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Research article

Existence of solutions of Dirichlet problems for one dimensional fractional equations

Armin Hadjian[1,*] **and Juan J. Nieto**[2]

[1] Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord 94531, Iran

[2] CITMAga, Institute of Mathematics, University of Santiago de Compostela, Santiago de Compostela 15782, Spain

* **Correspondence:** Email: hadjian83@gmail.com.

Abstract: We establish the existence of infinitely many solutions for some nonlinear fractional differential equations under suitable oscillating behaviour of the nonlinear term. These problems have a variational structure and we prove our main results by using a critical point theorem due to Ricceri.

Keywords: fractional differential equations; Caputo fractional derivatives; infinitely many solutions; variational methods
Mathematics Subject Classification: 34A08, 35A15, 26A33

1. Introduction

In this paper, we are eager to investigate the nonlinear fractional boundary value problem (BVP) of the form

$$\begin{cases} \dfrac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big) + \lambda f(t,u(t)) = 0 \quad \text{a.e. } t\in[0,T] \\ u(0)=u(T)=0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a real parameter, $\alpha \in (1/2, 1]$, ${}_0^cD_t^{\alpha}$ and ${}_t^cD_T^{\alpha}$ are the left and right Caputo fractional derivatives of order α, respectively, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional integrals of order $1-\alpha$, respectively, and $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a continuous function.

Fractional differential equations (i.e. $\alpha \in (1/2, 1)$) arise in real applications in many fields of sciences such as conservation laws, minimal surfaces, water waves and ultra-relativistic limits of quantum mechanics. Due to these applications, non-local fractional problems are extensively investigated. There has been remarkable progress in fractional differential equations; the interested reader may see the books [16, 18, 24, 26, 29].

Critical point theory has been very effective in specifying the existence and multiplicity of solutions for integer order differential equations provided that the equation has a variational construction on some appropriate Sobolev spaces, e.g., we refer to [12,20,21,23,27,30] and the references therein for detailed discussions. But until now, there are little consequences on the existence of solutions to fractional BVPs which were proved by the variational methods, since it is frequently hard to establish a proper space and variational functional for fractional differential equations. For instance, in [1–5, 13, 14, 17], variational methods are applied to study the existence and multiplicity of solutions for fractional BVPs.

An attractive physical case is considered in [17] where Jiao and Zhou, by applying the critical point theory, proved the existence and multiplicity of solutions for the problem

$$\begin{cases} \dfrac{d}{dt}\Big(\dfrac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big) + \nabla F(t,u(t)) = 0 \quad \text{a.e. } t \in [0,T] \\ u(0) = u(T) = 0, \end{cases}$$

where $\beta \in [0,1)$, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ (with $N \geq 1$) is a suitable given function and $\nabla F(t,x)$ is the gradient of F at x (see Remark 2.9 below for the relation of this problem with problem (1.1)).

Also, Bai in [5], applying a critical point result for differentiable functionals proved by Bonanno [7], discussed the existence of at least one non-zero solution for the problem

$$\begin{cases} \dfrac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big) + \lambda f(u(t)) = 0 \quad \text{a.e. } t \in [0,T] \\ u(0) = u(T) = 0. \end{cases} \tag{1.2}$$

The authors in [13] obtained, by using three critical point theorems, for the following BVP for fractional order differential equations

$$\begin{cases} \dfrac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big) + \lambda f(t,u(t)) + \mu g(t,u(t)) = 0 \quad \text{a.e. } t \in [0,T] \\ u(0) = u(T) = 0, \end{cases}$$

the existence of at least three solutions, where $\lambda, \mu > 0$ are two parameters.

More recently, Galewski and Molica Bisci in [14] studied the problem (1.2), in the case $\lambda = 1$. With an asymptotic behaviour of f at zero and using a critical point result [8], they proved the existence of one non-zero solution for the problem.

In this paper, we will prove the existence of infinitely many solutions for problem (1.1) with rather various hypotheses on the function f. We need that the potential F of f assures an appropriate oscillatory behavior either at infinity or at the origin.

Our results are based on the variational principle due to Ricceri [28]. We address the eager reader to the book [19] as a comprehensive reference on critical point theory adopted here.

Finally, we note that an interesting and careful analysis of fractional BVPs was extended in the nice and recent works [6, 9–11, 15, 22, 25, 32–34] and the references therein.

This paper is organized as follows. In Section 2, we state some preliminary definitions and properties of the fractional calculus that will be required in the paper. In Section 3, our principal result, Theorem 3.1, and some significant conclusions (see Corollaries 3.3, 3.4 and 3.6) are presented. Then Example 3.8 is given as an application of Corollary 3.3.

2. Preliminaries

In the present section, first we present several needful definitions and properties of the fractional calculus which are required further in this paper. Let $AC([a,b],\mathbb{R}) = AC^1([a,b],\mathbb{R})$ be the space of absolutely continuous functions on the interval $[a,b]$, where $a < b$ are real numbers (see [18]). Set $AC^n([a,b],\mathbb{R})$ the space of functions $u : [a,b] \to \mathbb{R}$ such that $u \in C^{n-1}([a,b],\mathbb{R})$ and $u^{(n-1)} \in AC([a,b],\mathbb{R})$. Here, $C^{n-1}([a,b],\mathbb{R})$ signifies the set of mappings that are $(n-1)$ times continuously differentiable on $[a,b]$.

