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Research article

A tighter M-eigenvalue localization set for partially symmetric tensors and its an application

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Abstract: In this paper, a new M-eigenvalue inclusion set for a partially symmetric tensor is provided. It is proved that the new set is tighter than some existing M-eigenvalue inclusion sets. Based on the obtained results, an upper bound of the largest M-eigenvalue is given and a modified WQZ-algorithm is established which guarantees the generated converges to the largest M-eigenvalue of the tensor faster.

Keywords: partially symmetric tensors; M-eigenvalues; localization sets; M-spectral radius
Mathematics Subject Classification: 15A18, 15A42, 15A69

1. Introduction

Let m and n be two positive integers with $m \geq 2$ and $n \geq 2$, $[n] = \{1, 2, \ldots, n\}$, $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^n$ be the set of all n-dimensional real vectors. Let $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. If a fourth-order tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ satisfies the properties

$$a_{ijkl} = a_{kjil} = a_{ilkj} = a_{klij}, \quad i, k \in [m], \quad j, l \in [n],$$

then we call $\mathcal{A}$ a partially symmetric tensor.

It is well know that the tensor of the elastic modulus of elastic materials is just partially symmetrical [11]. And the components of a fourth-order partially symmetric tensor $\mathcal{A}$ can be regarded as the coefficients of the following biquadratic homogeneous polynomial optimization problem [6, 19]:

$$\begin{aligned} &\max \ f(x, y) \equiv \mathcal{A}xyxy \equiv \sum_{i,k\in[m]} \sum_{j,l\in[n]} a_{ijkl} x_i y_j x_k y_l, \\ &\text{s.t.} \ \ x^\top x = 1, \ \ y^\top y = 1. \end{aligned} \tag{1.1}$$

The optimization problem plays a great role in the analysis of nonlinear elastic materials and the entanglement problem in quantum physics [5, 6, 8, 9, 26]. To solve the problem, we would establish a new version based on the following definition:

Definition 1.1. [11, 20, 21] *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor. If there are* $\lambda \in \mathbb{R}$*,* $x \in \mathbb{R}^m \setminus \{0\}$ *and* $y \in \mathbb{R}^n \setminus \{0\}$ *such that*

$$\mathcal{A}\cdot yxy = \lambda x, \quad \mathcal{A}xyx\cdot = \lambda y, \quad x^\top x = 1, \quad y^\top y = 1, \tag{1.2}$$

where

$$(\mathcal{A}\cdot yxy)_i = \sum_{k\in[m]}\sum_{j,l\in[n]} a_{ijkl}y_jx_ky_l, \ \ (\mathcal{A}xyx\cdot)_l = \sum_{i,k\in[m]}\sum_{j\in[n]} a_{ijkl}x_iy_jx_k,$$

then we call λ *an M-eigenvalue of* $\mathcal{A}$*,* x *and* y *the left and right M-eigenvectors associated with* λ*, respectively. Let* $\sigma(\mathcal{A})$ *be the set of all M-eigenvalues of* $\mathcal{A}$ *and* $\lambda_{\max}(\mathcal{A})$ *be the largest M-eigenvalue of* $\mathcal{A}$*, i.e.,*

$$\lambda_{\max}(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

In 2009, Wang, Qi and Zhang [24] pointed out that Problem (1.1) is equivalently transformed into calculating the largest M-eigenvalue of a fourth-order partially symmetric tensor. Based on this, Wang et al. [24] presented an algorithm (WQZ-algorithm) to find the largest M-eigenvalue of a fourth-order partially symmetric tensor.

WQZ-algorithm [24, Algorithm 4.1]:

Initial step: Input $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ and unfold it into a matrix $A = (A_{st}) \in \mathbb{R}^{[mn]\times[mn]}$ by mapping $A_{st} = a_{ijkl}$ with $s = n(i-1) + j, \quad t = n(k-1) + l$.

Substep 1: Take

$$\tau = \sum_{1\le s\le t\le mn} |A_{st}|, \tag{1.3}$$

and set

$$\overline{\mathcal{A}} = \tau\mathcal{I} + \mathcal{A}, \tag{1.4}$$

where $\mathcal{I} = (\delta_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ with $\delta_{ijkl} = 1$ if $i = k$ and $j = l$, otherwise, $\delta_{ijkl} = 0$. Then unfold $\overline{\mathcal{A}} = (\overline{a}_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ into a matrix $\overline{A} = (\overline{A}_{st}) \in \mathbb{R}^{[mn]\times[mn]}$.

Substep 2: Compute the unit eigenvector $w = (w_i)_{i=1}^{mn} \in \mathbb{R}^{mn}$ of matrix $\overline{A}$ associated with its largest eigenvalue, and fold vector w into the matrix $W = (W_{ij}) \in \mathbb{R}^{[m]\times[n]}$ in the following way:

$$W_{ij} = w_k,$$

set $i = \lceil k/n \rceil, \quad j = (k-1)\text{mod}n + 1, \quad \forall\, k = 1, 2, \cdots, mn$.

Substep 3: Compute the singular vectors u_1 and v_1 corresponding to the largest singular value σ_1 of the matrix W. Specifically, the singular value decomposition of W is

$$W = U^T\Sigma V = \sum_{i=1}^{r} \sigma_i u_i v_i^T,$$

where $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r$ and r is the rank of W.

Substep 4: Take $x_0 = u_1, y_0 = v_1$, and let $k = 0$.

Iterative step: Execute the following procedures alternatively until certain convergence criterion is satisfied and output x^*, y^* :

$$\begin{aligned}
&\overline{x}_{k+1} = \overline{\mathcal{A}} \cdot y_k x_k y_k, \qquad x_{k+1} = \frac{\overline{x}_{k+1}}{\|\overline{x}_{k+1}\|},\\
&\overline{y}_{k+1} = \overline{\mathcal{A}} x_{k+1} y_k x_{k+1} \cdot, \quad y_{k+1} = \frac{\overline{y}_{k+1}}{\|\overline{y}_{k+1}\|},\\
&k = k+1.
\end{aligned}$$

Final step: Output the largest M-eigenvalue of the tensor $\mathcal{A}$:

$$\lambda_{\max}(\mathcal{A}) = f(x^*, y^*) - \tau,$$

where

$$f(x^*, y^*) = \sum_{i,k\in[m]} \sum_{j,l\in[n]} \overline{a}_{ijkl} x_i^* y_j^* x_k^* y_l^*,$$

and the associated M-eigenvectors: x^*, y^*.

