



Research article

Ulam-Hyers stability for conformable fractional integro-differential impulsive equations with the antiperiodic boundary conditions

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Abstract: This paper focuses on the stability for a class of conformable fractional impulsive integro-differential equations with the antiperiodic boundary conditions. Firstly, the existence and uniqueness of solutions of the integro-differential equations are studied by using the fixed point theorem under the condition of nonlinear term increasing at most linearly. And then, the Ulam-Hyers stability and Ulam-Hyers-Rassias stability for the boundary value problems are discussed by using the nonlinear functional analysis method and constraining related parameters. Finally, an example is given out to illustrate the applicability and feasibility of our main conclusions. It is worth mentioning that the stability studied in this paper highlights the role of boundary conditions. This method of studying stability is effective and can be applied to the study of stability for many types of differential equations.

Keywords: conformable fractional derivative; integro-differential impulsive equations; antiperiodic boundary conditions; existence and uniqueness; Ulam-Hyers stability; Ulam-Hyers-Rassias stability

Mathematics Subject Classification: 34A08, 34B37, 34D20

1. Introduction

In this paper, we study the solvability and stability of the following conformable fractional integro-differential impulsive equations with antiperiodic boundary conditions:

$$\begin{cases} T_{t_k}^\alpha x(t) = f(t, x(t), Ax(t)), & t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \dots, m, \\ \Delta x(t)|_{t=t_k} = P_k(x(t_k)), & k = 1, 2, \dots, m, \\ \Delta x'(t)|_{t=t_k} = Q_k(x(t_k)), & k = 1, 2, \dots, m, \\ x(0) = -x(1), \quad T_0^\beta x(0) = -T_m^\beta x(1), \end{cases} \quad (1.1)$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, T_a^α is the conformable fractional derivatives of order α starting from a , $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$, $J = [0, 1]$, $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $P_k, Q_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \dots, m$,

$\Delta x(t)|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $\Delta x'(t)|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$, where $x(t_k^+)$, $x'(t_k^+)$ and $x(t_k^-)$, $x'(t_k^-)$ represent the right and left limits of $x(t)$ and $x'(t)$ at $t = t_k$ ($k = 1, 2, \dots, m$), respectively. A is an integral operator as $Ax(t) = \int_0^t h(s, x(s))ds$, $h \in C(J \times \mathbb{R}, \mathbb{R})$ is a given function.

As a generalization of integer order calculus, fractional calculus could be used to describe and simulate practical phenomena more accurately than integer order calculus. As a result, fractional differential equations are widely used to solve the problems in the fields of science and engineering, such as materials and mechanical systems, thermal and optical systems, control engineering and theory, signal processing and system identification and so on, see [1–9]. In [10], Khalil proposed a local limit-based operator which was a natural extension of general derivative and retained the basic properties of the classical derivative, and it could be called the conformable fractional derivative. In [11], Abdeljawad introduced the chain rule, mean value theorem, Grönwall inequality, exponential function and Laplace transform about the conformable fractional derivatives. Since the conformable fractional derivative has some properties that the Riemann-Liouville derivative and Caputo derivative do not have, it could be used to describe some special phenomena frequently. In some cases and conditions, the conformable fractional derivative can better establish the principle of action in some practical problems than the classical fractional derivative, and the derived formula is much simpler, see [12]. For more applications of conformable fractional derivative, see [13–20] and the references therein.

In many continuous gradual processes, the system abrupt state changes due to disturbances or external influences in some times, and this phenomenon is called impulsive effect. As the impulsive differential equations describing the abrupt phenomenon have played an important role in electronic technology and communication engineering, they have become an important object of researches in recent years. As a branch of pulse theory, the instantaneous pulse theory has also received extensive attention, see [21–26] and the references therein.

In [21], Ahmad et al. studied the following hybrid systems of non-linear conformable fractional impulsive differential equations with Dirichlet boundary conditions:

$$\begin{cases} T_\alpha^{t_k} x(t) = f(t, x(t)), & 1 < \alpha \leq 2, \quad k = 0, 1, 2, \dots, p, \quad t \in J', \\ \Delta x(t_k) = S_k(x(t_k)), \quad \Delta x'(t_k) = S_k^*(x(t_k)), & k = 1, 2, \dots, p, \\ x(0) = 0, \quad x(T) = 0, \end{cases}$$

where $T_\alpha^{t_k}$ is the conformable fractional derivative of order $1 < \alpha \leq 2$ starting from t_k , $J' = [0, T] \setminus \{t_i\}$. By applying Krasnoselskii's fixed point theorem and contraction mapping principle, the existence and uniqueness of solutions of the system are obtained.

The stability theory plays a more and more important role in control engineering and theory, error analysis and other fields. In [27], Agarwal et al. studied Mittag-Leffler stability for a class of impulsive Caputo fractional differential equations. The researches on Ulam stability could be traced back to 1940s, and till now, Ulam-Hyers (U-H) stability and Ulam-Hyers-Rassias (U-H-R) stability have become one of the most active research topics in differential equations, and have achieved a large number of research results, see [28–39] etc.

In [31], Li et al. studied the following conformable fractional order nonlinear differential equations with constant coefficients:

$$\begin{cases} D_\beta^\alpha y(x) = \lambda y(x) + g(x, y(x)), & x \in (a, b] \text{ or } (a, \infty), \quad 0 < \beta < 1, \\ y(a) = y_a, \end{cases}$$

where $D_{\beta}^{\alpha}y$ is the conformable fractional derivative of order β of the function y starting from a . By using the constant variation method, the existence of solutions for the given problem are obtained. And the results of U-H stability and U-H-R stability on finite time interval and infinite time interval are given out.

As far as we know, the studies on conformable fractional differential impulsive equations are basically carried out around existence and uniqueness under homogeneous boundary conditions, and the studies on Ulam-Hyers stability are carried out under initial value conditions. Based on this, we consider that in some cases the system may involve some averaging or accumulation, for which we add integral terms. In order to enrich the research in this field, we study the solvability and stability of (1.1). The existence, uniqueness, U-H stability and U-H-R stability for solutions of (1.1) are obtained. It is worth mentioning that the stability studied in this paper is related to the boundary conditions.