Definition 2.1 ([18, 26])**.** Let $u \in L^1([a,b],\mathbb{R})$. We denote by ${}_aD_t^{-\gamma}u(t)$ and ${}_tD_b^{-\gamma}u(t)$ the left and right Riemann-Liouville fractional integrals of order $\gamma > 0$ for function u, respectively, that are defined by

$$ {}_aD_t^{-\gamma}u(t) = \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}u(s)ds, $$

and

$$ {}_tD_b^{-\gamma}u(t) = \frac{1}{\Gamma(\gamma)}\int_t^b (s-t)^{\gamma-1}u(s)ds, $$

for every $t \in [a,b]$, while the right-hand sides are pointwise defined on $[a,b]$, where $\Gamma > 0$ is the standard gamma function given by

$$ \Gamma(\gamma) = \int_0^{+\infty} z^{\gamma-1}e^{-z}dz. $$

We note that ${}_aD_t^{-\gamma}$ and ${}_tD_b^{-\gamma}$ are linear and bounded operators from $L^1([a,b],\mathbb{R})$ into $L^1([a,b],\mathbb{R})$; see also Lemma 2.6 below.

Definition 2.2 ([18, 26])**.** Let $u \in AC^n([a,b],\mathbb{R})$. We denote by ${}_aD_t^{\gamma}u(t)$ and ${}_tD_b^{\gamma}u(t)$ the left and right Riemann-Liouville fractional derivatives of order γ ($n-1 \le \gamma < n$ and $n \in \mathbb{N}$) for function u, respectively, that are defined by

$$ {}_aD_t^{\gamma}u(t) = \frac{d^n}{dt^n}{}_aD_t^{\gamma-n}u(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\left(\int_a^t (t-s)^{n-\gamma-1}u(s)ds\right), $$

and

$$ {}_tD_b^{\gamma}u(t) = (-1)^n\frac{d^n}{dt^n}{}_tD_b^{\gamma-n}u(t) = \frac{1}{\Gamma(n-\gamma)}(-1)^n\frac{d^n}{dt^n}\left(\int_t^b (s-t)^{n-\gamma-1}u(s)ds\right), $$

where $t \in [a,b]$. Specially, if $0 \le \gamma < 1$, then

$$ {}_aD_t^{\gamma}u(t) = \frac{d}{dt}{}_aD_t^{\gamma-1}u(t) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\left(\int_a^t (t-s)^{-\gamma}u(s)ds\right), \qquad t \in [a,b], $$

and

$$ {}_tD_b^{\gamma}u(t) = -\frac{d}{dt}{}_tD_b^{\gamma-1}u(t) = -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\left(\int_t^b (s-t)^{-\gamma}u(s)ds\right), \qquad t \in [a,b]. $$

Definition 2.3 ([18])**.** Suppose that $\gamma \ge 0$ and $n \in \mathbb{N}$.

(i) Let $\gamma \in (n-1, n)$ and $u \in AC^n([a,b],\mathbb{R})$. We denote by ${}_a^cD_t^\gamma u(t)$ and ${}_t^cD_b^\gamma u(t)$ the left and right Caputo fractional derivatives of order γ for function u, respectively. These derivatives exist almost everywhere on $[a,b]$. ${}_a^cD_t^\gamma u(t)$ and ${}_t^cD_b^\gamma u(t)$ are illustrated by

$$ {}_a^cD_t^\gamma u(t) = {}_aD_t^{\gamma-n}u^{(n)}(t) = \frac{1}{\Gamma(n-\gamma)}\int_a^t (t-s)^{n-\gamma-1}u^{(n)}(s)ds, $$

and

$$ {}_t^cD_b^\gamma u(t) = (-1)^n {}_tD_b^{\gamma-n}u^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\gamma)}\int_t^b (s-t)^{n-\gamma-1}u^{(n)}(s)ds, $$

for every $t \in [a,b]$, respectively. Specially, if $0 < \gamma < 1$, then

$$ {}_a^cD_t^\gamma u(t) = {}_aD_t^{\gamma-1}u'(t) = \frac{1}{\Gamma(1-\gamma)}\int_a^t (t-s)^{-\gamma}u'(s)ds, \qquad t \in [a,b], $$

and

$$ {}_t^cD_b^\gamma u(t) = -{}_tD_b^{\gamma-1}u'(t) = -\frac{1}{\Gamma(1-\gamma)}\int_t^b (s-t)^{-\gamma}u'(s)ds, \qquad t \in [a,b]. $$

(ii) If $\gamma = n-1$ and $u \in AC^{n-1}([a,b],\mathbb{R})$, then ${}_a^cD_t^{n-1}u(t)$ and ${}_t^cD_b^{n-1}u(t)$ are illustrated by

$$ {}_a^cD_t^{n-1}u(t) = u^{(n-1)}(t), \quad \text{and} \quad {}_t^cD_b^{n-1}u(t) = (-1)^{(n-1)}u^{(n-1)}(t), $$

for every $t \in [a,b]$. Specially, ${}_a^cD_t^0u(t) = {}_t^cD_b^0u(t) = u(t),\ t \in [a,b]$.

If $u \in AC^n([a,b],\mathbb{R})$, then the relation between the Riemann-Liouville fractional derivative and the Caputo fractional derivative is expressed by the following

$$ {}_aD_t^\gamma u(t) = {}_a^cD_t^\gamma u(t) + \sum_{j=0}^{n-1}\frac{u^{(j)}(a)}{\Gamma(j-\gamma+1)}(t-a)^{j-\gamma}, \qquad t \in [a,b]. $$

Proposition 2.4 ([18]). *The left and right Riemann-Liouville fractional integral operators have the feature of a semigroup, that is*

$$ {}_aD_t^{-\gamma_1}({}_aD_t^{-\gamma_2}u(t)) = {}_aD_t^{-\gamma_1-\gamma_2}u(t) \quad \textit{and} \quad {}_tD_b^{-\gamma_1}({}_tD_b^{-\gamma_2}u(t)) = {}_tD_b^{-\gamma_1-\gamma_2}u(t), \quad \forall \gamma_1, \gamma_2 > 0 $$

at every point $t \in [a,b]$ for a continuous function u, and for a.e. point in $[a,b]$ if $u \in L^1([a,b],\mathbb{R})$.