The M-eigenvalues of tensors have a close relationship with the strong ellipticity condition in elasticity theory, which guarantees the existence of the solution to the fundamental boundary value problems of elastostatics [3, 5, 16]. However, when the dimensions m and n of tensors are large, it is not easy to calculate all M-eigenvalues. Thus, the problem of M-eigenvalue localization have attracted the attention of many researchers and many M-eigenvalue localization sets are given; see [2, 4, 13–15, 17, 18, 23, 27].

For this, Wang, Li and Che [23] presented the following M-eigenvalue localization set for a partially symmetric tensor:

Theorem 1.1. [23, Theorem 2.2] *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{H}(\mathcal{A}) = \bigcup_{i\in[m]} \bigcap_{k\in[m],k\neq i} \mathcal{H}_{i,k}(\mathcal{A}),$$

where

$$\begin{gathered}
\mathcal{H}_{i,k}(\mathcal{A}) = \left[\widehat{\mathcal{H}}_{i,k}(\mathcal{A}) \cup (\overline{\mathcal{H}}_{i,k}(\mathcal{A}) \cap \Gamma_i(\mathcal{A}))\right],\\
\widehat{\mathcal{H}}_{i,k}(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq R_i(\mathcal{A}) - R_i^k(\mathcal{A}),\ |z| \leq R_k^k(\mathcal{A})\},\\
\overline{\mathcal{H}}_{i,k}(\mathcal{A}) = \{z \in \mathbb{C} : (|z| - (R_i(\mathcal{A}) - R_i^k(\mathcal{A})))(|z| - R_k^k(\mathcal{A})) \leq R_i^k(\mathcal{A})(R_k(\mathcal{A}) - R_k^k(\mathcal{A}))\},\\
R_i(\mathcal{A}) = \sum_{k\in[m]} \sum_{j,l\in[n]} |a_{ijkl}|,\ \ R_i^k(\mathcal{A}) = \sum_{j,l\in[n]} |a_{ijkl}|.
\end{gathered}$$

From the set $\mathcal{H}(\mathcal{A})$ in Theorem 1.1, we can obtain an upper bound of the largest M-eigenvalue $\lambda_{\max}(\mathcal{A})$, which can be taken as an parameter τ in WQZ-algorithm. From Example 2 in [15], it can be seen that the smaller the upper bound of $\lambda_{\max}(\mathcal{A})$, the faster WQZ-algorithm converges. In view of this, this paper intends to provide a smaller upper bound based on a new inclusion set and take this new upper bound as a parameter τ to make WQZ-algorithm converges to $\lambda_{\max}(\mathcal{A})$ faster.

The remainder of this paper is organized as follows. In Section 2, we provide an M-eigenvalue localization set for a partially symmetric tensor $\mathcal{A}$ and prove that the new set is tighter than some existing M-eigenvalue localization sets. In Section 3, based on the new set, we provide an upper bound for the largest M-eigenvalue of $\mathcal{A}$. As an application, in order to make the sequence generated by WQZ-algorithm converge to the largest M-eigenvalue of $\mathcal{A}$ faster, we replace the parameter τ in WQZ-algorithm with the upper bound. In Section 4, we conclude this article.

2. A shaper M-eigenvalue localization set of a fourth-order partially symmetric tensor

In this section, we provide a new M-eigenvalue localization set of a fourth-order partially symmetric tensor and prove that the new M-eigenvalue localization set is tighter than that in Theorem 1.1, i.e., Theorem 2.2 in [23]. Before that, the following conclusion in [1, 25] is needed.

Lemma 2.1. *Let $x = (x_1, x_2, \dots, x_n)^\top \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n)^\top \in \mathbb{R}^n$. Then*

a) If $\| x \|_2 = 1$, then $|x_i||x_j| \le \frac{1}{2}$ for $i, j \in [n]$, $i \neq j$;

b) $\Big(\sum\limits_{i\in[n]} x_i y_i \Big)^2 \le \sum\limits_{i\in[n]} x_i^2 \sum\limits_{i\in[n]} y_i^2$.

Theorem 2.1. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. Then*

$$\sigma(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}) = \bigcup_{i\in[m]} \bigcap_{s\in[m], s\neq i} \Upsilon_{i,s}(\mathcal{A}),$$

where

$$\begin{aligned}
\Upsilon_{i,s}(\mathcal{A}) &= \left[\widehat{\Upsilon}_{i,s}(\mathcal{A}) \cup (\widetilde{\Upsilon}_{i,s}(\mathcal{A}) \cap \overline{\Upsilon}_{i,s}(\mathcal{A}))\right],\\
\widehat{\Upsilon}_{i,s}(\mathcal{A}) &= \{z \in \mathbb{R} : |z| < \widetilde{r}_i^s(\mathcal{A}),\ |z| < r_s^s(\mathcal{A})\},\\
\widetilde{\Upsilon}_{i,s}(\mathcal{A}) &= \{z \in \mathbb{R} : (|z| - \widetilde{r}_i^s(\mathcal{A}))(|z| - r_s^s(\mathcal{A})) \le r_i^s(\mathcal{A})\widetilde{r}_s^s(\mathcal{A})\},\\
\overline{\Upsilon}_{i,s}(\mathcal{A}) &= \{z \in \mathbb{R} : |z| < \widetilde{r}_i^s(\mathcal{A}) + r_i^s(\mathcal{A})\},
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{r}_t^s(\mathcal{A}) &= \frac{1}{2} \sum_{k\in[m],k\neq s} \sum_{j,l\in[n],j\neq l} |a_{tjkl}| + \sum_{k\in[m],k\neq s} \sqrt{\sum_{l\in[n]} a_{tlkl}^2},\\
r_t^s(\mathcal{A}) &= \frac{1}{2} \sum_{j,l\in[n],j\neq l} |a_{tjsl}| + \sqrt{\sum_{l\in[n]} a_{tlsl}^2}, \quad t \in [m].
\end{aligned}$$

