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Research article

Preinvexity of n-dimensional fuzzy number-valued functions: characterization, variational inequality and optimization problems

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Abstract: In this paper, the preinvexity of n-dimensional fuzzy number-valued functions are defined and discussed by means of the partial order relation in n-dimensional fuzzy number space which including preinvexity, weak preinvexity, strict preinvexity, weakly strict preinvexity, prequasiinvexity, weak prequasiinvexity, strict prequasiinvexity, weakly strict prequasiinvexity, and so on. In addition, their interrelations of the preinvexity of n-dimensional fuzzy number-valued functions are discussed, and some counterexamples are given. Furthermore, the two-parameter optimization problem, n-dimensional fuzzy variational-like inequality and optimality conditions related to n-dimensional preinvex fuzzy number-valued functions are discussed.

Keywords: fuzzy numbers; invex set; preinvexity fuzzy number-valued functions
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1. Introduction

It is well known that the convex analysis is closely linked with the development of optimization theory. Meanwhile, there is often uncertainty of parameters or dates in the process of the mathematical modeling of the specific optimization problems. In order to describe these uncertain parameters or dates in a mathematical modeling, a straightforward and effective way to think about whether it can be represented as fuzzy number in some sense. Therefore, fuzzy convex analysis theory and its corresponding fuzzy optimization problems have been studied by many researchers. In 1992, Nanda and Kar [14] set up a mapping from a vector space to the space of fuzzy numbers, and they introduced the definitions of convex fuzzy mapping, strictly convex fuzzy mapping, quasiconvex fuzzy mapping, strictly quasiconvex fuzzy mapping and logarithmic convex fuzzy mapping, and then applied their results to the problems of fuzzy convex optimization. Moreover, Furukawa [5] proposed the concepts of convex fuzzy mapping and local Lischitz continuous fuzzy mapping by using "fuzzy-max" order, and the basis theorem of fuzzy mapping local Lischitz continuity is given. In 1999, based on the partial order relation of interval number, Syau [17, 18] introduced the definitions of the convex fuzzy mapping and quasiconvex fuzzy mapping, and established characterization for convex fuzzy mapping. In 2000, Yang and Teo [27] investigated pseudoconvexity, invexity, pseudoinvexity for fuzzy mappings by considering the concept of ordering proposed by Goetsschel and Voxman [6], and discussed their interrelations. Meanwhile, based on the same order relation, Yan and Xu [28] introduced another concept of convex fuzzy mapping, and studied a kind of fuzzy convex optimization problems. In 2006, the operation of convex fuzzy mapping proposed by Nanda and Kar were investigated by Zhang and Yuan, and the important concepts of positive homogeneous fuzzy mapping, infimal convolution, convex hull were given, and the corresponding characterization theorem was presented by using the parameter of fuzzy number [31]. In 2008, based on the concept of differentiability of fuzzy mapping, Panigrahi and panda [16] gave the concepts of convexity, quasiconvexity, strictly quasiconvexity, strong quasiconvexity and pseudoconvexity of fuzzy mappings from R^n to the set of fuzzy numbers, and derived the Karush-Kuhn-Tucker optimization condition for a constrained fuzzy optimization problem. In 2013, Li and Noor [9] discussed the properties of the convex fuzzy mappings based on a linear ordering of fuzzy numbers proposed by Goestschel and Voxman. Furthermore, they obtained the judgement theorems of convex, strictly convex and semi-strictly convex fuzzy mapping under lower and upper semicontinuity condition, respectively. In addition, convexity and other related problems of the fuzzy mapping have been studied extensively [4, 10, 23, 25]. As a generalization of convex fuzzy mapping, in 1994, Noor [15]introduced the concepts of preinvex fuzzy mapping and invex set, and the minimization problem of preinvex fuzzy number-valued functions was described by using variational inequality. In 1999, Syau in [19] showed that the preinvexity given by Noor is too restrictive, redefined the preinvexity of η vector-valued functions, established two characterizations for the preinvex fuzzy mappings, and applied their results to the optimization theory. After that several investigators [1–3, 11, 12, 20, 21, 24, 30] also proposed and studied different types of the preinvexity and the generalized preinvexity for fuzzy mappings. However, all of the above works are discussed for 1-dimensional fuzzy number-valued functions. The main reason is that the partial ordered relation in n-dimensional fuzzy number space, the difference between n-dimensional fuzzy numbers, and convex analysis of high-dimensional fuzzy mapping have not been discussed. Until 2016, Gong Zengtai et al. [7] first introduced the partially ordered relation on n-dimensional fuzzy number space, the convexity of the n-dimensional fuzzy mapping, the differentiability, and the corresponding optimization theory. Based on the partially ordered relation in n-dimensional fuzzy number space, considering the convexity of vector-valued function, and combining with the characteristics of n-dimensional fuzzy mappings, they proposed and investigated the convexity of n-dimensional fuzzy number-valued functions, generalized convexity, upper semicontinuity, lower semicontinuity, and discussed their interrelations, and pointed out the local minimum point of convex fuzzy mapping is its global minimum point [8] . As a continuous research of [7, 8], in this paper, we introduce the preinvexity of n-dimensional fuzzy number-valued functions based on the partial order relation in n-dimensional fuzzy number space and some properties of them are discussed. In addition, some counterexamples are given. Then we present criteria for n-dimensional preinvex fuzzy number-valued functions under upper or lower semicontinuity conditions, respectively. Furthermore, the two-parameter optimization problem, n-dimensional fuzzy variational-like inequality problem and the optimal conditions related to n-dimensional preinvex fuzzy number-valued function are discussed.

2. Preliminaries

Let R^n denote the n-dimensional Euclidean space and $F(R^n)$ denote the set of all fuzzy subset on R^n. Fuzzy set $u \in F(R^n)$ is called a fuzzy number if u is a normal, convex fuzzy set, upper semi-continuous and $[u]^0 = \overline{\{x \in R^n, u(x) > 0\}}$ is compact. We denote E^n as n-dimensional fuzzy number space [22,26].

Let $u \in F(R^n)$. For $r \in (0,1]$, we denote $[u]^r = \{x \in R^n, u(x) \geq r\}$. The addition and non-negative scalar multiplication are defined as follows for fuzzy number $u, v \in E^n, \alpha \in R$, $k, k_1, k_2 \in R$, according to the Zadeh's extension principle:

(1) $k(u+v) = ku + kv$;

(2) $k_1(k_2u) = (k_1k_2)u$;

(3)$(k_1 + k_2)u = k_1u + k_2u$ when $k_1 \geq 0$ and $k_2 \geq 0$.

Given $u, v \in E^n$, the distance $D : E^n \times E^n \to [0, +\infty)$ between u and v is defined by the equation

$$D(u, v) = \sup_{r\in[0,1]} d([u]^r, [v]^r),$$

where d is the Hausdorff metric

$$\begin{aligned} d([u]^r, [v]^r) &= \inf\{\varepsilon : [u]^r \subset N([v]^r, \varepsilon), [v]^r \subset N([u]^r, \varepsilon)\} \\ &= \max\{\sup_{a\in[u]^r} \inf_{b\in[v]^r} \|a-b\|, \sup_{b\in[v]^r} \inf_{a\in[u]^r} \|a-b\|\}. \end{aligned}$$

For $u \in E^n$, we denote the centroid of $[u]^r$, $r \in [0,1]$ as

$$\left(\frac{\int\cdots\int_{[u]^r} x_1dx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}, \frac{\int\cdots\int_{[u]^r} x_2dx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}, \cdots, \frac{\int\cdots\int_{[u]^r} x_ndx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}\right)$$

where $\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n$ is the solidity of $[u]^r$, $r \in [0,1]$ and $\int\cdots\int_{[u]^r} x_idx_1dx_2\cdots dx_n$ $(i = 1, 2, \cdots, n)$ is the multiple integral of x_i on measurable sets $[u]^r$, $r \in [0,1]$, refer to [7].

Let $u \in E^n$, n-dimensional vector-valued function τ denote the centroid of the fuzzy number,

$$\tau(u) = \left(2\int_0^1 r\frac{\int\cdots\int_{[u]^r} x_1dx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}dr, 2\int_0^1 r\frac{\int\cdots\int_{[u]^r} x_2dx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}dr, \cdots, 2\int_0^1 r\frac{\int\cdots\int_{[u]^r} x_ndx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}dr\right)$$

where $\int_0^1 r\frac{\int\cdots\int_{[u]^r} x_idx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}dr$ $(i = 1, 2, \cdots, n)$ is the Lebesgue integral of $r\frac{\int\cdots\int_{[u]^r} x_idx_1dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1dx_1dx_2\cdots dx_n}$ $(i = 1, 2, \cdots, n)$ on $[0,1]$, refer to [7].

Definition 2.1. *(see [7]) Let $u, v \in E^n$, $C \subseteq R^n$ be a closed convex cone with $0 \in C$ and $C \neq R^n$. We say that $u \preceq_c v$ (u precedes v) if*

$$\tau(v) \in \tau(u) + C.$$

The order relation $\preceq_c$ is reflexive and transitive, and $\preceq_c$ is a partially ordered relation on E^n. For $u, v \in E^n$, if either $u \preceq_c v$, or $v \preceq_c u$, then we say u and v are comparable; otherwise, they are non-comparable. If $u, v \in E^1$, $C = [0, +\infty) \subseteq R$, then Definition 2.1 coincides with Definition 2.5 from [6].

Remark 2.1. *(see [7].) Let $u, v \in E^1$. If we write $\tau(u) = \frac{1}{2}\int_0^1 r(u^+(r) + u^-(r))dr$, then $u \preceq_c v$ in the sense of Goetschel [6] if and only if $\tau(u) \le \tau(v)$, i.e., $\tau(v) \in \tau(u) + [0, +\infty)$. Furthermore,*

$$\tau(\lambda_1 u + \lambda_2 v) = \lambda_1 \tau(u) + \lambda_1 \tau(v).$$

for $\lambda_1, \lambda_2 > 0$, where $[u]^r = [u^-(r), u^+(r)]$.

Based on Definition 2.1 and the vector-valued function τ, we say $u \prec_c v$ if $u \preceq_c v$ and $\tau(u) \ne \tau(v)$. Sometimes we may write $v \succeq_c u$(resp. $v \succ_c u$) instead of $u \preceq_c v$(resp. $u \prec_c v$).

Set-valued mappings $F_r : K \to P_k(R^n)$ are defined by $F_r(t) = [\widetilde{F}(t)]^r, r \in [0, 1]$, where $P_k(R^n)$ denotes the power set of R^n.

Definition 2.2. *(see [7].) Let $\widetilde{F} : K \to E^n$, $\tau_F : K \to R^n$ is defined by $\tau_F(t) = \tau(\widetilde{F}(t))$*
$$= (2\int_0^1 r\frac{\int\cdots\int_{F_r(t)} x_1 dx_1 dx_2 \cdots dx_n}{\int\cdots\int_{F_r(t)} 1 dx_1 dx_2 \cdots dx_n}dr, 2\int_0^1 r\frac{\int\cdots\int_{F_r(t)} x_2 dx_1 dx_2 \cdots dx_n}{\int\cdots\int_{F_r(t)} 1 dx_1 dx_2 \cdots dx_n}dr \cdots, 2\int_0^1 r\frac{\int\cdots\int_{F_r(t)} x_n dx_1 dx_2 \cdots dx_n}{\int\cdots\int_{F_r(t)} 1 dx_1 dx_2 \cdots dx_n}dr).$$

Obviously, the fuzzy number-valued function $\widetilde{F} : E^m \to E^n$ is said to be increasing if $\widetilde{F}(u) \preceq_c \widetilde{F}(v)$, whenever $u, v \in E^m$, and $u \preceq_c v$.

In this article, the epigraph of $\widetilde{F}$, which is denoted by $epi(\widetilde{F})$, is defined as

$$epi(\widetilde{F}) = \{(t, u) : t \in K, u \in E^n, \widetilde{F}(t) \preceq_c u\}.$$

The generalized difference (gH-difference for short, and refer to [8]) of two fuzzy numbers $\widetilde{u}, \widetilde{v} \in E^n$ is given by its level sets as

$$[\widetilde{u} \ominus_g \widetilde{v}]^r = cl(\text{conv}\bigcup_{\beta \ge r}([\widetilde{u}]^\beta \ominus_{gH} [\widetilde{v}]^\beta)),\ \forall r \in [0, 1],$$

where the gH-difference $\ominus_{gH}$ is with interval operands $[\widetilde{u}]^\beta$ and $[\widetilde{v}]^\beta$.

Definition 2.3. *Let $\widetilde{F} : K \to E^n$ be a fuzzy number-valued function on an invex set $K \subset R^n$, $K \ne \emptyset$, with respect to (w.r.t.) a function $\eta : K \times K \to R^n$. If for any $x, y \in K$, there exists a $\delta > 0$, such that the H-difference $\widetilde{F}(y + h\eta(x, y)) - \widetilde{F}(y)$ exists for any real number $h \in (0, \delta)$, and $u_\eta^i \in E^n$, $i = 1, 2, \cdots n$, such that*

$$\widetilde{\nabla}\widetilde{F}_\eta(y)\eta(x, y) = \lim_{h \to 0^+} \frac{\widetilde{F}(y + h\eta(x, y)) - \widetilde{F}(y)}{h},$$

then $\widetilde{F}$ is called fuzzy η-extended directionally differentiable at y. $\widetilde{\nabla}\widetilde{F}_\eta(y)\eta(x, y)$ is called the fuzzy η-extended directional derivative at y in the direction $\eta(x, y)$(denoted $\widetilde{\nabla}\widetilde{F}_\eta(y) = (u_\eta^1, u_\eta^2, \cdots u_\eta^n)$).

The example 4.4 illustrates the notion of fuzzy η-extended directional differentiability.

3. n-dimensional preinvex fuzzy number-valued functions

Since the space of the n-dimensional fuzzy numbers is a partially ordered set, two n-dimensional fuzzy numbers might not be comparable. For a fuzzy number-valued function $\widetilde{F} : K \to E^n$, $\widetilde{F}$ is said to be a comparable fuzzy number-valued function if for each pair $x, y \in K$ and $x \ne y$, $\widetilde{F}(x)$ and $\widetilde{F}(y)$ could be compared. In this paper, we assume that a fuzzy number-valued function $\widetilde{F} : K \to E^n$ involved is comparable.