Proposition 2.5 ([29]). *We have*

$$ \int_a^b [{}_aD_t^{-\gamma}u(t)]v(t)dt = \int_a^b [{}_tD_b^{-\gamma}v(t)]u(t)dt, \quad \gamma > 0, $$

with the condition that $u \in L^p([a,b],\mathbb{R})$, $v \in L^q([a,b],\mathbb{R})$ and $p \geq 1$, $q \geq 1$ and $1/p + 1/q \leq 1+\gamma$, or $p \neq 1$, $q \neq 1$ and $1/p + 1/q = 1+\gamma$.

For any $u \in L^2([0,T],\mathbb{R})$ and for every fixed $t \in [0,T]$, set

$$ \|u\|_{L^2([0,t])} := \left(\int_0^t |u(\xi)|^2 d\xi\right)^{1/2}, \qquad \|u\|_{L^2} := \left(\int_0^T |u(t)|^2 dt\right)^{1/2}, \qquad \|u\|_\infty := \max_{t\in[0,T]} |u(t)|. $$

Lemma 2.6 ([17, Lemma 3.1]). *Let $\alpha \in (0,1]$. For any $u \in L^2([0,T],\mathbb{R})$, we have*

$$\|{}_0D_\xi^{-\alpha}u\|_{L^2([0,t])} \leq \frac{t^\alpha}{\Gamma(\alpha+1)}\|u\|_{L^2([0,t])}, \qquad \text{for } \xi \in [0,t],\ t \in [0,T].$$

Let $C_0^\infty([0,T],\mathbb{R})$ be the collection of all functions $g \in C^\infty([0,T],\mathbb{R})$ with compact support contained in $(0,T)$. Then any function $g \in C_0^\infty([0,T],\mathbb{R})$ satisfies $g(0) = g(T) = 0$.

Definition 2.7. Suppose that $0 < \alpha \leq 1$. We define the fractional derivative space E_0^α by the closure of $C_0^\infty([0,T],\mathbb{R})$ with respect to the norm

$$\|u\| := \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt\right)^{1/2},$$

for every $u \in E_0^\alpha$.

Remark 2.8. (i) For any $u \in E_0^\alpha$, with the fact that $u(0) = 0$, one has ${}_0^cD_t^\alpha u(t) = {}_0D_t^\alpha u(t)$, $t \in [0,T]$ according to the equality (10) of [17].

(ii) According to Lemma 2.6, for every $u \in C_0^\infty([0,T],\mathbb{R})$, one has $u \in L^2([0,T],\mathbb{R})$ and ${}_0^cD_t^\alpha u \in L^2([0,T],\mathbb{R})$. Thus, it is obvious that E_0^α is the space of functions $u \in L^2([0,T],\mathbb{R})$ having an α-order Caputo fractional derivative ${}_0^cD_t^\alpha u \in L^2([0,T],\mathbb{R})$ and $u(0) = u(T) = 0$.

Remark 2.9. In view of Definition 2.3, for every $u \in AC([0,T],\mathbb{R})$, BVP (1.1) transforms to

$$\begin{cases} \dfrac{d}{dt}\left({}_0D_t^{-\frac{\beta}{2}}\left({}_0D_t^{-\frac{\beta}{2}}u'(t)\right) + {}_tD_T^{-\frac{\beta}{2}}\left({}_tD_T^{-\frac{\beta}{2}}u'(t)\right)\right) + \lambda f(t,u(t)) = 0 \quad \text{a.e. } t \in [0,T] \\ u(0) = u(T) = 0, \end{cases} \tag{2.1}$$

where $\beta := 2(1-\alpha) \in [0,1)$.

Furthermore by Proposition 2.4, it is clear that $u \in AC([0,T],\mathbb{R})$ is a solution of BVP (2.1) if and only if u is a solution of the problem

$$\begin{cases} \dfrac{d}{dt}\left({}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t))\right) + \lambda f(t,u(t)) = 0 \quad \text{a.e. } t \in [0,T] \\ u(0) = u(T) = 0. \end{cases} \tag{2.2}$$

For completeness we recall that a function $u \in AC([0,T],\mathbb{R})$ is named a solution of BVP (2.2) if:

(j) The map
$$t \mapsto {}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t)),$$
is differentiable for a.e. $t \in [0,T]$, and

(jj) The function u assures (2.2).

Proposition 2.10 ([31, Lemma 4.2]). *Suppose that $\alpha \in (0,1]$. The space E_0^α is a separable and reflexive Banach space.*

Lemma 2.11 ([17, Proposition 3.2]). *Suppose that $\alpha \in (1/2,1]$. For all $u \in E_0^\alpha$, one has*

$$\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0^cD_t^\alpha u\|_{L^2},$$
$$\|u\|_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\|{}_0^cD_t^\alpha u\|_{L^2}.$$

Hence, we can consider E_0^α equipped with the equivalent norm

$$\|u\|_\alpha := \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt\right)^{1/2} = \|{}_0^cD_t^\alpha u\|_{L^2}, \quad \forall\, u \in E_0^\alpha.$$

Lemma 2.12 ([17, Proposition 4.1])**.** *Suppose that* $\alpha \in (1/2, 1]$*. For every* $u \in E_0^\alpha$*, one has*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \le -\int_0^T {}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha u(t) dt \le \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2.$$

We refer our main results for $\alpha \in (1/2, 1]$ rather than $\alpha \in (0, 1/2]$, since by Lemmas 2.11 and 2.12, for $\alpha \in (1/2, 1]$, the space E_0^α is compactly embedded in $C([0, T], \mathbb{R})$ and the functional $-\int_0^T {}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha u(t)\, dt$ is coercive.

We formulate the following version of Ricceri's variational principle [28, Theorem 2.5], that is our principal tool for establishing the principal result of this paper.