Proof. Let λ be an M-eigenvalue of $\mathcal{A}$, $x \in \mathbb{R}^m\backslash\{0\}$ and $y \in \mathbb{R}^n\backslash\{0\}$ be its left and right M-eigenvectors, respectively. Then $x^\top x = 1$. Let $|x_t| = \max\limits_{i\in[m]} |x_i|$. Then $0 < |x_t| \le 1$. For any given $s \in [m]$ and $s \neq t$, by the t-th equation of (1.2), we have

$$\begin{aligned}
\lambda x_t &= \sum_{k\in[m]} \sum_{j,l\in[n]} a_{tjkl} y_j x_k y_l\\
&= \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{\substack{j,l\in[n],\\ j\neq l}} a_{tjkl} y_j x_k y_l + \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{l\in[n]} a_{tlkl} y_l x_k y_l + \sum_{\substack{j,l\in[n],\\ j\neq l}} a_{tjsl} y_j x_s y_l + \sum_{l\in[n]} a_{tlsl} y_l x_s y_l.
\end{aligned}$$

Taking the modulus of the above equation and using the triangle inequality and Lemma 2.1, one has

$$\begin{aligned}
|\lambda||x_t| \leq & \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{tjkl}||y_j||x_k||y_l| + \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{l\in[n]} |a_{tlkl}||y_l||x_k||y_l| + \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{tjsl}||y_j||x_s||y_l| + \sum_{l\in[n]} |a_{tlsl}||y_l||x_s||y_l| \\
\leq & \frac{1}{2} \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{tjkl}||x_t| + \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{l\in[n]} |a_{tlkl}||y_l||x_t| + \frac{1}{2} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{tjsl}||x_s| + \sum_{l\in[n]} |a_{tlsl}||y_l||x_s| \\
= & \frac{1}{2} \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{tjkl}||x_t| + |x_t| \sum_{\substack{k\in[m],\\ k\neq s}} \Big(\sum_{l\in[n]} |a_{tlkl}||y_l| \Big) + \frac{1}{2} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{tjsl}||x_s| + |x_s| \sum_{l\in[n]} |a_{tlsl}||y_l| \\
\leq & \frac{1}{2} \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{tjkl}||x_t| + |x_t| \sum_{\substack{k\in[m],\\ k\neq s}} \left(\sqrt{\sum_{l\in[n]} |a_{tlkl}|^2} \sqrt{\sum_{l\in[n]} |y_l|^2} \right) \\
& + \frac{1}{2} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{tjsl}||x_s| + |x_s| \sqrt{\sum_{l\in[n]} |a_{tlsl}|^2} \sqrt{\sum_{l\in[n]} |y_l|^2} \\
= & \frac{1}{2} \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{tjkl}||x_t| + |x_t| \sum_{\substack{k\in[m],\\ k\neq s}} \sqrt{\sum_{l\in[n]} a_{tlkl}^2} + \frac{1}{2} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{tjsl}||x_s| + |x_s| \sqrt{\sum_{l\in[n]} a_{tlsl}^2} \\
= & \left(\frac{1}{2} \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{tjkl}| + \sum_{\substack{k\in[m],\\ k\neq s}} \sqrt{\sum_{l\in[n]} a_{tlkl}^2} \right) |x_t| + \left(\frac{1}{2} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{tjsl}| + \sqrt{\sum_{l\in[n]} a_{tlsl}^2} \right) |x_s| \\
= & \widetilde{r_t^s}(\mathcal{A})|x_t| + r_t^s(\mathcal{A})|x_s|,
\end{aligned}$$

i.e.,

$$(|\lambda| - \widetilde{r_t^s}(\mathcal{A}))|x_t| \leq r_t^s(\mathcal{A})|x_s|. \tag{2.1}$$

By (2.1), we have $(|\lambda| - \widetilde{r_t^s}(\mathcal{A}))|x_t| \leq r_t^s(\mathcal{A})|x_t|$, which leads to that $|\lambda| \leq \widetilde{r_t^s}(\mathcal{A}) + r_t^s(\mathcal{A})$, i.e., $\lambda \in \overline{\Upsilon}_{t,s}(\mathcal{A})$.

If $|x_s| > 0$, then by the s-th equation of (1.2), we have

$$\begin{aligned}
\lambda x_s = & \sum_{k\in[m]} \sum_{j,l\in[n]} a_{sjkl} y_j x_k y_l \\
= & \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{\substack{j,l\in[n],\\ j\neq l}} a_{sjkl} y_j x_k y_l + \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{l\in[n]} a_{slkl} y_l x_k y_l + \sum_{\substack{j,l\in[n],\\ j\neq l}} a_{sjsl} y_j x_s y_l + \sum_{l\in[n]} a_{slsl} y_l x_s y_l.
\end{aligned}$$

Taking the modulus of the above equation and using the triangle inequality and Lemma 2.1 yield

$$\begin{aligned}
|\lambda||x_s| \leq & \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{sjkl}||y_j||x_k||y_l| + \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{l\in[n]} |a_{slkl}||y_l||x_k||y_l| + \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{sjsl}||y_j||x_s||y_l| + \sum_{l\in[n]} |a_{slsl}||y_l||x_s||y_l| \\
\leq & \frac{1}{2} \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{sjkl}||x_t| + \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{l\in[n]} |a_{slkl}||y_l||x_t| + \frac{1}{2} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{sjsl}||x_s| + \sum_{l\in[n]} |a_{slsl}||y_l||x_s| \\
= & \frac{1}{2} \sum_{\substack{k\in[m],\\ k\neq s}} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{sjkl}||x_t| + |x_t| \sum_{\substack{k\in[m],\\ k\neq s}} \left(\sum_{l\in[n]} |a_{slkl}||y_l| \right) + \frac{1}{2} \sum_{\substack{j,l\in[n],\\ j\neq l}} |a_{sjsl}||x_s| + |x_s| \sum_{l\in[n]} |a_{slsl}||y_l|
\end{aligned}$$