Refer to the definition of [29], a set $K \subseteq R^n$ is said to be an invex set w.r.t. a function $\eta : R^n \times R^n \to R^n$ if $x, y \in K$ implies that $y + \lambda\eta(x, y) \in K$ for $\lambda \in [0, 1]$.

Definition 3.1. *Let K be an invex set of R^n w.r.t. η, and $\widetilde{F}: K \to E^n$ be a fuzzy number-valued function.*
(1) $\widetilde{F}$ is said to be preinvex $(p.)$ on K if

$$\widetilde{F}(y + \lambda\eta(x, y)) \preceq_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y),$$

for any $x, y \in K$ and $\lambda \in [0, 1]$.
(2) $\widetilde{F}$ is said to be weakly preinvex $(w.p.)$ on K if there exists a $\lambda \in (0, 1)$ such that

$$\widetilde{F}(y + \lambda\eta(x, y)) \preceq_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y),$$

for any $x, y \in K$.
(3) $\widetilde{F}$ is said to be strictly preinvex $(s.p.)$ on K if

$$\widetilde{F}(y + \lambda\eta(x, y)) \prec_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y),$$

for any $x, y \in K$ with $x \neq y$ and $\lambda \in (0, 1)$.
(4) $\widetilde{F}$ is said to be weakly strictly preinvex $(w.s.p.)$ on K if there exists a $\lambda \in (0, 1)$ such that

$$\widetilde{F}(y + \lambda\eta(x, y)) \prec_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y),$$

for any $x, y \in K$ with $x \neq y$.
(5) $\widetilde{F}$ is said to be prequasiinvex $(q.p.)$ on K if

$$\widetilde{F}(y + \lambda\eta(x, y)) \preceq_c \max\{\widetilde{F}(x), \widetilde{F}(y)\},$$

for any $x, y \in K$ and $\lambda \in [0, 1]$.
(6) $\widetilde{F}$ is said to be weakly prequasiinvex $(w.q.p.)$ on K if there exists a $\lambda \in (0, 1)$ such that

$$\widetilde{F}(y + \lambda\eta(x, y)) \preceq_c \max\{\widetilde{F}(x), \widetilde{F}(y)\},$$

for any $x, y \in K$.
(7) $\widetilde{F}$ is said to be strictly prequasiinvex $(s.q.p.)$ on K if

$$\widetilde{F}(y + \lambda\eta(x, y)) \prec_c \max\{\widetilde{F}(x), \widetilde{F}(y)\},$$

for any $x, y \in K$ with $x \neq y$ and $\lambda \in (0, 1)$.
(8) $\widetilde{F}$ is said to be weakly strictly prequasiinvex $(w.s.q.p.)$ on K if there exists a $\lambda \in (0, 1)$ such that

$$\widetilde{F}(y + \lambda\eta(x, y)) \prec_c \max\{\widetilde{F}(x), \widetilde{F}(y)\},$$

for any $x, y \in K$ with $x \neq y$.

Remark 3.1. *Let $\widetilde{F}: K \to E^n$ be a preinvex fuzzy number-valued function, then $-\widetilde{F}$ is preincave on K.*

Remark 3.2. *Let $\widetilde{F}: K \to E^n$ be a strictly preinvex fuzzy number-valued function, then $-\widetilde{F}$ is strictly preincave on K.*

Remark 3.3. *In Definition 3.1, taking $\eta(x, y) = x - y$, $\widetilde{F}$ is said to be convex, weakly convex, strictly convex, weakly strictly convex, quasiconvex, weakly quasiconvex, strictly quasiconvex, and weakly strictly quasiconvex on K, respectively [7].*

4. The relationships of n-dimensional preinvex fuzzy number-valued functions

Theorem 4.1. *If $\widetilde{F} : K \to E^n$ is a preinvex fuzzy number-valued function, then $\widetilde{F}$ is prequasiinvex on K.*

Proof. If $\widetilde{F}$ is preinvex on K, then we obtain

$$\widetilde{F}(y + \lambda\eta(x, y)) \preceq_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y) \preceq_c \max\{\widetilde{F}(x)\widetilde{F}(y)\},$$

for any $x, y \in K$ and $\lambda \in [0, 1]$, This completes the proof.

Theorem 4.2. *If $\widetilde{F} : K \to E^n$ is a strictly preinvex fuzzy number-valued function, then $\widetilde{F}$ is strictly prequasiinvex on K.*

The proof is similar to the proof of Theorem 4.1.

Example 4.1. *Let $K = [1, 2] \cup [3, 4]$. It is easy to prove that K is invex w.r.t. $\eta : R^2 \times R^2 \to R^2$ defined by*

$$\eta(x, y) = \begin{cases} x - y, & (x, y) \in [1, 2]^2 \cup [3, 4]^2, \\ 1 - y, & (x, y) \in [3, 4] \times [1, 2], \\ 3 - y, & (x, y) \in [1, 2] \times [3, 4]. \end{cases}$$

In fact, if $(x, y) \in [1, 2]^2 \cup [3, 4]^2$, from the convexity of [1,2] and [3,4], we have

$$y + \lambda\eta(x, y) = y + \lambda(x - y) \in [1, 2] \cup [3, 4].$$

If $(x, y) \in [3, 4] \times [1, 2]$, $y + \lambda\eta(x, y) = y + \lambda(1 - y) = (1 - \lambda)y + \lambda$, we can choose $\lambda = 1$ and $\lambda = 0$, it follows that $1 \leq (1 - \lambda)y + \lambda \leq 2$, i.e., $y + \lambda\eta(x, y) \in [1, 2] \subset K$. Similarly, if $(x, y) \in [1, 2] \times [3, 4]$, we have $y + \lambda\eta(x, y) \in [3, 4] \subset K$.

Let the fuzzy number-valued function $\widetilde{F} : K \to E^2$ be defined by

$$\widetilde{F}(\xi)(x_1, x_2) = \begin{cases} \frac{2x_1+3\xi}{6\xi}, & -\frac{3}{2}\xi \leq x_1 \leq \frac{3}{2}\xi, 0 \leq x_2 \leq 2\xi, \\ 1, & \frac{3}{2}\xi \leq x_1 \leq 0, 0 \leq x_2 \leq 2\xi, \\ \frac{\sqrt{9\xi^2-3\xi x_1}}{3\xi}, & 0 \leq x_1 \leq 3\xi, 0 \leq x_2 \leq 2\xi, \\ 0, & otherwise. \end{cases}$$

and $C = R^{2+} \subseteq R^2$, where $R^{2+} = \{(x_1, x_2) \in R^2 : x_1 \geq 0, x_2 \geq 0\}$. Then

$$F_r(\xi) = \{(x_1, x_2) : 3\xi r - \frac{3}{2}\xi \leq x_1 \leq 3\xi(1 - r^2), 0 \leq x_2 \leq 2\xi\}, r \in [0, 1].$$

For any $\xi \in [1, 2] \cup [3, 4]$, from Definition 2.2, it follows that

$$\tau(\widetilde{F}(\xi)) = (\xi, \xi).$$

Therefore, for any $\xi_1, \xi_2 \in [1, 2] \cup [3, 4]$ and $(\xi_1, \xi_2) \in [1, 2]^2 \cup [3, 4]^2$, and for any $\lambda \in [0, 1]$, we have $\xi_2 + \lambda\eta(\xi_1, \xi_2) = \xi_2 + \lambda(\xi_1 - \xi_2)$. Thus,

$$\tau(\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2))) = (\xi_2 + \lambda\eta(\xi_1, \xi_2), \xi_2 + \lambda\eta(\xi_1, \xi_2)) = (\xi_2 + \lambda(\xi_1 - \xi_2), \xi_2 + \lambda(\xi_1 - \xi_2))$$

$$= (\lambda\xi_1 + (1-\lambda)\xi_2, \lambda\xi_1 + (1-\lambda)\xi_2) = \lambda(\xi_1,\xi_1) + (1-\lambda)(\xi_2,\xi_2) = \lambda\tau(\widetilde{F}(\xi_1)) + (1-\lambda)\tau(\widetilde{F}(\xi_2)).$$

In particular, from [7], we get $\lambda\tau(\widetilde{F}(\xi_1)) + (1-\lambda)\tau(\widetilde{F}(\xi_2)) = \tau(\lambda\widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2))$. *We find that, for any* $\xi_1, \xi_2 \in [1,2] \cup [3,4]$ *and* $(\xi_1,\xi_2) \in [1,2]^2 \cup [3,4]^2$, $\lambda \in [0,1]$,

$$\tau(\lambda\widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2))) = \tau(\widetilde{F}(\xi_2 + \lambda\eta(\xi_1,\xi_2))) \in \tau(\widetilde{F}(\xi_2 + \lambda\eta(\xi_1,\xi_2))) + C,$$

i.e., $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1,\xi_2)) \preceq_c \lambda\widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2)$.

For any $\xi_1,\xi_2 \in [1,2] \cup [3,4]$ *and* $(\xi_1,\xi_2) \in [3,4] \times [1,2]$,

$$\xi_2 + \lambda\eta(\xi_1,\xi_2) = \xi_2 + \lambda\eta(\xi_1,\xi_2) = \xi_2 + \lambda(1-\xi_2),$$

and $(\xi_1,\xi_1) \in (1,1) + C$, *it follows that*

$$\tau(\widetilde{F}(\xi_2 + \lambda\eta(\xi_1,\xi_2))) = (\xi_2 + \lambda(1-\xi_2), \xi_2 + \lambda(1-\xi_2)) = \lambda(1,1) + (1-\lambda)(\xi_2,\xi_2)$$
$$\preceq_c \lambda\tau(\widetilde{F}(\xi_1)) + (1-\lambda)\tau(\widetilde{F}(\xi_2)) = \tau(\lambda\widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2)))$$

i.e., $\widetilde{F}(t_2 + \lambda\eta(t_1,t_2)) \preceq_c \lambda\widetilde{F}(t_1) + (1-\lambda)\widetilde{F}(t_2)$.

For any $\xi_1,\xi_2 \in [1,2] \cup [3,4]$, *and*$(\xi_1,\xi_2) \in [1,2] \times [3,4]$,

$$\xi_2 + \lambda\eta(\xi_1,\xi_2) = \xi_2 + \lambda(3-\xi_2) = 3\lambda + (1-\lambda)\xi_2$$

and $(3,3) \in (\xi_1,\xi_1) + C$, *it follows that*

$$\tau(\widetilde{F}(\xi_2 + \lambda\eta(\xi_1,\xi_2))) = (\xi_2 + \lambda(3-\xi_2), \xi_2 + \lambda(3-\xi_2)) = \lambda(3,3) + (1-\lambda)(\xi_2,\xi_2)$$
$$\succeq_c \lambda\tau(\widetilde{F}(\xi_1)) + (1-\lambda)\tau(\widetilde{F}(\xi_2)) = \tau(\lambda\widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2))$$

i.e., $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1,\xi_2)) \succeq_c \lambda\widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2)$. *Above all, we denote* $K_1 = [1,2], K_2 = [3,4], K = K_1 \cup K_2$, *then*

(1) $\widetilde{F}$ *is preinvex on* K_1 *w.r.t.* η, *but it is not strictly preinvex.*

(2) $\widetilde{F}$ *is preinvex on* K_2 *w.r.t.* η, *but it is not strictly preinvex.*

(3) $\widetilde{F}$ *is not preinvex on K w.r.t.* η. *Since* $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1,\xi_2)) \preceq_c \lambda\widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2)$, *for* $\xi_1,\xi_2 \in K_1$, *or* $\xi_1,\xi_2 \in K_2$, *or* $\xi_1 \in K_2, \xi_2 \in K_1$. *However,* $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1,\xi_2)) \succeq_c \lambda\widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2)$, *for* $\xi_1 \in K_1, \xi_2 \in K_2$. *Also, it is not strictly preinvex.*

Example 4.2. *Let* $K = [1,2] \cup [3,4]$. *It is an invex set w.r.t.* $\eta : R^2 \times R^2 \to R^2$, $\eta(x,y)$ *is the same as Example 4.1. Let the fuzzy number-valued function* $\widetilde{F} : K \to E^2$ *be defined by*

$$\widetilde{F}(\xi)(x_1,x_2) = \begin{cases} \sqrt{1 - (\frac{x_1}{\ln 2\xi})^2}, & 0 \leq x_1 \leq \ln(2\xi), 0 \leq x_2 \leq 3, \\ \text{0}, & \text{otherwise}, \end{cases}$$

and $C = R^{2+} \subseteq R^2$. *Then,* $F_r(\xi) = \{(x_1,x_2) : 0 \leq x_1 \leq \ln 2\xi\sqrt{1-r^2}, 0 \leq x_2 \leq 3\}, r \in [0,1]$. *It is not difficult to calculate, for any* $\xi \in [1,2] \cup [3,4]$,

$$\tau(\widetilde{F}(\xi)) = (\frac{\ln 2\xi}{3}, \frac{3}{2}).$$

Thus, if for any $\xi_1, \xi_2 \in [1,2] \cup [3,4]$ *and* $(\xi_1, \xi_2) \in [1,2]^2 \cup [3,4]^2$, *we have* $\xi_2 + \lambda\eta(\xi_1, \xi_2) = \xi_2 + \lambda(\xi_1 - \xi_2)$, *In addition, for any* $\lambda \in [0,1]$,

$$\ln(2\xi_2 + \lambda(2\xi_1 - 2\xi_2)) \leq \max\{\ln 2\xi_1, \ln 2\xi_2\}.$$

Without loss of generality, we assume that $\ln(2\xi_2 + \lambda(2\xi_1 - 2\xi_2)) \leq \max\{\ln 2\xi_1, \ln 2\xi_2\} = \ln 2\xi_2$, *Thus*

$$\max\{\tau(\widetilde{F}(\xi_1)), \tau(\widetilde{F}(\xi_2))\} = \max\{(\frac{\ln 2\xi_1}{3}, \frac{3}{2}), (\frac{\ln 2\xi_2}{3}, \frac{3}{2})\} = (\frac{\ln 2\xi_2}{3}, \frac{3}{2})$$
$$\in (\frac{\ln(2\xi_2 + \lambda(2\xi_1 - 2\xi_2))}{3}, \frac{3}{2}) + C = \tau(\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2))) + C.$$

i.e., $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2)) \preceq_c \max\{\widetilde{F}(\xi_1), \widetilde{F}(\xi_2)\}$. *However, for any* $\xi_1, \xi_2 \in [1,2] \cup [3,4]$ *and* $(\xi_1, \xi_2) \in [1,2]^2 \cup [3,4]^2$, $\lambda \in [0,1]$, *we have* $\ln(2\xi_2 + \lambda(2\xi_1 - 2\xi_2)) \geq \lambda \ln 2\xi_1 + (1-\lambda)\ln 2\xi_2$. *Taking* $2\xi_1 = 2, 2\xi_2 = e, \lambda = \frac{1}{2}$, *then*

$$\tau(\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2))) = (\frac{\ln(1 + \frac{e}{2})}{3}, \frac{3}{2}) \in \frac{1}{2}(\frac{\ln 2}{3}, \frac{3}{2}) + \frac{1}{2}(\frac{\ln\, e}{3}, \frac{3}{2}) + C = \lambda\tau(\widetilde{F}(\xi_1)) + (1-\lambda)\tau(\widetilde{F}(\xi_2)) + C.$$

i.e., $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2)) \succeq_c \lambda\, \widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2)$.