The rest of the paper is organized as follows. In the second section, we introduce the symbols, spaces, definitions of conformable fractional derivatives and integrals, necessary lemmas and theorems to prove the main results, and the definitions of U-H stability and U-H-R stability. In the third section, we establish an integral equation equivalent to conformable fractional integro-differential impulsive equation, and obtain the existence and uniqueness of solutions of (1.1) by using Schauder fixed point theorem and Banach fixed point theorem. In the fourth section, we use the method of nonlinear functional analysis to discuss the U-H stability of (1.1), and the results for U-H stability and U-H-R stability are obtained. In the fifth section, an example is given out to illustrate the applicability and feasibility of our main conclusions.

2. Preliminaries

In this section, we introduce some symbols, spaces, definitions and necessary lemmas to prove the main results.

We denote $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, $J_2 = (t_2, t_3]$, \dots , $J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, 1]$, $J' = J \setminus \{t_1, t_2, \dots, t_m\}$, and

$$\lambda = \max_{1 \leq i \leq m+1} \{t_i - t_{i-1}\}.$$

Let

$$PC(J, \mathbb{R}) = \{x : J \rightarrow \mathbb{R} \mid x \in C(J', \mathbb{R}), x(t_k^+) \text{ and } x(t_k^-) \text{ exist, } x(t_k^-) = x(t_k), k = 1, 2, \dots, m\},$$

and with the norm $\|x\|_{PC} = \sup_{t \in J} |x(t)|$. Then $PC(J, \mathbb{R})$ is a Banach space.

Definition 2.1. (see [10]) Let $\gamma \in (n, n+1]$, $u : [a, +\infty) \rightarrow \mathbb{R}$, for any $t > a$, u is differentiable of order n , then the conformable fractional derivative of γ order of function u at $t > a$ is defined as

$$T_a^{\gamma}u(t) = \lim_{\varepsilon \rightarrow 0} \frac{u^{(n)}(t + \varepsilon(t-a)^{n+1-\gamma}) - u^{(n)}(t)}{\varepsilon}.$$

Remark 2.1. Given $\gamma \in (0, 1]$, if u is differentiable, then $T_a^{\gamma}u(t) = (t-a)^{1-\gamma}u'(t)$. If $\gamma \in (n, n+1]$, $k = 0, 1, 2, \dots, n$, then $T_a^{\gamma}(t-a)^k = 0$.

Definition 2.2. (see [11]) Let $\gamma \in (n, n + 1]$, $u : [a, +\infty) \rightarrow \mathbb{R}$, then the conformable fractional integral of γ order of function u at $t > a$ is defined as

$$I_a^\gamma u(t) = \frac{1}{n!} \int_a^t (t-s)^n (s-a)^{\gamma-n-1} u(s) ds.$$

Lemma 2.1. (see [11]) Let $\gamma \in (n, n + 1]$, $u : [a, \infty) \rightarrow \mathbb{R}$ such that $u^{(n)}(t)$ is continuous, then for all $t > a$, we have

$$T_a^\gamma I_a^\gamma u(t) = u(t).$$

Lemma 2.2. (see [11]) Let $\gamma \in (n, n + 1]$ and $u : [a, \infty) \rightarrow \mathbb{R}$ be $(n + 1)$ times differentiable for $t > a$, then for all $t > a$, we have

$$I_a^\gamma T_a^\gamma u(t) = u(t) - \sum_{k=0}^n \frac{u^{(k)}(a)(t-a)^k}{k!}.$$

Theorem 2.1 (Schauder fixed point theorem). (see [40]) Let X be a non-empty bounded closed convex set in Banach space E and $K : X \rightarrow X$ be a completely continuous operator, then K has a fixed point in X .

Theorem 2.2 (Banach fixed point theorem). (see [41]) Let X be a non-empty closed convex subset of a Banach space E and $S : X \rightarrow X$ be a contraction operator. Then there is a unique $u^* \in X$ with $Su^* = u^*$.

In the following, we will give the definitions of U-H stability and U-H-R stability for boundary value problem of the conformable fractional impulsive integro-differential Eq (1.1).

Definition 2.3. Boundary value problem (1.1) is said to be U-H stable, if there exists a constant $c_{f,m} > 0$ such that for any $\varepsilon > 0$ and any solution $z \in PC(J, \mathbb{R})$ of the following inequalities system

$$\begin{cases} |T_{t_k}^\alpha z(t) - f(t, z(t), Az(t))| \leq \varepsilon, & t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \dots, m, \\ |\Delta z(t)|_{t=t_k} - P_k(z(t_k))| \leq \varepsilon, & k = 1, 2, \dots, m, \\ |\Delta z'(t)|_{t=t_k} - Q_k(z(t_k))| \leq \varepsilon, & k = 1, 2, \dots, m, \\ z(0) = -z(1), \quad T_0^\beta z(0) = -T_{t_m}^\beta z(1), \end{cases} \quad (2.1)$$

there exists a unique solution $x \in PC(J, \mathbb{R})$ of boundary value problem (1.1) with

$$\|z - x\|_{PC} \leq c_{f,m} \varepsilon.$$

Definition 2.4. Boundary value problem (1.1) is said to be U-H-R stable, if there exist a continuous function $g : J \rightarrow (0, +\infty)$ and constants $\varphi, c_{f,m,g} > 0$ such that for any $\varepsilon > 0$ and any solution $z \in PC(J, \mathbb{R})$ of the following inequalities system

$$\begin{cases} |T_{t_k}^\alpha z(t) - f(t, z(t), Az(t))| \leq \varepsilon g(t), & t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \dots, m, \\ |\Delta z(t)|_{t=t_k} - P_k(z(t_k))| \leq \varepsilon \varphi, & k = 1, 2, \dots, m, \\ |\Delta z'(t)|_{t=t_k} - Q_k(z(t_k))| \leq \varepsilon \varphi, & k = 1, 2, \dots, m, \\ z(0) = -z(1), \quad T_0^\beta z(0) = -T_{t_m}^\beta z(1), \end{cases} \quad (2.2)$$

there exists a unique solution $x \in PC(J, \mathbb{R})$ of boundary value problem (1.1) with

$$\|z - x\|_{PC} \leq c_{f,m,g} \varepsilon (g(t) + \varphi).$$

Remark 2.2. A function $z \in PC(J, \mathbb{R})$ is a solution of inequalities (2.1) if and only if there exist a function $\phi \in PC(J, \mathbb{R})$ and sequences $\phi_k, \tilde{\phi}_k, k = 1, 2, \dots, m$, such that

