldsymbol{v}_h) = 0, \tag{3.3}$$
$$-(\boldsymbol{p}(u_h) - \boldsymbol{p}_h, \nabla w_h) + (\phi(y(u_h)) - \phi(y_h), w_h) = 0, \tag{3.4}$$
$$(A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \boldsymbol{v}_h) + (\nabla(z(u_h) - z_h), \boldsymbol{v}_h) = -(\boldsymbol{p}(u_h) - \boldsymbol{p}_h, \boldsymbol{v}_h), \tag{3.5}$$
$$-(\boldsymbol{q}(u_h) - \boldsymbol{q}_h, \nabla w_h) + (\phi'(y(u_h))z(u_h) - \phi'(y_h)z_h, w_h) = (y(u_h) - y_h, w_h), \tag{3.6}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$.

Since $\nabla W_h \subset \boldsymbol{V}_h$, with the aid of (2.19), we rewrite (3.3)–(3.6) as

$$\begin{aligned}(A^{-1}(\Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \boldsymbol{v}_h) + (\nabla(R_h y(u_h) - y_h), \boldsymbol{v}_h) = {} & -(\nabla(y(u_h) - R_h y(u_h)), \boldsymbol{v}_h) \\ & - (A^{-1}(\boldsymbol{p}(u_h) - \Pi_h\boldsymbol{p}(u_h)), \boldsymbol{v}_h),\end{aligned} \tag{3.7}$$
$$-(\Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h, \nabla w_h) + (\phi(y(u_h)) - \phi(y_h), w_h) = 0, \tag{3.8}$$
$$\begin{aligned}(A^{-1}(\Pi_h\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \boldsymbol{v}_h) + (\nabla(R_h z(u_h) - z_h), \boldsymbol{v}_h) = {} & -(A^{-1}(\boldsymbol{q}(u_h) - \Pi_h\boldsymbol{q}(u_h)), \boldsymbol{v}_h) \\ & - (\Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h, \boldsymbol{v}_h) - (\nabla(z(u_h) - R_h z(u_h)), \boldsymbol{v}_h),\end{aligned} \tag{3.9}$$
$$-(\Pi_h\boldsymbol{q}(u_h) - \boldsymbol{q}_h, \nabla w_h) + (\phi'(y(u_h))z(u_h) - \phi'(y_h)z_h, w_h) = (y(u_h) - y_h, w_h), \tag{3.10}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$.

Choosing $\boldsymbol{v}_h = \Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h$ in (3.7) and $w_h = R_h y(u_h) - y_h$ in (3.8), respectively. Then adding the two equations to get

$$(A^{-1}(\Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h) + (\phi(R_h y(u_h)) - \phi(y_h), R_h y(u_h) - y_h)$$

$$= -(\nabla(y(u_h)-R_hy(u_h)),\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h)-(A^{-1}(\boldsymbol{p}(u_h)-\Pi_h\boldsymbol{p}(u_h)),\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h)$$
$$-(\phi(y(u_h))-\phi(R_hy(u_h)),R_hy(u_h)-y_h). \tag{3.11}$$

Using Cauchy inequality, (3.11), (2.18), (2.21), the assumption on $A$ and $\phi' \geq 0$, we find that

$$\|\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h\| \leq Ch(\|y(u_h)\|_2+\|\boldsymbol{p}(u_h)\|_1)+\varepsilon\|R_hy(u_h)-y_h\|, \tag{3.12}$$

where $\varepsilon$ is an arbitrary small positive constant.

Letting $\boldsymbol{v}_h = \nabla(R_hy(u_h)-y_h)$ in (3.7), using Cauchy inequality, (2.18) and (2.21), we find that

$$\|\nabla(R_hy(u_h)-y_h)\| \leq Ch(\|y(u_h)\|_2+\|\boldsymbol{p}(u_h)\|_1)+C\|\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h\|. \tag{3.13}$$

For sufficiently small $\varepsilon$, combining (3.12), (3.13), (2.18), (2.21), triangle inequality and Poincare's inequality, we have

$$\|\nabla(y(u_h)-y_h)\|+\|\boldsymbol{p}(u_h)-\boldsymbol{p}_h\| \leq Ch. \tag{3.14}$$

Similarly, choosing $\boldsymbol{v}_h = \Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h$ in (3.9) and $w_h = R_hz(u_h)-z_h$ in (3.10), respectively, it is easy to see that

$$(A^{-1}(\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h)+(\phi'(y(u_h)(R_hz(u_h)-z_h),R_hz(u_h)-z_h)$$
$$= -(\phi'(y(u_h)(z(u_h)-R_hz(u_h)),R_hz(u_h)-z_h)-((\phi'(y(u_h)-\phi'(y_h))z_h,R_hz(u_h)-z_h)$$
$$-(\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h,\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h)-(A^{-1}(\boldsymbol{q}(u_h)-\Pi_h\boldsymbol{q}(u_h)),\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h)$$
$$-(\nabla(z(u_h)-R_hz(u_h)),\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h)+(y(u_h)-y_h,R_hz(u_h)-z_h). \tag{3.15}$$

Using Cauchy inequality, (3.15), (2.18), (2.21), the assumption on $A$ and $\phi' \geq 0$, we find that

$$\|\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h\| \leq Ch(\|z(u_h)\|_2+\|\boldsymbol{q}(u_h)\|_1)+C\|\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h\|$$
$$+C\|y(u_h)-y_h\|+\varepsilon\|R_hz(u_h)-z_h\|. \tag{3.16}$$

Letting $\boldsymbol{v}_h = \nabla(R_hz(u_h)-z_h)$ in (3.9), using Cauchy inequality, (2.18) and (2.21), we find that

$$\|\nabla(R_hz(u_h)-z_h)\| \leq C\|\Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h\|+C\|\Pi_h\boldsymbol{q}(u_h)-\boldsymbol{q}_h\|$$
$$+Ch(\|z(u_h)\|_2+\|\boldsymbol{q}(u_h)\|_1). \tag{3.17}$$

Combining (2.18), (2.21), (3.14), (3.16), (3.17) and triangle inequality, it is easy to get the following result

$$\|\nabla(z(u_h)-z_h)\|+\|\boldsymbol{q}(u_h)-\boldsymbol{q}_h\| \leq Ch. \tag{3.18}$$