$$
\begin{aligned}
\le&\frac{1}{2}\sum_{\substack{k\in[m],\\k\neq s}}\sum_{\substack{j,l\in[n],\\j\neq l}}|a_{sjkl}||x_t| + |x_t|\sum_{\substack{k\in[m],\\k\neq s}}\left(\sqrt{\sum_{l\in[n]}|a_{slkl}|^2}\sqrt{\sum_{l\in[n]}|y_l|^2}\right)\\
&+\frac{1}{2}\sum_{\substack{j,l\in[n],\\j\neq l}}|a_{sjsl}||x_s| + |x_s|\sqrt{\sum_{l\in[n]}|a_{slsl}|^2}\sqrt{\sum_{l\in[n]}|y_l|^2}\\
=&\frac{1}{2}\sum_{\substack{k\in[m],\\k\neq s}}\sum_{\substack{j,l\in[n],\\j\neq l}}|a_{sjkl}||x_t| + |x_t|\sum_{\substack{k\in[m],\\k\neq s}}\sqrt{\sum_{l\in[n]}a_{slkl}^2} + \frac{1}{2}\sum_{\substack{j,l\in[n],\\j\neq l}}|a_{sjsl}||x_s| + |x_s|\sqrt{\sum_{l\in[n]}a_{slsl}^2}\\
=&\left(\frac{1}{2}\sum_{\substack{k\in[m],\\k\neq s}}\sum_{\substack{j,l\in[n],\\j\neq l}}|a_{sjkl}| + \sum_{\substack{k\in[m],\\k\neq s}}\sqrt{\sum_{l\in[n]}a_{slkl}^2}\right)|x_t| + \left(\frac{1}{2}\sum_{\substack{j,l\in[n],\\j\neq l}}|a_{sjsl}| + \sqrt{\sum_{l\in[n]}a_{slsl}^2}\right)|x_s|\\
=&\widetilde{r}_s^s(\mathcal{A})|x_t| + r_s^s(\mathcal{A})|x_s|,
\end{aligned}
$$

i.e.,

$$
(|\lambda| - r_s^s(\mathcal{A}))|x_s| \le \widetilde{r}_s^s(\mathcal{A})|x_t|. \tag{2.2}
$$

When $|\lambda| \ge \widetilde{r}_t^s(\mathcal{A})$ or $|\lambda| \ge r_s^s(\mathcal{A})$, multiplying (2.1) and (2.2) and eliminating $|x_t||x_s| > 0$, we have

$$
(|\lambda| - \widetilde{r}_t^s(\mathcal{A}))(|\lambda| - r_s^s(\mathcal{A})) \le r_t^s(\mathcal{A})\widetilde{r}_s^s(\mathcal{A}), \tag{2.3}
$$

which implies that

$$
\lambda \in (\widetilde{\Upsilon}_{t,s}(\mathcal{A}) \cap \overline{\Upsilon}_{t,s}(\mathcal{A})). \tag{2.4}
$$

When $|\lambda| < \widetilde{r}_t^s(\mathcal{A})$ and $|\lambda| < r_s^s(\mathcal{A})$, it holds that

$$
\lambda \in \widehat{\Upsilon}_{t,s}(\mathcal{A}). \tag{2.5}
$$

It follows from (2.4) and (2.5) that

$$
\lambda \in \left[\widehat{\Upsilon}_{t,s}(\mathcal{A}) \cup (\widetilde{\Upsilon}_{t,s}(\mathcal{A}) \cap \overline{\Upsilon}_{t,s}(\mathcal{A}))\right] = \Upsilon_{t,s}(\mathcal{A}). \tag{2.6}
$$

If $|x_s| = 0$ in (2.1), then $|\lambda| \le \widetilde{r}_t^s(\mathcal{A})$. When $|\lambda| = \widetilde{r}_t^s(\mathcal{A})$, then (2.3) holds and consequently, (2.4) holds. When $|\lambda| < \widetilde{r}_t^s(\mathcal{A})$, if $|\lambda| \ge r_s^s(\mathcal{A})$, then (2.3) and (2.4) hold. If $|\lambda| < r_s^s(\mathcal{A})$, then (2.5) holds. Hence, (2.6) holds. By the arbitrariness of $s \in [m]$, and $s \neq t$, we have

$$
\lambda \in \bigcap_{t\neq s}\Upsilon_{t,s}(\mathcal{A}) \subseteq \bigcup_{t\in[m]}\bigcap_{t\neq s}\Upsilon_{t,s}(\mathcal{A}),
$$

therefore, the assertion is proved. □

Next, we give the relationship between the localization set $\mathcal{H}(\mathcal{A})$ given in Theorem 1.1 and the set $\Upsilon(\mathcal{A})$ given in Theorem 2.1.

Theorem 2.2. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. Then*

$$
\Upsilon(\mathcal{A}) \subseteq \mathcal{H}(\mathcal{A}).
$$

Proof. For any $i, s \in [m]$ and $i \neq s$, it holds that

$$\widetilde{r}_i^s(\mathcal{A}) = \frac{1}{2}\sum_{\substack{k\in[m],\\ k\neq s}}\sum_{\substack{j,l\in[n],\\ j\neq l}}|a_{ijkl}| + \sum_{\substack{k\in[m],\\ k\neq s}}\sqrt{\sum_{l\in[n]}a_{ilkl}^2} \leq \sum_{\substack{k\in[m],\\ k\neq s}}\sum_{j,l\in[n]}|a_{ijkl}| = R_i(\mathcal{A}) - R_i^s(\mathcal{A}); \tag{2.7}$$

and

$$r_i^s(\mathcal{A}) = \frac{1}{2}\sum_{\substack{j,l\in[n],\\ j\neq l}}|a_{ijsl}| + \sqrt{\sum_{l\in[n]}a_{ilsl}^2} \leq \sum_{j,l\in[n]}|a_{ijsl}| = R_i^s(\mathcal{A}). \tag{2.8}$$

Let $z \in \Upsilon(\mathcal{A})$. By Theorem 2.1, there is an index $i \in [m]$ such that for any $s \in [m]$, $i \neq s$, $z \in \Upsilon_{i,s}(\mathcal{A})$, which means that $z \in \widehat{\Upsilon}_{i,s}(\mathcal{A})$, or $z \in \widetilde{\Upsilon}_{i,s}(\mathcal{A})$ and $z \in \overline{\Upsilon}_{i,s}(\mathcal{A})$.

Let $z \in \widehat{\Upsilon}_{i,s}(\mathcal{A})$, i.e., $|z| < \widetilde{r}_i^s(\mathcal{A})$ and $|z| < r_s^s(\mathcal{A})$. By (2.7) and (2.8), we have $|z| \leq R_i(\mathcal{A}) - R_i^s(\mathcal{A})$ and $|z| \leq R_s^s(\mathcal{A})$, therefore, $z \in \widehat{\mathcal{H}}_{i,s}(\mathcal{A})$.