- (1) $|\phi(t)| \leq \varepsilon, |\phi_k| \leq \varepsilon, |\tilde{\phi}_k| \leq \varepsilon, t \in J, k = 1, 2, \dots, m;$
- (2) $T_{t_k}^\alpha z(t) = f(t, z(t), Az(t)) + \phi(t), t \in (t_k, t_{k+1}), k = 0, 1, 2, \dots, m;$
- (3) $\Delta z(t)|_{t=t_k} = P_k(z(t_k)) + \phi_k, k = 1, 2, \dots, m;$
- (4) $\Delta z'(t)|_{t=t_k} = Q_k(z(t_k)) + \tilde{\phi}_k, k = 1, 2, \dots, m;$
- (5) $z(0) = -z(1), T_0^\beta z(0) = -T_{t_m}^\beta z(1).$

Remark 2.3. A function $z \in PC(J, \mathbb{R})$ is a solution of inequalities (2.2) if and only if there exist a function $\phi \in PC(J, \mathbb{R})$ and sequences $\phi_k, \tilde{\phi}_k, k = 1, 2, \dots, m$, such that

- (1) $|\phi(t)| \leq \varepsilon g(t), |\phi_k| \leq \varepsilon \varphi, |\tilde{\phi}_k| \leq \varepsilon \varphi, t \in J, k = 1, 2, \dots, m;$
- (2) $T_{t_k}^\alpha z(t) = f(t, z(t), Az(t)) + \phi(t), t \in (t_k, t_{k+1}), k = 0, 1, 2, \dots, m;$
- (3) $\Delta z(t)|_{t=t_k} = P_k(z(t_k)) + \phi_k, k = 1, 2, \dots, m;$
- (4) $\Delta z'(t)|_{t=t_k} = Q_k(z(t_k)) + \tilde{\phi}_k, k = 1, 2, \dots, m;$
- (5) $z(0) = -z(1), T_0^\beta z(0) = -T_{t_m}^\beta z(1).$

3. Existence and uniqueness of solutions

In this section, we obtain the existence and uniqueness for the solution of boundary value problem of the conformable fractional impulsive integro-differential Eq (1.1).

Lemma 3.1. Let $y \in PC(J, \mathbb{R}), p_k, q_k \in \mathbb{R}$, then the following boundary value problem of linear conformable fractional differential impulsive equation

$$\begin{cases} T_{t_k}^\alpha x(t) = y(t), & t \in (t_k, t_{k+1}), k = 0, 1, 2, \dots, m, \\ \Delta x(t)|_{t=t_k} = p_k, & k = 1, 2, \dots, m, \\ \Delta x'(t)|_{t=t_k} = q_k, & k = 1, 2, \dots, m, \\ x(0) = -x(1), & T_0^\beta x(0) = -T_{t_m}^\beta x(1) \end{cases} \quad (3.1)$$

has a unique solution

$$x(t) = \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} G(t, s)(s - t_{i-1})^{\alpha-2} y(s) ds + \frac{1}{2} \sum_{i=1}^m \chi(t, t_i) p_i + \sum_{i=1}^m G(t, t_i) q_i, t \in J_k, k = 0, 1, \dots, m,$$

where

$$G(t, s) = \begin{cases} -\frac{s}{2}, & t > s, \\ \frac{s}{2} - t, & t \leq s, \end{cases} \quad \chi(t, s) = \begin{cases} 1, & t > s, \\ -1, & t \leq s. \end{cases}$$

Proof. Assuming that $x = x(t)$ is the solution of the problem (3.1), then there exist constants $c_{0,0}, c_{0,1} \in \mathbb{R}$, for any $t \in J_0$,

$$x(t) = \int_0^t (t-s)s^{\alpha-2} y(s) ds + c_{0,0} + c_{0,1} t, \quad (3.2)$$

$$x'(t) = \int_0^t s^{\alpha-2} y(s) ds + c_{0,1}. \quad (3.3)$$

Similarly, there exist constants $c_{1,0}, c_{1,1} \in \mathbb{R}$, for any $t \in J_1$,

$$x(t) = \int_{t_1}^t (t-s)(s-t_1)^{\alpha-2} y(s) ds + c_{1,0} + c_{1,1}(t-t_1), \quad (3.4)$$

$$x'(t) = \int_{t_1}^t (s-t_1)^{\alpha-2} y(s) ds + c_{1,1}. \quad (3.5)$$

From (3.2)–(3.5), we have

$$\begin{aligned} x(t_1^-) &= \int_0^{t_1} (t_1-s)s^{\alpha-2} y(s) ds + c_{0,0} + c_{0,1}t_1, & x(t_1^+) &= c_{1,0}, \\ x'(t_1^-) &= \int_0^{t_1} s^{\alpha-2} y(s) ds + c_{0,1}, & x'(t_1^+) &= c_{1,1}. \end{aligned}$$

From impulsive conditions $\Delta x(t)|_{t=t_1} = p_1$, $\Delta x'(t)|_{t=t_1} = q_1$, we have

$$c_{1,0} = \int_0^{t_1} (t_1-s)s^{\alpha-2} y(s) ds + c_{0,0} + c_{0,1}t_1 + p_1, \quad c_{1,1} = \int_0^{t_1} s^{\alpha-2} y(s) ds + c_{0,1} + q_1.$$

So, for any $t \in J_1$, we have

$$x(t) = \int_{t_1}^t (t-s)(s-t_1)^{\alpha-2} y(s) ds + \int_0^{t_1} (t-s)s^{\alpha-2} y(s) ds + p_1 + (t-t_1)q_1 + c_{0,0} + c_{0,1}t,$$

and

$$x'(t) = \int_{t_1}^t (s-t_1)^{\alpha-2} y(s) ds + \int_0^{t_1} s^{\alpha-2} y(s) ds + q_1 + c_{0,1}.$$

Repeat the above process, for any $t \in J_k$, $k = 1, 2, \dots, m$, and we get

$$\begin{aligned} x(t) &= \int_{t_k}^t (t-s)(s-t_k)^{\alpha-2} y(s) ds + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t-s)(s-t_{i-1})^{\alpha-2} y(s) ds + \sum_{i=1}^k p_i \\ &\quad + \sum_{i=1}^k (t-t_i)q_i + c_{0,0} + c_{0,1}t, \end{aligned}$$

and

$$T_{t_k}^\beta x(t) = (t-t_k)^{1-\beta} \left(\int_{t_k}^t (s-t_k)^{\alpha-2} y(s) ds + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (s-t_{i-1})^{\alpha-2} y(s) ds + \sum_{i=1}^k q_i + c_{0,1} \right).$$