Theorem 2.13. *Suppose that X is a reflexive real Banach space, and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two Gâteaux differentiable functionals such that* Φ *is sequentially weakly lower semicontinuous, strongly continuous and coercive, and* Ψ *is sequentially weakly upper semicontinuous. For any* $r > \inf_X \Phi$*, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\left(\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}.$$

Put

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \textit{and} \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, the following properties hold:

(a) *If* $\gamma < +\infty$*, then for each* $\lambda \in (0, 1/\gamma)$*, the following alternative holds: Either*

  (a$_1$) I_λ *possesses a global minimum, or*

  (a$_2$) *there is a sequence* $\{u_n\}$ *of critical points* (*local minima*) *of* I_λ *such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(b) *If* $\delta < +\infty$*, then for each* $\lambda \in (0, 1/\delta)$*, the following alternative holds: Either*

  (b$_1$) *there is a global minimum of* Φ *which is a local minimum of* I_λ*, or*

  (b$_2$) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points* (*local minima*) *of* I_λ *that converges weakly to a global minimum of* Φ*, with* $\lim_{n\to+\infty} \Phi(u_n) = \inf_{u\in X} \Phi(u)$*.*

3. Main results

In the present section, we state and prove our principal result. Let

$$\kappa := \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha - 1}},$$

$$C(T,\alpha) \;:=\; \int_0^{T/4} t^{2-2\alpha}\,dt + \int_{T/4}^{3T/4}\left[t^{1-\alpha} - \left(t-\frac{T}{4}\right)^{1-\alpha}\right]^2 dt + \int_{3T/4}^{T}\left[t^{1-\alpha} - \left(t-\frac{T}{4}\right)^{1-\alpha} - \left(t-\frac{3T}{4}\right)^{1-\alpha}\right]^2 dt,$$

and

$$B^\infty := \limsup_{\xi\to+\infty} \frac{\int_{T/4}^{3T/4} F(t,\xi)\,dt}{\xi^2},$$

where F is the potential of f defined by

$$F(t,\xi) := \int_0^\xi f(t,x)\,dx, \qquad (t,\xi)\in[0,T]\times\mathbb{R}.$$

We suppose that the following condition holds:

(f_1) $F(t,\xi)\ge 0$ for any $(t,\xi)\in([0,\frac{T}{4}]\cup[\frac{3T}{4},T])\times\mathbb{R}$.

Our principal result reads as follows.

Theorem 3.1. *Suppose that $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function whose potential satisfies* (f_1). *Assume that there exist real sequences $\{a_n\}$ and $\{b_n\}$ in $(0,+\infty)$, with $\lim\limits_{n\to+\infty} b_n = +\infty$, such that*:

(h_1) *For some $n_0\in\mathbb{N}$ we have $a_n < \dfrac{T|\cos(\pi\alpha)|\Gamma(2-\alpha)}{4\kappa\sqrt{C(T,\alpha)}} b_n$ for each $n\ge n_0$;*

(h_2) $\displaystyle \mathcal{A}_\infty := \lim_{n\to+\infty} \frac{\int_0^T \max_{|\xi|\le b_n} F(t,\xi)\,dt - \int_{T/4}^{3T/4} F(t,a_n)\,dt}{T^2|\cos(\pi\alpha)|^2\Gamma^2(2-\alpha)b_n^2 - 16\kappa^2 a_n^2 C(T,\alpha)} < \frac{B^\infty}{16\kappa^2 C(T,\alpha)}.$

Then, for each

$$\lambda\in\left]\frac{16C(T,\alpha)}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|B^\infty}, \frac{1}{\kappa^2T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|\mathcal{A}_\infty}\right[,$$

problem (1.1) *admits an unbounded sequence of solutions in E_0^α.*

Proof. We want to apply Theorem 2.13 to problem (1.1). For this, we define the functionals Φ,Ψ : $X\to\mathbb{R}$ by

$$\Phi(u) := -\int_0^T {}_0^cD_t^\alpha u(t)\cdot {}_t^cD_T^\alpha u(t)\,dt, \qquad \Psi(u) := \int_0^T F(t,u(t))\,dt,$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

for every $u\in X := E_0^\alpha$.

Obviously, Φ and Ψ are Gâteaux differentiable functionals whose derivatives at $u\in E_0^\alpha$ are

$$\Phi'(u)(v) = -\int_0^T ({}_0^cD_t^\alpha u(t)\cdot {}_t^cD_T^\alpha v(t) + {}_t^cD_T^\alpha u(t)\cdot {}_0^cD_t^\alpha v(t))\,dt,$$

$$\Psi'(u)(v) = \int_0^T f(t, u(t))v(t)\,dt = -\int_0^T \int_0^t f(s, u(s))\,ds \cdot v'(t)\,dt,$$

for every $v \in E_0^\alpha$. By Definition 2.3 and (2.1), we have

$$\Phi'(u)(v) = \int_0^T ({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))) \cdot v'(t)\,dt.$$

Hence, $I_\lambda = \Phi - \lambda\Psi \in C^1(E_0^\alpha, \mathbb{R})$ and Φ and Ψ are sequentially weakly lower and upper semicontinuous, respectively.

Also by applying Lemma 2.12, we deduce that the functional Φ is coercive. Indeed, we have

$$\Phi(u) \geq |\cos(\pi\alpha)|\|u\|_\alpha^2 \to +\infty,$$

as $\|u\|_\alpha \to +\infty$.