Let $z \in \widetilde{\Upsilon}_{i,s}(\mathcal{A})$ and $z \in \overline{\Upsilon}_{i,s}(\mathcal{A})$, i.e.,

$$(|z| - \widetilde{r}_i^s(\mathcal{A}))(|z| - r_s^s(\mathcal{A})) \leq r_i^s(\mathcal{A})\widetilde{r}_s^s(\mathcal{A}), \tag{2.9}$$

and

$$|z| < \widetilde{r}_i^s(\mathcal{A}) + r_i^s(\mathcal{A}). \tag{2.10}$$

By (2.7), (2.8) and (2.10), one has $|z| < \widetilde{r}_i^s(\mathcal{A}) + r_i^s(\mathcal{A}) \leq R_i(\mathcal{A})$, which means that $z \in \Gamma_i(\mathcal{A})$. When $|z| \geq R_i(\mathcal{A}) - R_i^s(\mathcal{A})$ and $|z| \geq R_s^s(\mathcal{A})$, by (2.7), (2.8) and (2.9), we have

$$|z| - \widetilde{r}_i^s(\mathcal{A}) \geq |z| - (R_i(\mathcal{A}) - R_i^s(\mathcal{A})) \geq 0,\ |z| - r_s^s(\mathcal{A}) \geq |z| - R_s^s(\mathcal{A}) \geq 0,$$

then

$$\begin{aligned}(|z| - (R_i(\mathcal{A}) - R_i^s(\mathcal{A})))(|z| - R_s^s(\mathcal{A})) &\leq (|z| - \widetilde{r}_i^s(\mathcal{A}))(|z| - r_s^s(\mathcal{A})) \\ &\leq r_i^s(\mathcal{A})\widetilde{r}_s^s(\mathcal{A}) \leq R_i^s(\mathcal{A})(R_s(\mathcal{A}) - R_s^s(\mathcal{A})),\end{aligned}$$

i.e.,

$$(|z| - (R_i(\mathcal{A}) - R_i^s(\mathcal{A})))(|z| - R_s^s(\mathcal{A})) \leq R_i^s(\mathcal{A})(R_s(\mathcal{A}) - R_s^s(\mathcal{A})), \tag{2.11}$$

which means that $z \in \overline{\mathcal{H}}_{i,s}(\mathcal{A})$. Thus, whether $R_i(\mathcal{A}) - R_i^s(\mathcal{A}) \leq |z| \leq R_s^s(\mathcal{A})$ or $R_s^s(\mathcal{A}) \leq |z| \leq R_i(\mathcal{A}) - R_i^s(\mathcal{A})$, (2.11) also holds. When $|z| \leq R_i(\mathcal{A}) - R_i^s(\mathcal{A})$ and $|z| \leq R_s^s(\mathcal{A})$, it follows that $z \in \widehat{\mathcal{H}}_{i,s}(\mathcal{A})$. i.e.,

$$z \in \left[\widehat{\mathcal{H}}_{i,s}(\mathcal{A}) \cup (\overline{\mathcal{H}}_{i,s}(\mathcal{A}) \cap \Gamma_i(\mathcal{A}))\right] = \mathcal{H}_{i,s}(\mathcal{A}).$$

From the arbitrariness of $s \in [m]$, and $s \neq i$, we have

$$z \in \bigcap_{s\in[m], s\neq i}\mathcal{H}_{i,s}(\mathcal{A}) \subseteq \bigcup_{i\in[m]}\bigcap_{s\in[m], s\neq i}\mathcal{H}_{i,s}(\mathcal{A}),$$

i.e., $z \in \mathcal{H}(\mathcal{A})$. Therefore, $\Upsilon(\mathcal{A}) \subseteq \mathcal{H}(\mathcal{A})$. □

In order to show the validity of the set $\Upsilon(\mathcal{A})$ given in Theorem 2.1, we present a running example.

Example 1. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[2]\times[2]\times[2]\times[2]}$ be a partially symmetric tensor with entries*

$$\begin{aligned} &a_{1111} = 1,\ a_{1112} = 2,\ a_{1121} = 2,\ a_{1212} = 3,\\ &a_{1222} = 5,\ a_{1211} = 2,\ a_{1122} = 4,\ a_{1221} = 4,\\ &a_{2111} = 2,\ a_{2112} = 4,\ a_{2121} = 3,\ a_{2122} = 5,\\ &a_{2211} = 4,\ a_{2212} = 5,\ a_{2221} = 5,\ a_{2222} = 6. \end{aligned}$$

By Theorem 1.1, we have

$$\mathcal{H}(\mathcal{A}) = \bigcup_{i\in[m]} \bigcap_{k\in[m],k\neq i} \mathcal{H}_{i,k}(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 29.4765\}.$$

By Theorem 2.1, we have

$$\Upsilon(\mathcal{A}) = \bigcup_{i\in[m]} \bigcap_{s\in[m],s\neq i} \Upsilon_{i,s}(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 20.0035\}.$$

It is easy to see that $\Upsilon(\mathcal{A}) \subseteq \mathcal{H}(\mathcal{A})$ and all M-eigenvalues are in $[-20.0035, 20.0035]$. In fact, all different M-eigenvalues of $\mathcal{A}$ are -1.2765, 0.0710, 0.1242, 0.2765, 0.3437 and 15.2091.

3. A sharp upper bound for the M-spectral radius of a partially symmetric tensor

In this section, based on the set in Theorem 2.1, we provide an upper bound for the largest M-eigenvalue of a fourth-order partially symmetric tensor $\mathcal{A}$. As an application, we apply the upper bound as a parameter τ to the WQZ-algorithm to make the sequence generated by the WQZ-algorithm converges to the largest M-eigenvalue of $\mathcal{A}$ faster.