If $\xi_1, \xi_2 \in [1,2] \cup [3,4]$ *and* $(\xi_1, \xi_2) \in [3,4] \times [1,2]$, *then* $\xi_2 + \lambda\eta(\xi_1, \xi_2) = \xi_2 + \lambda(1 - \xi_2) \leq \xi_2 + \lambda(\xi_1 - \xi_2)$. *In addition, for any* $\lambda \in [0,1]$,

$$\ln(2\xi_2 + \lambda(2\xi_1 - 2\xi_2)) \leq \max\{\ln 2\xi_1, \ln 2\xi_2\} = \ln 2\xi_1$$

Thus, $\max\{\tau(\widetilde{F}(\xi_1)), \tau(\widetilde{F}(\xi_2))\} \in \tau(\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2)) + C$. *i.e.,* $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2)) \preceq_c \max\{\widetilde{F}(\xi_1), \widetilde{F}(\xi_2)\}$. *For any* $\xi_1, \xi_2 \in [1,2] \cup [3,4]$ *and* $(\xi_1, \xi_2) \in [3,4] \times [1,2]$, *for any* $\lambda \in [0,1]$, $\ln(2\xi_2 + \lambda(2 - 2\xi_2)) \leq \lambda \ln 2\xi_1 + (1-\lambda)\ln 2\xi_2$, *it follows that,*

$$\lambda\tau(\widetilde{F}(\xi_1)) + (1-\lambda)\tau(\widetilde{F}(\xi_2)) = \lambda(\frac{\ln 2\xi_1}{3}, \frac{3}{2}) + (1-\lambda)(\frac{\ln 2\xi_2}{3}, \frac{3}{2})$$
$$\in (\frac{\ln(2\xi_2 + \lambda(2 - 2\xi_2))}{3}, \frac{3}{2}) + C = \tau(\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2)) + C,$$

i.e., $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2)) \preceq_c \lambda\widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2)$.

If $\xi_1, \xi_2 \in [1,2] \cup [3,4]$ *and* $(\xi_1, \xi_2) \in [1,2] \times [3,4]$, *then* $\xi_2 + \lambda\eta(\xi_1, \xi_2) = \xi_2 + \lambda(3 - \xi_2)$. *It is easy to verify that* $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2)) \preceq_c \max\{\widetilde{F}(\xi_1), \widetilde{F}(\xi_2)\}$. *However, when* $2\xi_1 = 2, 2\xi_2 = 7, \lambda = \frac{4}{5}$, *we obtain*

$$\tau(\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2))) = (\frac{\ln \frac{31}{5}}{3}, \frac{3}{2}) \in \frac{1}{5}(\frac{\ln 6}{3}, \frac{3}{2}) + \frac{4}{5}(\frac{\ln 2}{3}, \frac{3}{2}) + C = \lambda\tau(\widetilde{F}(\xi_1)) + (1-\lambda)\tau(\widetilde{F}(\xi_2)) + C.$$

i.e. $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2)) \succeq_c \lambda\, \widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2)$.

Above all, we denote $K_1 = [1,2], K_2 = [3,4], K = K_1 \cup K_2$, *then* $\widetilde{F}$ *is prequasiinvex on* K *w.r.t.* η, *but* $\widetilde{F}$ *is not preinvex on* K *w.r.t.* η. *Since we have* $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2)) \preceq_c \lambda\widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2)$, *for* $\xi_1 \in K_2, \xi_2 \in K_1$ *and* $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1, \xi_2)) \succeq_c \lambda\widetilde{F}(\xi_1) + (1-\lambda)\widetilde{F}(\xi_2)$, *for* $\xi_1, \xi_2 \in K_1$, *or* $\xi_1, \xi_2 \in K_2$, *or* $\xi_1 \in K_1, \xi_2 \in K_2$.

Example 4.3. *Let $K = [-2,-1] \cup [1,3]$. It is easy to prove that K is an invex set w.r.t. $\eta : R^2 \times R^2 \to R^2$ defined by*

$$\eta(x,y) = \begin{cases} x - y, & (x,y) \in [-2,-1]^2 \cup [1,3]^2, \\ -2 - y, & (x,y) \in [1,3] \times [-2,-1], \\ 1 - y, & (x,y) \in [-2,-1] \times [1,3] \end{cases}$$

Let a fuzzy number-valued function $\widetilde{F} : K \to E^2$ be defined by

$$\widetilde{F}(\xi)(x_1,x_2) = \begin{cases} \frac{1}{e^{\xi}} \sqrt{e^{2\xi} - {x_1}^2 - {x_2}^2}, & 1 \le \xi \le 3, {x_1}^2 + {x_2}^2 \le e^{2\xi}, x_1 \ge 0, x_2 \ge 0 \\ 1, & -2 \le \xi \le -1, -1 \le x_1 \le 0, -2 \le x_2 \le 0 \\ 0, & \text{otherwise}. \end{cases}$$

and $C = R^{2+} \subseteq R^2$. Then,

$F_r(\xi) = \{(x_1,x_2) : -1 \le x_1 \le 0, -2 \le x_2 \le 0\}$, *when* $-2 \le \xi \le -1$;

$F_r(\xi) = \{(x_1,x_2) : {x_1}^2 + {x_2}^2 \le e^{2\xi}(1 - r^2), x_1 \ge 0, x_2 \ge 0\}$, *when* $1 \le \xi \le 3$.

According to Definition 2.2, we obtain

$\tau(\widetilde{F}(\xi)) = (-\frac{1}{2}, -1)$, *when* $-2 \le \xi \le -1$;

$\tau(\widetilde{F}(\xi)) = (\frac{8e^{\xi}}{9\pi}, \frac{8e^{\xi}}{9\pi})$, *when* $1 \le \xi \le 3$.

Then, we have

$$\max\{\tau(\widetilde{F}(\xi_1)), \tau(\widetilde{F}(\xi_2))\} \in \tau(\widetilde{F}(\xi_2 + \lambda\eta(\xi_1,\xi_2))) + C,$$

for any $\xi_1, \xi_2 \in [-2,-1] \cup [1,3]$ and for any $\lambda \in [0,1]$. i.e., $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1,\xi_2)) \preceq_c \max\{\widetilde{F}(\xi_1), \widetilde{F}(\xi_2)\}$.

Above all, we denote $K_1 = [-2,-1], K_2 = [1,3], K = K_1 \cup K_2$, then $\widetilde{F}$ is prequasiinvex on K w.r.t. η, but $\widetilde{F}$ is not strictly prequasiinvex on K w.r.t. η. In fact, since we have $\widetilde{F}(\xi_2 + \lambda\eta(\xi_1,\xi_2)) \prec_c \max\{\widetilde{F}(\xi_1), \widetilde{F}(\xi_2)\}$, for any $\xi_1 \in K_2, \xi_2 \in K_1, \xi_2 + \lambda\eta(\xi_1,\xi_2) \in K_1$ or $\xi_1, \xi_2 \in K_2, \xi_2 + \lambda\eta(\xi_1,\xi_2) \in K_2$ or $\xi_1 \in K_1$, $\xi_2 \in K_2, \xi_2 + \lambda\eta(\xi_1,\xi_2) \in K_2$, with $\xi_1 \ne \xi_2$ and for any $\lambda \in (0,1)$. However, $\tau(\widetilde{F}(\xi_2 + \lambda\eta(\xi_1,\xi_2))) = \max\{\tau(\widetilde{F}(\xi_1)), \tau(\widetilde{F}(\xi_2))\}$, for $\xi_1, \xi_2 \in K_1, \xi_2 + \lambda\eta(\xi_1,\xi_2) \in K_1$ with $\xi_1 \ne \xi_2$.

In order to discuss the relationships of preinvex and prequasiinvex fuzzy number-valued functions, we get the following special function η and the fuzzy number-valued functions according to the discussion of [13].

Let $\eta : R^n \times R^n \to R^n$, we say that a function η satisfies the condition C *if*

$$C_1 : \quad \eta(y, y + \lambda\eta(x,y)) = -\lambda\eta(x,y),$$

$$C_2 : \quad \eta(x, y + \lambda\eta(x,y)) = (1 - \lambda)\eta(x,y),$$

for any $x, y \in R^n$, $\lambda \in [0,1]$ (refer to [13]).

A fuzzy number-valued function $\widetilde{F} : K \to E^n$ satisfies Condition D*, if $K \subset R^n$ is an invex set w.r.t. $\eta : R^n \times R^n \to R^n$, for any $x, y \in K$, we have*

$$\widetilde{F}(y + \eta(x,y)) \preceq_c \widetilde{F}(x).$$

In order to include singletons in R^n as an invex sets, we assume that for all $x \in R^n$,

$$\eta(x,x) = O,$$

where O being the origin of R^n.

Example 4.4. *Let* $K = [-1, \frac{1}{2}] \cup [\frac{1}{4}, 1]$. *It is easy to prove that K is invex w.r.t.* $\eta : R^2 \times R^2 \to R^2$ *defined by*

$$\eta(x,y) = \begin{cases} x - y, & (x,y) \in [-1, \frac{1}{2}]^2 \cup [\frac{1}{4}, 1]^2, \\ -1 - y, & (x,y) \in [\frac{1}{4}, 1] \times [-1, \frac{1}{2}], \\ \frac{1}{4} - y, & (x,y) \in [-1, \frac{1}{2}] \times [\frac{1}{4}, 1]. \end{cases}$$

In fact, if $(x,y) \in [-1, \frac{1}{2}]^2 \cup [\frac{1}{4}, 1]^2$, *from the convexity of* $[-1, \frac{1}{2}]$ *and* $[\frac{1}{4}, 1]$, *we have*

$$y + \lambda\eta(x,y) = y + \lambda(x-y) \in [-1, \frac{1}{2}] \cup [\frac{1}{4}, 1].$$

If $(x,y) \in [\frac{1}{4}, 1] \times [-1, \frac{1}{2}]$, $y + \lambda\eta(x,y) = y + \lambda(1-y) = (1-\lambda)y + \lambda$, *we can choose* $\lambda = 1$ *and* $\lambda = 0$), *it follows that* $-1 \le (1-\lambda)y + \lambda \le \frac{1}{2}$, *i.e.,* $y + \lambda\eta(x,y) \in [-1, \frac{1}{2}] \subset K$. *Similarly, if* $(x,y) \in [-1, \frac{1}{2}] \times [\frac{1}{4}, 1]$, *we have* $y + \lambda\eta(x,y) \in [\frac{1}{4}, 1] \subset K$.

Let the fuzzy-number-valued function $\widetilde{F} : K \to E^2$ *be defined as*

$$\widetilde{F}(\xi)(x_1, x_2) = \begin{cases} \frac{x_1+1+|\xi|}{1+|\xi|}, & -1-|\xi| \le x_1 \le 0,\ -|\xi| \le x_2 \le |\xi|, \\ \frac{-x_1+1+|\xi|}{1+|\xi|}, & 0 \le x_1 \le 1+|\xi|,\ -|t| \le x_2 \le |\xi|, \\ 0, & \textit{otherwise}. \end{cases}$$

Then, for any $r \in [0,1]$,

$$\begin{aligned} F_r(\xi) &= \{(x_1, x_2) :\ (-1+r)(1+|\xi|) \le x_1 \le (1-r)(1+|\xi|),\ -|\xi| \le x_2 \le |\xi|\} \\ &= [(-1+r)(1+|\xi|), (1-r)(1+|\xi|)] \times [-|\xi|, |\xi|]. \end{aligned}$$

Since

$$\begin{aligned} &[\widetilde{F}(h\eta(x,y)) \ominus_g \widetilde{F}(0)]^r \\ &= [\inf_{\beta \ge r} \min\{(-1+\beta)|h\eta(x,y)|, (1-\beta)|h\eta(x,y)|\}, \sup_{\beta \ge r} \max\{(-1+\beta)|h\eta(x,y)|, (1-\beta)|h\eta(x,y)|\}] \\ &\times [\inf_{\beta \ge r} \min\{-|h\eta(x,y)|, |h\eta(x,y)|\}, \sup_{\beta \ge r} \max\{-|h\eta(x,y)|, |h\eta(x,y)|\}] \\ &= [(-1+r)|h\eta(x,y)|, (1-r)|h\eta(x,y)|] \times [-|h\eta(x,y)|, |h\eta(x,y)|]. \end{aligned}$$

Thus, $[\frac{\widetilde{F}(0+h\eta(x,y)) \ominus_g \widetilde{F}(0)}{h}]^r = \frac{[\widetilde{F}(h\eta(x,y)) \ominus_g \widetilde{F}(0)]^r}{h} = [-1+r, 1-r]|\eta(x,y)| \times [-1,1]|\eta(x,y)|$ *for any* $r \in [0,1]$. *Assume that*

$$\widetilde{u}(x_1, x_2) = \begin{cases} x_1 + 1, & -1 \le x_1 \le 0,\ -1 \le x_2 \le 1, \\ -x_1 + 1, & 0 \le x_1 \le 1,\ -1 \le x_2 \le 1, \\ 0, & \textit{otherwise}, \end{cases}$$

we have $[\widetilde{u}]^r = [-1+r, 1-r] \times [-1,1]$ *for any* $r \in [0,1]$. *Then,*

$$\widetilde{\nabla}\widetilde{F}_\eta(0)\eta(x,y) = \lim_{h \to 0^+} \frac{\widetilde{F}(0 + h\eta(x,y)) - \widetilde{F}(0)}{h} = |\eta(x,y)|\widetilde{u}$$

$\widetilde{F}$ *is fuzzy* η*-extended directionally differentiable at* 0, *and* $\widetilde{\nabla}\widetilde{F}_\eta(0)$ *is the fuzzy* η*-extended directional derivative at* 0 *in the direction* $\eta(x,y)$*(denoted* $\widetilde{\nabla}\widetilde{F}_\eta(0) = (-1)\widetilde{u}$.