Further, we prove that a critical point of I_λ is a solution of (1.1). For this, if $u_* \in E_0^\alpha$ is a critical point of I_λ, then

$$\begin{aligned} 0 = I'_\lambda(u_*)(v) = \int_0^T \Big(& {}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u_*(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u_*(t)) \\ & + \lambda \int_0^t f(s, u_*(s))\,ds \Big) \cdot v'(t)\,dt, \end{aligned} \tag{3.1}$$

for every $v \in E_0^\alpha$. The Du Bois-Reymond Lemma and (3.1) imply

$${}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u_*(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u_*(t)) + \lambda \int_0^t f(s, u_*(s))\,ds = m \tag{3.2}$$

a.e. on $[0, T]$ for some $m \in \mathbb{R}$. By (3.2), it is obvious to prove that $u_* \in E_0^\alpha$ is a solution of (1.1).

By Lemma 2.11, when $\alpha > 1/2$, for each $u \in E_0^\alpha$ one has

$$\|u\|_\infty \leq \kappa \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,dt \right)^{1/2} = \kappa\|u\|_\alpha. \tag{3.3}$$

First we establish that $\lambda < 1/\gamma$, for any fixed λ as in the conclusion. For this, put

$$r_n := \frac{|\cos(\pi\alpha)|}{\kappa^2} b_n^2, \qquad \forall\, n \in \mathbb{N}. \tag{3.4}$$

Then, for all $u \in E_0^\alpha$ with $\Phi(u) < r_n$, by applying Lemma 2.12, we see that

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \leq \Phi(u) < r_n,$$

which implies

$$\|u\|_\alpha^2 < \frac{r_n}{|\cos(\pi\alpha)|}. \tag{3.5}$$

Thus, by (3.3)–(3.5) we obtain

$$\|u\|_\infty \leq b_n, \qquad (\forall\, n \in \mathbb{N})$$

for any $u \in E_0^\alpha$ with the condition $\Phi(u) < r_n$. Then, for every $n \in \mathbb{N}$, we get that

$$\varphi(r_n) \le \inf_{\Phi(u)<r_n} \frac{\displaystyle\int_0^T \max_{|\xi|\le b_n} F(t,\xi)\,dt - \int_0^T F(t,u(t))\,dt}{\dfrac{|\cos(\pi\alpha)|}{\kappa^2} b_n^2 + \displaystyle\int_0^T {}_0^cD_t^\alpha u(t)\cdot {}_t^cD_T^\alpha u(t)\,dt}.$$

Let w_n be defined by

$$w_n(t) := \begin{cases} \dfrac{4a_n}{T} t & t \in [0, T/4) \\ a_n & t \in [T/4, 3T/4] \\ \dfrac{4a_n}{T}(T-t) & t \in (3T/4, T] \end{cases}$$

for each $n \in \mathbb{N}$.

Clearly, we can investigate that $w_n(0) = w_n(T) = 0$ and $w_n \in L^2([0, T])$. Moreover, w_n is Lipschitz continuous on $[0, T]$, and therefore w_n is absolutely continuous on $[0, T]$. We have

$${}_0^cD_t^\alpha w_n(t) = \begin{cases} \dfrac{4a_n}{T\Gamma(2-\alpha)} t^{1-\alpha} & t \in [0, T/4) \\ \dfrac{4a_n}{T\Gamma(2-\alpha)}\Big[t^{1-\alpha} - \Big(t - \dfrac{T}{4}\Big)^{1-\alpha}\Big] & t \in [T/4, 3T/4] \\ \dfrac{4a_n}{T\Gamma(2-\alpha)}\Big[t^{1-\alpha} - \Big(t - \dfrac{T}{4}\Big)^{1-\alpha} - \Big(t - \dfrac{3T}{4}\Big)^{1-\alpha}\Big] & t \in (3T/4, T]. \end{cases}$$

Obviously, the function ${}_0^cD_t^\alpha w_n$ is continuous in $[0, T]$, and

$$\begin{aligned} \int_0^T |{}_0^cD_t^\alpha w_n(t)|^2\,dt &= \frac{16a_n^2}{T^2\Gamma^2(2-\alpha)}\Big\{\int_0^{T/4} t^{2-2\alpha}\,dt + \int_{T/4}^{3T/4}\Big[t^{1-\alpha} - \Big(t-\frac{T}{4}\Big)^{1-\alpha}\Big]^2 dt \\ &\quad + \int_{3T/4}^{T}\Big[t^{1-\alpha} - \Big(t-\frac{T}{4}\Big)^{1-\alpha} - \Big(t-\frac{3T}{4}\Big)^{1-\alpha}\Big]^2 dt\Big\} \\ &= \frac{16a_n^2}{T^2\Gamma^2(2-\alpha)} C(T,\alpha). \end{aligned}$$

Therefore,

$$\Phi(w_n) \le \frac{1}{|\cos(\pi\alpha)|}\|w_n\|_\alpha^2 = \frac{16a_n^2}{T^2|\cos(\pi\alpha)|\Gamma^2(2-\alpha)} C(T,\alpha).$$

Hence, by (h_1), one has $\Phi(w_n) < r_n$ for all $n \ge n_0$. Moreover, by (f_1), we also have

$$\Psi(w_n) \ge \int_{T/4}^{3T/4} F(t, a_n)\,dt,$$

for each $n \in \mathbb{N}$.

Then, it follows that

$$\varphi(r_n) \le \frac{\displaystyle\int_0^T \max_{|\xi|\le b_n} F(t,\xi)\,dt - \int_{T/4}^{3T/4} F(t,a_n)\,dt}{\dfrac{|\cos(\pi\alpha)|}{\kappa^2} b_n^2 - \dfrac{16C(T,\alpha)a_n^2}{T^2|\cos(\pi\alpha)|\Gamma^2(2-\alpha)}},$$

for every $n \geq n_0$.

Hence, by the hypothesis (h_2), we get that

$$0 \leq \gamma \leq \lim_{n\to+\infty} \varphi(r_n) \leq \kappa^2 T^2 \Gamma^2(2-\alpha)|\cos(\pi\alpha)|\mathcal{A}_\infty < +\infty.$$

By the above relation, since

$$\lambda < \frac{1}{\kappa^2 T^2 \Gamma^2(2-\alpha)|\cos(\pi\alpha)|\mathcal{A}_\infty},$$

we also have $\lambda < 1/\gamma$.