This completes the proof of the theorem. □

Now, we are in the position of deriving the estimate for $\|y(u_h)-y_h\|$ and $\|z(u_h)-z_h\|$, we need a priori regularity estimate for the following auxiliary problems:

$$-\operatorname{div}(A\nabla\xi)+\Phi\xi = F_1, \quad x\in\Omega,\ \xi|_{\partial\Omega} = 0, \tag{3.19}$$
$$-\operatorname{div}(A\nabla\zeta)+\phi'(y(u_h))\zeta = F_2, \quad x\in\Omega,\ \zeta|_{\partial\Omega} = 0, \tag{3.20}$$

where

$$\Phi = \begin{cases} \dfrac{\phi(y(u_h)) - \phi(y_h)}{y(u_h) - y_h}, & y(u_h) \neq y_h, \\ \phi'(y_h), & y(u_h) = y_h. \end{cases}$$

The next lemma gives the desired a priori estimate. (see [28])

**Lemma 3.2.** *[28] Let $\xi$ and $\zeta$ be the solutions for (3.19) and (3.20), respectively. Assume that $\Omega$ is convex. Then we have*

$$\|\xi\|_{H^2(\Omega)} \leq C\|F_1\|_{L^2(\Omega)}, \tag{3.21}$$

$$\|\zeta\|_{H^2(\Omega)} \leq C\|F_2\|_{L^2(\Omega)}. \tag{3.22}$$

**Lemma 3.3.** *Let $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h)) \in (\boldsymbol{V} \times W)^2$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) \in (\boldsymbol{V}_h \times W_h)^2$ be the solutions of (2.33)–(2.36) and (2.37)–(2.40) with $\tilde{u} = u_h$ respectively. Assume that*

$$\boldsymbol{p}(u_h),\ \boldsymbol{q}(u_h) \in (H^1(\Omega))^2 \text{ and } y(u_h),\ z(u_h) \in W^{1,\infty}(\Omega),$$

*then we have*

$$\|y(u_h) - y_h\| \leq Ch^2, \tag{3.23}$$

$$\|z(u_h) - z_h\| \leq Ch^2. \tag{3.24}$$

*Proof.* Let $\xi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of (3.19) with $F_1 = y(u_h) - y_h$. We can see from (3.3) and (3.4)

$$\begin{aligned} \|y(u_h) - y_h\|^2 &= (y(u_h) - y_h, -\text{div}(A\nabla\xi)) + (y(u_h) - y_h, \Phi\xi) \\ &= (A\nabla\xi, \nabla(y(u_h) - y_h)) + (y(u_h) - y_h, \Phi\xi) \\ &= (A\nabla\xi - \Pi_h(A\nabla\xi), \nabla(y(u_h) - y_h)) - (A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \Pi_h(A\nabla\xi)) \\ &\quad + (\Phi(y(u_h) - y_h), \xi) \\ &= (A\nabla\xi - \Pi_h(A\nabla\xi), \nabla(y(u_h) - y_h)) + (A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), A\nabla\xi - \Pi_h(A\nabla\xi)) \\ &\quad - (\boldsymbol{p}(u_h) - \boldsymbol{p}_h, \nabla(\xi - R_h\xi)) + (\phi(y(u_h)) - \phi(y_h), \xi - R_h\xi) \\ &\leq Ch\|A\|_{1,\infty}\|\xi\|_2\|\nabla(y(u_h) - y_h)\| + Ch\|A^{-1}\|_{0,\infty}\|A\|_{1,\infty}\|\xi\|_2\|\boldsymbol{p}(u_h) - \boldsymbol{p}_h\| \\ &\quad + Ch\|\xi\|_2\|\boldsymbol{p}(u_h) - \boldsymbol{p}_h\| + Ch^2\|\phi\|_{1,\infty}\|\xi\|_2\|y(u_h) - y_h\| \\ &\leq Ch\|\xi\|_2(\|\nabla(y(u_h) - y_h)\| + \|\boldsymbol{p}(u_h) - \boldsymbol{p}_h\| + h\|y(u_h) - y_h\|), \end{aligned} \tag{3.25}$$

where we used the estimates (2.18) and (2.21).

Substituting (3.1) into (3.25) , using (3.21), for sufficiently small $h$, we obtain

$$\|y(u_h) - y_h\| \leq Ch^2. \tag{3.26}$$

At last, let $\zeta \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of (3.20) with $F_2 = z(u_h) - z_h$. We can see from (3.5) and (3.6).

$$\|z(u_h) - z_h\|^2 = (z(u_h) - z_h, -\text{div}(A\nabla\zeta)) + (z(u_h) - z_h, \phi'(y(u_h))\zeta)$$

$$
\begin{aligned}
&= (A\nabla\zeta, \nabla(z(u_h) - z_h)) + (z(u_h) - z_h, \phi'(y(u_h))\zeta)\\
&= (\nabla(z(u_h) - z_h), A\nabla\zeta - \Pi_h(A\nabla\zeta)) - (A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \Pi_h(A\nabla\zeta))\\
&\quad - (\boldsymbol{p}(u_h) - \boldsymbol{p}_h, \Pi_h(A\nabla\zeta)) + (z(u_h) - z_h, \phi'(y(u_h))\zeta)\\
&= (\nabla(z(u_h) - z_h), A\nabla\zeta - \Pi_h(A\nabla\zeta)) - (A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \Pi_h(A\nabla\zeta) - A\nabla\zeta)\\
&\quad - (\boldsymbol{q}(u_h) - \boldsymbol{q}_h, \nabla(\zeta - R_h\zeta)) - (\phi'(y(u_h))z(u_h) - \phi'(y_h)z_h, R_h\zeta - \zeta)\\
&\quad + (y(u_h) - y_h, R_h\zeta) + (z_h(\phi'(y_h) - \phi'(y(u_h))), \zeta) - (\boldsymbol{p}(u_h) - \boldsymbol{p}_h, \Pi_h(A\nabla\zeta))\\
&=: \sum_{i=1}^{7} I_i. \qquad (3.27)
\end{aligned}
$$

Now, we will estimate $I_1$-$I_7$ one by one. For the first term $I_1$ and the second term $I_2$, by (2.21), we get

$$I_1 \leq Ch\|A\|_{1,\infty}\|\zeta\|_2\|\nabla(z(u_h) - z_h)\| \qquad (3.28)$$

and

$$I_2 \leq Ch\|A^{-1}\|_{0,\infty}\|A\|_{1,\infty}\|\zeta\|_2\|\boldsymbol{q}(u_h) - \boldsymbol{q}_h)\|. \qquad (3.29)$$