Theorem 4.3. *Let $K \subset R^n$ be an invex set w.r.t. η, η satisfy Condition* C, *and $\widetilde{F} : K \to E^n$ be a preinvex fuzzy number-valued function. If $\widetilde{F}$ is weakly strictly preinvex on K, i.e., there exists a $\lambda_0 \in (0, 1)$ such that*

$$\widetilde{F}(y + \lambda_0\eta(x, y)) \prec_c \lambda_0\widetilde{F}(x) + (1 - \lambda_0)\widetilde{F}(y), \tag{4.1}$$

for any $x, y \in K$, with $x \neq y$, then $\widetilde{F}$ is strictly preinvex on K.

Proof. Assume that $\widetilde{F}$ is not strictly preinvex on K, then $x, y \in K$ with $x \neq y$ and for any $\lambda \in (0, 1)$ such that

$$\widetilde{F}(y + \lambda\eta(x, y)) \succeq_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y). \tag{4.2}$$

Choose $\lambda_1, \lambda_2 \in (0, 1)$ such that $\lambda = \lambda_0\lambda_1 + (1 - \lambda_0)\lambda_2$ and by taking $\overline{x} = y + \lambda_1\eta(x, y)$,$\overline{y} = y + \lambda_2\eta(x, y)$, from Condition C, we have

$$\begin{aligned}\overline{y} + \lambda_0\eta(\overline{x}, \overline{y}) &= y + \lambda_2\eta(x, y) + \lambda_0\eta(y + \lambda_1\eta(x, y), y + \lambda_2\eta(x, y))\\ &= y + \lambda_2\eta(x, y) + \lambda_0(\lambda_1 - \lambda_2)\eta(x, y)\\ &= y + (\lambda_0\lambda_1 + (1 - \lambda_0)\lambda_2)\eta(x, y) = y + \lambda\eta(x, y).\end{aligned}$$

From the preinvexity of $\widetilde{F}$, we find that

$$\widetilde{F}(\overline{x}) \preceq_c \lambda_1\widetilde{F}(x) + (1 - \lambda_1)\widetilde{F}(y), \quad \widetilde{F}(\overline{y}) \preceq_c \lambda_2\widetilde{F}(x) + (1 - \lambda_2)\widetilde{F}(y).$$

From (4.1), it follows that

$$\begin{aligned}\widetilde{F}(y + \lambda\eta(x, y)) &= \widetilde{F}(\overline{y} + \lambda_0\eta(\overline{x}, \overline{y})) \prec_c \lambda_0\widetilde{F}(\overline{x}) + (1 - \lambda_0)\widetilde{F}(\overline{y})\\ &\preceq_c \lambda_0[\lambda_1\widetilde{F}(x) + (1 - \lambda_1)\widetilde{F}(y)] + (1 - \lambda_0)[\lambda_2\widetilde{F}(x) + (1 - \lambda_2)\widetilde{F}(y)]\\ &= [\lambda_0\lambda_1 + (1 - \lambda_0)\lambda_2]\widetilde{F}(x) + [1 - \lambda_0\lambda_1 - (1 - \lambda_0)\lambda_2]\widetilde{F}(y)\\ &= \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y).\end{aligned}$$

It is a contradiction to (4.2), i.e., $\widetilde{F}$ is strictly preinvex on K.

Lemma 4.1. *Let $K \subset R^n$ be an invex set w.r.t. η, η satisfy Condition* C, *and $\widetilde{F} : K \to E^n$ satisfy Condition* D. *If there exists a $\alpha \in (0, 1)$ such that*

$$\widetilde{F}(y + \alpha\eta(x, y)) \preceq_c \alpha\widetilde{F}(x) + (1 - \alpha)\widetilde{F}(y),$$

for any $x, y \in K$, then the set

$$A = \{\lambda \in [0, 1] : \widetilde{F}(y + \lambda\eta(x, y)) \preceq_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y)\}$$

is dense in $[0, 1]$.

Proof. It is obvious that $0 \in A$, from Condition D, it follows that $1 \in A$, i.e., $A \neq \emptyset$ and A is not a single point set. Suppose that A is not dense in $[0, 1]$, then there exists a $\lambda_0 \in (0, 1)$ such that $U \cap A = \emptyset$, and where U is a δ-neighborhood $N_\delta(\lambda_0)$ of λ_0. Now, we denote

$$\lambda_1 = \inf\{\lambda \in A : \lambda \geq \lambda_0\}, \quad \lambda_2 = \sup\{\lambda \in A : \lambda \leq \lambda_0\},$$

then, we have $0 \leq \lambda_2 \leq \lambda_1 \leq 1$. Due to $\alpha, (1-\alpha) \in (0,1)$, we can choose $u_1, u_2 \in A$ such that

$$\max\{\alpha(u_1 - u_2), (1-\alpha)(u_1 - u_2)\} < \lambda_1 - \lambda_2,$$

and take $u_1 \geq \lambda_1$, $u_2 \leq \lambda_2$. Let $\overline{\lambda} = \alpha\, u_1 + (1-\alpha)u_2$, from Condition C, for any $x, y \in K$, we have

$$y + u_2\eta(x,y) + \alpha\eta(y + u_1\eta(x,y), y + u_2\eta(x,y)) = y + (u_2 + \alpha(u_1 - u_2))\eta(x,y) = y + \overline{\lambda}\eta(x,y).$$

According to the fact that $u_1, u_2 \in A$, we find that

$$\begin{aligned} f(y + \overline{\lambda}\eta(x,y)) &= f(y + u_2\eta(x,y) + \alpha\eta(y + u_1\eta(x,y), y + u_2\eta(x,y))) \\ &\preceq_c \alpha f(y + u_1\eta(x,y)) + (1-\alpha) f(y + u_2\eta(x,y)) \\ &\preceq_c \alpha(u_1 f(x) + (1-u_1) f(y)) + (1-\alpha)(u_2 f(x) + (1-u_2) f(y)) \\ &= \overline{\lambda} f(x) + (1-\overline{\lambda}) f(y). \end{aligned}$$

Then, it follows that $\overline{\lambda} \in A$. If $\overline{\lambda} \geq \lambda_0$, from the definition of λ_1, we get $\lambda_1 \leq \overline{\lambda}$. In addition, we have

$$\overline{\lambda} - u_2 = \alpha(u_1 - u_2) < \lambda_1 - \lambda_2,$$

moreover,

$$\lambda_1 > \overline{\lambda} - u_2 + \lambda_2 \geq \overline{\lambda} - \lambda_2 + \lambda_2 = \overline{\lambda}.$$

It is a contradiction. Similar to that $\overline{\lambda} \leq \lambda_0$. Thus, A is dense in $[0,1]$.

Theorem 4.4. *Let $K \subset R^n$ be an invex set w.r.t. η, η satisfy Condition* C, *$\widetilde{F} : K \to E^n$ satisfy Condition* D, *and $\widetilde{F}$ be a prequasiinvex fuzzy number-valued function. If $\widetilde{F}$ is weakly preinvex on K, i.e., there exists a $\lambda_0 \in (0,1)$ such that*

$$\widetilde{F}(y + \lambda_0\eta(x,y)) \prec_c \lambda_0\widetilde{F}(x) + (1-\lambda_0)\widetilde{F}(y) \tag{4.3}$$

for any $x, y \in K$, then $\widetilde{F}$ is preinvex on K.

Proof. Assume that $\widetilde{F}$ is not preinvex on K, then for $x, y \in K$, there exists a $\lambda \in [0,1]$ such that

$$\widetilde{F}(y + \lambda\eta(x,y)) \succ_c \lambda\widetilde{F}(x) + (1-\lambda)\widetilde{F}(y) \tag{4.4}$$

If $\widetilde{F}(x) = \widetilde{F}(y)$, then we have $\widetilde{F}(y + \lambda\eta(x,y)) \succ_c \widetilde{F}(x)$. Choose $\lambda_1, \lambda_2 \in [0,1]$, such that $\lambda = \lambda_0\lambda_1 + (1-\lambda_0)\lambda_2$ and by taking $\overline{x} = y + \lambda_1\eta(x,y), \overline{y} = y + \lambda_2\eta(x,y)$, from Condition C, we get $y + \lambda\eta(x,y) = \overline{y} + \lambda_0\eta(\overline{x}, \overline{y})$. From the prequasiinvexity of $\widetilde{F}$, it follows that

$$\widetilde{F}(\overline{x}) \preceq_c \max\{\widetilde{F}(x), \widetilde{F}(y)\}, \quad \widetilde{F}(\overline{y}) \preceq_c \max\{\widetilde{F}(x), \widetilde{F}(y)\}.$$

From (4.3), we find that

$$\widetilde{F}(y + \lambda\eta(x,y)) = \widetilde{F}(\overline{y} + \lambda_0\eta(\overline{x}, \overline{y})) \prec_c \lambda_0\widetilde{F}(\overline{x}) + (1-\lambda_0)\widetilde{F}(\overline{y}) \preceq_c \max\{\widetilde{F}(x), \widetilde{F}(y)\} = \widetilde{F}(x),$$

which is a contradiction to (4.4), i.e., $\widetilde{F}$ is preinvex on K.

Otherwise, let $\widetilde{F}(x) \prec_c \widetilde{F}(y)$. Since $\widetilde{F}$ is weakly preinvex on K, then, according to Lemma 4.1, there exists a $\lambda_1 \in A$ with $\lambda_1 < \lambda$, such that

$$\lambda_1 \widetilde{F}(x) + (1-\lambda_1)\widetilde{F}(y) \prec_c \widetilde{F}(y+\lambda\eta(x,y)).$$

Thus,

$$\widetilde{F}(y+\lambda_1\eta(x,y)) \preceq_c \lambda_1 \widetilde{F}(x) + (1-\lambda_1)\widetilde{F}(y) \prec_c \widetilde{F}(y+\lambda\eta(x,y)). \tag{4.5}$$

Choose $\lambda_2 = \frac{\lambda-\lambda_1}{1-\lambda_1}$ and by taking $\overline{x} = x, \overline{y} = y + \lambda_1\eta(x,y)$, then from Condition C, we have

$$\begin{aligned}\overline{y} + \lambda_2\eta(\overline{x},\overline{y}) &= y + \lambda_1\eta(x,y) + \lambda_2\eta(x, y+\lambda_1\eta(x,y))\\ &= y + \lambda_1\eta(x,y) + \lambda_2(1-\lambda_1)\eta(x,y) = y + \lambda\eta(x,y).\end{aligned}$$

If $\widetilde{F}(\overline{x}) \preceq_c \widetilde{F}(\overline{y})$, then from the prequasiinvexity of $\widetilde{F}$, we obtain

$$\widetilde{F}(y+\lambda\eta(x,y)) = \widetilde{F}(\overline{y}+\lambda_2\eta(\overline{x},\overline{y})) \preceq_c \max\{\widetilde{F}(\overline{x}), \widetilde{F}(\overline{y})\} = \widetilde{F}(\overline{y}) = \widetilde{F}(y+\lambda_1\eta(x,y)),$$

which is a contradiction to (4.5), i.e., $\widetilde{F}$ is preinvex on K.

If $\widetilde{F}(\overline{x}) \succ_c \widetilde{F}(\overline{y})$, then from the prequasiinvexity of $\widetilde{F}$, we obtain

$$\widetilde{F}(y+\lambda\eta(x,y)) = \widetilde{F}(\overline{y}+\lambda_2\eta(\overline{x},\overline{y})) \preceq_c \max\{\widetilde{F}(\overline{x}), \widetilde{F}(\overline{y})\} = \widetilde{F}(\overline{x}) = \widetilde{F}(x) \prec_c \lambda\widetilde{F}(x) + (1-\lambda)\widetilde{F}(y).$$

which is a contradiction to (4.4), i.e., $\widetilde{F}$ is preinvex on K.

According to Theorem 4.3 and Theorem 4.4, we have the following conclusion.

Theorem 4.5. *Let $K \subset R^n$ be an invex set w.r.t. η, η satisfy Condition* C*, $\widetilde{F} : K \to E^n$ satisfy Condition* D*, and $\widetilde{F}$ be a prequasiinvex fuzzy number-valued function. If $\widetilde{F}$ is weakly strictly preinvex on K, i.e., there exists a $\lambda_0 \in (0,1)$ such that*

$$\widetilde{F}(y+\lambda_0\eta(x,y)) \prec_c \lambda_0\widetilde{F}(x) + (1-\lambda_0)\widetilde{F}(y)$$

for any $x, y \in K$ with $x \neq y$, then $\widetilde{F}$ is strictly preinvex on K.