Theorem 3.1. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. Then*

$$\rho(\mathcal{A}) \leq \Omega(\mathcal{A}) = \max_{i\in[m]} \min_{s\in[m],i\neq s} \Omega_{i,s}(\mathcal{A}),$$

where

$$\Omega_{i,s}(\mathcal{A}) = \max\left\{\min\{\widetilde{r}_i^s(\mathcal{A}), r_s^s(\mathcal{A})\}, \min\{\widetilde{r}_i^s(\mathcal{A}) + r_i^s(\mathcal{A}), \widehat{\Omega}_{i,s}(\mathcal{A})\}\right\},$$

and

$$\widehat{\Omega}_{i,s}(\mathcal{A}) = \frac{1}{2}\left\{\widetilde{r}_i^s(\mathcal{A}) + r_s^s(\mathcal{A}) + \sqrt{(r_s^s(\mathcal{A}) - \widetilde{r}_i^s(\mathcal{A}))^2 + 4r_i^s(\mathcal{A})\widetilde{r}_s^s(\mathcal{A})}\right\}.$$

Proof. By Theorem 2.1 and $\rho(\mathcal{A}) \in \sigma(\mathcal{A})$, it follows that there exists an index $i \in [m]$ such that for any $s \in [m]$ and $s \neq i$, $\rho(\mathcal{A}) \in \widehat{\Upsilon}_{i,s}(\mathcal{A})$, or $\rho(\mathcal{A}) \in (\widetilde{\Upsilon}_{i,s}(\mathcal{A}) \cap \overline{\Upsilon}_{i,s}(\mathcal{A}))$. If $\rho(\mathcal{A}) \in \widehat{\Upsilon}_{i,s}(\mathcal{A})$, that is, $\rho(\mathcal{A}) < \widetilde{r}_i^s(\mathcal{A})$ and $\rho(\mathcal{A}) < r_s^s(\mathcal{A})$, then

$$\rho(\mathcal{A}) < \min\{\widetilde{r}_i^s(\mathcal{A}), r_s^s(\mathcal{A})\}. \tag{3.1}$$

If $\rho(\mathcal{A}) \in (\widetilde{\Upsilon}_{i,s}(\mathcal{A}) \cap \overline{\Upsilon}_{i,s}(\mathcal{A}))$, that is,

$$\rho(\mathcal{A}) < \widetilde{r}_i^s(\mathcal{A}) + r_i^s(\mathcal{A}) < \min\{\widetilde{r}_i^s(\mathcal{A}) + r_i^s(\mathcal{A})\}, \tag{3.2}$$

and

$$(\rho(\mathcal{A}) - \widetilde{r}_i^s(\mathcal{A}))(\rho(\mathcal{A}) - r_s^s(\mathcal{A})) \leq r_i^s(\mathcal{A})\widetilde{r}_s^s(\mathcal{A}). \tag{3.3}$$

Solving Inequality (3.3), we have

$$\rho(\mathcal{A}) \leq \widehat{\Omega}_{i,s}(\mathcal{A}) \leq \min\{\widehat{\Omega}_{i,s}(\mathcal{A})\}. \tag{3.4}$$

Combining (3.2) and (3.4), we have

$$\rho(\mathcal{A}) \leq \min\{\widetilde{r}_i^s(\mathcal{A}) + r_i^s(\mathcal{A}), \widehat{\Omega}_{i,s}(\mathcal{A})\}. \tag{3.5}$$

Hence, by (3.1) and (3.5), we have

$$\rho(\mathcal{A}) \leq \max\left\{\min\{\widetilde{r}_i^s(\mathcal{A}), r_s^s(\mathcal{A})\}, \min\{\widetilde{r}_i^s(\mathcal{A}) + r_i^s(\mathcal{A}), \widehat{\Omega}_{i,s}(\mathcal{A})\}\right\} = \Omega_{i,s}(\mathcal{A}).$$

Furthermore, by the arbitrariness of s, we have

$$\rho(\mathcal{A}) \leq \min_{s\in[m], i\neq s} \Omega_{i,s}(\mathcal{A}).$$

Since we do not know which i is appropriate to $\rho(\mathcal{A})$, we can only conclude that

$$\rho(\mathcal{A}) \leq \max_{i\in[m]} \min_{s\in[m], i\neq s} \Omega_{i,s}(\mathcal{A}).$$

This proof is complete. □

Remark 3.1. *In Theorem 3.1, we obtain an upper bound $\Omega(\mathcal{A})$ for the largest M-eigenvalue of a fourth order partially symmetric tensor $\mathcal{A}$. Now, we take $\Omega(\mathcal{A})$ as the parameter τ in WQZ-algorithm to obtain a modified WQZ-algorithm. That is, the only difference between WQZ-algorithm and the modified WQZ-algorithm is the selection of τ, in particular, $\tau = \sum\limits_{1\leq s\leq t\leq mn} |A_{st}|$ in WQZ-algorithm and $\tau = \Omega(\mathcal{A})$ in the modified WQZ-algorithm.*

Next, we take $\Omega(\mathcal{A})$ and some existing upper bounds of the largest M-eigenvalue as τ in WQZ-algorithm to calculate the largest M-eigenvalue of a fourth-order partially symmetric tensor $\mathcal{A}$.

Example 2. Consider the tensor $\mathcal{A}$ in Example 4.1 of [24], where

$$\mathcal{A}(:,:,1,1) = \begin{bmatrix} -0.9727 & 0.3169 & -0.3437 \\ -0.6332 & -0.7866 & 0.4257 \\ -0.3350 & -0.9896 & -0.4323 \end{bmatrix},$$

$$\mathcal{A}(:,:,2,1) = \begin{bmatrix} -0.6332 & -0.7866 & 0.4257 \\ 0.7387 & 0.6873 & -0.3248 \\ -0.7986 & -0.5988 & -0.9485 \end{bmatrix},$$

$$\mathcal{A}(:,:,3,1)=\begin{bmatrix} -0.3350 & -0.9896 & -0.4323 \\ -0.7986 & -0.5988 & -0.9485 \\ 0.5853 & 0.5921 & 0.6301 \end{bmatrix},$$

$$\mathcal{A}(:,:,1,2)=\begin{bmatrix} 0.3169 & 0.6158 & -0.0184 \\ -0.7866 & 0.0160 & 0.0085 \\ -0.9896 & -0.6663 & 0.2559 \end{bmatrix},$$

$$\mathcal{A}(:,:,2,2)=\begin{bmatrix} -0.7866 & 0.0160 & 0.0085 \\ 0.6873 & 0.5160 & -0.0216 \\ -0.5988 & 0.0411 & 0.9857 \end{bmatrix},$$

$$\mathcal{A}(:,:,3,2)=\begin{bmatrix} -0.9896 & -0.6663 & 0.2559 \\ -0.5988 & 0.0411 & 0.9857 \\ 0.5921 & -0.2907 & -0.3881 \end{bmatrix},$$

$$\mathcal{A}(:,:,1,3)=\begin{bmatrix} -0.3437 & -0.0184 & 0.5649 \\ 0.4257 & 0.0085 & -0.1439 \\ -0.4323 & 0.2559 & 0.6162 \end{bmatrix},$$

$$\mathcal{A}(:,:,2,3)=\begin{bmatrix} 0.4257 & 0.0085 & -0.1439 \\ -0.3248 & -0.0216 & -0.0037 \\ -0.9485 & 0.9857 & -0.7734 \end{bmatrix},$$

$$\mathcal{A}(:,:,3,3)=\begin{bmatrix} -0.4323 & 0.2559 & 0.6162 \\ -0.9485 & 0.9857 & -0.7734 \\ 0.6301 & -0.3881 & -0.8526 \end{bmatrix}.$$