For the third term $I_3$, by (2.18), we have

$$I_3 \leq Ch\|\zeta\|_2\|\boldsymbol{q}(u_h) - \boldsymbol{q}_h)\|. \qquad (3.30)$$

Note that

$$\phi'(y(u_h))z(u_h) - \phi'(y_h)z_h = z(u_h)(\phi'(y(u_h)) - \phi'(y_h)) + \phi'(y_h)(z(u_h) - z_h). \qquad (3.31)$$

Then, by (2.18), (3.26) and the assumption on $\phi$, we find that

$$
\begin{aligned}
I_4 \leq & C\|z(u_h)\|_{0,\infty}\|\phi\|_{2,\infty}\|y(u_h) - y_h\| \cdot \|\zeta - R_h\zeta\|\\
& + C\|\phi\|_{1,\infty}\|z(u_h) - z_h\| \cdot \|\zeta - R_h\zeta\|\\
\leq & Ch^4\|z(u_h)\|_{0,\infty}\|\phi\|_{2,\infty}\|\zeta\|_2\\
& + Ch^2\|\phi\|_{1,\infty}\|\zeta\|_2\|z(u_h) - z_h\|. \qquad (3.32)
\end{aligned}
$$

For the fifth term $I_5$, by (3.26), we find that

$$I_5 \leq Ch^2\|\zeta\|_2. \qquad (3.33)$$

For the sixth term $I_6$, by (3.26), we obtain

$$I_6 \leq Ch^2\|\phi\|_{2,\infty}\|\zeta\|_2\|z_h\|, \qquad (3.34)$$

where

$$\|z_h\| \leq \|z(u_h) - z_h\| + \|z(u_h)\|. \qquad (3.35)$$

For the last term $I_7$, by (2.21), (3.1), (3.3), (3.26), we have

$$\begin{aligned} I_7 =&(\boldsymbol{p}(u_h) - \boldsymbol{p}_h, A\nabla\zeta - \Pi_h(A\nabla\zeta)) - (A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), A^2\nabla\zeta) \\ =&(\boldsymbol{p}(u_h) - \boldsymbol{p}_h, A\nabla\zeta - \Pi_h(A\nabla\zeta)) - (A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), A^2\nabla\zeta - \Pi_h(A^2\nabla\zeta)) \\ &+ (\nabla(y(u_h) - y_h), \Pi_h(A^2\nabla\zeta) - A^2\nabla\zeta) - (y(u_h) - y_h, \mathrm{div}(A^2\nabla\zeta)) \\ \leq& Ch^2\|\zeta\|_2. \end{aligned} \tag{3.36}$$

Substituting the estimates $I_1$-$I_7$ into (3.27), using (3.2) and (3.22), for sufficiently small $h$, we easily get

$$\|z(u_h) - z_h\| \leq Ch^2. \tag{3.37}$$

The proof is ended. □

**Lemma 3.4.** *Let* $(\boldsymbol{p}(P_hu), y(P_hu), \boldsymbol{q}(P_hu), z(P_hu))$ *and* $(\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u))$ *be the solutions of (2.33)–(2.36) with* $\tilde{u} = P_hu$ *and* $\tilde{u} = u$*, respectively. Assume that* $u \in H^1(\Omega)$*. Then we have*

$$\|y(u) - y(P_hu)\| + \|\boldsymbol{p}(u) - \boldsymbol{p}(P_hu)\| + \|\nabla(y(u) - y(P_hu))\| \leq Ch^2, \tag{3.38}$$

$$\|z(u) - z(P_hu)\| + \|\boldsymbol{q}(u) - \boldsymbol{q}(P_hu)\| + \|\nabla(z(u) - z(P_hu))\| \leq Ch^2. \tag{3.39}$$

*Proof.* First,we choose $\tilde{u} = P_hu$ and $\tilde{u} = u$ in (2.33)–(2.36) respectively, then we can easily obtain the following error equations

$$(A^{-1}(\boldsymbol{p}(u) - \boldsymbol{p}(P_hu)), \boldsymbol{v}) + (\nabla(y(u) - y(P_hu)), \boldsymbol{v}) = 0, \tag{3.40}$$

$$-(\boldsymbol{p}(u) - \boldsymbol{p}(P_hu), \nabla w) + (\phi(y(u)) - \phi(y(P_hu)), w) = (u - P_hu, w), \tag{3.41}$$

$$(A^{-1}(\boldsymbol{q}(u) - \boldsymbol{q}(P_hu)), \boldsymbol{v}) + (\nabla(z(u) - z(P_hu)), \boldsymbol{v}) = -(\boldsymbol{p}(u) - \boldsymbol{p}(P_hu), \boldsymbol{v}), \tag{3.42}$$

$$-(\boldsymbol{q}(u) - \boldsymbol{q}(P_hu), \nabla w) + (\phi'(y(u)z(u) - \phi'(y(P_hu))z(P_hu), w) = (y(u) - y(P_hu), w), \tag{3.43}$$

for any $\boldsymbol{v} \in \boldsymbol{V}$ and $w \in W$.

Choosing $\boldsymbol{v} = \boldsymbol{p}(u) - \boldsymbol{p}(P_hu)$ in (3.40) and $w = y(u) - y(P_hu)$ in (3.41), respectively, then adding the two equations to get

$$\begin{aligned} &(A^{-1}(\boldsymbol{p}(u) - \boldsymbol{p}(P_hu)), \boldsymbol{p}(u) - \boldsymbol{p}(P_hu)) + (\phi(y(u)) - \phi(y(P_hu)), y(u) - y(P_hu)) \\ =&(u - P_hu, y(u) - y(P_hu)), \end{aligned} \tag{3.44}$$

where

$$(\phi(y(u)) - \phi(y(P_hu)), y(u) - y(P_hu)) = (\phi'(\tilde{y})(y(u) - y(P_hu)), y(u) - y(P_hu)) \geq 0.$$

From (3.40), we can see that

$$\boldsymbol{p}(u) - \boldsymbol{p}(P_hu) = -A\nabla(y(u) - y(P_hu)). \tag{3.45}$$

It follows from (2.23), (3.45) and the assumption on $u$ that

$$(u - P_hu, y(u) - y(P_hu)) \leq C|u - P_hu|_{-1}|y - y(P_hu)|_1$$

$$\leq Ch^2\|u\|_1\|\boldsymbol{p}(u)-\boldsymbol{p}(P_hu)\|. \tag{3.46}$$

Thus, using (3.44)–(3.46), the assumption on A, Cauchy inequality and Poincare's inequality, we derive (3.38).