Theorem 4.6. *Let $K \subset R^n$ be an invex set w.r.t. η, η satisfy Condition* C*, and $\widetilde{F} : K \to E^n$ be a prequasiinvex fuzzy number-valued function. If $\widetilde{F}$ is weakly strictly prequasiinvex on K, i.e., there exists a $\lambda_0 \in (0,1)$ such that*

$$\widetilde{F}(y+\lambda_0\eta(x,y)) \prec_c \max\{\widetilde{F}(x), \widetilde{F}(y)\} \tag{4.6}$$

for any $x, y \in K$ with $x \neq y$, then $\widetilde{F}$ is strictly prequasiinvex on K.

Proof. Assume that $\widetilde{F}$ is not strictly prequasiinvex on K. Then for $x, y \in K$ with $x \neq y$, there exists a $\lambda \in (0,1)$ such that

$$\widetilde{F}(y+\lambda\eta(x,y)) \succeq_c \max\{\widetilde{F}(x), \widetilde{F}(y)\}. \tag{4.7}$$

Choose $\lambda_1, \lambda_2 \in (0,1)$, such that $\lambda = \lambda_0\lambda_1 + (1-\lambda_0)\lambda_2$ and by taking $\overline{x} = y+\lambda_1\eta(x,y), \overline{y} = y+\lambda_2\eta(x,y)$, then, using Condition C, we have $\overline{y} + \lambda_0\eta(\overline{x},\overline{y}) = y + \lambda\eta(x,y)$. According to the prequasiinvex of $\widetilde{F}$, it follows that

$$\widetilde{F}(\overline{x}) \preceq_c \max\{\widetilde{F}(x), \widetilde{F}(y)\}, \quad \widetilde{F}(\overline{y}) \preceq_c \max\{\widetilde{F}(x), \widetilde{F}(y)\}.$$

From (4.6), we get

$$\widetilde{F}(y+\lambda\eta(x,y)) = \widetilde{F}(\overline{y}+\lambda_0\eta(\overline{x},\overline{y})) \prec_c \max\{\widetilde{F}(\overline{x}),\widetilde{F}(\overline{y})\} \preceq_c \max\{\widetilde{F}(x),\widetilde{F}(y)\},$$

which is a contradiction to (4.7). i.e., $\widetilde{F}$ is strictly prequasiinvex on K.

It is similar to Theorem 4.5, we have the following result.

Theorem 4.7. *Let $K \subset R^n$ be an invex set w.r.t. η, η satisfy Condition* C*, $\widetilde{F} : K \to E^n$ satisfy Condition* D*, and $\widetilde{F}$ be a strictly prequasiinvex fuzzy number-valued function. If $\widetilde{F}$ is weakly strictly preinvex on K, i.e., there exists a $\lambda_0 \in (0,1)$ such that*

$$\widetilde{F}(y+\lambda_0\eta(x,y)) \prec_c \lambda_0\widetilde{F}(x) + (1-\lambda_0)\widetilde{F}(y)$$

for any $x, y \in K$ with $x \neq y$, then $\widetilde{F}$ is strictly preinvex on K.

The above relationships of preinvexity, weak preinvexity, strict preinvexity, weakly strict preinvexity, prequasiinvexity, weak prequasiinvexity, strict prequasiinvexity, weakly strict prequasiinvexity can be summarized in the following diagram (D refers to Condition D*, C refers to Condition* C*).*

5. Properties of n-dimensional preinvex, prequasiinvex fuzzy number-valued functions

In this section, we introduce the properties of n-dimensional preinvex, prequasiinvex fuzzy number-valued functions, and their applications in the fuzzy optimization problems.

Theorem 5.1. *Let K be an invex set of R^n w.r.t. η, and $\widetilde{F} : K \to E^n$ be a preinvex fuzzy number-valued function. Then the epigraph*

$$epi(\widetilde{F}) = \{(x,u) : \ x \in K, u \in E^n, \widetilde{F}(x) \preceq_c u\} \tag{5.1}$$

of $\widetilde{F}$ is an invex set of $K \times E^n$ w.r.t. the function

$$\eta' : epi(\widetilde{F}) \times epi(\widetilde{F}) \to K \times E^n,$$

defined by

$$\eta'((x,u),(y,v)) = (\eta(x,y), u + (-1)v) \tag{5.2}$$

for $(x,u),(y,v) \in epi(\widetilde{F})$ with $x, y \in K$ and $u, v \in E^n$. Here $epi(\widetilde{F})$ is an invex set of $K \times E^n$ means that $(y,v) + \lambda\eta'((x,u),(y,v)) \in epi(\widetilde{F})$ for any $(x,u),(y,v) \in epi(\widetilde{F})$ with $x, y \in K$.

Proof. If $epi(\widetilde{F})$ is the empty set or a singleton, then it is obvious that it is an invex set w.r.t. η'. Let (x,u), $(y,v) \in epi(\widetilde{F})$, where $x, y \in K$ and $u, v \in E^n$. Then, from (5.1), we have

$$\widetilde{F}(x) \preceq_c u \ \text{ and } \ \widetilde{F}(y) \preceq_c v.$$

From the preinvexity of $\widetilde{F}$, for any $\lambda \in [0,1]$, we have

$$\widetilde{F}(y+\lambda\eta(x,y)) \preceq_c \lambda\widetilde{F}(x) + (1-\lambda)\widetilde{F}(y) \preceq_c \lambda u + (1-\lambda)v,$$

which implies that for any $\lambda \in [0,1]$,

$$(y,v) + \lambda\eta'((x,u),(y,v)) = (y,v) + \lambda(\eta(x,y), u + (-1)v) = (y + \lambda\eta(x,y), \lambda u + (1-\lambda)v) \in epi(\widetilde{F}).$$

This proves that $epi(\widetilde{F})$ is an invex set of $K \times E^n$ w.r.t. the function η' defined by (5.2).

Theorem 5.2. *Let $\widetilde{F} : K \to E^l$ be a preinvex fuzzy number-valued function.*

(1) If $\widetilde{G} : E^l \to E^n$ is convex and increasing, then $\widetilde{G} \circ \widetilde{F} : K \to E^n$ is a preinvex fuzzy number-valued function;

(2) If $\widetilde{G} : E^l \to E^n$ is a positively homogeneous, increasing and sub-addition, then $\widetilde{G} \circ \widetilde{F} : K \to E^n$ is a preinvex fuzzy number-valued function.

Proof. Let $x, y \in K, \lambda \in [0,1]$, since $\widetilde{F} : K \to E^l$ is a preinvex, we have

$$\widetilde{F}(y + \lambda\eta(x,y)) \preceq_c \lambda\widetilde{F}(x) + (1-\lambda)\widetilde{F}(y).$$

(1) Since $\widetilde{G} : E^l \to E^n$ is an increasing, it follows that

$$\widetilde{G}(\widetilde{F}(y + \lambda\eta(x,y))) \preceq_c \widetilde{G}(\lambda\widetilde{F}(x) + (1-\lambda)\widetilde{F}(y)).$$

In addition, since $\widetilde{G}$ is a convex fuzzy mapping, it follows that

$$\widetilde{G}(\lambda\widetilde{F}(x) + (1-\lambda)\widetilde{F}(y)) \preceq_c \lambda\widetilde{G}(\widetilde{F}(x)) + (1-\lambda)\widetilde{G}(\widetilde{F}(y)).$$

From the above arguments, we have for $x, y \in K$ and $\lambda \in [0,1]$,

$$\widetilde{G}(\widetilde{F}(y + \lambda\eta(x,y))) \preceq_c \lambda\widetilde{G}(\widetilde{F}(x)) + (1-\lambda)\widetilde{G}(\widetilde{F}(y)),$$

which proves that $\widetilde{G} \circ \widetilde{F} : K \to E^n$ is a preinvex mapping on K.

(2) Since $\widetilde{G} : E^l \to E^n$ is a positively homogeneous, increasing and sub-addition fuzzy mapping, it follows that

$$\begin{aligned}\widetilde{G}(\widetilde{F}(y + \lambda\eta(x,y))) &\preceq_c \widetilde{G}(\lambda\widetilde{F}(x) + (1-\lambda)\widetilde{F}(y)) \\ &\preceq_c \widetilde{G}(\lambda\widetilde{F}(x)) + \widetilde{G}((1-\lambda)\widetilde{F}(y)) \\ &= \lambda\widetilde{G}(\widetilde{F}(x)) + (1-\lambda)\widetilde{G}(\widetilde{F}(y)).\end{aligned}$$

Theorem 5.3. *Let $\widetilde{F}_j : K \to E^n, j = 1, 2 \cdots l$ be preinvex fuzzy number-valued functions. For $k_1, k_2, \cdots, k_l > 0$, The fuzzy mapping $\widetilde{F} : K \to E^n$ defined by*

$$\widetilde{F}(x) = \sum_{j=1}^{l} k_j\widetilde{F}_j(x), \quad for \;\; each \;\; x \in K \tag{5.3}$$

is a preinvex fuzzy number-valued function.

Proof. Since $\widetilde{F}_j : K \to E^n, j = 1, 2 \cdots l$ is preinvex for each $j = 1, 2 \cdots l$, we have for $x, y \in K$ and $\lambda \in [0,1]$,

$$\widetilde{F}_j(y + \lambda\eta(x,y)) \preceq_c \lambda\widetilde{F}_j(x) + (1-\lambda)\widetilde{F}_j(y).$$

From (5.3), it follows that for $x, y \in K$ and $\lambda \in [0,1]$,

$$\widetilde{F}(y + \lambda\eta(x,y)) = (\sum_{j=1}^{l} k_j\widetilde{F}_j)(y + \lambda\eta(x,y))$$

$$
\begin{aligned}
&= \sum_{j=1}^{l} k_j \widetilde{F}_j(y+\lambda\eta(x,y)) \\
&\preceq_c \sum_{j=1}^{l} k_j(\lambda\widetilde{F}_j(x)+(1-\lambda)\widetilde{F}_j(y)) \\
&= \lambda\sum_{j=1}^{l} k_j\widetilde{F}_j(x)+(1-\lambda)\sum_{j=1}^{l} k_j\widetilde{F}_j(y) \\
&= \lambda(\sum_{j=1}^{l} k_j\widetilde{F}_j)(x)+(1-\lambda)(\sum_{j=1}^{l} k_j\widetilde{F}_j)(y) \\
&= \lambda\widetilde{F}(x)+(1-\lambda)\widetilde{F}(y),
\end{aligned}
$$

which proves that $\widetilde{F} : K \to E^n$ is a preinvex fuzzy number-valued function.

Theorem 5.4. *Let $\widetilde{F} : K \to E^n$ be a fuzzy number-valued function. Then $\widetilde{F}$ is preinvex w.r.t. η if and only if $\widetilde{F}(y+\lambda\eta(x,y)) \prec_c \lambda u+(1-\lambda)v$ for any $x, y \in K$ satisfying $\widetilde{F}(x) \prec_c u$, $\widetilde{F}(y) \prec_c v$ and $\lambda \in [0,1]$.*

Proof. Necessity is easy to prove.

Conversely, let for any $\widetilde{\varepsilon} \succ_c \widetilde{0}$, we have $\widetilde{F}(x) \prec_c \widetilde{F}(x)+\widetilde{\varepsilon}$, $\widetilde{F}(y) \prec_c \widetilde{F}(y)+\widetilde{\varepsilon}$, such that

$$
\begin{aligned}
\widetilde{F}(y+\lambda\eta(x,y)) &\prec_c \lambda(\widetilde{F}(x)+\widetilde{\varepsilon})+(1-\lambda)(\widetilde{F}(y)+\widetilde{\varepsilon}) \\
&= \lambda\widetilde{F}(x)+(1-\lambda)\widetilde{F}(y)+\widetilde{\varepsilon}.
\end{aligned}
$$

Since $\widetilde{\varepsilon}$ is an arbitrary positive fuzzy number, then for any $\lambda \in [0,1]$, we obtain

$$
\widetilde{F}(y+\lambda\eta(x,y)) \preceq_c \lambda\widetilde{F}(x)+(1-\lambda)\widetilde{F}(y),
$$

This completes the proof.

Theorem 5.5. *Let $\widetilde{F} : K \to E^n$ be a preinvex fuzzy number-valued function w.r.t. η. Then for $u \in E^n$, the lower u-level set*

$$
K_u(\widetilde{F}) = \{x | x \in K, \ \ \widetilde{F}(x) \preceq_c u\}
$$

of $\widetilde{F}$ is an invex set.

Proof. For any $x, y \in K_u(\widetilde{F})$, we have $\widetilde{F}(x) \preceq_c u$ and $\widetilde{F}(y) \preceq_c u$. Then, by the preinvexity of $\widetilde{F}$, it follows that, for any $\lambda \in [0,1]$,

$$
\widetilde{F}(y+\lambda\eta(x,y)) \preceq_c \lambda\widetilde{F}(x)+(1-\lambda)\widetilde{F}(y) \preceq u,
$$

which implies that

$$
y+\lambda\eta(x,y) \in K_u(\widetilde{F}).
$$

Definition 5.1. *Let $S \subset R^n \times E^n$, S is said to be G-invex set, if there exists a function $\eta : R^n \times R^n \to R^n$, for any (x,u), $(y,v) \in S$, $(y+\lambda\eta(x,y), \lambda u+(1-\lambda)v) \in S$, $\quad 0 \leq \lambda \leq 1$.*

Theorem 5.6. *Assume K is an invex set, then $\widetilde{F} : K \to E^n$ is a preinvex fuzzy number-valued function on K if and only if $epi(\widetilde{F})$ is G-invex set of $R^n \times E^n$.*

Proof. Assume that $\widetilde{F}$ is a preinvex fuzzy number-valued function on K. For $(x, u), (y, v) \in epi(\widetilde{F})$, $\lambda \in [0, 1]$, it follows that $y + \lambda\eta(x, y) \in K$ and

$$\widetilde{F}(y + \lambda\eta(x, y)) \preceq_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y) \preceq_c \lambda u + (1 - \lambda)v.$$

Thus,

$$(y + \lambda\eta(x, y), \lambda u + (1 - \lambda)v) \in epi(\widetilde{F}),$$

which implies that $epi(\widetilde{F})$ is G-invex set of $R^n \times E^n$ w.r.t. a given function $\eta \times \eta_o$, where $\eta_o : E^n \times E^n \to E^n$, $(u, v) \to u - v$.