Now, we claim that I_λ is unbounded from below. By the relation

$$\frac{1}{\lambda} < \frac{T^2|\cos(\pi\alpha)|\Gamma^2(2-\alpha)B^\infty}{16C(T,\alpha)},$$

taking into account the definition of B^∞, there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ with the conditions $\lim_{n\to+\infty} \eta_n = +\infty$ and

$$\frac{1}{\lambda} < \tau < \frac{T^2|\cos(\pi\alpha)|\Gamma^2(2-\alpha)}{16C(T,\alpha)} \frac{\int_{T/4}^{3T/4} F(t,\eta_n)\,dt}{\eta_n^2},$$

for every $n \in \mathbb{N}$ large enough.

For every $n \in \mathbb{N}$, suppose that $s_n \in X$ is defined by

$$s_n(t) := \begin{cases} \dfrac{4\eta_n}{T} t & t \in [0, T/4) \\ \eta_n & t \in [T/4, 3T/4] \\ \dfrac{4\eta_n}{T}(T-t) & t \in (3T/4, T]. \end{cases}$$

Thus, we obtain

$$\begin{aligned} I_\lambda(s_n) &= \Phi(s_n) - \lambda\Psi(s_n) \\ &\leq \frac{16C(T,\alpha)}{T^2|\cos(\pi\alpha)|\Gamma^2(2-\alpha)}\eta_n^2 - \lambda \int_{T/4}^{3T/4} F(t,\eta_n)\,dt \\ &< \frac{16C(T,\alpha)}{T^2|\cos(\pi\alpha)|\Gamma^2(2-\alpha)}\eta_n^2(1-\lambda\tau), \end{aligned}$$

for all $n \in \mathbb{N}$ large enough. By the relation $\lambda\tau > 1$ and $\lim_{n\to+\infty} \eta_n = +\infty$, we have

$$\lim_{n\to+\infty} I_\lambda(s_n) = -\infty.$$

Therefore, I_λ is unbounded from below, and so, we get that I_λ has no global minimum. Then, by applying Theorem 2.13, part (b), there exists a sequence $\{u_n\}$ of critical points of I_λ with

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

So by Lemma 2.12, we have

$$\lim_{n\to+\infty} \|u_n\|_\alpha = +\infty.$$

The proof is complete. □

Put

$$B^0 := \limsup_{\xi \to 0^+} \frac{\int_{T/4}^{3T/4} F(t,\xi)\,dt}{\xi^2}.$$

Discussing as in the proof of Theorem 3.1 and exploiting part (c) of Theorem 2.13, we arrive the following.

Theorem 3.2. *Suppose that* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function whose potential satisfies* (f_1). *Assume that there exist real sequences* $\{c_n\}$ *and* $\{d_n\}$ *in* $(0,+\infty)$*, with* $\lim_{n\to+\infty} d_n = 0$*, such that*:

(h_3) *For some* $n_0 \in \mathbb{N}$ *we have* $c_n < \dfrac{T|\cos(\pi\alpha)|\Gamma(2-\alpha)}{4\kappa\sqrt{C(T,\alpha)}} d_n$ *for each* $n \geq n_0$;

(h_4) $\mathcal{A}_0 := \displaystyle\lim_{n\to+\infty} \frac{\int_0^T \max_{|\xi|\leq d_n} F(t,\xi)\,dt - \int_{T/4}^{3T/4} F(t,c_n)\,dt}{T^2|\cos(\pi\alpha)|^2\Gamma^2(2-\alpha)d_n^2 - 16\kappa^2 c_n^2 C(T,\alpha)} < \frac{B^0}{16\kappa^2 C(T,\alpha)}.$

Then, for each

$$\lambda \in \left] \frac{16C(T,\alpha)}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|B^0}, \frac{1}{\kappa^2 T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|\mathcal{A}_0} \right[,$$

problem (1.1) *has a sequence of non-zero solutions which strongly converges to zero in* E_0^α.

At the present, we state some remarkable consequences of Theorem 3.1. Let

$$A_\infty := \liminf_{\xi \to +\infty} \frac{\int_0^T \max_{|x|\leq\xi} F(t,x)\,dt}{\xi^2}.$$

Corollary 3.3. *Suppose that* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function whose potential satisfies* (f_1). *Assume that*

(h_5) $A_\infty < \dfrac{T^2|\cos(\pi\alpha)|^2\Gamma^2(2-\alpha)}{16\kappa^2 C(T,\alpha)} B^\infty.$

Then, for each

$$\lambda \in \left] \frac{16C(T,\alpha)}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|B^\infty}, \frac{|\cos(\pi\alpha)|}{\kappa^2 A_\infty} \right[,$$

problem (1.1) *has an unbounded sequence of solutions in* E_0^α.

Proof. Assume that $\{b_n\}$ be a sequence of positive numbers, with $\lim_{n\to+\infty} b_n = +\infty$, such that

$$\lim_{n\to+\infty} \frac{\int_0^T \max_{|\xi|\leq b_n} F(t,\xi)\,dt}{b_n^2} = A_\infty.$$

Taking $a_n = 0$ for all $n \geq n_0$, by applying Theorem 3.1, we have the outcome. □

A specific case of Corollary 3.3 is the following.

Corollary 3.4. *Assume that $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function whose potential satisfies* (f_1). *Assume that*

(h_6) $A_\infty < \dfrac{|\cos(\pi\alpha)|}{\kappa^2}$ *and* $B^\infty > \dfrac{16C(T,\alpha)}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|}$.