Similar to (3.44), we have

$$\begin{aligned}&(A^{-1}(\boldsymbol{q}(u)-\boldsymbol{q}(P_hu)),\boldsymbol{q}(u)-\boldsymbol{q}(P_hu))+(\phi'(y(u))(z(u)-z(P_hu)),z(u)-z(P_hu))\\&=(y(u)-y(P_hu),z(u)-z(P_hu))-(\boldsymbol{p}(u)-\boldsymbol{p}(P_hu),\boldsymbol{q}(u)-\boldsymbol{q}(P_hu))\\&\quad-(z(P_hu)(\phi'(y(u))-\phi'(y(P_hu))),z(u)-z(P_hu)),\end{aligned} \tag{3.47}$$

where

$$\|z(P_hu)(\phi'(y(u))-\phi'(y(P_hu)))\|\leq C\|\phi\|_{2,\infty}\|y(u)-y(P_hu)\|. \tag{3.48}$$

Using Cauchy inequality, the assumption on A and $\phi'\geq 0$, we conclude that

$$\|\boldsymbol{q}(u)-\boldsymbol{q}(P_hu)\|\leq C\|y(u)-y(P_hu)\|+C\|\boldsymbol{p}(u)-\boldsymbol{p}(P_hu)\|+\varepsilon\|z(u)-z(P_hu)\|, \tag{3.49}$$

where $\varepsilon$ is an arbitrary small positive constant.

Letting $\boldsymbol{v}=\nabla(z(u)-z(P_hu))$ in (3.42) and using Cauchy inequality, we have

$$\|\nabla(z(u)-z(P_hu))\|\leq C\|\boldsymbol{q}(u)-\boldsymbol{q}(P_hu)\|+C\|\boldsymbol{p}(u)-\boldsymbol{p}(P_hu)\|. \tag{3.50}$$

Combining (3.38), (3.49), (3.50) and Poincare's inequality, we derive (3.39). □

**Lemma 3.5.** *Let $(\boldsymbol{p}(P_hu),y(P_hu),\boldsymbol{q}(P_hu),z(P_hu))$ and $(\boldsymbol{p}(u_h),y(u_h),\boldsymbol{q}(u_h),z(u_h))$ be the solutions of (2.33)–(2.36) with $\tilde{u}=P_hu$ and $\tilde{u}=u_h$, respectively. Assume that $u\in H^1(\Omega)$. Then we have*

$$\|y(u_h)-y(P_hu)\|+\|\boldsymbol{p}(u_h)-\boldsymbol{p}(P_hu)\|+\|\nabla(y(u_h)-y(P_hu))\|\leq C\|P_hu-u_h\|, \tag{3.51}$$

$$\|z(u_h)-z(P_hu)\|+\|\boldsymbol{q}(u_h)-\boldsymbol{q}(P_hu)\|+\|\nabla(z(u_h)-z(P_hu))\|\leq C\|P_hu-u_h\|. \tag{3.52}$$

*Proof.* We choose $\tilde{u}=P_hu$ and $\tilde{u}=u_h$ in (2.33)–(2.36) respectively, then we can easily obtain the following error equations

$$(A^{-1}(\boldsymbol{p}(P_hu)-\boldsymbol{p}(u_h)),\boldsymbol{v})+(\nabla(y(P_hu)-y(u_h)),\boldsymbol{v})=0, \tag{3.53}$$

$$-(\boldsymbol{p}(P_hu)-\boldsymbol{p}(u_h),\nabla w)+(\phi(y(P_hu))-\phi(y(u_h)),w)=(P_hu-u_h,w), \tag{3.54}$$

$$(A^{-1}(\boldsymbol{q}(P_hu)-\boldsymbol{q}(u_h)),\boldsymbol{v})+(\nabla(z(P_hu)-z(u_h)),\boldsymbol{v})=-(\boldsymbol{p}(P_hu)-\boldsymbol{p}(u_h),\boldsymbol{v}), \tag{3.55}$$

$$-(\boldsymbol{q}(P_hu)-\boldsymbol{q}(u_h),\nabla w)+(\phi'(y(P_hu))z(P_hu)-\phi'(y(u_h)z(u_h),w)=(y(P_hu)-y(u_h),w), \tag{3.56}$$

for any $\boldsymbol{v}\in\boldsymbol{V}$ and $w\in W$.

The proof can be completed by using the stability analysis as in Lemma 3.4. □

Let $(\boldsymbol{p}(u),y(u))$ be the solutions of (2.5)–(2.7) and $J(\cdot):L^2(\Omega)\to\mathbb{R}$ be a $G$-differential convex functional near the solution $u$ which satisfies the following form:

$$J(u)=\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2. \tag{3.57}$$

It can be shown that

$$(J'(u), v) = (\nu u + z, v), \tag{3.58}$$

$$(J'(u_h), v) = (\nu u_h + z(u_h), v), \tag{3.59}$$

$$(J'(P_h u), v) = (\nu P_h u + z(P_h u), v). \tag{3.60}$$

In many applications, $J(\cdot)$ is uniform convex near the solution $u$. The convexity of $J(\cdot)$ is closely related to the second order sufficient conditions of the control problem, which are assumed in many studies on numerical methods of the problem (see [1, 2]). Then, there exists a constant $c > 0$, independent of $h$, such that

$$(J'(P_h u) - J'(u_h), P_h u - u_h) \geq c\|P_h u - u_h\|^2, \tag{3.61}$$

where $u$ and $u_h$ are solutions of (2.8)–(2.12) and (2.27)–(2.31) respectively, $P_h u$ is the orthogonal projection of $u$ which is defined in (2.22). We shall assume that the above inequality throughout this paper.

Now, we will discuss the superclose for the control variable.