By boundary conditions $x(0) = -x(1)$, $T_0^\beta x(0) = -T_{t_m}^\beta x(1)$, we get

$$c_{0,0} = \frac{1}{2} \left(\sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} s(s-t_{i-1})^{\alpha-2} y(s) ds - \sum_{i=1}^m p_i + \sum_{i=1}^m t_i q_i \right),$$

$$c_{0,1} = - \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (s - t_{i-1})^{\alpha-2} y(s) ds - \sum_{i=1}^m q_i.$$

Therefore, for any $t \in J_k$, $k = 0, 1, 2, \dots, m$,

$$\begin{aligned} x(t) &= \int_{t_k}^t (t-s)(s-t_k)^{\alpha-2} y(s) ds + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t-s)(s-t_{i-1})^{\alpha-2} y(s) ds + \sum_{i=1}^k p_i + \sum_{i=1}^k (t-t_i) q_i \\ &\quad + \frac{1}{2} \left(\sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} s(s-t_{i-1})^{\alpha-2} y(s) ds - \sum_{i=1}^m p_i + \sum_{i=1}^m t_i q_i \right) \\ &\quad - t \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (s-t_{i-1})^{\alpha-2} y(s) ds - t \sum_{i=1}^m q_i \\ &= \int_{t_k}^t (t-s)(s-t_k)^{\alpha-2} y(s) ds + \int_{t_k}^t \left(\frac{s}{2} - t \right) (s-t_k)^{\alpha-2} y(s) ds - \frac{1}{2} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} s(s-t_{i-1})^{\alpha-2} y(s) ds \\ &\quad + \int_t^{t_{k+1}} \left(\frac{s}{2} - t \right) (s-t_k)^{\alpha-2} y(s) ds + \sum_{i=k+2}^{m+1} \int_{t_{i-1}}^{t_i} \left(\frac{s}{2} - t \right) (s-t_{i-1})^{\alpha-2} y(s) ds \\ &\quad + \frac{1}{2} \sum_{i=1}^m \chi(t, t_i) p_i + \sum_{i=1}^m G(t, t_i) q_i \\ &= - \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{s}{2} (s-t_{i-1})^{\alpha-2} y(s) ds - \int_{t_k}^t \frac{s}{2} (s-t_k)^{\alpha-2} y(s) ds + \int_t^{t_{k+1}} \left(\frac{s}{2} - t \right) (s-t_k)^{\alpha-2} y(s) ds \\ &\quad + \sum_{i=k+2}^{m+1} \int_{t_{i-1}}^{t_i} \left(\frac{s}{2} - t \right) (s-t_{i-1})^{\alpha-2} y(s) ds + \frac{1}{2} \sum_{i=1}^m \chi(t, t_i) p_i + \sum_{i=1}^m G(t, t_i) q_i \\ &= \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} G(t, s) (s-t_{i-1})^{\alpha-2} y(s) ds + \frac{1}{2} \sum_{i=1}^m \chi(t, t_i) p_i + \sum_{i=1}^m G(t, t_i) q_i. \end{aligned}$$

The proof is completed. \square

By the definition of $G(t, s)$, it is easy to prove that

$$|G(t, s)| \leq \frac{1}{2}, \text{ for all } t, s \in [0, 1]. \quad (3.6)$$

For any $x \in PC(J, \mathbb{R})$, let

$$\begin{aligned} \Lambda x(t) &= \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} G(t, s) (s-t_{i-1})^{\alpha-2} f(s, x(s), Ax(s)) ds + \frac{1}{2} \sum_{i=1}^m \chi(t, t_i) P_i(x(t_i)) \\ &\quad + \sum_{i=1}^m G(t, t_i) Q_i(x(t_i)), \quad t \in J_k, \quad k = 0, 1, 2, \dots, m. \end{aligned} \quad (3.7)$$

It is easy to show that $\Lambda : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R})$, and boundary value problem (1.1) is equivalent to integral Eq (3.7), that is, $x = x(t)$ is a solution of (1.1) if and only if $x \in PC(J, \mathbb{R})$ is a fixed point of operator Λ .

For convenience, we give the following hypotheses.

(H1) There exist functions $\omega, \mu, \kappa, \xi, \zeta \in C(J, [0, +\infty))$ such that, for any $u, v \in \mathbb{R}, t \in J$,

$$|f(t, u, v)| \leq \omega(t) + \mu(t)|u| + \kappa(t)|v|, \quad |h(t, u)| \leq \xi(t) + \zeta(t)|u|,$$

and denote

$$\omega^* = \sup_{t \in J} \omega(t), \quad \mu^* = \sup_{t \in J} \mu(t), \quad \kappa^* = \sup_{t \in J} \kappa(t), \quad \xi^* = \sup_{t \in J} \xi(t), \quad \zeta^* = \sup_{t \in J} \zeta(t);$$

(H2) There exist constants $M^*, N^*, F^*, G^* > 0$ such that, for any $u \in \mathbb{R}$,

$$|P_k(u)| \leq M^*|u| + N^*, \quad |Q_k(u)| \leq F^*|u| + G^*, \quad k = 1, 2, \dots, m;$$

(H3) There exist constants $M_f, N_f, L > 0$ such that, for any $u, v, \bar{u}, \bar{v} \in \mathbb{R}, t \in J$,

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq M_f|u - \bar{u}| + N_f|v - \bar{v}|, \quad |h(t, u) - h(t, \bar{u})| \leq L|u - \bar{u}|;$$

(H4) There exist constants $l, l^* > 0$ such that, for any $u, \bar{u} \in \mathbb{R}$,

$$|P_k(u) - P_k(\bar{u})| \leq l|u - \bar{u}|, \quad |Q_k(u) - Q_k(\bar{u})| \leq l^*|u - \bar{u}|, \quad k = 1, 2, \dots, m.$$

Theorem 3.1. Suppose (H1) and (H2) hold. If

$$(m+1)\lambda^{\alpha-1}(\mu^* + \kappa^*\zeta^*) + m(\alpha-1)(M^* + F^*) < 2(\alpha-1),$$

then boundary value problem (1.1) has at least one solution.