Conversely, since $epi(\widetilde{F})$ is a G-invex set of $R^n \times E^n$, $(x, \widetilde{F}(x)) \in epi(\widetilde{F})$ and $(y, \widetilde{F}(y)) \in epi(\widetilde{F})$ for any $x, y \in K, \lambda \in [0, 1]$. Thus we have

$$(y + \lambda\eta(x, y), \widetilde{F}(y) + \lambda\eta_0(\widetilde{F}(x), \widetilde{F}(y)) = (y + \lambda\eta(x, y), \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y)) \in epi(\widetilde{F}),$$

which implies that

$$\widetilde{F}(y + \lambda\eta(x, y)) \preceq_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y).$$

This shows that $\widetilde{F}$ is preinvex fuzzy number-valued function on K.

Theorem 5.7. *Let $\{S_i\}_{i\in I}$ be a finite or infinite collection of G-invex sets of $R^n \times E^n$, where I is index set. Then $S = \bigcap_{i\in I} S_i$ is a G-invex set.*

Proof. Let for any $(x, u), (y, v) \in S$, we have, for any $i \in I$, $(x, u), (y, v) \in S_i$, then

$$(y + \lambda\eta(x, y), \lambda u + (1 - \lambda)v) \in S_i, \quad \forall\lambda \in [0, 1].$$

Therefore, we obtain

$$(y + \lambda\eta(x, y), \lambda u + (1 - \lambda)v) \in \bigcap_{i\in I} S_i = S, \quad \forall\lambda \in [0, 1].$$

That is, $S = \bigcap_{i\in I} S_i$ is a G-invex set.

Theorem 5.8. *Let $K \subseteq R^n$ be an invex set w.r.t. η, $\{\widetilde{F}_i\}_{i\in I}$ be a set of n-dimensional preinvex fuzzy number-valued functions on K. If $\sup\{\widetilde{F}_i(x)|i \in I\}$ exists in E^n for any $x \in K$, then $\widetilde{F}(x) = \sup\{\widetilde{F}_i(x)|i \in I\}$ is a n-dimensional preinvex fuzzy number-valued function on K.*

Proof. Since each $\widetilde{F}_i (i \in I)$ is n-dimensional preinvex fuzzy number-valued function on K, then by Theorem 5.6, we know that

$$epi(\widetilde{F}_i) = \{(x, u) \in k \times E^n : \widetilde{F}_i(x) \preceq_c u\}$$

is a G-invex set of $R^n \times E^n$. By Theorem 5.7, we have

$$\bigcap_{i\in I} epi(\widetilde{F}_i) = \{(x, u) \in k \times E^n : \widetilde{F}_i(x) \preceq_c u, \forall i \in I\}$$

is a G-invex set of $R^n \times E^n$. It is an easy matter to verify that

$$\bigcap_{i\in I} epi(\widetilde{F}_i) = \{(x, u) \in k \times E^n : \widetilde{F}_i(x) \preceq_c u, \forall i \in I\}$$

$$= \{(x,u) \in k \times E^n : \widetilde{F}(x) \preceq_c u\}$$
$$= epi(\widetilde{F}).$$

Thus, $epi(\widetilde{F})$ is a G-invex set of $R^n \times E^n$. By Theorem 5.6, we find that $\widetilde{F}$ is a n-dimensional preinvex fuzzy number-valued function on K.

Theorem 5.9. *Let $K \subseteq R^n$ be an invex set w.r.t. η. Then $\widetilde{F} : K \to E^n$ is a prequasiinvex fuzzy number-valued function on K if and only if the lower u-level set*

$$L_u(\widetilde{F}) = \{x | x \in K,\ \widetilde{F}(x) \preceq_c u\}$$

of $\widetilde{F}$ is an invex set w.r.t. η for each $u \in E^n$.

Proof. Necessity is easy to prove.

Conversely, assume that $L_u(\widetilde{F})$ is an invex set for each $u \in E^n$. Let $x, y \in K$, without loss of generality, we may assume that $\widetilde{F}(x) \preceq_c \widetilde{F}(y)$. Let $u = \widetilde{F}(y)$, since $\preceq_c$ is reflexive and transitive, we have

$$\widetilde{F}(x) \preceq_c u \quad and \quad \widetilde{F}(y) \preceq_c u,$$

which implies that

$$x, y \in L_u(\widetilde{F}).$$

We have $y + \lambda\eta(x,y) \in L_u(\widetilde{F})$, which implies that

$$\widetilde{F}(y + \lambda\eta(x,y)) \preceq_c u = \max\{\widetilde{F}(x), \widetilde{F}(u)\},$$

which completes the proof.

Theorem 5.10. *Let $\widetilde{F} : K \to E^n$ be a prequasiinvex fuzzy number-valued function. Then $t\widetilde{F}$ is prequasiinvex fuzzy number-valued function on K for any $t > 0$.*

Proof.

$$\begin{aligned} k\widetilde{F}(y + \lambda\eta(x,y)) &= k(\widetilde{F}(y + \lambda\eta(x,y))) \\ &\preceq_c k \max\{\widetilde{F}(x), \widetilde{F}(y)\} \\ &= \max\{k\widetilde{F}(x), k\widetilde{F}(y)\}. \end{aligned}$$

Theorem 5.11. *Let $\widetilde{F} : K \to E^n$ be prequasiinvex fuzzy number-valued function w.r.t. η, and $\overline{x} \in K$ be the global minimizer of $\widetilde{F}$ on K. Then, the set*

$$\Omega = \{x \in K : \widetilde{F}(x) = \widetilde{F}(\overline{x})\}$$

is an invex set w.r.t. η.

Proof. If Ω is the empty set or singleton, then it is obvious an invex set. Assume that $x, y \in \Omega$, then

$$\widetilde{F}(x) = \widetilde{F}(\overline{x}) \quad and \quad \widetilde{F}(y) = \widetilde{F}(\overline{x}).$$

Since $\widetilde{F} : K \to E^n$ is a prequasiinvex fuzzy number-valued function w.r.t. η, we have

$$\widetilde{F}(y + \lambda\eta(x, y)) \preceq_c \max\{\widetilde{F}(x), \widetilde{F}(y)\} = \widetilde{F}(\overline{x})$$

for any $\lambda \in [0, 1]$. Consider $\overline{x} \in K$ is a global minimizer of $\widetilde{F}$, it follows that

$$\widetilde{F}(y + \lambda\eta(x, y)) = \widetilde{F}(\overline{x})$$

for $\lambda \in [0, 1]$. It implies that $y + \lambda\eta(x, y) \in \Omega$ for $\lambda \in [0, 1]$. Thus, Ω is an invex set w.r.t. η. This completes the proof.

Theorem 5.12. *Let $\widetilde{F} : K \to E^n$ be preinvex fuzzy number-valued function w.r.t. η, and $\overline{x} \in K$ satisfying $\widetilde{F}(\overline{x}) = \min_{x\in K} \widetilde{F}(x)$. If $u = \min_{x\in K} \widetilde{F}(x)$, then the set*

$$\Omega = \{x \in K : \widetilde{F}(x) = u\}$$

is an invex set w.r.t. η.

6. Semicontinuity and preinvexity of fuzzy number-valued functions

In this section, we present several practical criteria for preinvex fuzzy number-valued functions under the lower or upper semicontinuity conditions.

Definition 6.1. *(see [7].) Let $\widetilde{F} : K \to E^n$ be a fuzzy number-valued function*

(1) $\widetilde{F}$ is said to be lower semicontinuous(l.c.) at $x_0 \in K$ if for any $\widetilde{\epsilon} \succ_c 0$, a neighborhood U of x_0 exists when $x \in K$, and we have

$$\widetilde{F}(x_0) \prec_c \widetilde{F}(x) + \widetilde{\epsilon}.$$

(2) $\widetilde{F}$ is said to be upper semicontinuous(u.c.) at $x_0 \in K$ if for any $\widetilde{\epsilon} \succ_c 0$, a neighborhood U of x_0 exists when $x \in K$, and we have

$$\widetilde{F}(x) \prec_c \widetilde{F}(x_0) + \widetilde{\epsilon}.$$

A fuzzy number-valued function $\widetilde{F} : K \to E^n$ is continuous at $x_0 \in K$ if it is both l.c. and u.c. at x_0, and that it is continuous at every point of K.

Theorem 6.1. *Let $K \subset R^n$ be an invex set w.r.t. η, η satisfy Condition* C*, $\widetilde{F} : K \to E^n$ be a lower semicontinuous fuzzy number-valued function, and $\widetilde{F}$ satisfy Condition* D*. If $\widetilde{F}$ is weakly preinvex on K, i.e., there exists a $\lambda_0 \in (0, 1)$ such that*

$$\widetilde{F}(y + \lambda_0\eta(x, y)) \preceq_c \lambda_0\widetilde{F}(x) + (1 - \lambda_0)\widetilde{F}(y)$$

for any $x, y \in K$, then $\widetilde{F}$ is preinvex on K.

Proof. Assume that $\widetilde{F}$ is not preinvex on K, then, $x, y \in K$ and there exists a $\lambda \in [0, 1]$ such that

$$\widetilde{F}(y + \lambda\eta(x, y)) \succ_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y). \tag{6.1}$$

By the weak preinvexity of $\widetilde{F}$ and Lemma 4.1, we can choose a sequence $\lambda_n \in A(n = 1, 2, \cdots)$ with $\lambda_n \to \lambda(n \to \infty)$ and

$$\widetilde{F}(y + \lambda_n\eta(x, y)) \preceq_c \lambda_n\widetilde{F}(x) + (1 - \lambda_n)\widetilde{F}(y). \tag{6.2}$$

From the lower semicontinuity of $\widetilde{F}$, for any $\widetilde{\varepsilon} \succ_c \widetilde{0}$, an $N > 0$ exists when $n > N$ and we have

$$\widetilde{F}(y + \lambda\eta(x, y)) \prec_c \widetilde{F}(y + \lambda_n\eta(x, y)) + \widetilde{\varepsilon}. \tag{6.3}$$

Since $\widetilde{\varepsilon}$ is an arbitrary positive fuzzy number, by taking the limit as $n \to \infty$, and by combining with (6.2) and (6.3), we have

$$\widetilde{F}(y + \lambda\eta(x, y)) \preceq_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y).$$

This contradicts the fact that (6.1), i.e., $\widetilde{F}$ is preinvex on K.

Theorem 6.2. *Let $K \subset R^n$ be an invex set w.r.t. η, η satisfy Condition* C, *$\widetilde{F} : K \to E^n$ be a upper semicontinuous fuzzy number-valued function, and $\widetilde{F}$ satisfy Condition* D. *If $\widetilde{F}$ is weakly preinvex on K, i.e., there exists a $\lambda_0 \in (0, 1)$ such that*

$$\widetilde{F}(y + \lambda_0\eta(x, y)) \preceq_c \lambda_0\widetilde{F}(x) + (1 - \lambda_0)\widetilde{F}(y)$$

for any $x, y \in K$, then $\widetilde{F}$ is preinvex on K.

Proof. Assume that $\widetilde{F}$ is not preinvex on K, then, $x, y \in K$ and there exists a $\lambda \in [0, 1]$ such that

$$\widetilde{F}(y + \lambda\eta(x, y)) \succ_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y). \tag{6.4}$$

By the weak preinvexity of $\widetilde{F}$ and Lemma 4.1, we can choose a sequence $\lambda_n \in A(n = 1, 2, \cdots)$ with $\lambda_n \to \lambda(n \to \infty)$ and

$$\widetilde{F}(\overline{y} + \lambda_n\eta(\overline{x}, \overline{y})) \preceq_c \lambda_n\widetilde{F}(\overline{x}) + (1 - \lambda_n)\widetilde{F}(\overline{y}). \tag{6.5}$$

for any $\overline{x}, \overline{y} \in K$, by taking $\overline{x} = x \in K$, and $\overline{y} = y + \frac{\lambda-\lambda_n}{1-\lambda_n}\eta(x, y) \in K$, using Condition C

$$\begin{aligned}
\overline{y} + \lambda_n\eta(\overline{x}, \overline{y}) &= y + \frac{\lambda - \lambda_n}{1 - \lambda_n}\eta(x, y) + \lambda_n\eta(x, y + \frac{\lambda - \lambda_n}{1 - \lambda_n}\eta(x, y)) \\
&= y + \frac{\lambda - \lambda_n}{1 - \lambda_n}\eta(x, y) + \lambda_n(1 - \frac{\lambda - \lambda_n}{1 - \lambda_n})\eta(x, y) \\
&= y + \lambda\eta(x, y).
\end{aligned}$$

and $\overline{y} \to y(n \to \infty)$. According to the upper semicontinuity of $\widetilde{F}$, for any $\widetilde{\varepsilon} \succ_c \widetilde{0}$, there exists an $N > 0$ when $n > N$ and we get

$$\widetilde{F}(\overline{y}) \prec_c \widetilde{F}(y) + \widetilde{\varepsilon}. \tag{6.6}$$

Since $\widetilde{\varepsilon}$ is an arbitrary positive fuzzy number, and by combining with (6.5) and (6.6), we have

$$\begin{aligned}
\widetilde{F}(y + \lambda\eta(x, y)) &= \widetilde{F}(\overline{y} + \lambda_n\eta(\overline{x}, \overline{y})) \\
&\preceq_c \lambda_n\widetilde{F}(\overline{x}) + (1 - \lambda_n)\widetilde{F}(\overline{y}) \\
&\preceq_c \lambda_n\widetilde{F}(x) + (1 - \lambda_n)\widetilde{F}(y).
\end{aligned}$$

By taking the limit as $n \to \infty$, we have

$$\widetilde{F}(y + \lambda\eta(x, y)) \preceq_c \lambda\widetilde{F}(x) + (1 - \lambda)\widetilde{F}(y).$$

This contradicts the fact that (6.4), i.e., $\widetilde{F}$ is preinvex on K.

By combining Theorem 6.1 and Theorem 6.2, we have the following result.