**Lemma 3.6.** *Let $u$ be the solution of (2.8)–(2.12) and $u_h$ be the solution of (2.27)–(2.31), respectively. Assume that $\boldsymbol{p}(u_h)$, $\boldsymbol{q}(u_h) \in (H^1(\Omega))^2$ and $u, z \in W^{1,\infty}(\Omega)$. Then, we have*

$$\|P_h u - u_h\| \leq Ch^2. \tag{3.62}$$

*Proof.* We choose $\tilde{u} = u_h$ in (2.12) and $\tilde{u}_h = P_h u$ in (2.31) to get the following two inequalities:

$$(\nu u + z, u_h - u) \geq 0 \tag{3.63}$$

and

$$(\nu u_h + z_h, P_h u - u_h) \geq 0. \tag{3.64}$$

Note that $u_h - u = u_h - P_h u + P_h u - u$. Adding the two inequalities (3.63) and (3.64), we have

$$(\nu u_h + z_h - \nu u - z, P_h u - u_h) + (\nu u + z, P_h u - u) \geq 0. \tag{3.65}$$

Thus, by (3.61), (3.65) and (2.22), we find that

$$\begin{aligned}
c\|P_h u - u_h\|^2 \leq & (J'(P_h u) - J'(u_h), P_h u - u_h) \\
= & \nu(P_h u - u_h, P_h u - u_h) + (z(P_h u) - z(u_h), P_h u - u_h) \\
= & \nu(P_h u - u, P_h u - u_h) + \nu(u - u_h, P_h u - u_h) \\
& + (z(P_h u) - z(u_h), P_h u - u_h) \\
\leq & (z_h - z, P_h u - u_h) + (\nu u + z, P_h u - u) \\
& + (z(P_h u) - z(u_h), P_h u - u_h) \\
= & (z_h - z(u_h), P_h u - u_h) + (\nu u + z, P_h u - u) \\
& + (z(P_h u) - z(u), P_h u - u_h).
\end{aligned} \tag{3.66}$$

By Lemmas 3.3 and 3.4, we find that

$$(z_h - z(u_h), P_h u - u_h) \leq Ch^4 + \frac{c}{4}\|P_h u - u_h\|^2 \tag{3.67}$$

and

$$(z(P_h u) - z(u), P_h u - u_h) \leq Ch^4 + \frac{c}{4}\|P_h u - u_h\|^2. \tag{3.68}$$

For the second term at the right side of (3.66), from (2.13), obviously, we obtain

$$\nu u + z = \max\{0, \bar{z}\} = constant. \tag{3.69}$$

Hence,

$$(\nu u + z, P_h u - u) = (\nu u + z)\int_\Omega (P_h u - u)dx = 0. \tag{3.70}$$

Combining (3.67)–(3.70), we complete the proof of (3.62). □

Now, we can obtain the optimal a priori error estimates by use of Lemma 3.1, Lemmas 3.3–3.6 and triangle inequality.

**Theorem 3.1.** *Let $u$ and $u_h$ be the solutions of (2.8)–(2.12) and (2.27)–(2.31), respectively. Assume that all the conditions in Lemma 3.6 are valid and $u \in W^{1,\infty}(\Omega)$. Then we have*

$$\|\nabla(y - y_h)\| + \|\nabla(z - z_h)\| + \|\boldsymbol{p} - \boldsymbol{p}_h\| + \|\boldsymbol{q} - \boldsymbol{q}_h\| \leq Ch, \tag{3.71}$$

$$\|y - y_h\| + \|z - z_h\| \leq Ch^2. \tag{3.72}$$

## 4. Two-grid discretization for optimal control problems

In this section, we shall present a two-grid algorithm for optimal control problems. The basic mechanisms in our approach are two triangulations of $\Omega$, $\mathcal{T}_H$, and $\mathcal{T}_h$, with two different mesh sizes $H$ and $h(H > h)$, and the corresponding finite element spaces $\boldsymbol{V}_H \times W_H$ and $U_H$, $\boldsymbol{V}_h \times W_h$ and $U_h$ which will be called coarse and fine spaces, respectively. Suppose the coarse mesh $\mathcal{T}_H$ is given and let $\mathcal{T}_h$ be obtained from $\mathcal{T}_H$ via regular refinement. Based on $\mathcal{T}_H$ and $\mathcal{T}_h$, we have $\boldsymbol{V}_H \times W_H \subset \boldsymbol{V}_h \times W_h$, $U_H \subset U_h$.

**Two-grid algorithm** :

1) On a coarse mesh, find $(\boldsymbol{p}_H, y_H, \boldsymbol{q}_H, z_H, u_H) \in (\boldsymbol{V}_H \times W_H)^2 \times U_H$ such that

$$(A^{-1}\boldsymbol{p}_H, \boldsymbol{v}_H) + (\nabla y_H, \boldsymbol{v}_H) = 0, \ \forall\, \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{4.1}$$

$$-(\boldsymbol{p}_H, \nabla w_H) + (\phi(y_H), w_H) = (f + u_H, w_H), \ \forall\, w_H \in W_H, \tag{4.2}$$

$$(A^{-1}\boldsymbol{q}_H, \boldsymbol{v}_H) + (\nabla z_H, \boldsymbol{v}_H) = -(\boldsymbol{p}_H - \boldsymbol{p}_d, \boldsymbol{v}_H), \ \forall\, \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{4.3}$$

$$-(\boldsymbol{q}_H, \nabla w_H) + (\phi'(y_H) z_H, w_H) = (y_H - y_d, w_H), \ \forall\, w_H \in W_H, \tag{4.4}$$

$$(\nu u_H + z_H, \tilde{u}_H - u_H) \geq 0, \ \forall\, \tilde{u}_H \in U_H. \tag{4.5}$$

2) On a fine mesh, find $(\boldsymbol{p}_h^*, y_h^*, \boldsymbol{q}_h^*, z_h^*, u_h^*) \in (\boldsymbol{V}_h \times W_h)^2 \times U_h$ such that

$$(\nu u_h^* + z_H, \tilde{u}_h - u_h^*) \geq 0, \ \forall\, \tilde{u}_h \in U_h, \tag{4.6}$$

$$(A^{-1}\boldsymbol{p}_h^*, \boldsymbol{v}_h) + (\nabla y_h^*, \boldsymbol{v}_h) = 0,\ \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.7}$$

$$-(\boldsymbol{p}_h^*, \nabla w_h) + (\phi(y_H) + \phi'(y_H)(y_h^* - y_H), w_h) = (f + u_h^*, w_h),\ \forall\ w_h \in W_h, \tag{4.8}$$

$$(A^{-1}\boldsymbol{q}_h^*, \boldsymbol{v}_h) + (\nabla z_h^*, \boldsymbol{v}_h) = -(\boldsymbol{p}_h^* - \boldsymbol{p}_d, \boldsymbol{v}_h),\ \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.9}$$

$$-(\boldsymbol{q}_h^*, \nabla w_h) + (\phi'(y_h^*)z_h^*, w_h) = (y_h^* - y_d, w_h),\ \forall\ w_h \in W_h. \tag{4.10}$$