Proof. We take $r \geq \frac{(m+1)\lambda^{\alpha-1}(\omega^* + \kappa^*\xi^*) + m(\alpha-1)(N^* + G^*)}{2(\alpha-1) - ((m+1)\lambda^{\alpha-1}(\mu^* + \kappa^*\zeta^*) + m(\alpha-1)(M^* + F^*))}$, and let

$$B_r = \{x \in PC(J, \mathbb{R}) : \|x\|_{PC} \leq r\}.$$

(1) We prove that $\Lambda : B_r \rightarrow B_r$.

For any $x \in B_r$, and any $t \in J_k, k = 0, 1, 2, \dots, m$, from the condition (H1), we have

$$\begin{aligned} & |f(t, x(t), Ax(t))| \\ & \leq \omega(t) + \mu(t)|x(t)| + \kappa(t)|Ax(t)| \leq \omega^* + \mu^*|x(t)| + \kappa^* \int_0^t (\xi(s) + \zeta(s)|x(s)|)ds \\ & \leq \omega^* + \mu^*|x(t)| + \kappa^* \int_0^t (\xi^* + \zeta^*|x(s)|)ds \leq \omega^* + \mu^*\|x\|_{PC} + \kappa^*(\xi^* + \zeta^*\|x\|_{PC}) \\ & \leq \omega^* + \kappa^*\xi^* + (\mu^* + \kappa^*\zeta^*)r, \end{aligned}$$

then, by (3.6) and (3.7), we get

$$\begin{aligned} & |\Lambda x(t)| \\ & \leq \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} |G(t, s)|(s - t_{i-1})^{\alpha-2} |f(s, x(s), Ax(s))| ds + \frac{1}{2} \sum_{i=1}^m |\chi(t, t_i)P_i(x(t_i))| + \sum_{i=1}^m |G(t, t_i)Q_i(x(t_i))| \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{1}{2} (s - t_{i-1})^{\alpha-2} |f(s, x(s), Ax(s))| ds + \frac{1}{2} \sum_{i=1}^m |P_i(x(t_i))| + \frac{1}{2} \sum_{i=1}^m |Q_i(x(t_i))| \\
&\leq \frac{1}{2} (\omega^* + \kappa^* \xi^* + (\mu^* + \kappa^* \xi^*)r) \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (s - t_{i-1})^{\alpha-2} ds + \frac{1}{2} \sum_{i=1}^m (M^* r + N^*) + \frac{1}{2} \sum_{i=1}^m (F^* r + G^*) \\
&\leq \frac{1}{2} (\omega^* + \kappa^* \xi^* + (\mu^* + \kappa^* \xi^*)r) \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha-1}}{\alpha - 1} + \frac{m}{2} (M^* r + N^*) + \frac{m}{2} (F^* r + G^*) \\
&\leq \frac{(m+1)\lambda^{\alpha-1}}{2(\alpha-1)} (\omega^* + \kappa^* \xi^*) + \frac{m}{2} N^* + \frac{m}{2} G^* + \left(\frac{(m+1)\lambda^{\alpha-1}}{2(\alpha-1)} (\mu^* + \kappa^* \xi^*) + \frac{m}{2} M^* + \frac{m}{2} F^* \right) r \\
&\leq r,
\end{aligned}$$

which implies that $\|\Lambda x\|_{PC} \leq r$.

(2) We prove that Λ is completely continuous.

Let $x_n, x \in B_r$, $n = 1, 2, \dots$, and $\|x_n - x\|_{PC} \rightarrow 0$, $n \rightarrow \infty$, that is, we have $\sup_{t \in [0,1]} |x_n(t) - x(t)| \rightarrow 0$ as $n \rightarrow \infty$. Since f, P_i, Q_i , $i = 1, 2, \dots, m$ are continuous functions, then for any $\varepsilon > 0$, there exists a positive integer N such that

$$|f(s, x_n(s), Ax_n(s)) - f(s, x(s), Ax(s))| < \varepsilon, |P_i(x_n(t_i)) - P_i(x(t_i))| < \varepsilon, |Q_i(x_n(t_i)) - Q_i(x(t_i))| < \varepsilon,$$

for $n \geq N$ and $s \in [0, 1]$.

Hence, for any $t \in J_k$, $k = 0, 1, 2, \dots, m$, we have

$$\begin{aligned}
|\Lambda x_n(t) - \Lambda x(t)| &\leq \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} |G(t, s)|(s - t_{i-1})^{\alpha-2} |f(s, x_n(s), Ax_n(s)) - f(s, x(s), Ax(s))| ds \\
&\quad + \frac{1}{2} \sum_{i=1}^m |\chi(t, t_i)| |P_i(x_n(t_i)) - P_i(x(t_i))| + \sum_{i=1}^m |G(t, t_i)| |Q_i(x_n(t_i)) - Q_i(x(t_i))| \\
&\leq \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha-1}}{2(\alpha-1)} \varepsilon + \frac{m}{2} \varepsilon + \frac{m}{2} \varepsilon \\
&\leq \frac{(m+1)\lambda^{\alpha-1} + 2m(\alpha-1)}{2(\alpha-1)} \varepsilon.
\end{aligned}$$

Then $\Lambda : B_r \rightarrow B_r$ is continuous.