Then, the problem

$$\begin{cases} \dfrac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\Big) + f(t,u(t)) = 0 \quad \text{a.e. } t\in[0,T] \\ u(0)=u(T)=0, \end{cases}$$

has an unbounded sequence of solutions in E_0^α.

Remark 3.5. We point out that when f is a nonnegative function, hypothesis (f_1) preserves and assumption (h_5) becomes

$$(h_5')\ A'_\infty := \liminf_{\xi\to+\infty}\frac{\int_0^T F(t,\xi)\,dt}{\xi^2} < \frac{T^2|\cos(\pi\alpha)|^2\Gamma^2(2-\alpha)}{16\kappa^2 C(T,\alpha)}B^\infty.$$

In this occasion, (h_5') ensures that for all

$$\lambda\in\left]\frac{16C(T,\alpha)}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|B^\infty},\frac{|\cos(\pi\alpha)|}{\kappa^2A'_\infty}\right[,$$

problem (1.1) has an unbounded sequence of solutions in E_0^α.

Corollary 3.6. *Assume that $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function whose potential satisfies* (f_1). *Assume that there exist real sequences $\{a_n\}$ and $\{b_n\}$ in $(0,+\infty)$, with $\lim_{n\to+\infty} b_n = +\infty$, such that* ($h_1$) *holds and*

(h_7) $\displaystyle\int_{T/4}^{3T/4} F(t,a_n)\,dt = \int_0^T \max_{|\xi|\le b_n} F(t,\xi)\,dt$ *for all* $n\in\mathbb{N}$.

If $B^\infty>0$, then, for all

$$\lambda > \frac{16C(T,\alpha)}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|B^\infty},$$

problem (1.1) *has an unbounded sequence of solutions in* E_0^α.

Proof. By (h_7) we get $\mathcal{A}_\infty = 0$. Therefore, since $B^\infty>0$, condition (h_2) of Theorem 3.1 holds and the result is obtained. □

Remark 3.7. By Theorem 3.2 we get the identical conclusions of Theorem 3.1. Namely, substituting $\xi\to+\infty$ with $\xi\to0^+$, assertions such as Corollaries 3.3, 3.4 and 3.6 can be established. We omit the details.

To conclude, we give an example of application of the main results.

Example 3.8. Consider the problem

$$\begin{cases} \dfrac{d}{dt}\Big({}_0D_t^{-0.3}({}_0^cD_t^{0.7}u(t)) - {}_tD_1^{-0.3}({}_t^cD_1^{0.7}u(t))\Big) + \lambda f(u(t)) = 0 & \text{a.e. } t \in [0,1] \\ u(0) = u(1) = 0, \end{cases} \tag{3.6}$$

where $f : \mathbb{R} \to \mathbb{R}$ is the continuous function defined by

$$f(x) := \begin{cases} x\,(2 - \cos(\ln|x|) - 2\sin(\ln|x|)) & x \in \mathbb{R} \setminus \{0\} \\ 0 & x = 0. \end{cases}$$

A direct calculation shows

$$F(x) = \begin{cases} x^2\,(1 - \sin(\ln|x|)) & x \in \mathbb{R} \setminus \{0\} \\ 0 & x = 0. \end{cases}$$

So, F satisfies (f_1) and we have

$$A_\infty = \liminf_{\xi \to +\infty}(1 - \sin(\ln|\xi|)) = 0, \qquad B^\infty = \limsup_{\xi \to +\infty} \frac{1}{2}(1 - \sin(\ln|\xi|)) = 1,$$

$$|\cos(0.7\pi)| \approx 0.58779, \qquad C(1, 0.7) \approx 0.13429, \qquad \Gamma^2(1.3) \approx 0.805454,$$

$$\frac{16 * 0.13429}{0.805454 * 0.58779} = 4.5384.$$

The above calculations are done using MAPLE. Hence, using Corollary 3.3, for each $\lambda \in]4.5384, +\infty[$, problem (3.6) has an unbounded sequence of solutions in $E_0^{0.7}$.

4. Conclusions

Taking advantage of a critical point theorem obtained by Ricceri [28], the existence of infinitely many solutions for a nonlinear fractional BVP with a parameter is established. More precisely, a concrete interval of positive parameters, for which the treated problem admits infinitely many solutions, is determined without any symmetry or monotonicity assumptions on the nonlinear data. Our goal was achieved by requiring an appropriate oscillatory behavior of the nonlinear term either at infinity or at zero, without any additional conditions.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