**Theorem 4.1.** *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ be the solution of (2.8)–(2.12) and $(\boldsymbol{p}_h^*, y_h^*, \boldsymbol{q}_h^*, z_h^*, u_h^*)$ be the solution of (4.1)–(4.10) respectively. Then we have*

$$\|u - u_h^*\| + \|\nabla(y - y_h^*)\| + \|\boldsymbol{p} - \boldsymbol{p}_h^*\| + \|\nabla(z - z_h^*)\| + \|\boldsymbol{q} - \boldsymbol{q}_h^*\| \leq C(h + H^2). \tag{4.11}$$

*Proof.* Since $P_h u \in U_h$, choosing $\tilde{u} = u_h^*$ in (2.12) and $\tilde{u}_h = P_h u$ in (4.6), we have

$$(\nu u + z, u_h^* - u) \geq 0 \tag{4.12}$$

and

$$(\nu u_h^* + z_H, P_h u - u_h^*) \geq 0. \tag{4.13}$$

Note that $P_h u - u_h^* = P_h u - u + u - u_h^*$, adding two inequalities (4.12) and (4.13), we have

$$(\nu u_h^* + z_H - \nu u - z, u - u_h^*) + (\nu u_h^* + z_H, P_h u - u) \geq 0. \tag{4.14}$$

It follows from (3.70), (4.14) and Young's inequality that

$$\begin{aligned}
\nu\|u - u_h^*\|^2 =& \nu(u - u_h^*, u - u_h^*)\\
\leq& (z_H - z, u - u_h^*) + (\nu u_h^* + z_H, P_h u - u)\\
=& (z_H - z, u - u_h^*) + (\nu u_h^* - \nu u, P_h u - u)\\
&+ (z_H - z, P_h u - u) + (\nu u + z, P_h u - u)\\
\leq& C\|z_H - z\|^2 + \frac{\nu}{2}\|u - u_h^*\|^2 + C\|u - P_h u\|^2.
\end{aligned} \tag{4.15}$$

Thus, it can be derived from (4.15), (3.72) and (2.23) that

$$\|u - u_h^*\| \leq C(h + H^2). \tag{4.16}$$

Subtracting (4.7)–(4.10) from (2.8)–(2.11), using $\nabla W_h \subset \boldsymbol{V}_h$ and (2.19), we have the following error equations

$$\begin{aligned}
(A^{-1}(\Pi_h\boldsymbol{p} - \boldsymbol{p}_h^*), \boldsymbol{v}_h) + (\nabla(R_h y - y_h^*), \boldsymbol{v}_h) =& -(\nabla(y - R_h y), \boldsymbol{v}_h)\\
&- (A^{-1}(\boldsymbol{p} - \Pi_h\boldsymbol{p}), \boldsymbol{v}_h),
\end{aligned} \tag{4.17}$$

$$-(\Pi_h\boldsymbol{p} - \boldsymbol{p}_h^*, \nabla w_h) + (\phi(y) - \phi(y_H) - \phi'(y_H)(y_h^* - y_H), w_h) = (u - u_h^*, w_h), \tag{4.18}$$

$$\begin{aligned}
(A^{-1}(\Pi_h\boldsymbol{q} - \boldsymbol{q}_h^*), \boldsymbol{v}_h) + (\nabla(R_h z - z_h^*), \boldsymbol{v}_h) =& -(A^{-1}(\boldsymbol{q} - \Pi_h\boldsymbol{q}), \boldsymbol{v}_h) - (\boldsymbol{p} - \boldsymbol{p}_h^*, \boldsymbol{v}_h)\\
&- (\nabla(z - R_h z), \boldsymbol{v}_h),
\end{aligned} \tag{4.19}$$

$$-(\Pi_h\boldsymbol{q} - \boldsymbol{q}_h^*, \nabla w_h) + (\phi'(y)z - \phi'(y_h^*)z_h^*, w_h) = (y - y_h^*, w_h), \tag{4.20}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$.

Using the Taylor expansion for function $\phi(y)$ as follows

$$\phi(y) = \phi(y_H) + \phi'(y_H)(y - y_H) + \frac{1}{2}\phi''(\tilde{y})(y - y_H)^2,$$

for some function $\tilde{y}$. Then we have

$$\begin{aligned}
&\phi(y) - \phi(y_H) - \phi'(y_H)(y_h^* - y_H) \\
=&\phi'(y_H)(y - y_h^*) + \frac{1}{2}\phi''(\tilde{y})(y - y_H)^2 \\
=&\phi'(y_H)(y - R_h y + R_h y - y_h^*) + \frac{1}{2}\phi''(\tilde{y})(y - y_H)^2.
\end{aligned} \tag{4.21}$$

Choosing $\boldsymbol{v}_h = \Pi_h \boldsymbol{p} - \boldsymbol{p}_h^*$ in (4.17) and $w_h = R_h y - y_h^*$ in (4.18), respectively. Then adding the two equations to get

$$\begin{aligned}
&(A^{-1}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h^*), \Pi_h \boldsymbol{p} - \boldsymbol{p}_h^*) + (\phi'(y_H)(R_h y - y_h^*), R_h y - y_h^*) \\
=& - (\nabla(y - R_h y), \Pi_h \boldsymbol{p} - \boldsymbol{p}_h^*) - (A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \Pi_h \boldsymbol{p} - \boldsymbol{p}_h^*) + (u - u_h^*, R_h y - y_h^*) \\
&- (\phi'(y_H)(y - R_h y), R_h y - y_h^*) - (\frac{1}{2}\phi''(\tilde{y})(y - y_H)^2, R_h y - y_h^*).
\end{aligned} \tag{4.22}$$

Using Cauchy inequality, (3.72), (2.18), (2.21), the assumption on $A$ and $\phi' \geq 0$, we find that

$$\|\Pi_h \boldsymbol{p} - \boldsymbol{p}_h^*\| \leq Ch(\|y\|_2 + \|\boldsymbol{p}\|_1) + C\|u - u_h^*\| + CH^4 + \varepsilon\|R_h y - y_h^*\|, \tag{4.23}$$

where $\varepsilon$ is an arbitrary small positive constant.