For any $x \in B_r$ and $\tau_1, \tau_2 \in J_k$, $k = 0, 1, 2, \dots, m$, $\tau_1 < \tau_2$, we have

$$\begin{aligned}
&|\Lambda x(\tau_2) - \Lambda x(\tau_1)| \\
&\leq \sum_{i=1}^{m+1} \int_{t_{i-1}}^{\tau_i} |G(\tau_2, s) - G(\tau_1, s)|(s - t_{i-1})^{\alpha-2} |f(s, x(s), Ax(s))| ds + \sum_{i=1}^m |G(\tau_2, t_i) - G(\tau_1, t_i)| |Q_i(x(t_i))| \\
&\leq (\tau_2 - \tau_1) \sum_{i=1}^{m+1} \int_{t_{i-1}}^{\tau_i} (s - t_{i-1})^{\alpha-2} |f(s, x(s), Ax(s))| ds + (\tau_2 - \tau_1) \sum_{i=1}^m |Q_i(x(t_i))| \\
&\leq (\tau_2 - \tau_1) (\omega^* + \kappa^* \xi^* + (\mu^* + \kappa^* \xi^*)r) \sum_{i=1}^{m+1} \int_{t_{i-1}}^{\tau_i} (s - t_{i-1})^{\alpha-2} ds + (\tau_2 - \tau_1) \sum_{i=1}^m (F^* r + G^*)
\end{aligned}$$

$$\leq \frac{(\tau_2 - \tau_1)((\omega^* + \kappa^* \zeta^* + (\mu^* + \kappa^* \zeta^*)r)(m+1)\lambda^{\alpha-1} + m(\alpha-1)(F^*r + G^*))}{\alpha - 1}.$$

Therefore,

$$|\Lambda x(\tau_2) - \Lambda x(\tau_1)| \rightarrow 0, \quad \tau_1 \rightarrow \tau_2, \quad \tau_1, \tau_2 \in J_k, \quad k = 0, 1, 2, \dots, m,$$

which implies that the operator Λ is equicontinuous on $\Lambda(B_r)$. By Arzela-Ascoli theorem, we know that the operator Λ is compact, so the operator Λ is completely continuous.

By Schauder fixed point theorem, Λ has a fixed point on B_r , which implies that boundary value problem (1.1) has at least one solution.

The proof is completed. \square

Theorem 3.2. *Suppose (H3) and (H4) hold. If*

$$(m+1)\lambda^{\alpha-1}(M_f + N_f L) + m(\alpha-1)(l + l^*) < 2(\alpha-1) \quad (3.8)$$

holds, then boundary value problem (1.1) has a unique solution.

Proof. For any $x_1, x_2 \in PC(J, \mathbb{R})$, and any $t \in J_k, k = 0, 1, 2, \dots, m$, from the condition (H3), we get

$$\begin{aligned} & |f(s, x_2(s), Ax_2(s)) - f(s, x_1(s), Ax_1(s))| \\ & \leq M_f |x_2(s) - x_1(s)| + N_f |Ax_2(s) - Ax_1(s)| \\ & \leq M_f |x_2(s) - x_1(s)| + N_f \left| \int_0^s h(\tau, x_2(\tau)) d\tau - \int_0^s h(\tau, x_1(\tau)) d\tau \right| \\ & \leq M_f |x_2(s) - x_1(s)| + N_f L \int_0^s |x_2(\tau) - x_1(\tau)| d\tau \\ & \leq (M_f + N_f L) \|x_2 - x_1\|_{PC}, \end{aligned}$$

then we have

$$\begin{aligned} |\Lambda x_2(t) - \Lambda x_1(t)| & \leq \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} |G(t, s)|(s - t_{i-1})^{\alpha-2} |f(s, x_2(s), Ax_2(s)) - f(s, x_1(s), Ax_1(s))| ds \\ & \quad + \frac{1}{2} \sum_{i=1}^m |\chi(t, t_i)| |P_i(x_2(t_i)) - P_i(x_1(t_i))| + \sum_{i=1}^m |G(t, t_i)| |Q_i(x_2(t_i)) - Q_i(x_1(t_i))| \\ & \leq \frac{1}{2} (M_f + N_f L) \|x_2 - x_1\|_{PC} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (s - t_{i-1})^{\alpha-2} ds + \frac{m}{2} l \|x_2 - x_1\|_{PC} \\ & \quad + \frac{m}{2} l^* \|x_2 - x_1\|_{PC} \\ & \leq \frac{1}{2} \left(\sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha-1}}{\alpha - 1} (M_f + N_f L) + ml + ml^* \right) \|x_2 - x_1\|_{PC} \\ & \leq \frac{1}{2(\alpha - 1)} ((m+1)\lambda^{\alpha-1}(M_f + N_f L) + m(\alpha-1)(l + l^*)) \|x_2 - x_1\|_{PC}. \end{aligned}$$

Therefore,

$$\|\Lambda x_2 - \Lambda x_1\|_{PC} \leq \frac{1}{2(\alpha - 1)} ((m+1)\lambda^{\alpha-1}(M_f + N_f L) + m(\alpha-1)(l + l^*)) \|x_2 - x_1\|_{PC}.$$

By (3.8), we can obtain Λ is contractive. According to Banach fixed point theorem, Λ has a unique fixed point on $PC(J, \mathbb{R})$, that is, boundary value problem (1.1) has a unique solution.

The proof is completed. \square

4. Ulam stability analysis

In views of Lemma 3.1, we can get the following lemma.

Lemma 4.1. *Let $\phi \in PC(J, \mathbb{R})$, $\phi_k, \tilde{\phi}_k \in \mathbb{R}$, $k = 1, 2, \dots, m$, then the following boundary value problem of the conformable fractional integro-differential impulsive equation*

$$\begin{cases} T_k^\alpha z(t) = f(t, z(t), Az(t)) + \phi(t), & t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \dots, m, \\ \Delta z(t)|_{t=t_k} = P_k(z(t_k)) + \phi_k, & k = 1, 2, \dots, m, \\ \Delta z'(t)|_{t=t_k} = Q_k(z(t_k)) + \tilde{\phi}_k, & k = 1, 2, \dots, m, \\ z(0) = -z(1), \quad T_0^\beta z(0) = -T_m^\beta z(1) \end{cases} \quad (4.1)$$

is equivalent to integral equation

$$\begin{aligned} z(t) = & \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} G(t, s)(s - t_{i-1})^{\alpha-2} (f(s, z(s), Az(s)) + \phi(s)) ds + \frac{1}{2} \sum_{i=1}^m \chi(t, t_i)(P_i(z(t_i)) + \phi_i) \\ & + \sum_{i=1}^m G(t, t_i)(Q_i(z(t_i)) + \tilde{\phi}_i), \quad t \in J_k, \quad k = 0, 1, 2, \dots, m. \end{aligned} \quad (4.2)$$

Theorem 4.1. *Assume that all the conditions of Theorem 3.2 are satisfied, then boundary value problem (1.1) is U-H stable.*

Proof. Since all the conditions of Theorem 3.2 are satisfied, then there exists a unique solution $x \in PC(J, \mathbb{R})$ of boundary value problem (1.1).