By (1.3), we have $\tau = \sum_{1\le s\le t\le 9} |A_{st}| = 23.3503$. By Corollary 1 of [17], we have

$$\rho(\mathcal{A}) \le 16.6014.$$

By Theorem 3.5 of [23], we have

$$\rho(\mathcal{A}) \le 15.4102.$$

By Corollary 2 of [17], we have

$$\rho(\mathcal{A}) \le 14.5910.$$

By Corollary 1 of [15], where $S_m = S_n = 1$, we have

$$\rho(\mathcal{A}) \le 13.8844.$$

By Corollary 2 of [15], where $S_m = S_n = 1$, we have

$$\rho(\mathcal{A}) \le 11.7253.$$

By Theorem 3.1, we have

$$\rho(\mathcal{A}) \le 8.2342.$$

From [24], it can be seen that $\lambda_{\max}(\mathcal{A}) = 2.3227$.

Taking $\tau = 23.3503$, 16.6014, 15.4102, 14.5910, 13.8844, 11.7253 and 8.2342 respectively, numerical results obtained by the WQZ-algorithm are shown in Figure 1.

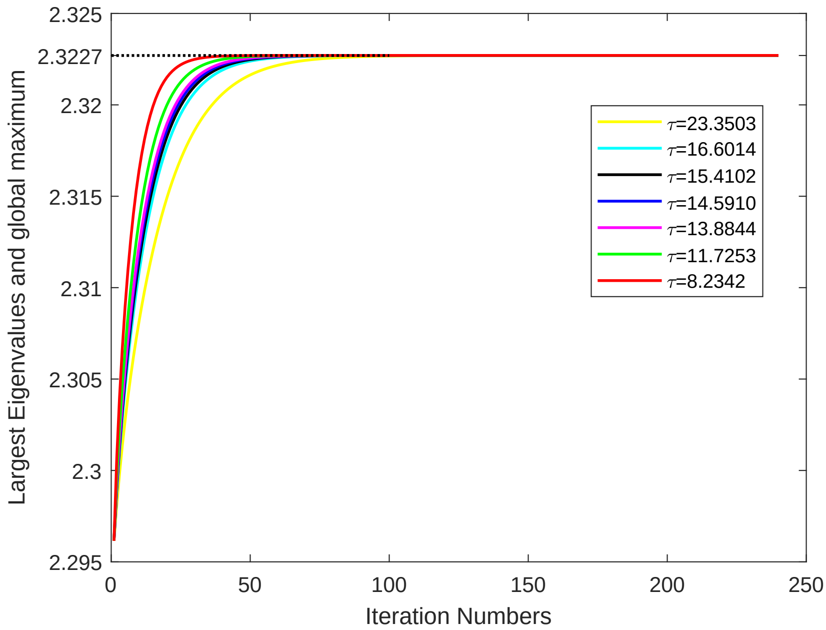

Figure 1. Numerical results for the WQZ-algorithm with different τ.

Numerical results in Figure 1 shows that :

1) When we take $\tau = 8.2342$, the sequence more rapidly converges to the largest M-eigenvalue $\lambda_{\max}(\mathcal{A})$ than taking $\tau = 23.3503$, $\tau = 16.6014$, $\tau = 15.4102$, $\tau = 14.5910$, $\tau = 13.8844$ and $\tau = 11.7253$, respectively.

2) When we take $\tau = 23.3503$, 16.6014, 15.4102, 14.5910, 13.8844, 11.7253 and 8.2342, the WQZ-algorithm can get the largest M-eigenvalue $\lambda_{\max}(\mathcal{A})$ after finite iterations. However, under the same stopping criterion, if we take $\tau = 23.3503$, 16.6014, 15.4102, 14.5910, 13.8844 and 11.7253, it can be seen that the WQZ-algorithm needs more iterations to obtain the largest M-eigenvalue, and when $\tau = 8.2342$, WQZ-algorithm can obtain the largest M-eigenvalue $\lambda_{\max}(\mathcal{A})$ faster.

3) The choice of the parameter τ in WQZ-algorithm has a significant impact on the convergence speed of the WQZ-algorithm. When τ is larger, the convergence speed of WQZ-algorithm is slower. When τ is smaller and τ is greater than the largest M-eigenvalue, the WQZ-algorithm converges faster. In other words, the faster the largest M-eigenvalue can be obtained.

4) The numerical result of the upper bound of the M-spectral radius obtained by Theorem 3.1 is of great help to the WQZ-algorithm. Therefore, it shows that the results we get have a certain effect. □

Now, we consider a real elasticity tensor, which is derived from the study of self-anisotropic materials [10] for explanation.

In anisotropy materials, the components of the tensor of elastic moduli $C = (c_{ijkl}) \in \mathbb{R}^{[3]\times[3]\times[3]\times[3]}$ satisfy the following symmetry:

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{jilk}, \quad c_{ijkl} = c_{klij}, \quad \forall\ 1 \le i, j, k, l \le 3,$$

which is also called an elasticity tensor. After a lot of research, we know that there are many anisotropic materials, of which crystal is one of its typical examples. We classify from the crystal homologues [22], the elasticity tensor $C = (c_{ijkl}) \in \mathbb{R}^{[3]\times[3]\times[3]\times[3]}$ of some crystals for trigonal system, such as $CaCO_3$ and HgS also satisfy

$$\begin{aligned}
&c_{1112} = c_{2212} = c_{3323} = c_{3331} = c_{3312} = c_{2331} = 0,\\
&c_{2222} = c_{1111},\ c_{3131} = c_{2323},\ c_{2233} = c_{1133},\ c_{2223} = -c_{1123},\\
&c_{2231} = -c_{1131},\ c_{3112} = \sqrt{2}c_{1123},\ c_{2312} = -\sqrt{2}c_{1131},\ c_{1212} = c_{1111} - c_{1122}.
\end{aligned}$$