1. G. A. Afrouzi, A. Hadjian, A variational approach for boundary value problems for impulsive fractional differential equations, *Fract. Calc. Appl. Anal.*, **21** (2018), 1565–1584. https://doi.org/10.1515/fca-2018-0082
2. G. A. Afrouzi, S. M. Kolagar, A. Hadjian, J. Xu, A variational approach for fractional boundary value systems depending on two parameters, *Filomat*, **32** (2018), 517–530. https://doi.org/10.2298/FIL1802517A
3. D. Averna, A. Sciammetta, E. Tornatore, Infinitely many solutions to boundary value problem for fractional differential equations, *Fract. Calc. Appl. Anal.*, **21** (2018), 1585–1597. https://doi.org/10.1515/fca-2018-0083
4. D. Averna, S. Tersian, E. Tornatore, On the existence and multiplicity of solutions for Dirichlet's problem for fractional differential equations, *Fract. Calc. Appl. Anal.*, **19** (2016), 253–266. https://doi.org/10.1515/fca-2016-0014
5. C. Bai, Existence of solutions for a nonlinear fractional boundary value problem via a local minimum theorem, *Electron. J. Differ. Eq.*, **2012** (2012), 176.
6. A. S. Berdyshev, B. J. Kadirkulov, J. J. Nieto, Solvability of an elliptic partial differential equation with boundary condition involving fractional derivatives, *Complex Var. Elliptic Equ.*, **59** (2014), 680–692. https://doi.org/10.1080/17476933.2013.777711
7. G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Anal.*, **75** (2012), 2992–3007. https://doi.org/10.1016/j.na.2011.12.003
8. G. Bonanno, G. Molica Bisci, Infinitely many solutions for a Dirichlet problem involving the p-Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A*, **140** (2010), 737–752. https://doi.org/10.1017/S0308210509000845
9. J. F. Bonder, Z. Cheng, H. Mikayelyan, Optimal rearrangement problem and normalized obstacle problem in the fractional setting, *Adv. Nonlinear Anal.*, **9** (2020), 1592–1606. https://doi.org/10.1515/anona-2020-0067
10. M. J. Ceballos-Lira, A. Pérez, Global solutions and blowing-up solutions for a nonautonomous and nonlocal in space reaction-diffusion system with Dirichlet boundary conditions, *Fract. Calc. Appl. Anal.*, **23** (2020), 1025–1053. https://doi.org/10.1515/fca-2020-0054
11. P. Chen, Z. Cao, S. Chen, X. Tang, Ground states for a fractional reaction-diffusion system, *J. Appl. Anal. Comput.*, **11** (2021), 556–567. https://doi.org/10.11948/20200349
12. J. N. Corvellec, V. V. Motreanu, C. Saccon, Doubly resonant semilinear elliptic problems via nonsmooth critical point theory, *J. Differ. Equations*, **248** (2010), 2064–2091. https://doi.org/10.1016/j.jde.2009.11.005
13. M. Ferrara, A. Hadjian, Variational approach to fractional boundary value problems with two control parameters, *Electron. J. Differ. Eq.*, **2015** (2015), 138.

14. M. Galewski, G. Molica Bisci, Existence results for one-dimensional fractional equations, *Math. Methods Appl. Sci.*, **39** (2016), 1480–1492. https://doi.org/10.1002/mma.3582

15. B. Ge, V. Rădulescu, J. C. Zhang, Infinitely many positive solutions of fractional boundary value problems, *Topol. Methods Nonlinear Anal.*, **49** (2017), 647–664. https://doi.org/10.12775/TMNA.2017.001

16. R. Hilfer, *Applications of fractional calculus in physics*, Singapore: World Scientific, 2000. https://doi.org/10.1142/3779

17. F. Jiao, Y. Zhou, Existence of solutions for a class of fractional boundary value problems via critical point theory, *Comput. Math. Appl.*, **62** (2011), 1181–1199. https://doi.org/10.1016/j.camwa.2011.03.086

18. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, In: *North-holland mathematics studies*, Vol. 204, Amsterdam: Elsevier, 2006.

19. A. Kristály, V. D. Rădulescu, C. G. Varga, *Variational principles in mathematical physics, geometry, and economics: Qualitative analysis of nonlinear equations and unilateral problems*, Encyclopedia of Mathematics and its Applications, No. 136, Cambridge: Cambridge University Press, 2010. https://doi.org/10.1017/CBO9780511760631

20. V. Lakshmikantham, A. S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.*, **69** (2008), 2677–2682. https://doi.org/10.1016/j.na.2007.08.042

21. F. Li, Z. Liang, Q. Zhang, Existence of solutions to a class of nonlinear second order two-point boundary value problems, *J. Math. Anal. Appl.*, **312** (2005), 357–373. https://doi.org/10.1016/j.jmaa.2005.03.043

22. J. Manimaran, L. Shangerganesh, A. Debbouche, Finite element error analysis of a time-fractional nonlocal diffusion equation with the Dirichlet energy, *J. Comput. Appl. Math.*, **382** (2021), 113066. https://doi.org/10.1016/j.cam.2020.113066

23. J. Mawhin, M. Willem, *Critical point theorey and Hamiltonian systems*, New York: Springer, 1989. https://doi.org/10.1007/978-1-4757-2061-7

24. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: Wiley, 1993.

25. T. Mukherjee, K. Sreenadh, On Dirichlet problem for fractional p-Laplacian with singular non-linearity, *Adv. Nonlinear Anal.*, **8** (2019), 52–72. https://doi.org/10.1515/anona-2016-0100

26. I. Podlubny, Fractional differential equations, In: *Mathematics in science and engineering*, Vol. 198, New York: Academic Press, 1999.

27. P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, Rhode Island: American Mathematical Society, 1986.

28. B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.*, **113** (2000), 401–410. https://doi.org/10.1016/S0377-0427(99)00269-1

29. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Switzerland: Gordon and Breach, 1993.

30. C. L. Tang, X. P. Wu, Some critical point theorems and their applications to periodic solution for second order Hamiltonian systems, *J. Differ. Equations*, **248** (2010), 660–692. https://doi.org/10.1016/j.jde.2009.11.007
31. Y. Tian, J. J. Nieto, The applications of critical-point theory to discontinuous fractional-order differential equations, *Proc. Edinburgh Math. Soc.*, **60** (2017), 1021–1051. https://doi.org/10.1017/S001309151600050X
32. M. P. Tran, T. N. Nguyen, New gradient estimates for solutions to quasilinear divergence form elliptic equations with general Dirichlet boundary data, *J. Differ. Equations*, **268** (2020), 1427–1462. https://doi.org/10.1016/j.jde.2019.08.052
33. M. Xiang, B. Zhang, V. D. Rădulescu, Superlinear Schrödinger-Kirchhoff type problems involving the fractional p-Laplacian and critical exponent, *Adv. Nonlinear Anal.*, **9** (2020), 690–709. https://doi.org/10.1515/anona-2020-0021
34. M. Zhen, B. Zhang, V. D. Rădulescu, Normalized solutions for nonlinear coupled fractional systems: Low and high perturbations in the attractive case, *Discrete Contin. Dyn. Syst.*, **41** (2021), 2653–2676. https://doi.org/10.3934/dcds.2020379