Letting $\boldsymbol{v}_h = \nabla(R_h y - y_h^*)$ in (4.17), using Cauchy inequality, (2.18) and (2.21), we find that

$$\|\nabla(R_h y - y_h^*)\| \leq Ch(\|y\|_2 + \|\boldsymbol{p}\|_1) + C\|\Pi_h \boldsymbol{p} - \boldsymbol{p}_h^*\|. \tag{4.24}$$

For sufficiently small $\varepsilon$, combining (4.23), (4.24), (4.16), (2.18), (2.21), triangle inequality and Poincare's inequality, we have

$$\|\nabla(y - y_h^*)\| + \|\boldsymbol{p} - \boldsymbol{p}_h^*\| \leq C(h + H^2). \tag{4.25}$$

Similarly, choosing $\boldsymbol{v}_h = \Pi_h \boldsymbol{q} - \boldsymbol{q}_h^*$ in (4.19) and $w_h = R_h z - z_h^*$ in (4.20), it is easy to see that

$$\begin{aligned}
&(A^{-1}(\Pi_h \boldsymbol{q} - \boldsymbol{q}_h^*), \Pi_h \boldsymbol{q} - \boldsymbol{q}_h^*) + (\phi'(y)(R_h z - z_h^*), R_h z - z_h^*) \\
=& - (\phi'(y)(z - R_h z), R_h z - z_h^*) - ((\phi'(y) - \phi'(y_h^*))z_h^*, R_h z - z_h^*) \\
&- (\boldsymbol{p} - \boldsymbol{p}_h^*, \Pi_h \boldsymbol{q} - \boldsymbol{q}_h^*) - (A^{-1}(\boldsymbol{q} - \Pi_h \boldsymbol{q}), \Pi_h \boldsymbol{q} - \boldsymbol{q}_h^*) \\
&- (\nabla(z - R_h z), \Pi_h \boldsymbol{q} - \boldsymbol{q}_h^*) + (y - y_h^*, R_h z - z_h^*).
\end{aligned} \tag{4.26}$$

Using Cauchy inequality, (4.26), (2.18), (2.21), the assumption on $A$ and $\phi' \geq 0$, we find that

$$\|\Pi_h \boldsymbol{q} - \boldsymbol{q}_h^*\| \leq Ch(\|z\|_2 + \|\boldsymbol{q}\|_1) + C\|\boldsymbol{p} - \boldsymbol{p}_h^*\| + C\|y - y_h^*\| + \varepsilon\|R_h z - z_h^*\|. \tag{4.27}$$

Letting $\boldsymbol{v}_h = \nabla(R_h z - z_h^*)$ in (4.19), using Cauchy inequality, (2.18) and (2.21), we find that

$$\|\nabla(R_h z - z_h^*)\| \leq Ch(\|z\|_2 + \|\boldsymbol{q}\|_1) + C\|\Pi_h \boldsymbol{q} - \boldsymbol{q}_h^*\| + C\|\boldsymbol{p} - \boldsymbol{p}_h^*\|. \tag{4.28}$$

Combining (2.18), (2.21), (4.25), (4.27), (4.28) and triangle inequality, we obtain

$$\|\nabla(z - z_h^*)\| + \|\boldsymbol{q} - \boldsymbol{q}_h^*\| \leq C(h + H^2). \tag{4.29}$$

This completes the proof of the theorem. □

## 5. Numerical experiments

In this section, we present below an example to illustrate the theoretical results. The discretization was already described in previous sections: the control function $u$ was discretized by piecewise constant functions, whereas the state $(y, \boldsymbol{p})$ and the co-state $(z, \boldsymbol{q})$ were approximated by $P_0^2$-$P_1$ element. All the numerical results are computed by Matlab software.

**Example.** We consider the optimal control problems on the domain $(0, 1)^2$ with $A(x) = I$, $\nu = 1$ and $\phi(y) = y^3$, where $I$ is the identity matrix. The exact solutions are as follows:

$$\begin{aligned}
y &= \sin(\pi x_1)\sin(\pi x_2),\\
z &= \sin(2\pi x_1)\sin(2\pi x_2),\\
u &= \max\{0, \bar{z}\} - z,\\
f &= \mathrm{div}\boldsymbol{p} + y^3 - u,\\
y_d &= -\mathrm{div}\boldsymbol{q} - 3y^2 z + y,\\
\boldsymbol{q} &= -\begin{pmatrix} 2\pi\cos(2\pi x_1)\sin(2\pi x_2)\\ 2\pi\sin(2\pi x_1)\cos(2\pi x_2)\end{pmatrix},\\
\boldsymbol{p} = \boldsymbol{p}_d &= -\begin{pmatrix} \pi\cos(\pi x_1)\sin(\pi x_2)\\ \pi\sin(\pi x_1)\cos(\pi x_2)\end{pmatrix}.
\end{aligned}$$

In Tables 1–2, the errors and the convergence orders of $\|P_h u - u_h\|$, $\|y - y_h\|$, $\|z - z_h\|$, $\|\nabla(y - y_h)\|$, $\|\nabla(z - z_h)\|$, $\|\boldsymbol{p} - \boldsymbol{p}_h\|$ and $\|\boldsymbol{q} - \boldsymbol{q}_h\|$ are provided by mixed finite element method (MFEM) with different $h$. It is obvious that the convergence results coincide with our theoretical analysis in Lemma 3.6 and Theorem 3.1.

Tables 3–4 show the errors and the convergence rates of $\|u - u_h^*\|$, $\|\nabla(y - y_h^*)\|$, $\|\boldsymbol{p} - \boldsymbol{p}_h^*\|$, $\|\nabla(z - z_h^*\|$ and $\|\boldsymbol{q} - \boldsymbol{q}_h^*\|$ by two-grid method (TGM) with the choice $h = H^2$. We can see that the two-grid solution can achieve the same accuracy as the mixed finite element solution. This is consistent with the theoretical analysis in Theorem 4.1.

Finally, we compare the CPU time of the mixed finite element method and the two-grid method in Table 5. It is shown that the computing cost for two-grid method is significantly less than that for mixed finite element method. Therefore, the two-grid algorithm proposed in this paper has great advantages in large-scale numerical calculations.