This shows that the triangular system of anisotropic materials has only 7 elasticities. In fact, $CaMg(CO_3)_2$-dolomite and $CaCO_3$-calcite have similar crystal structures, in which the atoms along any triplet are alternated with magnesium and calcium. In [22], we can know that the elasticity tensor of $CaMg(CO_3)_2$-dolomite is as follows.

$$\begin{aligned}
&c_{2222} = c_{1111} = 196.6,\ c_{3131} = c_{2323} = 83.2,\ c_{2233} = c_{1133} = 54.7,\ c_{2223} = -c_{1123} = 31.7,\\
&c_{2231} = -c_{1131} = -25.3,\ c_{3112} = 44.8,\ c_{2312} = -35.84,\ c_{1212} = 132.2,\ c_{3333} = 110,\\
&c_{1122} = 64.4.
\end{aligned}$$

Next, we transform the elastic tensor C into a partially symmetric tensor $\mathcal{A}$ through the following double mapping, and the M-eigenvalue of $\mathcal{A}$ after transformation is the same as the M-eigenvalue of C [7, 12]:

$$a_{ijkl} = a_{ikjl},\quad 1 \le i, j, k, l \le 3.$$

In order to illustrate the validity of the results we obtained, we take the above-mentioned partial symmetry tensor of the $CaMg(CO_3)_2$-dolomite elasticity tensor transformation as an example.

Example 3. Consider the tensor $\mathcal{A}_2 = (a_{ijkl}) \in \mathbb{R}^{[3]\times[3]\times[3]\times[3]}$ in Example 3 of [17], where

$$\begin{aligned}
&a_{2222} = a_{1111} = 196.6,\ a_{3311} = a_{2233} = 83.2,\ a_{2323} = a_{3232} = a_{1313} = a_{3131} = 54.7,\\
&a_{2223} = a_{2232} = -a_{1213} = -a_{2131} = -31.7,\ a_{3333} = 110,\ a_{1212} = a_{2121} = 64.4,\\
&a_{1122} = 132.2,\ a_{2321} = a_{1232} = -a_{1311} = -a_{1131} = -25.3,\ a_{3112} = a_{1321} = 44.8,\\
&a_{2132} = a_{1223} = -35.84,
\end{aligned}$$

and other $a_{ijkl} = 0$.

The data results of Example 2 show that the upper bound of the largest M-eigenvalue in Theorem 3.1 is sharper than the existing results. Here, we only calculate the upper bound of the largest M-eigenvalue of $\mathcal{A}_2$ by Theorem 3.1, and use it as the parameter τ in the WQZ-algorithm to calculate the largest M-eigenvalue of $\mathcal{A}_2$. Here, in order to distinguish different values of τ, we calculate the result by Theorem 3.1 and record it as τ_2, that is, WQZ-algorithm $\tau = \tau_2$.

By Theorem 3.1, we can get $\tau_2 = 647.6100$.

By Eq (1.3), we can get

$$\tau = \sum_{1\le s\le t\le 9} |A_{st}| = 1998.6000.$$

In the WQZ-algorithm, when we take $\tau = 1998.6000$ and 647.6100 respectively, the numerical results we get are shown in Figure 2.

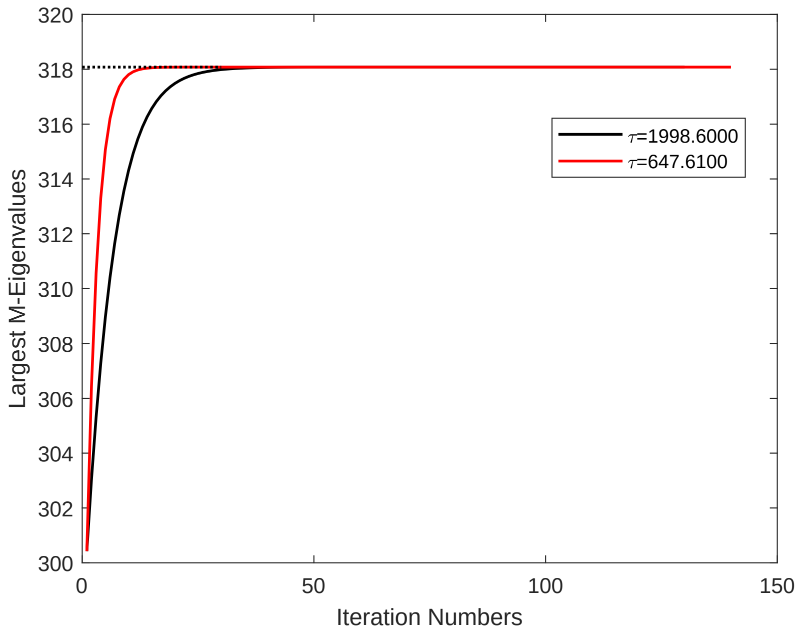

Figure 2. Numerical results for the WQZ-algorithm with different τ.

As we can see in Figure 2, in the WQZ-algorithm, when we regard τ_2 as τ, it makes the convergence sequence in the WQZ-algorithm converges faster than $\tau = \sum_{1\leq s\leq t\leq 9} |A_{st}|$, so that the largest M-eigenvalue can be calculated faster. That is to say, in this article, the result we provide as the parameter τ in the WQZ-algorithm can speed up the convergence speed, so that the largest M-eigenvalue can be calculated quickly. □

4. Conclusions

In this paper, we first in Theorem 2.1 provided an M-eigenvalue localization set $\Upsilon(\mathcal{A})$ for a fourth-order partially symmetric tensor $\mathcal{A}$, and then proven that the set $\Upsilon(\mathcal{A})$ is tighter than the set $\mathcal{H}(\mathcal{A})$ in Theorem 2.2 of [23]. Secondly, based on the set $\Upsilon(\mathcal{A})$, we derived an upper bound for the M-spectral radius of $\mathcal{A}$. As an application, we took the upper bound of the M-spectral radius as a parameter τ in the WQZ-algorithm to make the sequence generated by this algorithm converge to the largest M-eigenvalue of $\mathcal{A}$ faster. Finally, two numerical examples are given to show the effectiveness of the set $\Upsilon(\mathcal{A})$ and the upper bound $\Omega(\mathcal{A})$.

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Conflict of interest

The author declares no conflict of interest.

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