Suppose $z \in PC(J, \mathbb{R})$ is a solution of inequalities system (2.1). By Lemma 4.1 and Remark 2.2, for any $t \in J_k$, $k = 0, 1, 2, \dots, m$, we have

$$\begin{aligned} |z(t) - x(t)| = & \left| \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} G(t, s)(s - t_{i-1})^{\alpha-2} (f(s, z(s), Az(s)) + \phi(s)) ds \right. \\ & + \frac{1}{2} \sum_{i=1}^m \chi(t, t_i)(P_i(z(t_i)) + \phi_i) + \sum_{i=1}^m G(t, t_i)(Q_i(z(t_i)) + \tilde{\phi}_i) \\ & - \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} G(t, s)(s - t_{i-1})^{\alpha-2} f(s, x(s), Ax(s)) ds - \frac{1}{2} \sum_{i=1}^m \chi(t, t_i)P_i(x(t_i)) \\ & \left. - \sum_{i=1}^m G(t, t_i)Q_i(x(t_i)) \right| \\ \leq & \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{1}{2}(s - t_{i-1})^{\alpha-2} |\phi(s)| ds + \frac{1}{2} \sum_{i=1}^m |\phi_i| + \sum_{i=1}^m \frac{1}{2} |\tilde{\phi}_i| \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} |G(t, s)|(s - t_{i-1})^{\alpha-2} |f(s, z(s), Az(s)) - f(s, x(s), Ax(s))| ds \\
& + \frac{1}{2} \sum_{i=1}^m |\chi(t, t_i)| |P_i(z(t_i)) - P_i(x(t_i))| + \sum_{i=1}^m |G(t, t_i)| |Q_i(z(t_i)) - Q_i(x(t_i))| \\
\leq & \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{1}{2} (s - t_{i-1})^{\alpha-2} |\phi(s)| ds + \frac{1}{2} \sum_{i=1}^m |\phi_i| + \sum_{i=1}^m \frac{1}{2} |\tilde{\phi}_i| \\
& + \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{1}{2} (s - t_{i-1})^{\alpha-2} (M_f |z(s) - x(s)| + N_f |Az(s) - Ax(s)|) ds \\
& + \frac{1}{2} \sum_{i=1}^m |P_i(z(t_i)) - P_i(x(t_i))| + \sum_{i=1}^m \frac{1}{2} |Q_i(z(t_i)) - Q_i(x(t_i))| \\
\leq & \frac{\varepsilon}{2} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (s - t_{i-1})^{\alpha-2} ds + \frac{m}{2} \varepsilon + \frac{m}{2} \varepsilon \\
& + \frac{1}{2} (M_f + N_f L) \|z - x\|_{PC} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (s - t_{i-1})^{\alpha-2} ds \\
& + \frac{1}{2} l \sum_{i=1}^m |z(t_i) - x(t_i)| + \frac{1}{2} l^* \sum_{i=1}^m |z(t_i) - x(t_i)| \\
\leq & \frac{1}{2} \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha-1}}{\alpha - 1} \varepsilon + m \varepsilon + \frac{1}{2} ((M_f + N_f L) \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha-1}}{\alpha - 1} + ml + ml^*) \|z - x\|_{PC} \\
\leq & \left(\frac{(m+1)\lambda^{\alpha-1}}{2(\alpha-1)} + m \right) \varepsilon + \frac{1}{2} \left(\frac{(m+1)\lambda^{\alpha-1}}{\alpha-1} (M_f + N_f L) + m(l + l^*) \right) \|z - x\|_{PC}.
\end{aligned}$$

Hence, we obtain

$$\|z - x\|_{PC} \leq \left(\frac{(m+1)\lambda^{\alpha-1}}{2(\alpha-1)} + m \right) \varepsilon + \frac{1}{2} \left(\frac{(m+1)\lambda^{\alpha-1}}{\alpha-1} (M_f + N_f L) + m(l + l^*) \right) \|z - x\|_{PC}.$$

By (3.8), we have

$$\|z - x\|_{PC} \leq \frac{(m+1)\lambda^{\alpha-1} + 2m(\alpha-1)}{2(\alpha-1) - ((m+1)\lambda^{\alpha-1}(M_f + N_f L) + m(\alpha-1)(l + l^*))} \varepsilon.$$

Let

$$c_{f,m} := \frac{(m+1)\lambda^{\alpha-1} + 2m(\alpha-1)}{2(\alpha-1) - ((m+1)\lambda^{\alpha-1}(M_f + N_f L) + m(\alpha-1)(l + l^*))}.$$

Therefore,

$$\|z - x\|_{PC} \leq c_{f,m} \varepsilon.$$

According to Definition 2.3, boundary value problem (1.1) is U-H stable.

The proof is completed. \square

Theorem 4.2. *Suppose that all the conditions of Theorem 3.2 are satisfied, and there exist a continuous function $g(t) > 0$ on $t \in J$, and a constant $c_g > 0$, such that, the inequalities*

$$\int_{t_{i-1}}^{t_i} (s - t_{i-1})^{\alpha-2} g(s) ds \leq c_g g(t), \quad t \in J, \quad i = 1, 2, \dots, m+1$$

hold, then boundary value problem (1.1) is U-H-R stable.

Proof. Since all the conditions of Theorem 3.2 are satisfied, then there exists a unique solution $x \in PC(J, \mathbb{R})$ of boundary value problem (1.1).

Suppose $z \in PC(J, \mathbb{R})$ is a solution of inequalities (2.2). Since the Lemma 4.1 and Remark 2.3, similar to Theorem 4.1, for any $t \in J_k$, $k = 0, 1, 2, \dots, m$, we have

$$\begin{aligned} |z(t) - x(t)| &\leq \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{1}{2} (s - t_{i-1})^{\alpha-2} |\phi(s)| ds + \frac{1}{2} \sum_{i=1}^m |\phi_i| + \sum_{i=1}^m \frac{1}{2} |\tilde{\phi}_i| \\ &\quad + \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{1}{2} (s - t_{i-1})^{\alpha-2} (M_f |z(s) - x(s)| + N_f |Az(s) - Ax(s)|) ds \\ &\quad + \frac{1}{2} \sum_{i=1}^m |P_i(z(t_i)) - P_i(x(t_i))| + \sum_{i=1}^m \frac{1}{2} |Q_i(z(t_i)) - Q_i(x(t_i))| \\ &\leq \frac{\varepsilon}{2} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (s - t_{i-1})^{\alpha-2} g(s) ds + \frac{m}{2} \varphi \varepsilon + \frac{m}{2} \varphi \varepsilon \\ &\quad + \frac{1}{2} (M_f + N_f L) \|z - x\|_{PC} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (s - t_{i-1})^{\alpha-2} ds \\ &\quad + \frac{1}{2} l \sum_{i=1}^m |z(t_i) - x(t_i)| + \frac{1}{2} l^* \sum_{i=1}^m |z(t_i) - x(t_i)| \\ &\leq \left(\frac{m+1}{2} c_g + m \right) \varepsilon (g(t) + \varphi) + \frac{1}{2} \left(\frac{(m+1)\lambda^{\alpha-1}}{\alpha-1} (M_f + N_f L) + m(l + l^*) \right) \|z - x\|_{PC}. \end{aligned}$$