Corollary 6.1. *Let $K \subset R^n$ be an invex set w.r.t. η, η satisfy Condition* C*, $\widetilde{F} : K \to E^n$ satisfy Condition* D*, and $\widetilde{F}$ be a lower semicontinuous or upper semicontinuous fuzzy number-valued function. Then $\widetilde{F}$ is preinvex on K if and only if $\widetilde{F}$ is weakly preinvex on K.*

Theorem 6.3. *Let $K \subset R^n$ be an invex set w.r.t. η, η satisfy Condition* C*, $\widetilde{F} : K \to E^n$ satisfy Condition* D*, and $\widetilde{F}$ be a lower (resp. an upper) semicontinuous fuzzy number-valued function. If $\widetilde{F}$ is weakly strictly preinvex on K, i.e., there exists a $\lambda_0 \in (0, 1)$ such that*

$$\widetilde{F}(y + \lambda_0\eta(x, y)) \prec_c \lambda_0\widetilde{F}(x) + (1 - \lambda_0)\widetilde{F}(y)$$

for any $x, y \in K$ with $x \neq y$, then $\widetilde{F}$ is strictly preinvex on K.

By combining Definition 3.1 and Theorem 6.3, we have the following result.

Corollary 6.2. *Let $K \subset R^n$ be an invex set w.r.t. η, η satisfy Condition* C*, $\widetilde{F} : K \to E^n$ satisfy Condition* D*, and $\widetilde{F}$ be a lower semicontinuous or upper semicontinuous fuzzy number-valued function. Then $\widetilde{F}$ is strictly preinvex on K if and only if $\widetilde{F}$ is weakly strictly preinvex on K.*

Lemma 6.1. *Let $K \subset R^n$ be an invex set w.r.t. η, η satisfy Condition* C*, and $\widetilde{F} : K \to E^n$ satisfy Condition* D*. If there exists a $\alpha \in (0, 1)$ such that*

$$\widetilde{F}(y + \alpha\eta(x, y)) \preceq_c \max\{\widetilde{F}(x), \widetilde{F}(y)\}$$

for any $x, y \in K$, then the set

$$A = \{\lambda \in [0, 1] : \widetilde{F}(y + \alpha\eta(x, y)) \preceq_c \max\{\widetilde{F}(x), \widetilde{F}(y)\}\}$$

is dense in $[0, 1]$.

This proof is similar to the proof of Lemma 4.1.

Theorem 6.4. *Let $K \subset R^n$ be an invex set w.r.t. η, and η satisfy Condition* C*, $\widetilde{F} : K \to E^n$ satisfy Condition* D*, and $\widetilde{F}$ be an upper semicontinuous fuzzy number-valued function. If $\widetilde{F}$ is weakly prequasiinvex on K, i.e., there exists a $\lambda_0 \in (0, 1)$ such that*

$$\widetilde{F}(y + \alpha_0\eta(x, y)) \preceq_c \max\{\widetilde{F}(x), \widetilde{F}(y)$$

for any $x, y \in K$, then $\widetilde{F}$ is prequasiinvex on K.

Proof. Assume that $\widetilde{F}$ is not prequasiinvex on K. Then, for any $x, y \in K$ and there exists a $\lambda \in [0, 1]$ such that

$$\widetilde{F}(y + \lambda\eta(x, y)) \succ_c \max\{\widetilde{F}(x), \widetilde{F}(y)\}. \tag{6.7}$$

By the weak prequasiinvexity of $\widetilde{F}$ and Lemma 6.1, we can choose a sequence $\lambda_n \in A (n = 1, 2, \cdots)$ with $\lambda_n \to \lambda (n \to \infty)$ and

$$\widetilde{F}(\overline{y} + \lambda_n\eta(\overline{x}, \overline{y})) \preceq_c \max\{\widetilde{F}(\overline{x}), \widetilde{F}(\overline{y})\}. \tag{6.8}$$

for any $\overline{x}, \overline{y} \in K$, by taking $\overline{x} = x \in K$, and $\overline{y} = y + \frac{\lambda-\lambda_n}{1-\lambda_n}\eta(x, y) \in K$, using Condition C, we have $y + \lambda\eta(x, y) = \overline{y} + \lambda_n\eta(\overline{x}, \overline{y})$, and $\overline{y} \to y (n \to \infty)$. From the upper semicontinuity of $\widetilde{F}$, for any $\widetilde{\varepsilon} \succ_c \widetilde{0}$, there exists an $N > 0$ when $n > N$ and we obtain

$$\widetilde{F}(\overline{y}) \prec_c \widetilde{F}(y) + \widetilde{\varepsilon}. \tag{6.9}$$

Since $\widetilde{\varepsilon}$ is an arbitrary positive fuzzy number, and by combining with (6.8) and (6.9), it follows that

$$\begin{aligned}\widetilde{F}(y + \lambda\eta(x, y)) &= \widetilde{F}(\overline{y} + \lambda_n\eta(\overline{x}, \overline{y}))\\ &\preceq_c \max\{\widetilde{F}(\overline{x}), \widetilde{F}(\overline{y})\}\\ &\preceq_c \max\{\widetilde{F}(x), \widetilde{F}(y)\}.\end{aligned}$$

This contradicts to the fact that (6.7), i.e., $\widetilde{F}$ is prequasiinvex on K.

According to Theorem 4.6 and Theorem 6.4, we have the following result.

Theorem 6.5. *Let $K \subset R^n$ be an invex set w.r.t. η, and η satisfy Condition* C*, $\widetilde{F} : K \to E^n$ satisfy Condition* D*, and $\widetilde{F}$ be an upper semicontinuous fuzzy number-valued function. If $\widetilde{F}$ is weakly strictly prequasiinvex on K, i.e., there exists a $\lambda_0 \in (0, 1)$ such that*

$$\widetilde{F}(y + \alpha_0\eta(x, y)) \prec_c \max\{\widetilde{F}(x), \widetilde{F}(y)$$

for any $x, y \in K$ with $x \neq y$, then $\widetilde{F}$ is strictly prequasiinvex on K.

7. Two-parameter optimization problem

In this section, two types of the parameter optimization problems are investigated. They are widely applied in the optimization theory of consumers and producers. In which the optimal value of objective function depends on the values of the parameters. Therefore, the optimal solution and optimal values are all functions of parameters. The central task of economic analysis is to clarify the character of these functions. Two-parameters optimization problems is shown as follows.

$$P(\alpha): \quad \max \widetilde{F}(x), \quad x \in S = \{x \in X \subset R^n : \widetilde{G}(x, \alpha) \preceq_c \widetilde{0}\} \quad \alpha \in A \subset R^n;$$

$$P(\beta): \quad \min \widetilde{F}(x, \beta), \quad x \in S = \{x \in X \subset R^n : \widetilde{G}(x) \preceq_c \widetilde{0}\} \quad \beta \in B \subset R^n.$$

where $X, A, B \subset R^n$ are invex sets w.r.t. $\eta : R^n \times R^n \to R^n$, $\widetilde{F} : X \to E^n$, $\widetilde{G} : X \to E^n$. In the problem $P(\alpha)$, the parameter appear in the fuzzy constraint function, and the parameter appear in the fuzzy objective function in the problem $P(\beta)$. We always assume that the problems $P(\alpha)$ and $P(\beta)$ have the optimal solution for any fixed parameters α, β respectively, and write $\widetilde{Z}(\alpha)$ and $\widetilde{\psi}(\beta)$ as the optimal objective values for $P(\alpha)$ and $P(\beta)$ respectively.

Theorem 7.1. *Consider the problem $P(\alpha)$, if $\widetilde{G}(x, \alpha)$ is a n-dimensional preincave fuzzy number-valued function on A w.r.t. $\eta : R^n \times R^n \to R^n$, then $\widetilde{Z}(\alpha)$ is a n-dimensional prequasiinvex fuzzy number-valued function on A w.r.t. the same function η.*

Proof. For $\alpha_1, \alpha_2 \in A$ and for any $\lambda \in [0, 1]$, let x_λ be a optimal solution for $P(\alpha_2 + \lambda\eta(\alpha_1, \alpha_2))$. From the preincavity of $\widetilde{G}$ w.r.t. α, we obtain

$$\widetilde{0} \succeq_c \widetilde{G}(x_\lambda, \alpha_2 + \lambda\eta(\alpha_1, \alpha_2)) \succeq_c \lambda\widetilde{G}(x_\lambda, \alpha_1) + (1 - \lambda)\widetilde{G}(x_\lambda, \alpha_2).$$

Since λ and $(1-\lambda)$ are all non-negative, it follows that, $\widetilde{G}(x_\lambda, \alpha_1)$ and $\widetilde{G}(x_\lambda, \alpha_2)$ at least one non-positive. Without loss of generality, we assume that

$$\widetilde{G}(x_\lambda, \alpha_1) \preceq_c \widetilde{0},$$

it follows that x_λ is a feasible solution for $P(\alpha_1)$ and $\widetilde{Z}(\alpha_1) \succeq_c \widetilde{F}(x_\lambda)$. Then, we have, for any $\lambda \in [0,1]$,

$$\max\{\widetilde{Z}(\alpha_1), \widetilde{Z}(\alpha_2)\} \succeq_c \widetilde{Z}(\alpha_1) \succeq_c \widetilde{F}(x_\lambda) = \widetilde{Z}(\alpha_2 + \lambda\eta(\alpha_1, \alpha_2))$$

i.e., $\widetilde{Z}(\alpha)$ is a n-dimensional prequasiinvex fuzzy number-valued function on A w.r.t. the same function η.

Theorem 7.2. *Consider $P(\beta)$, if $\widetilde{F}(x,\beta)$ is a n-dimensional preincave fuzzy number-valued function on B w.r.t. parameter β and $\eta : R^n \times R^n \to R^n$, then $\widetilde{\psi}(\beta)$ is a n-dimensional preincave fuzzy number-valued function on B w.r.t. the same function η.*

Proof. For $\alpha_1, \alpha_2 \in B$ and for any $\lambda \in [0,1]$, let x_λ be a optimal solution for $P(\beta_2 + \lambda\eta(\beta_1,\beta_2))$. Since $\widetilde{G}(x_\lambda) \preceq_c \widetilde{0}$, it follows that, x_λ is the feasible solution to $P(\beta_1)$ and $P(\beta_2)$, which implies

$$\widetilde{F}(x_\lambda,\beta_1) \succeq_c \widetilde{\psi}(\beta_1) \quad and \quad \widetilde{F}(x_\lambda,\beta_2) \succeq_c \widetilde{\psi}(\beta_2).$$

From the preincavity of $\widetilde{F}$ w.r.t. β, we obtain

$$\begin{aligned}\widetilde{\psi}(\beta_2 + \lambda\eta(\beta_1,\beta_2)) &= \widetilde{F}(x_\lambda,\beta_2 + \lambda\eta(\beta_1,\beta_2))\\ &\succeq_C \lambda\widetilde{F}(x_\lambda,\beta_1) + (1-\lambda)\widetilde{F}(x_\lambda,\beta_2)\\ &\succeq_C \lambda\widetilde{\psi}(\beta_1) + (1-\lambda)\widetilde{\psi}(\beta_2),\end{aligned}$$

i.e., $\widetilde{\psi}(\beta)$ is a n-dimensional preincave fuzzy number-valued function on B w.r.t. the same function η.

Example 7.1. *The optimization problem in consumer theory. A consumer is an economic entity that uses available resources (income) to purchase goods and obtains satisfaction from the consumption of goods. The problem of the consumer is how to select the consumption bundle so that the consumer can get the maximum satisfaction from his consumption under the constraint that the total expenditure is not greater than the income of the consumer. Let $\widetilde{p} = (\widetilde{p}_1, \cdots, \widetilde{p}_n)^T$ be the price vector, where $\widetilde{p}_i \succ_c 0 (i = 1, \cdots, n)$ is the price of the good i, and let $x = (x_1, \cdots, x_n)^T$ be the consumption bundle, where $x_i \geqslant 0 (i = 1, \cdots, n)$ is the quantity of the good i consumed. If the total expenditure $p^T x$ is not greater than the consumer's income m, there must be a constraint $p^T x \leqslant m$, which is the consumer's budget constraint. The set $S = \{x \in R^n_+ : \widetilde{p}^T x \preceq_c m\}$ of all feasible consumption bundles is called the budget set. The consumer has different satisfaction for different consumption bundles, which is called consumer preference. We use utility functions to describe this preference. The quantity is only an estimated quantity, then using a fuzzy-valued function to express the quantity is more appropriate than using a crisp quantity. Specifically, a utility function $\widetilde{U} : R^n_+ \to E^n_+$ is a nonnegative fuzzy-valued function satisfying the following specification: $\widetilde{U}(x) \succ_c \widetilde{U}(y)$ means that consumption bundle x is better than consumption bundle y; $\widetilde{U}(x) = \widetilde{U}(y)$ means that consumption bundle x equals consumption bundle y; $\widetilde{U}(x) \succeq_c \widetilde{U}(y)$ means that consumption bundle x is not worse than consumption bundle y.*

The utility maximization problem of consumers can be formalized into the following optimization model:

$$(M)\begin{cases} \max \quad \widetilde{U}(x), \\ s.t. \quad x \in S = \{x \in R^n_+ : \widetilde{p}^T x \preceq_c m\}, \end{cases}$$

where $\widetilde{p} \succ_c 0$, $m > 0$. Obviously, this is a $P(\alpha)$ type optimization problem, where the parameters $(\widetilde{p}, m)$ appear in the fuzzy constraint function. Let $v(\widetilde{p}, m) = \widetilde{U}(x(\widetilde{p}, m))$ for every $\widetilde{p} \succ +c0$ and $m > 0$, *and $v(\widetilde{p}, m)$ is said to be an indirect utility function. Notice that the function $g(x; \widetilde{p}, m) = \widetilde{p}^T x - m$ is a preincave fuzzy number-valued function w.r.t. $(\widetilde{p}, m)$ and $\eta = x - y$. According to Theorem 7.1, $v(\widetilde{p}, m) = \widetilde{U}(x(\widetilde{p}, m))$ a n-dimensional prequasiinvex fuzzy number-valued function on R^n_+ w.r.t. the same function η.*

8. n-dimensional fuzzy variational-like inequality

Here, we present the fuzzy variational-like inequality and discuss the relationships between the fuzzy variational-like inequality problem and the unconstrained fuzzy vector optimization problem.