**Table 1.** The errors and the convergence orders of $\|P_h u - u_h\|$, $\|y - y_h\|$ and $\|z - z_h\|$.

| $h$ | $\|P_h u - u_h\|$ | order | $\|y - y_h\|$ | order | $\|z - z_h\|$ | order |
|---|---|---|---|---|---|---|
| $\frac{1}{8}$ | 9.4375e-2 | - | 2.0728e-2 | - | 8.1275e-2 | - |
| $\frac{1}{16}$ | 2.5247e-2 | 1.90 | 5.2937e-3 | 1.96 | 2.1677e-2 | 1.90 |
| $\frac{1}{32}$ | 6.4216e-3 | 1.97 | 1.3308e-3 | 1.99 | 5.5100e-3 | 1.97 |
| $\frac{1}{64}$ | 1.6124e-3 | 1.99 | 3.3316e-4 | 1.99 | 1.3833e-3 | 1.99 |
| $\frac{1}{128}$ | 4.0354e-4 | 1.99 | 8.3323e-5 | 1.99 | 3.4618e-4 | 1.99 |
| $\frac{1}{256}$ | 1.0091e-4 | 2.00 | 2.0836e-5 | 1.99 | 8.6572e-5 | 2.00 |

**Table 2.** The errors and the convergence orders of $||\nabla(y - y_h)||$, $||\nabla(z - z_h)||$, $||\boldsymbol{p} - \boldsymbol{p}_h||$ and $||\boldsymbol{q} - \boldsymbol{q}_h||$.

| $h$ | $\|\|\nabla(y-y_h)\|\|$ | order | $\|\|\nabla(z-z_h)\|\|$ | order | $\|\|\boldsymbol{p}-\boldsymbol{p}_h\|\|$ | order | $\|\|\boldsymbol{q}-\boldsymbol{q}_h\|\|$ | order |
|---|---|---|---|---|---|---|---|---|
| $\frac{1}{8}$ | 4.3193e-1 | - | 1.6719e-0 | - | 4.3193e-1 | - | 1.7848e-0 | - |
| $\frac{1}{16}$ | 2.1756e-1 | 0.98 | 8.6296e-1 | 0.95 | 2.1756e-1 | 0.98 | 9.1976e-1 | 0.95 |
| $\frac{1}{32}$ | 1.0898e-1 | 0.99 | 4.3499e-1 | 0.98 | 1.0898e-1 | 0.99 | 4.6345e-1 | 0.98 |
| $\frac{1}{64}$ | 5.4514e-2 | 0.99 | 2.1794e-1 | 0.99 | 5.4514e-2 | 0.99 | 2.3217e-1 | 0.99 |
| $\frac{1}{128}$ | 2.7260e-2 | 1.00 | 1.0903e-1 | 0.99 | 2.7260e-2 | 1.00 | 1.1614e-1 | 0.99 |
| $\frac{1}{256}$ | 1.3630e-2 | 1.00 | 5.4520e-2 | 1.00 | 1.3630e-2 | 1.00 | 5.8079e-2 | 1.00 |

**Table 3.** The errors and the convergence orders of two-grid method with $h = H^2$.

| $(H, h)$ | $\|\|u-u_h^*\|\|$ | order | $\|\|\nabla(y-y_h^*)\|\|$ | order | $\|\|\boldsymbol{p}-\boldsymbol{p}_h^*\|\|$ | order |
|---|---|---|---|---|---|---|
| $(\frac{1}{4}, \frac{1}{16})$ | 8.6276e-2 | - | 1.0930e-1 | - | 1.0930e-1 | - |
| $(\frac{1}{8}, \frac{1}{64})$ | 8.2092e-2 | 0.03 | 5.5154e-2 | 0.49 | 5.5154e-2 | 0.49 |
| $(\frac{1}{16}, \frac{1}{256})$ | 2.1919e-2 | 0.95 | 1.3814e-2 | 0.99 | 1.3814e-2 | 0.99 |
| $(\frac{1}{32}, \frac{1}{1024})$ | 5.5725e-3 | 0.98 | 3.4550e-3 | 1.00 | 3.4550e-3 | 0.99 |

**Table 4.** The errors and the convergence orders of two-grid method with $h = H^2$.

| $(H, h)$ | $\|\|\nabla(z-z_h^*)\|\|$ | order | $\|\|\boldsymbol{q}-\boldsymbol{q}_h^*\|\|$ | order |
|---|---|---|---|---|
| $(\frac{1}{4}, \frac{1}{16})$ | 4.3509e-1 | - | 4.6345e-1 | - |
| $(\frac{1}{8}, \frac{1}{64})$ | 2.1811e-1 | 0.49 | 2.3217e-1 | 0.50 |
| $(\frac{1}{16}, \frac{1}{256})$ | 5.4568e-2 | 0.99 | 5.8079e-2 | 1.99 |
| $(\frac{1}{32}, \frac{1}{1024})$ | 1.3643e-2 | 1.00 | 1.4520e-2 | 1.00 |

**Table 5.** The CPU time of mixed finite element method and two-grid method.

| $(H, h)$ | MFEM time (s) | TGM time (s) |
|---|---|---|
| $(\frac{1}{4}, \frac{1}{16})$ | 1.1782 | 0.2117 |
| $(\frac{1}{8}, \frac{1}{64})$ | 18.4429 | 3.4445 |
| $(\frac{1}{16}, \frac{1}{256})$ | 314.4770 | 26.8162 |

## 6. Conclusions

In this paper, we constructed a two-grid $P_0^2$-$P_1$ mixed finite element scheme for semilinear elliptic optimal control problem (1.1)–(1.4). Our theoretical analysis for this class of optimal control problems discretized by $P_0^2$-$P_1$ mixed finite element seems to be new. In the proposed two-grid algorithm, we first solve a nonlinear system on the coarse-grid space, then we use the coarse grid solution to extrapolate the solution on the fine grid. On the fine grid we need to solve a decoupled system of linear equations, this is the main finding of our article. If the coarse and fine mesh sizes satisfy $h = H^2$, the two-grid solution can achieve the same accuracy as $P_0^2$-$P_1$ mixed finite element solution. In our future work, we shall design a two-grid algorithm of mixed finite element method for optimal control problems governed by nonlinear elliptic equations.

## Acknowledgments

This work was supported by the National Natural Science Foundation of China (No. 11601014).

## Conflict of interest

The authors declare no conflict of interest.

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