And by (3.8), we have,

$$\|z - x\|_{PC} \leq c_{f,m,g} \varepsilon (g(t) + \varphi),$$

where

$$c_{f,m,g} = \frac{(m+1)(\alpha-1)c_g + 2m(\alpha-1)}{2(\alpha-1) - ((m+1)\lambda^{\alpha-1}(M_f + N_f L) + m(\alpha-1)(l + l^*))}.$$

According to Definition 2.4, boundary value problem (1.1) is U-H-R stable.

The proof is completed. \square

5. Illustrations

In this section, We consider the following boundary value problems of fractional integro-differential impulsive equations:

$$\begin{cases} T_{t_k}^{\frac{3}{2}}x(t) = \frac{1}{3}\left(t + \frac{x(t)}{1+t}\right) + \int_0^t \frac{\sin s}{2}x(s)ds, & t \in (t_k, t_{k+1}), \quad k = 0, 1, \\ \Delta x(t)|_{t=\frac{1}{2}} = \frac{|x(\frac{1}{2})|}{6 + |x(\frac{1}{2})|}, \quad \Delta x'(t)|_{t=\frac{1}{2}} = \frac{|x(\frac{1}{2})|}{9 + |x(\frac{1}{2})|}, \\ x(0) = -x(1), \quad T_0^\beta x(0) = -T_{\frac{1}{2}}^\beta x(1). \end{cases} \quad (5.1)$$

Corresponding to Eq (5.1), where $\alpha = \frac{3}{2}$, $m = 1$, $t_1 = \frac{1}{2}$, $f(t, x(t), Ax(t)) = \frac{1}{3}\left(t + \frac{x(t)}{1+t}\right) + \int_0^t \frac{\sin s}{2}x(s)ds$, $P_1(x(\frac{1}{2})) = \frac{|x(\frac{1}{2})|}{6 + |x(\frac{1}{2})|}$, $Q_1(x(\frac{1}{2})) = \frac{|x(\frac{1}{2})|}{9 + |x(\frac{1}{2})|}$.

Let $\omega(t) = t$, $\mu(t) = \frac{1}{3(1+t)}$, $\kappa(t) = \frac{1}{3}$, $\xi(t) = t$, $\zeta(t) = \frac{\sin t}{2}$, and $M^* = \frac{1}{6}$, $N^* = 1$, $F^* = \frac{1}{9}$, $G^* = 1$, $M_f = N_f = \frac{1}{3}$, $L = \frac{1}{2}$, $l = \frac{1}{6}$, $l^* = \frac{1}{9}$.

For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$, and any $t \in [0, 1]$, we have

$$|f(t, u, v)| \leq t + \frac{1}{3(1+t)}|u| + \frac{1}{3}|v|, \quad |h(t, u)| \leq t + \frac{\sin t}{2}|u|, \quad |P_1(u)| \leq \frac{1}{6}|u| + 1, \quad |Q_1(u)| \leq \frac{1}{9}|u| + 1,$$

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq \frac{1}{3}(|u - \bar{u}| + |v - \bar{v}|), \quad |h(t, u) - h(t, \bar{u})| \leq \frac{1}{2}|u - \bar{u}|,$$

$$|P_1(u) - P_1(\bar{u})| \leq \frac{1}{6}|u - \bar{u}|, \quad |Q_1(u) - Q_1(\bar{u})| \leq \frac{1}{9}|u - \bar{u}|.$$

So the conditions (H1)–(H4) hold. And due to that

$$(m + 1)\lambda^{\alpha-1}(\mu^* + \kappa^* \zeta^*) + m(\alpha - 1)(M^* + F^*) \approx 0.81 < 1 = 2(\alpha - 1),$$

and all the conditions of the Theorem 3.1 are satisfied, boundary value problem (5.1) has at least one solution. Since

$$(m + 1)\lambda^{\alpha-1}(M_f + N_f L) + m(\alpha - 1)(l + l^*) \approx 0.85 < 1 = 2(\alpha - 1),$$

it follows from Theorem 3.2 that boundary value problem (5.1) has a unique solution. And by Theorem 4.1, boundary value problem (5.1) is U-H stable.

Let $g(t) = t + 1$, $t \in [0, 1]$, we have

$$\int_0^{\frac{1}{2}} s^{\frac{3}{2}-2}(s+1)ds \approx 1.65 \leq 2(t+1), \quad \int_{\frac{1}{2}}^1 \left(s - \frac{1}{2}\right)^{\frac{3}{2}-2}(s+1)ds \approx 2.36 \leq 3(t+1),$$

and let $c_g = 3$. All the conditions of Theorem 4.2 are satisfied, then boundary value problem (5.1) is U-H-R stable.

6. Conclusions

Since the stability plays an important role in control theory and error analysis, it is very necessary to discuss the stability for solutions of differential equations. In this paper, a class of conformable fractional integro-differential impulsive equations with the antiperiodic boundary conditions are studied. After discussing the existence and uniqueness of solutions, we focus on the U-H stability and U-H-R stability of solutions of problems (1.1). Finally, an example is given out to illustrate the effectiveness and applicability of our results.

A further extension of this paper is to study the motion states of vibrators in systems described in boundary value problems (1.1) and the existence and stability of solutions to boundary value problems with other boundary conditions. In addition, according to the conclusion of stability obtained in this paper, we can further study the approximate solution of differential equation and make error analysis.

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Conflict of interest

The authors declare no conflicts of interest regarding this article.

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