Let $K \subseteq R^n$, $\eta(x, \overline{x}) : K \times K \to R^n$, $\widetilde{F} : K \to E^n$ be a n-dimensional fuzzy number-valued function. Then the fuzzy variational-like inequality problem is to be found $\overline{x} \in K$, $u \in E^n$, such that

$$(FVLI) \qquad u\eta(x, \overline{x}) \succ_c \widetilde{0}, \qquad \forall x \in K.$$

Consider the unconstrained fuzzy vector optimization problem

$$(P) \qquad \min_{x \in K} \widetilde{F}(x)$$

where $K \subseteq R^n$ is an invex set w.r.t. η, $\widetilde{F} : K \to E^n$ is a n-dimensional fuzzy number-valued function.

A point $x_0 \in K$ is called a local minimum of $\widetilde{F}$, if $x_0 \in K$ and there exists a δ-neighborhood $N_\delta(x_0)$ around x_0, such that for any $x \in K \cap N_\delta(x_0)$, $\widetilde{F}(x_0) \preceq_c \widetilde{F}(x)$. Similarly, if $x_0 \in K$ and there exists a δ-neighborhood $N_\delta(x_0)$ around x_0, such that for any $x \in K \cap N_\delta(x_0)$, with $x \neq x_0$, $\widetilde{F}(x_0) \prec_c \widetilde{F}(x)$, then x_0 is called a strict local minimum point of $\widetilde{F}$.

Theorems 8.1-8.3 show the relationship between the fuzzy variational-like inequality problem and the preinvexity of n-dimensional fuzzy number-valued function.

Theorem 8.1. *Let K be an invex set of R^n w.r.t. η, $\overline{x} \in K$, $\widetilde{F} : K \to E^n$ a preinvex fuzzy number-valued function w.r.t. η and $\widetilde{F}$ fuzzy η-extended directionally differentiable on K. If $(\overline{x}, \widetilde{\nabla}\widetilde{F}_\eta(\overline{x}))$ is a solution of $(FVLI)$, then $\overline{x}$ is a strict local optimal solution of (P).*

Proof. Let $(\overline{x}, \widetilde{\nabla}\widetilde{F}_\eta(\overline{x}))$ be a solution of $(FVLI)$. Suppose that there exists an $x^* \in K \cap N_\delta(\overline{x})$, such that

$$\widetilde{F}(x^*) \preceq_c \widetilde{F}(\overline{x}). \tag{8.1}$$

Since $\widetilde{F}$ is a preinvex fuzzy number-valued function, it follows that

$$\frac{\widetilde{F}(\overline{x} + \lambda\eta(x^*, \overline{x})) - \widetilde{F}(\overline{x})}{\lambda} \preceq_c \widetilde{F}(x^*) - \widetilde{F}(\overline{x}) \quad \forall \lambda \in [0, 1].$$

From the η-extended directionally differentiability of $\widetilde{F}$, and taking the limit as $\lambda \to 0^+$, we find that

$$\widetilde{\nabla}\widetilde{F}_\eta(\overline{x})\eta(x^*, \overline{x}) \preceq_c \widetilde{F}(x^*) - \widetilde{F}(\overline{x}). \tag{8.2}$$

According to (8.1) and (8.2), we obtain

$$\widetilde{\nabla}\widetilde{F}_\eta(\overline{x})\eta(x^*, \overline{x}) \preceq_c \widetilde{0}.$$

This contradicts the fact that $(\overline{x}, \widetilde{\nabla}\widetilde{F}_\eta(\overline{x}))$ is a solution of $(FVLI)$.

Theorem 8.2. *Let K be an invex set of R^n w.r.t. η, $\overline{x} \in K$, $\widetilde{F} : K \to E^n$ a preincave fuzzy number-valued function w.r.t. η and $\widetilde{F}$ fuzzy η-extended directionally differentiable on K. If $\overline{x}$ be a strict local optimal solution of (P), then $(\overline{x}, \widetilde{\nabla}\widetilde{F}_\eta(\overline{x}))$ is a solution of $(FVLI)$.*

Proof. Let $\overline{x}$ be a strict local optimal solution of (P). Suppose that there exists an $x^* \in K$, such that

$$\widetilde{\nabla}\widetilde{F}_\eta(\overline{x})\eta(x^*, \overline{x}) \preceq_c \widetilde{0}.$$

Since $\widetilde{F}$ is a preincave fuzzy number-valued function, it follows that

$$\widetilde{F}(x^*) - \widetilde{F}(\overline{x}) \preceq_c \frac{\widetilde{F}(\overline{x} + \lambda\eta(x^*, \overline{x})) - \widetilde{F}(\overline{x})}{\lambda} \quad \forall \lambda \in [0, 1].$$

By the η-extended directionally differentiability of $\widetilde{F}$, and taking the limit as $\lambda \to 0^+$, we obtain

$$\widetilde{F}(x^*) - \widetilde{F}_\eta(\overline{x}) \preceq_c \widetilde{\nabla}\widetilde{F}_\eta(\overline{x})\eta(x^*, \overline{x}).$$

Therefore, we have

$$\widetilde{F}(x^*) \preceq_c \widetilde{F}(\overline{x}).$$

This contradicts the fact that $\overline{x}$ is a strict local optimal solution of (P).

Theorem 8.3. *Let K be an invex set of R^n w.r.t. η, $\overline{x} \in K$, $\widetilde{F} : K \to E^n$ be a strictly preincave fuzzy number-valued function w.r.t. η and $\widetilde{F}$ be fuzzy η-extended directionally differentiable on K. If $\overline{x}$ be an optimal solution of (P), then $(\overline{x}, \widetilde{\nabla}\widetilde{F}_\eta(\overline{x}))$ is a solution of $(FVLI)$.*

Proof. Let $\overline{x}$ be an optimal solution of (P). Suppose that there exists an $x^* \in K$, such that

$$\widetilde{\nabla}\widetilde{F}_\eta(\overline{x})\eta(x^*, \overline{x}) \preceq_c \widetilde{0}.$$

Since $\widetilde{F}$ is a strictly preincave fuzzy number-valued function, it follows that

$$\widetilde{F}(x^*) - \widetilde{F}(\overline{x}) \prec_c \frac{\widetilde{F}(\overline{x} + \lambda\eta(x^*, \overline{x})) - \widetilde{F}(\overline{x})}{\lambda} \quad \forall \lambda \in [0, 1].$$

From the η-extended directionally differentiability of $\widetilde{F}$. Taking the limit as $\lambda \to 0^+$, we obtain

$$\widetilde{F}(x^*) - \widetilde{F}_\eta(\overline{x}) \prec_c \widetilde{\nabla}\widetilde{F}(\overline{x})\eta(x^*, \overline{x}).$$

Therefore, we have

$$\widetilde{F}(x^*) \prec_c \widetilde{F}(\overline{x}).$$

This contradicts the fact that $\overline{x}$ is a optimal solution of (P).

9. Kuhn-Tucker conditions for a multiobjective fuzzy programming problem

Consider a multiobjective fuzzy programming problem,

$$(P)\begin{cases} \min \widetilde{F}(x) \\ x \in S = \{x \in X : \quad \widetilde{G}_i(x) \preceq_c \widetilde{0}, \quad i = 1, 2, \cdots m\} \end{cases}$$

where $\widetilde{F}: K \to E^n$, $\widetilde{G}_i: K \to E^n$, $(i = 1, 2, \cdots m)$, $X \subset R^n$ is an invex set w.r.t. η.

Let S be the set of the feasible solution for (P), x_0 is a feasible point for (P). For a feasible point x_0, we denote

$$I(x_0) = \{i \in \{1, 2, \cdots m\} : \ \widetilde{G}_i(x_0) = \widetilde{0}\},$$

$$I'(x_0) = \{i \in \{1, 2, \cdots m\} : \ \widetilde{G}_i(x_0) \prec_c \widetilde{0}\}.$$

Then,

$$I(x_0) \cup I'(x_0) = \{1, 2, \cdots m\}.$$

Let $\widetilde{F}$ be a n-dimensional fuzzy number-valued function defined on X and $\widetilde{G}$ be a m-dimensional fuzzy number-valued function defined on X, $X \subset R^n$ is an invex set w.r.t. η. ($\widetilde{G} = (\widetilde{G}_i)_{i=1}^m$, means $\widetilde{G}_i$ is a n-dimensional fuzzy number-valued function, for each $i = 1, 2, \cdots m$).

Define the n-dimensional lagrangian fuzzy function as

$$\widetilde{L}(x, \lambda) = \widetilde{F}(x) + \lambda^t \widetilde{G}(x).$$

The Kuhn-Tucker stationary point of a n-dimensional fuzzy optimal problem is to find a $x \in X$, $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m) \in R^m$, if they exists, such that

$$\widetilde{\nabla}_x \widetilde{L}(x, \lambda) = \widetilde{\nabla} \widetilde{F}_\eta(x) + \lambda^t \widetilde{\nabla} \widetilde{G}_\eta(x) = \widetilde{0},$$

$$\widetilde{G}(x) \preceq_c \widetilde{0},$$

$$\lambda^t \widetilde{G}(x) = \Sigma_{i=1}^m \lambda_i \widetilde{G}_i(x) = \widetilde{0},$$

$$\lambda \geq 0.$$

In what follows, we show the connection between the Kuhn-Tucker stationary point for a n-dimensional fuzzy optimal problem and the optimal solution of (P).

Theorem 9.1. *Let x_0 be a feasible solution of (P), and let $\widetilde{F}: K \to E^n$ be a preinvex fuzzy number-valued function at x_0 w.r.t. η and $\widetilde{F}$ be fuzzy η-extended directionally differentiable at x_0 on K. Let $\widetilde{G}_i: K \to E^n(1, 2, \cdots m)$ be a prequasiinvex fuzzy number-valued function at x_0 w.r.t. the same function η and $\widetilde{G}_i(1, 2, \cdots m)$ be fuzzy η-extended directionally differentiable at x_0 on K. Moreover, if there exist $\lambda_i \geq 0(1, 2, \cdots m)$, such that*

$$\begin{cases} \widetilde{\nabla} \widetilde{F}_\eta(x_0) + \Sigma_{i=1}^m \lambda_i \widetilde{\nabla} \widetilde{G}_{i\eta}(x_0) = \widetilde{0} \\ \lambda_i \widetilde{G}_i(x_0) = \widetilde{0} \qquad 1, 2, \cdots m \end{cases}$$

Then, x_0 is the global minimum point of (P).

Proof. Assume that x_0 is not a global minimum point of (P). Then, there exists $\overline{x} \in S$ such that

$$\widetilde{F}(\overline{x}) \prec_c \widetilde{F}(x_0).$$

Since $\widetilde{F}$ is a preinvex fuzzy number- valued function at x_0, and $\widetilde{F}$ is η-extended directionally differentiable at x_0, we have

$$\widetilde{\nabla} \widetilde{F}_\eta(x_0) \eta(\overline{x}, x_0) \preceq_c \widetilde{F}(\overline{x}) - \widetilde{F}(x_0) \prec_c \widetilde{0}.$$

Also, since $\widetilde{G}_i(\overline{x}) \preceq_c \widetilde{0} = \widetilde{G}_i(x_0), i \in I(x_0)$ and the η-extended directionally differentiability of $\widetilde{G}_i$ at x_0, for any $\lambda \in [0, 1]$, we get

$$\widetilde{\nabla}\widetilde{G}_{i\eta}(x_0)\eta(\overline{x}, x_0) = \lim_{\lambda \to 0^+} \frac{\widetilde{G}_i(x_0 + \lambda\eta(\overline{x}, x_0)) - \widetilde{G}_i(x_0)}{\lambda}. \tag{9.1}$$

$\widetilde{G}_i$ is a prequasiinvex fuzzy number-valued function at x_0 w.r.t. the same function η, then

$$\widetilde{G}_i(x_0 + \lambda\eta(\overline{x}, x_0)) \preceq_c \max\{\widetilde{G}_i(x_0), \widetilde{G}_i(\overline{x})\} \preceq_c \widetilde{G}_i(x_0) \quad i \in I(x_0). \tag{9.2}$$

By combining (9.1) and (9.2), we find that

$$\widetilde{\nabla}\widetilde{G}_{i\eta}(x_0)\eta(\overline{x}, x_0) \preceq_c \widetilde{0} \quad i \in I(x_0).$$

According to $\lambda_i\widetilde{G}_i(x_0) = \widetilde{0}$, it follows that, $\forall i \in I'(x_0), \lambda_i = 0$.
From above discussion, we have

$$\widetilde{\nabla}\widetilde{F}_\eta(x_0)\eta(\overline{x}, x_0) + \Sigma_{i=1}^m \lambda_i \widetilde{\nabla}\widetilde{G}_{i\eta}(x_0)\eta(\overline{x}, x_0) \prec_c \widetilde{0},$$

which is a contradiction to the condition

$$\widetilde{\nabla}\widetilde{F}_\eta(x_0) + \Sigma_{i=1}^m \lambda_i \widetilde{\nabla}\widetilde{G}_{i\eta}(x_0) = \widetilde{0}.$$

It completes the proof.

10. Conclusions

In this paper, we first introduce the concept of the preinvexity of n-dimensional fuzzy number-valued functions based on the partial order relation in n-dimensional fuzzy number space and their properties are discussed. In addition, some counterexamples are given to show the proposed concepts and their relationships. Then we present the criteria theorems for n-dimensional preinvex fuzzy number-valued functions under the upper or lower semicontinuity conditions, respectively. Furthermore, the two-parameter optimization problem, n-dimensional fuzzy variational-like inequality problem, and the optimality conditions related to n-dimensional preinvex fuzzy number-valued function are discussed. These results can be applied in many fields, such as fuzzy optimization, fuzzy control, engineering science, fuzzy-making problems and so on.

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Conflict of interest

The authors declare that they have no conflict of interest.

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